Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics,

a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

The Institute of Mathematical Sciences

IMSc

January-March 2017

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Chapter 7

Heaps in statistical mechanics

(3)

(slides: second part)

q-Bessel functions in physics

IMSc, Chennai

16 March 2017

complements

q-Bessel functions
and SOS models

(Solid on solid)

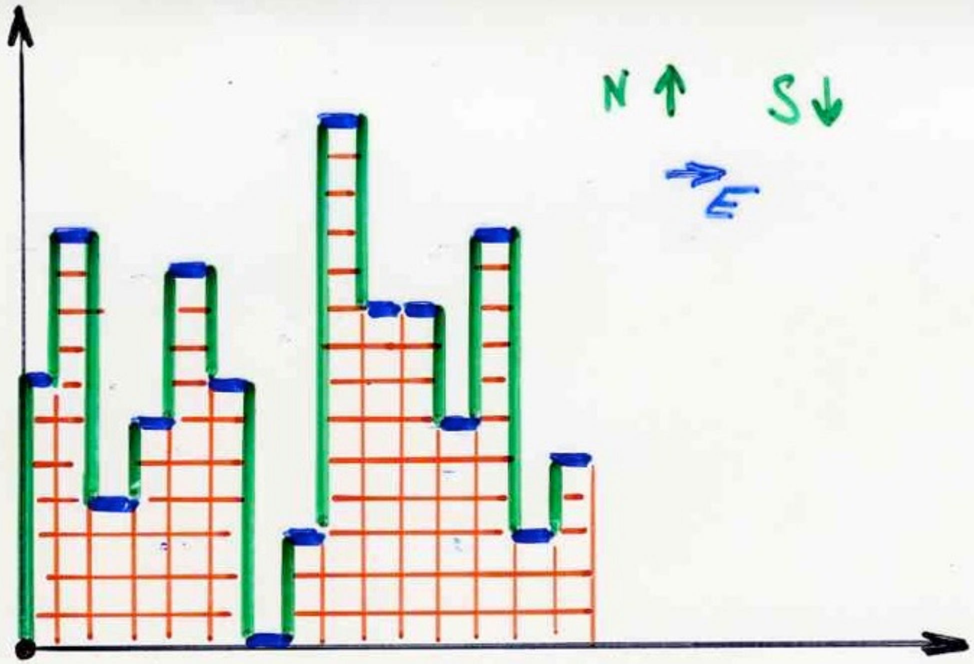
discrete (1+1)-dimensional

SOS model with

- magnetic field
- boundary potential
- surface interactions

exact solution

A. Owczarek, T. Prellberg (1993)



Partially directed
self-avoiding walks
(paths)

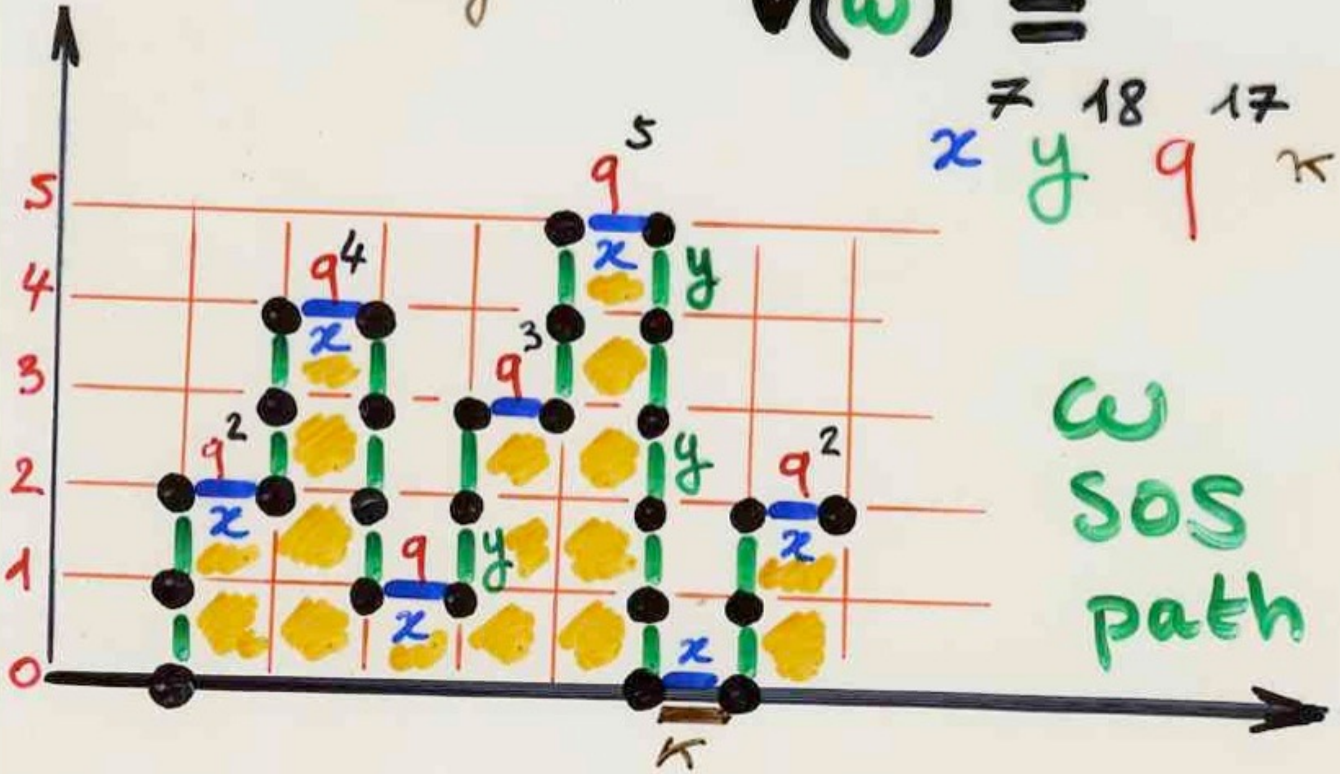
$$G(x,y,q,\kappa) = \sum_{\substack{\omega \\ \text{SOS path}}} v(\omega)$$

<u>weight</u> $\uparrow$ N $\quad \downarrow$ S $\quad y$

level j $\quad \bullet\!\rightarrow\!\bullet$ E $\quad \begin{cases} x q^{j} & \text{if } j>0 \\ x \kappa & \text{if } j=0 \end{cases}$

A. Owczarek, T. Prellberg (1993)

weight $v(\omega) = x^{7} y^{18} q^{17} \kappa$



ω
SOS
path

ou encore :

$$\sum_{\omega} v(\omega) = x \, \frac{H\left(qy^2,\, q,\, x(1-y^2)q\right)}{H\left(y^2,\, q,\, x(1-y^2)\right)}$$

ω chemins SOS

arrivant au niveau 0

Owczarek, Prellberg (1993)

<u>notations</u> : $$H(u,q,t) = \sum_{n \geq 0} \frac{(-t)^n \, q^{\binom{n}{2}}}{(u,q)_n \, (q,q)_n}$$

avec $$(u,q)_n = (u)_n = (1-u)(1-uq)\cdots(1-uq^{n-1})$$

$$J_0 = H(uq, q, xq)$$

$$J_1 = H(uq, q, xq) - H(uq, q, xq^2)$$

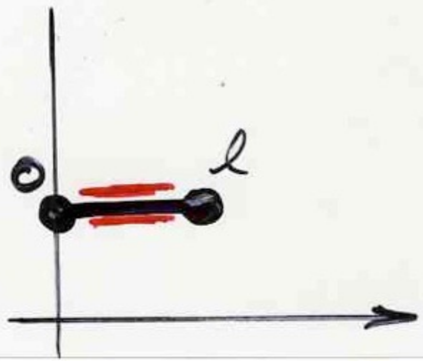
$$\frac{J_1}{J_0} \quad \text{or} \quad \frac{H(uq,\ q,\ xq^2)}{H(uq,\ q,\ xq)}$$

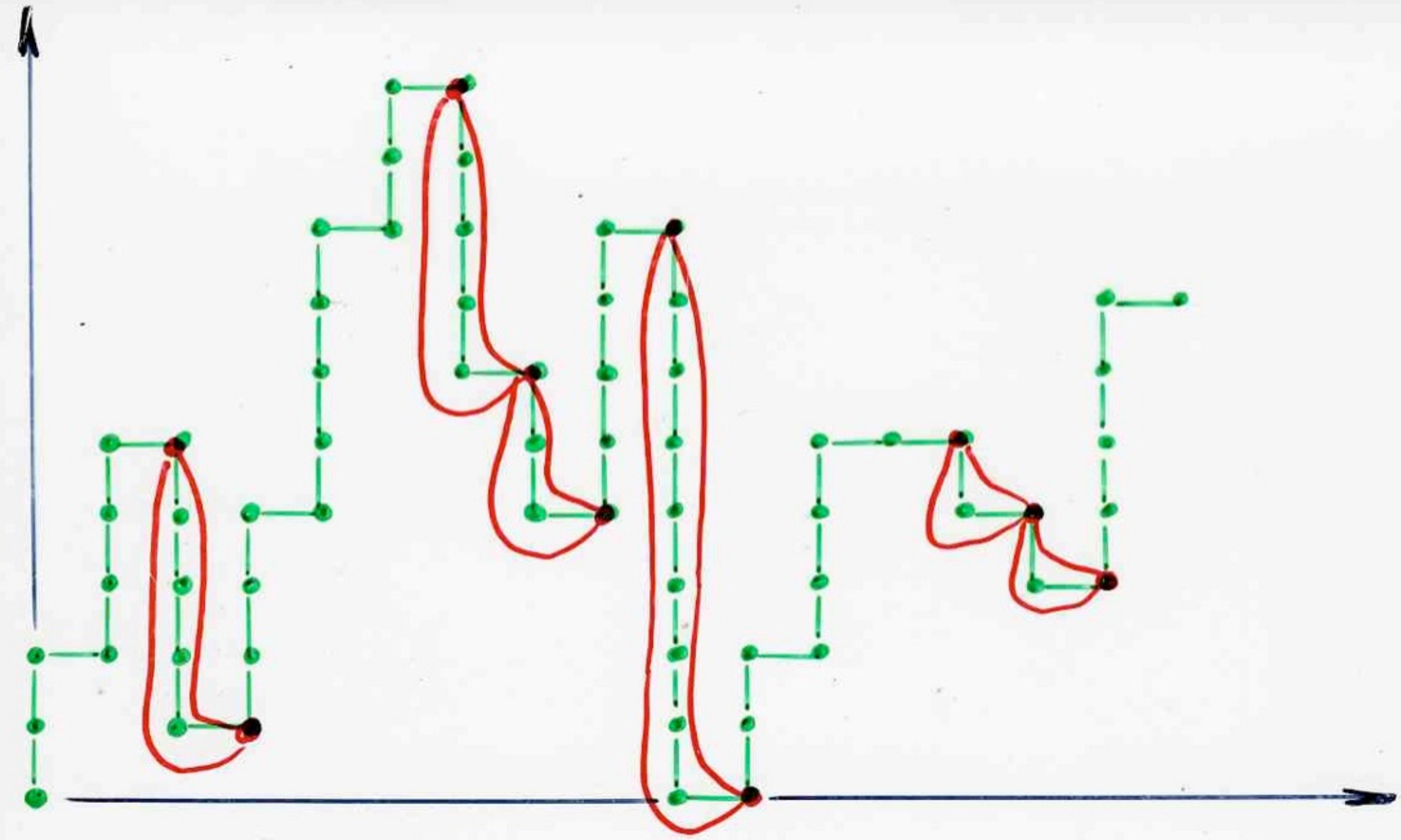
$$\frac{N}{D}$$

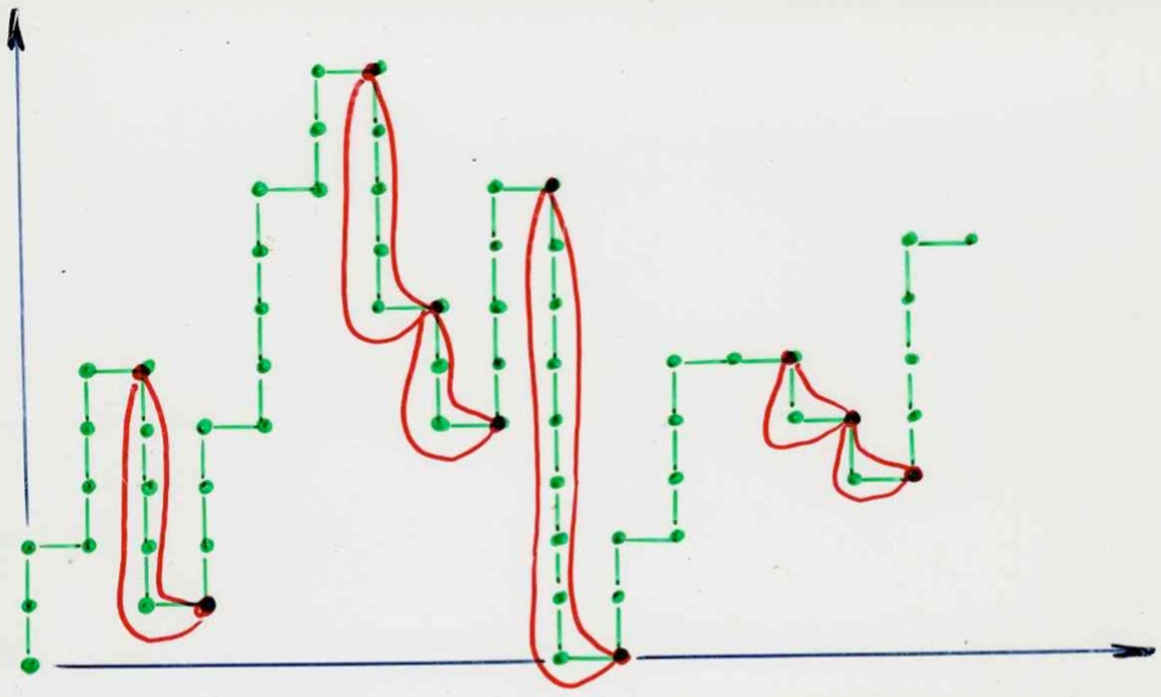
$$= \sum_{P} v(P)$$

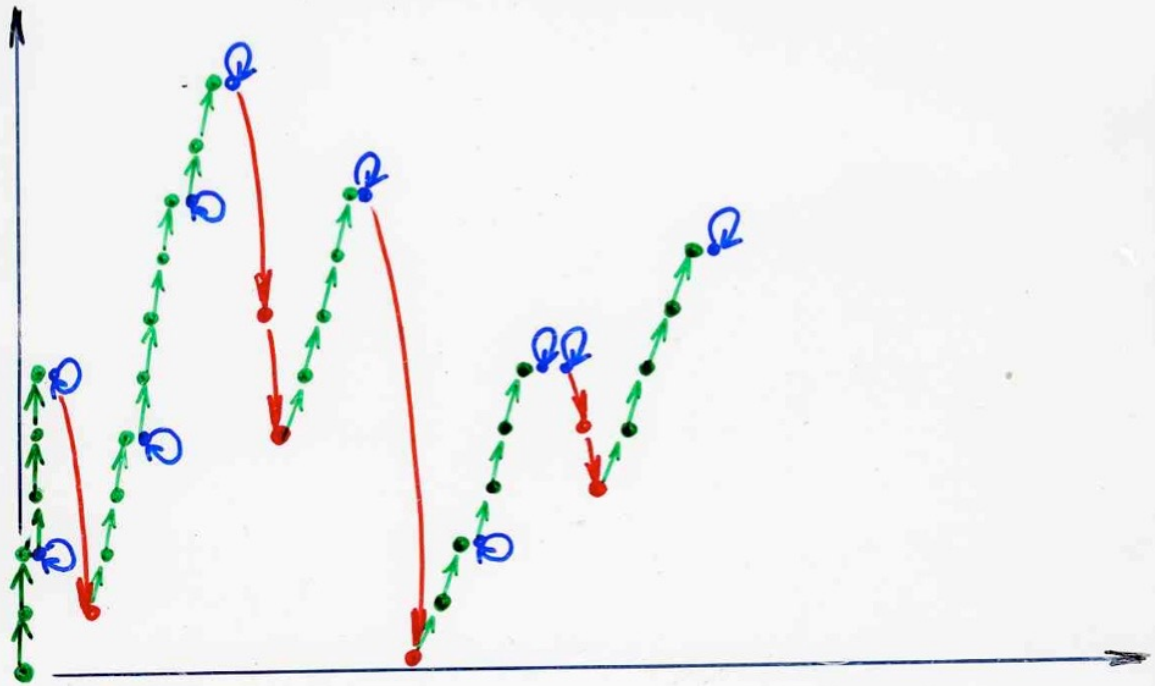
P pyramid

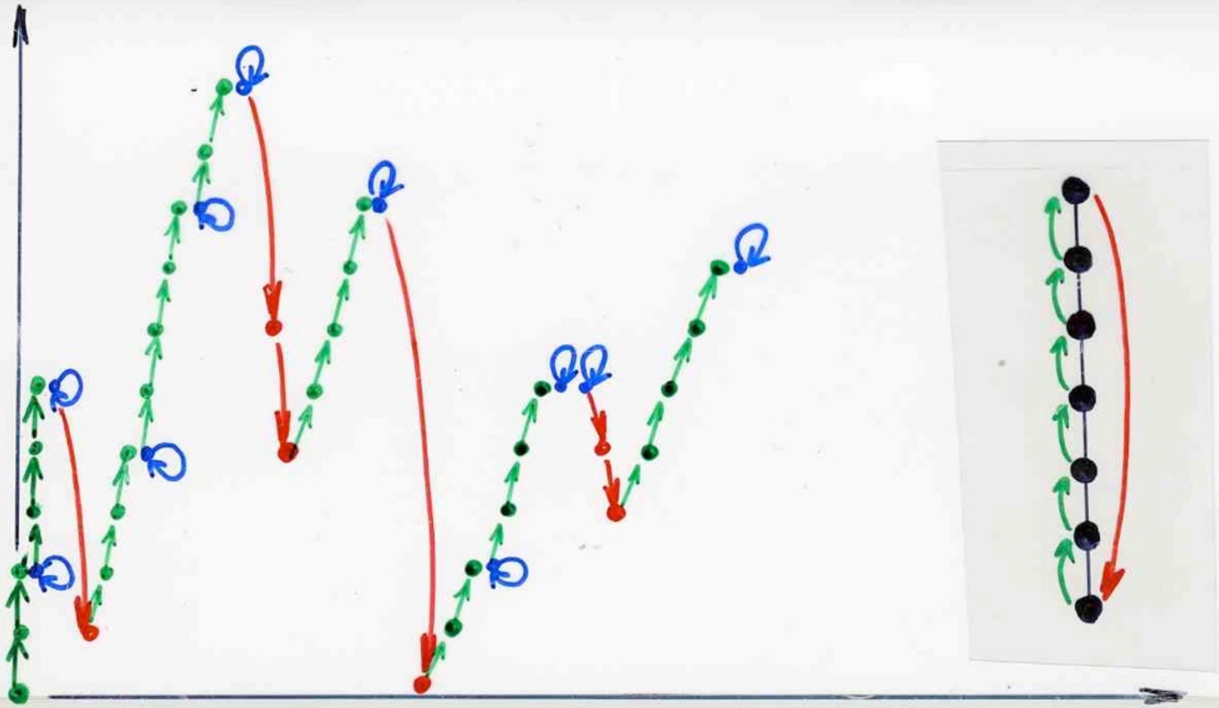
maximal piece is

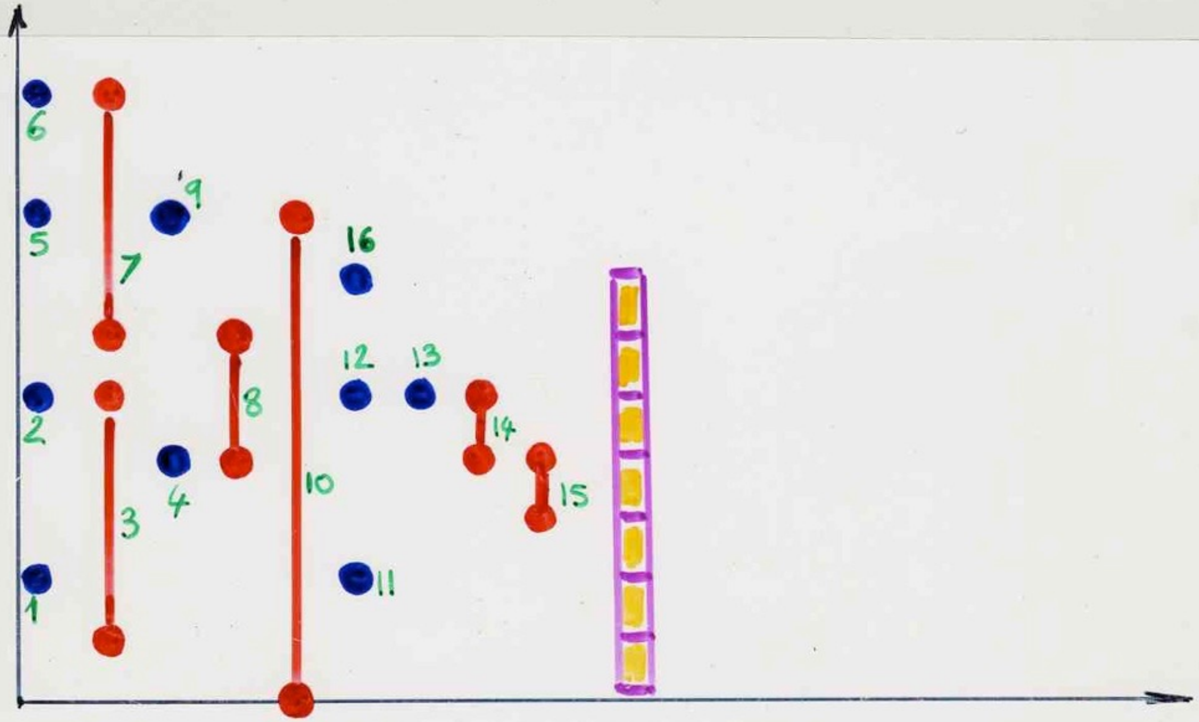












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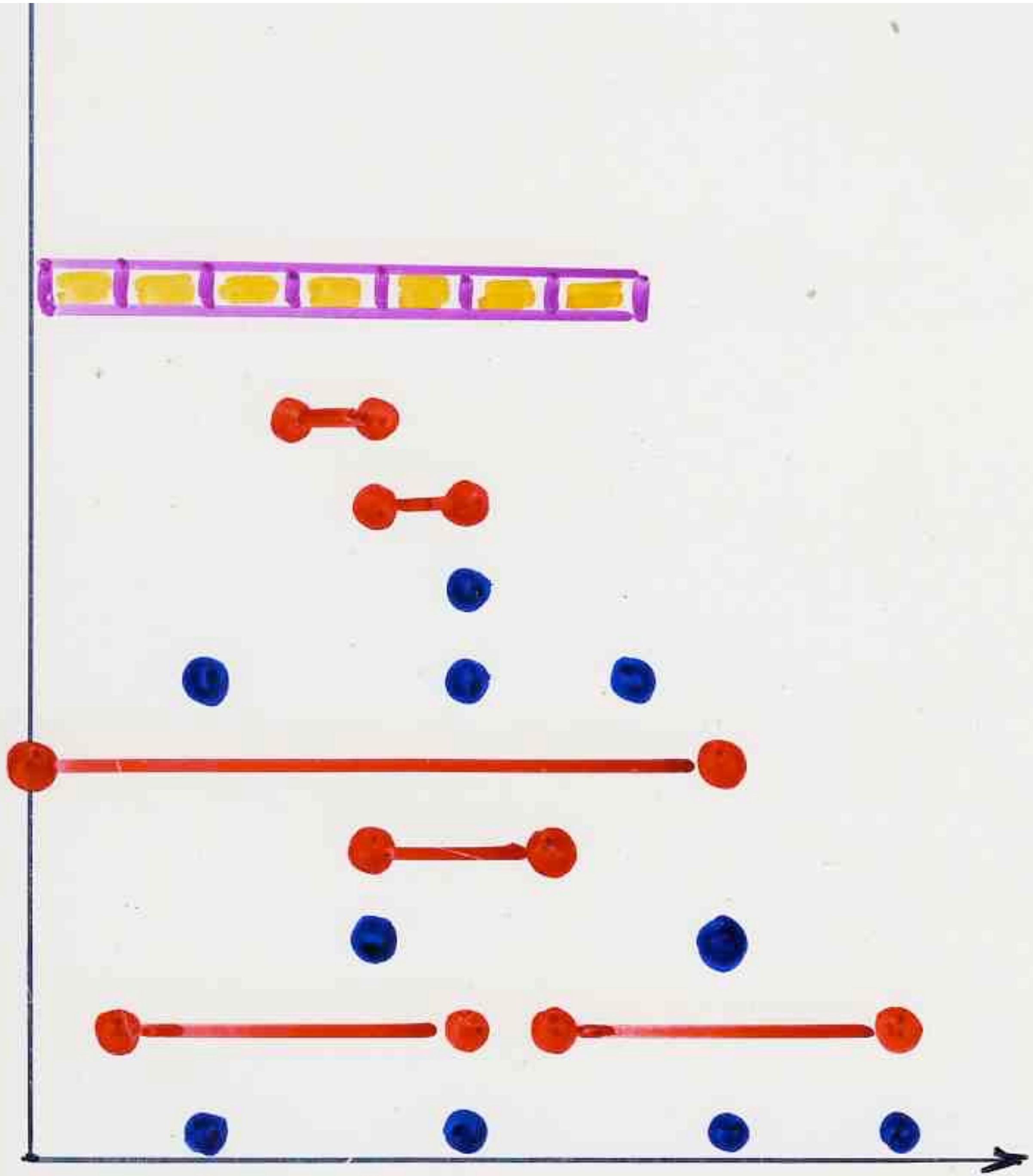
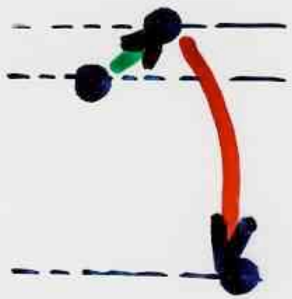


$$v_g([i,j]) = t\,u^{(j-i)}\,q^{i}$$

$0 \le i \le j$

Paths with no **peaks**



$$t \leftarrow x(1-y^2)$$

- q-Bessel

chemins partiellement dirigés avec interaction

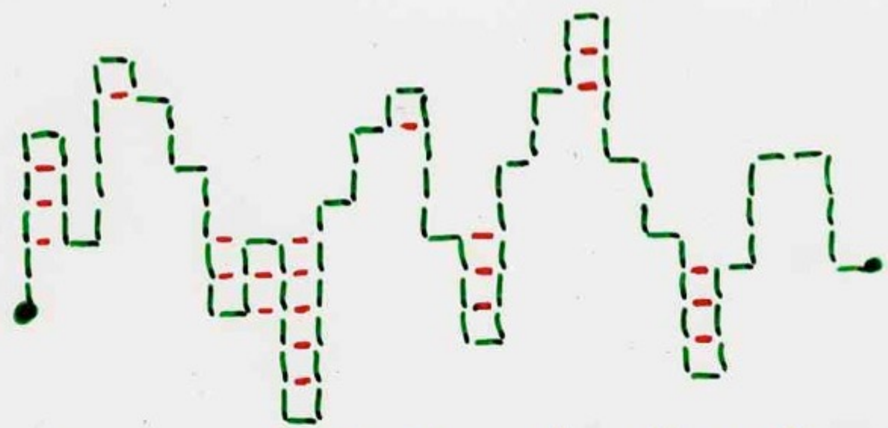
"effondrement" des polymères

Brak, Guttmann, Whittington 1992

Owczarek, Prellberg, Brak 1993

Zwanzig, Lauritzen 1968, 1970

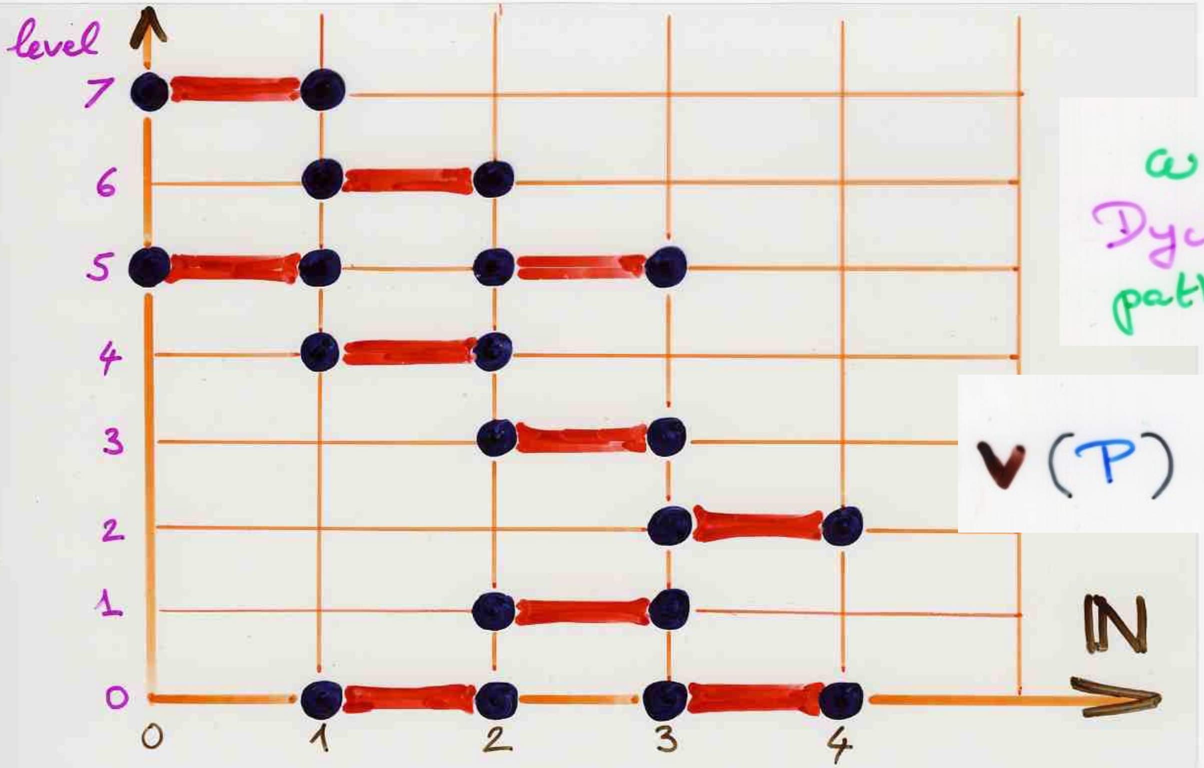
autres familles de polyominos convexes



chemin partiellement dirigé avec interactions

particular case:
heaps of dimers
and
Ramanujan contined fraction

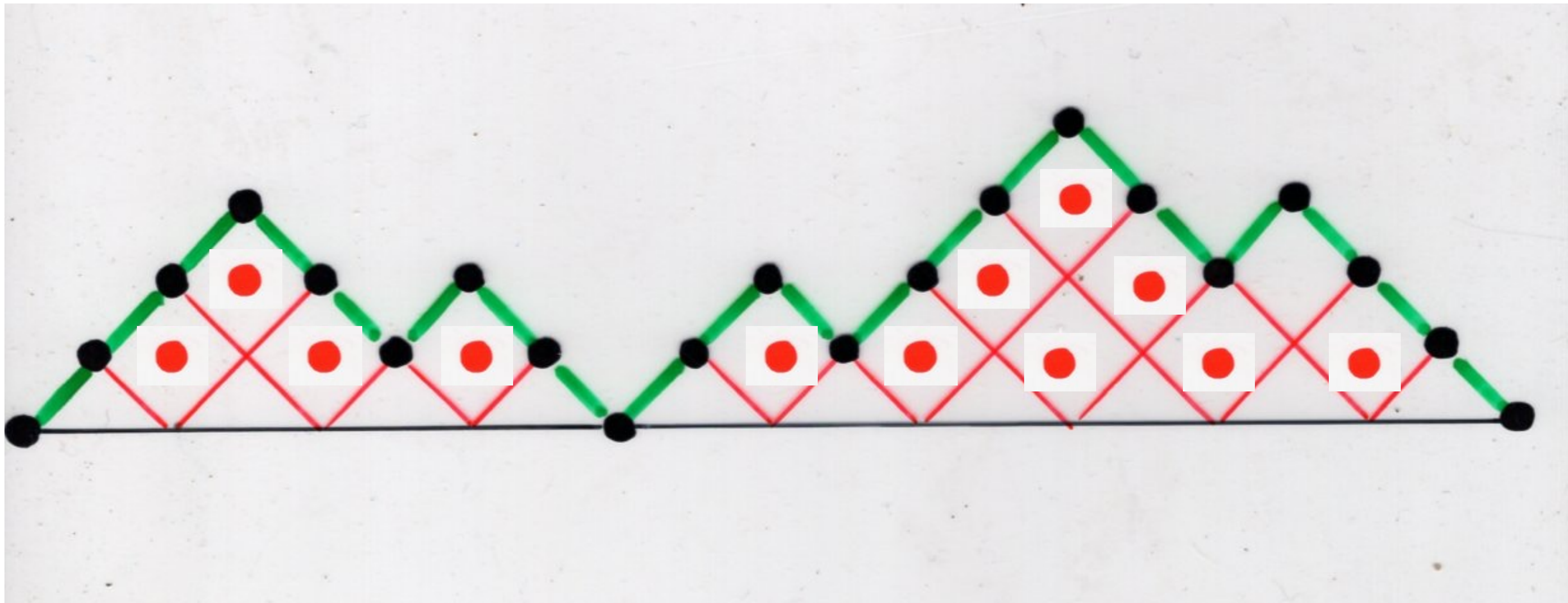
$$v([k-1, k]) = q^k t$$



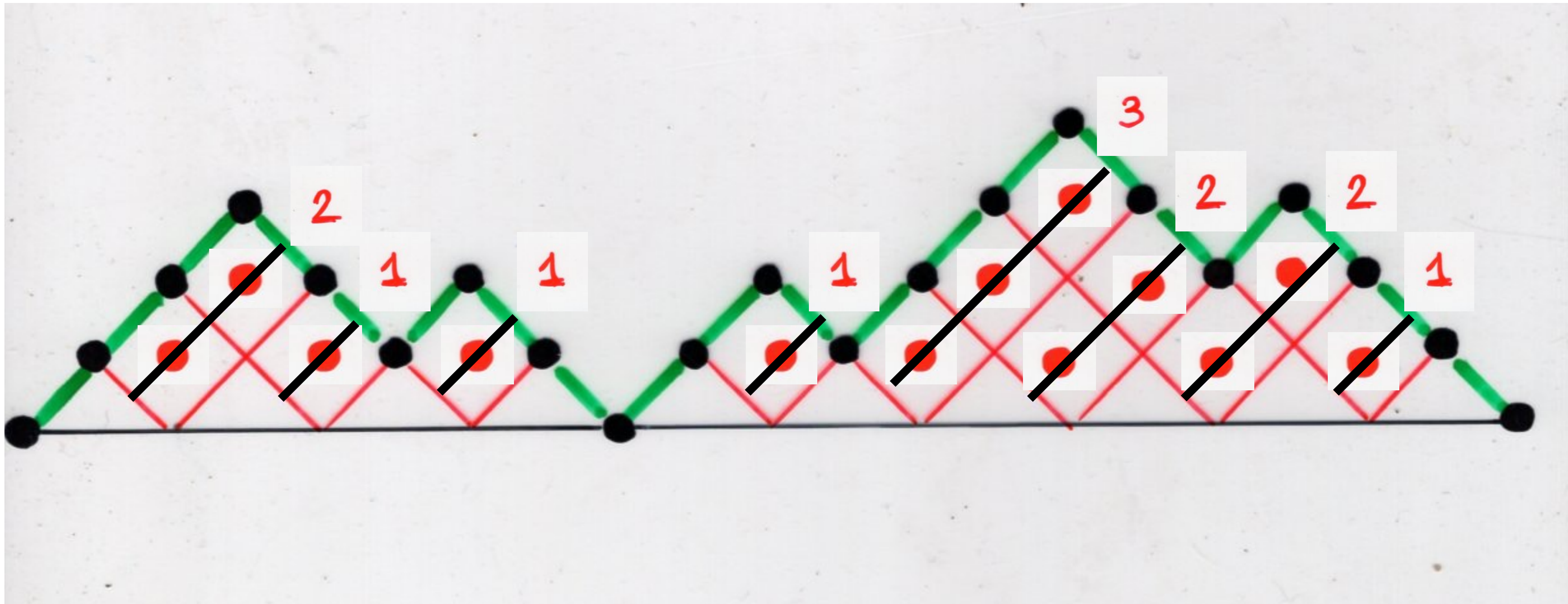
ω Dyck path $\longrightarrow P$ semi-pyramid of dimers on $\mathbb{N}$

$$v(P) = q^{|\omega|/2 + \mathrm{area}(\omega)} t^{|\omega|/2}$$

area = 13



area = 13



$$v([k-1, k]) = q^{k} t$$

ω Dyck path $\longrightarrow$ P semi-pyramid of dimers on $\mathbb{N}$

$$v(P) = q^{|\omega|/2 + \mathrm{area}(\omega)}\, t^{|\omega|/2}$$

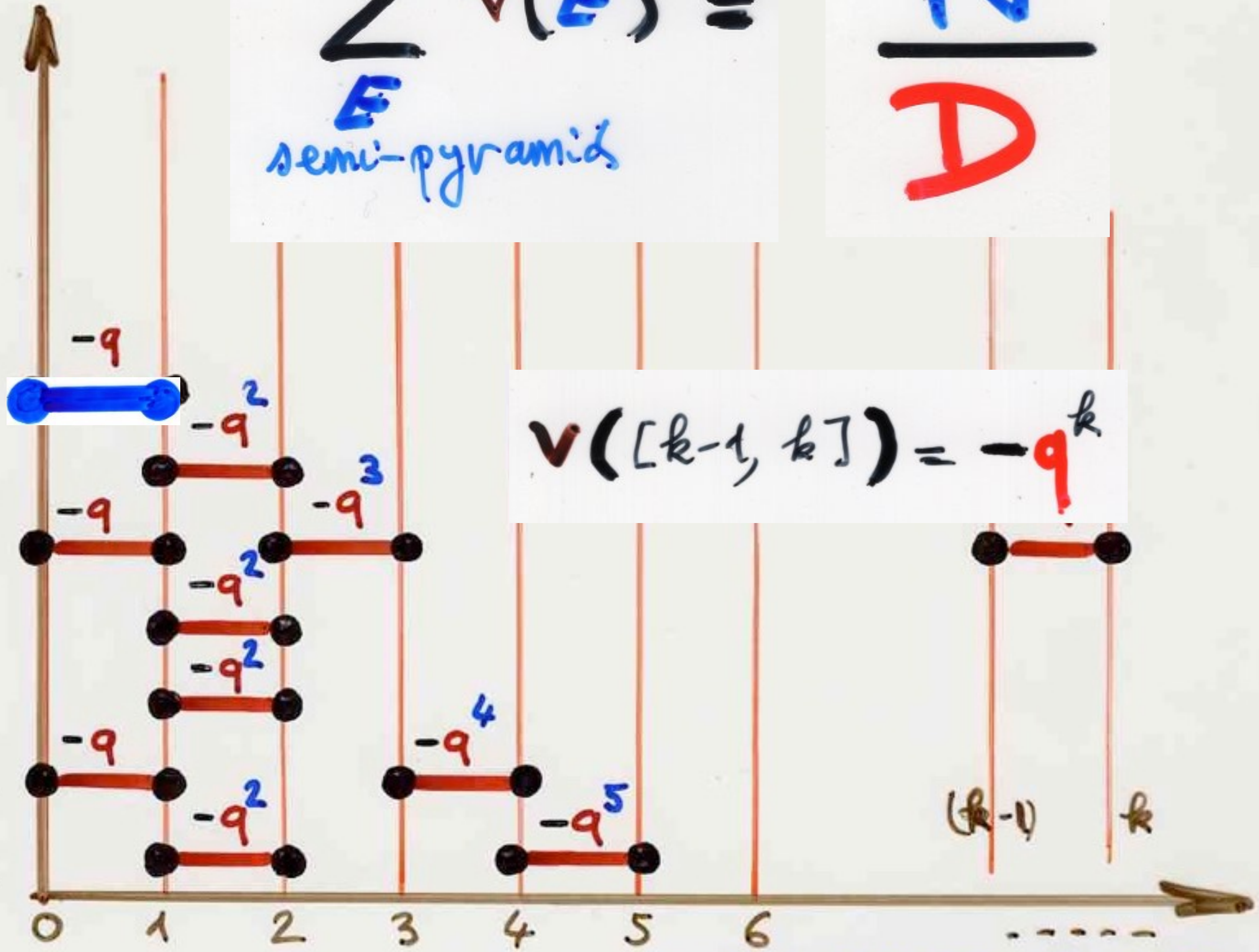
$$v([k-1, k]) = q^{k-1} t$$

$$v(P) = q^{\mathrm{area}(\omega)}\, t^{|\omega|/2}$$

weighted heap $v(E)$

$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E) = \frac{N}{D}$$

$$v([k-1, k]) = -q^k$$



total weight $(-1)^{10}\, q^{1+1+1+2+2+2+2+3+4+5} = q^{23}$

Rogers-Ramanujan identities

Rogers - Ramanujan identities

$$R_{I} \quad \sum_{n\geqslant 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \mathrm{mod}\ 5}} \frac{1}{(1-q^i)}$$

$$R_{II} \quad \sum_{n\geqslant 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \mathrm{mod}\ 5}} \frac{1}{(1-q^i)}$$

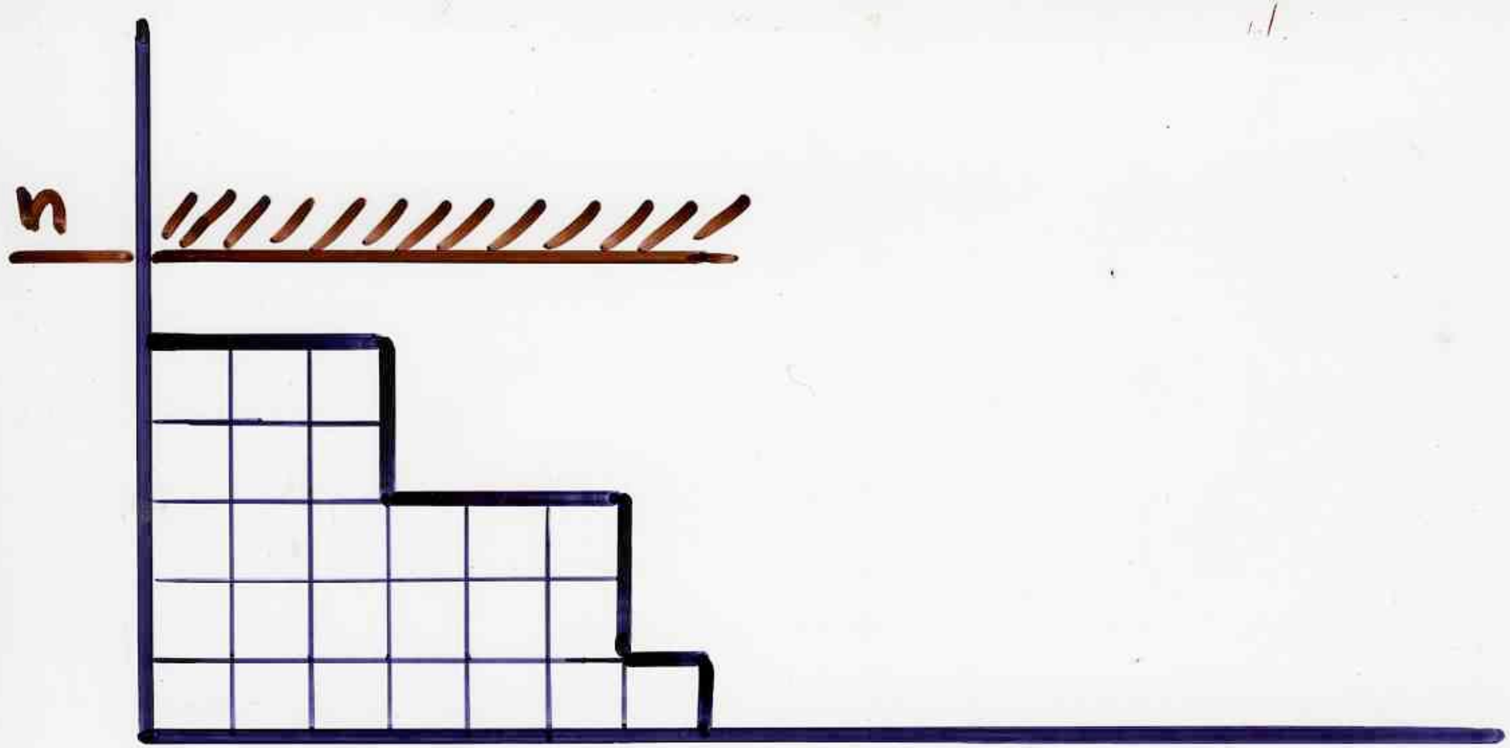
D- partition $\lambda_i - \lambda_{i+1} \geq 2$

$\lambda = (\lambda_1, \ldots, \lambda_k)$ $(1 \leq i < k)$

generating function for D-partitions

$$\sum_{m \geq 0} \frac{q^{(m^2)}}{(1-q)(1-q^2)\cdots(1-q^m)}$$

Partition ayant au plus n parts		D-partition ayant exactement n parts
$0 \le (\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n)$	$\longleftrightarrow$	$(1+\lambda_1, 3+\lambda_2, \dots, (2n-1)+\lambda_n)$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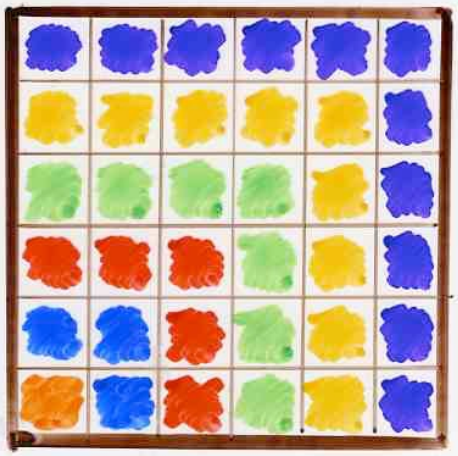
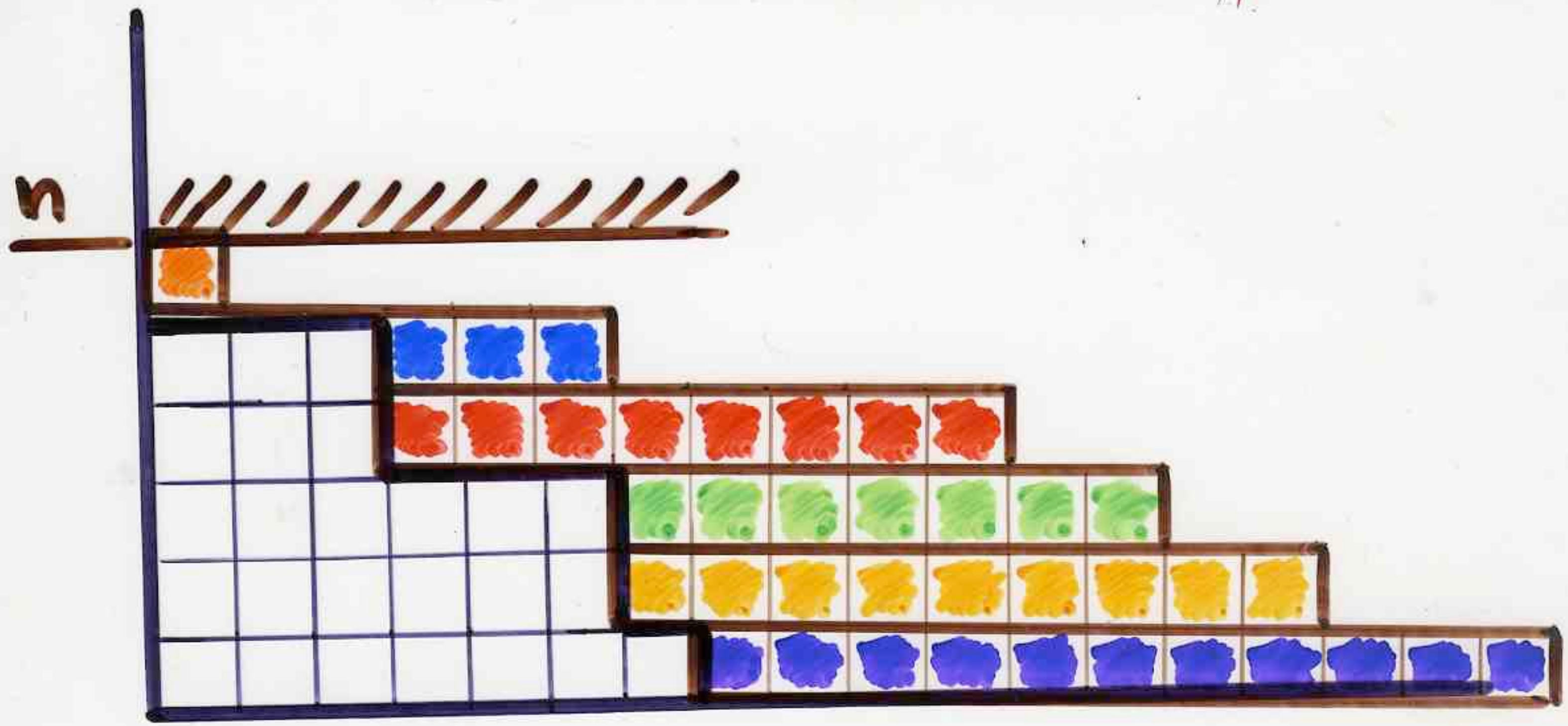


$$n^2 = 1 + 3 + .. + (2n-1)$$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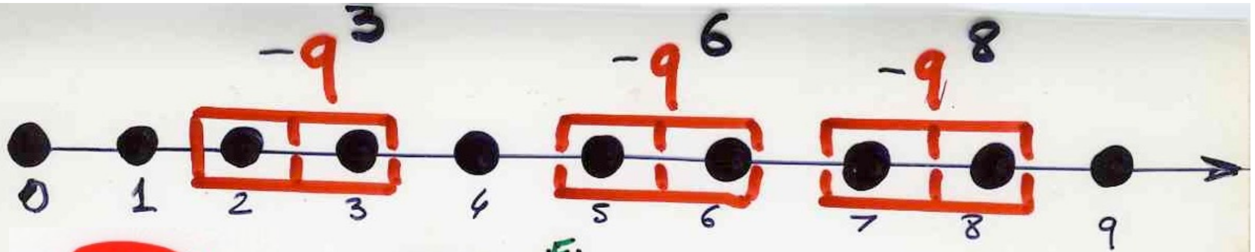
Rogers-Ramanujan 1st identity

$$D = \sum_{n \geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}$$



$$D = \sum_{E} (-1)^{|E|} v(E)$$

E trivial heaps of dimers on $\mathbb{N}$

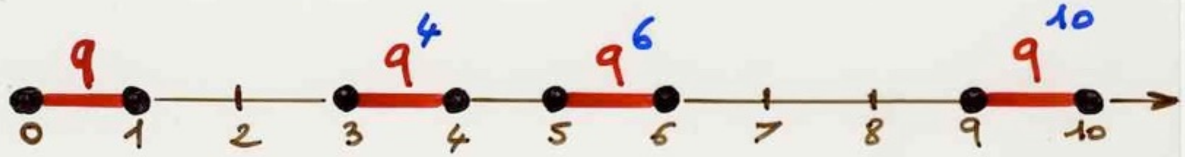
$$v([k-1, k]) = -q^k$$

Rogers-Ramanujan 1st identity

$$D = \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

poids total

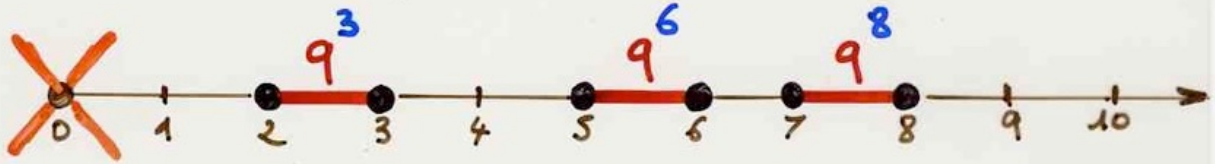
q^{21}



D-partition $\lambda = (10, 6, 4, 1)$

$21 = 10 + 6 + 4 + 1$

q^{17}

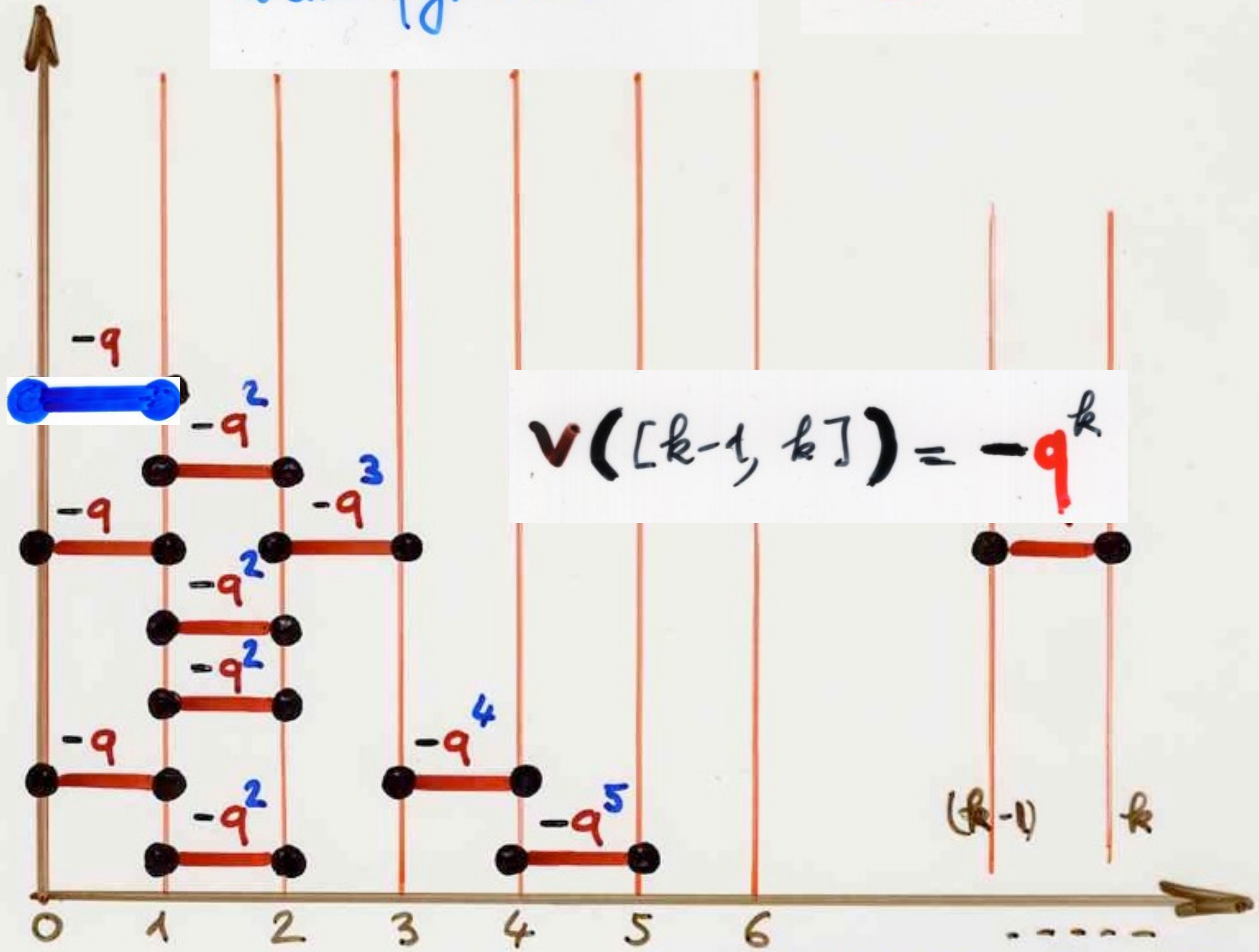


$\lambda = (8, 6, 3)$

$17 = 8 + 6 + 3$

$$N = \sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \delta D$$

$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E) = \frac{N}{D}$$



$$v([k-1, k]) = -q^{k}$$

total weight $(-1)^{10}\, q^{1+1+1+2+2+2+2+3+4+5} = q^{23}$

Rogers - Ramanujan identities

$$R_I \quad \sum_{n\geqslant 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \mathrm{mod}\ 5}} \frac{1}{(1-q^i)}$$

partitions

parts $\equiv 1,4$ mod 5

$$\begin{cases} 9 \\ 4+4+1 \\ 6+1+1+1 \\ 4+1+1+1+1+1 \end{cases} \qquad \begin{cases} 1+\ldots+1 \end{cases}$$

Rogers - Ramanujan identities

R_I

$$\sum_{n\geqslant 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

D-partitions

$$\left\{\begin{array}{l} 9 \\ 8+1 \\ 7+2 \\ 6+3 \\ 5+3+1 \end{array}\right.$$

?

partitions

parts $\equiv 1,4$ mod 5

$$\left\{\begin{array}{l} 9 \\ 4+4+1 \\ 6+1+1+1 \\ 4+1+1+1+1+1 \end{array}\right. \left\{ 1+\dots+1 \right.$$

$$R_{II} \quad \sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \bmod 5}} \frac{1}{(1-q^i)}$$

D-partitions

parts $\neq 1$

$$\begin{cases} 7+2 \\ 6+3 \\ 9 \end{cases}$$

?

Partitions

parts $\equiv 2, 3 \bmod 5$

$2+2+2+3$

$3+3+3$

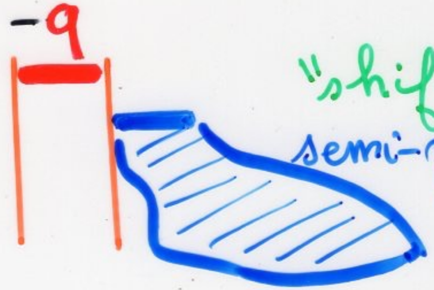
$7+2$

Ramanujan contined fraction

$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E) = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots}}}}}$$

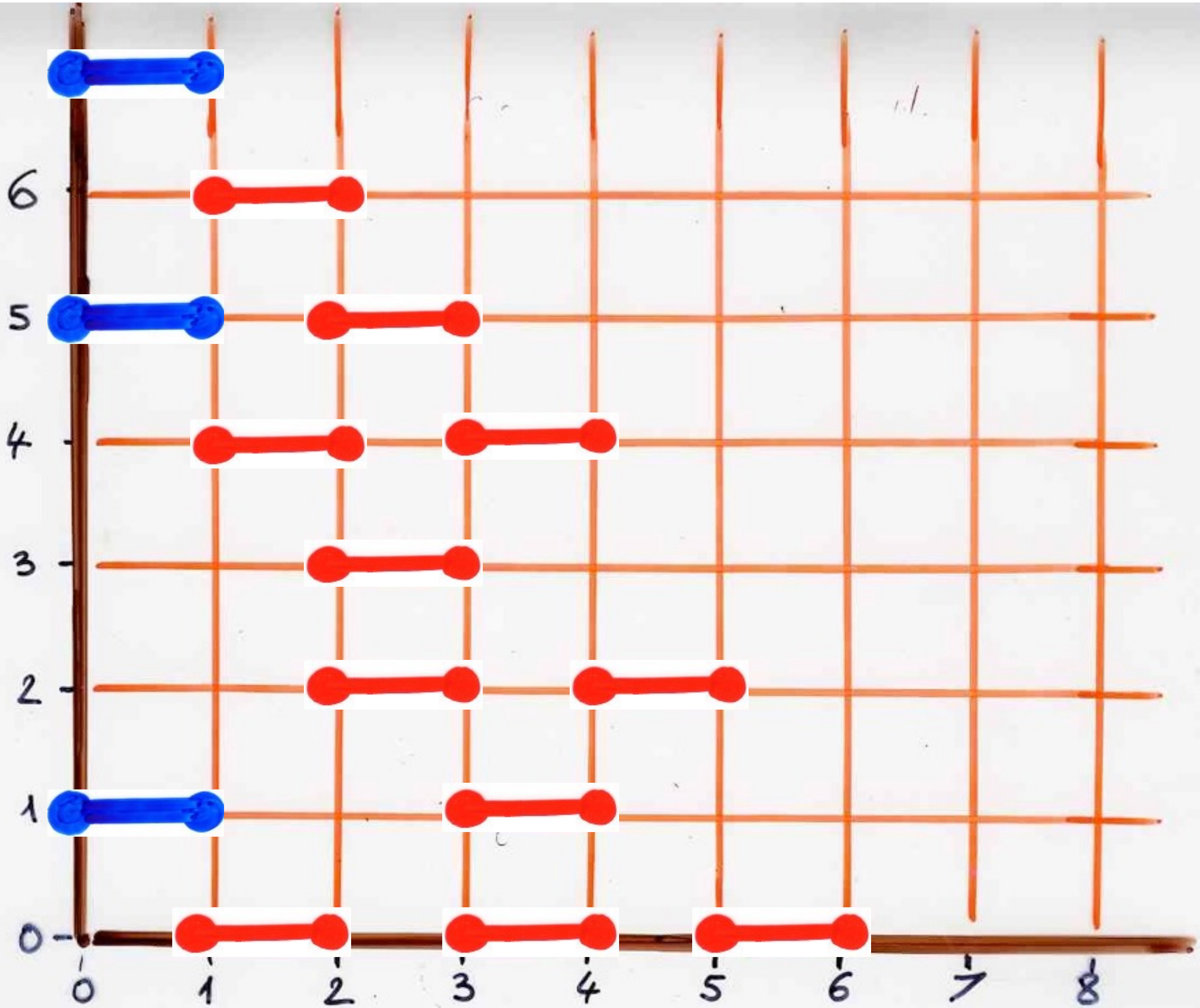
Semi-pyramid

= sequence of "primitive" semi-pyramids

"Primitive" semi-pyramid = 

"shifted" semi-pyramid

weight: $q^{i} \xrightarrow{\delta} q^{i+1}$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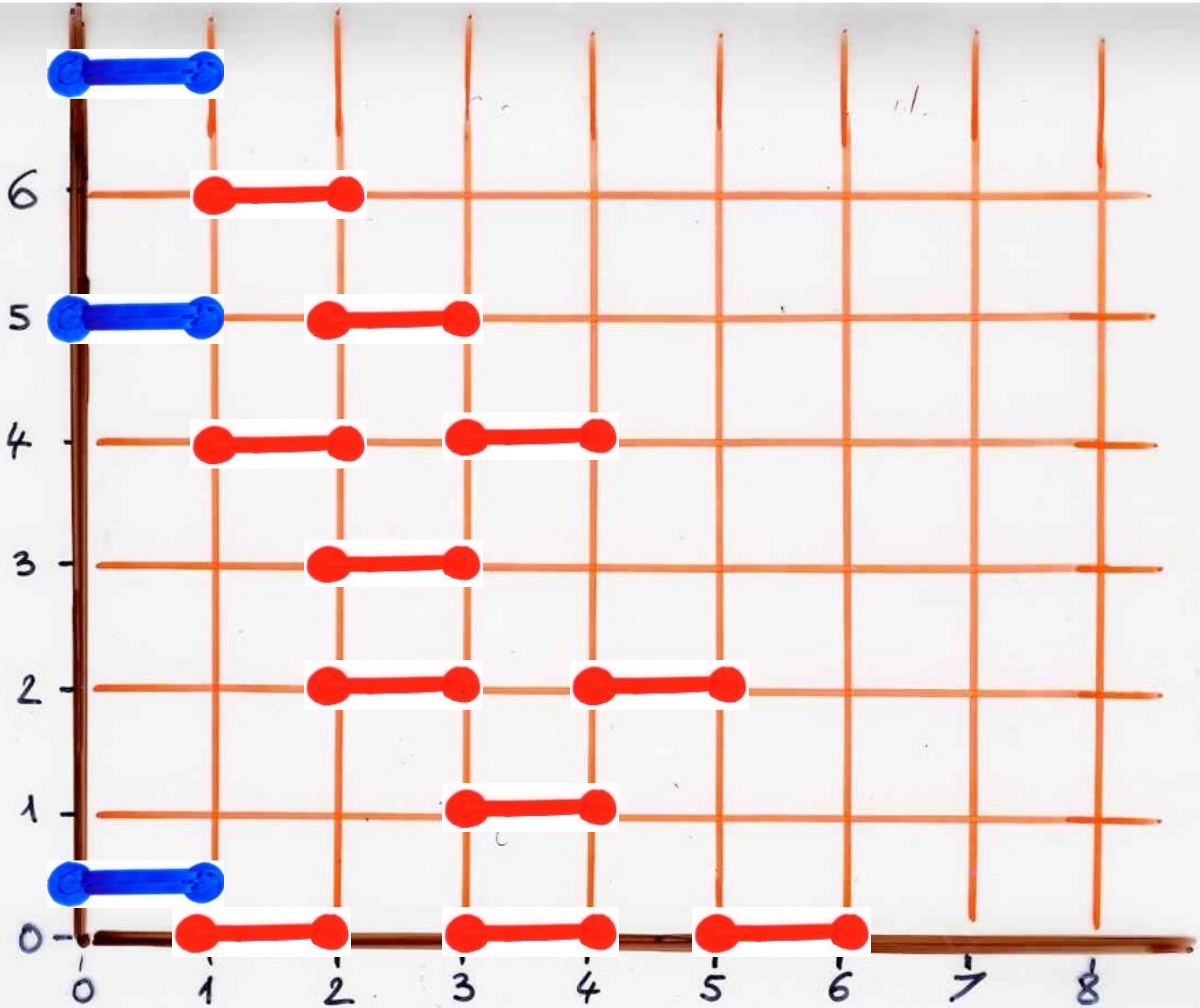
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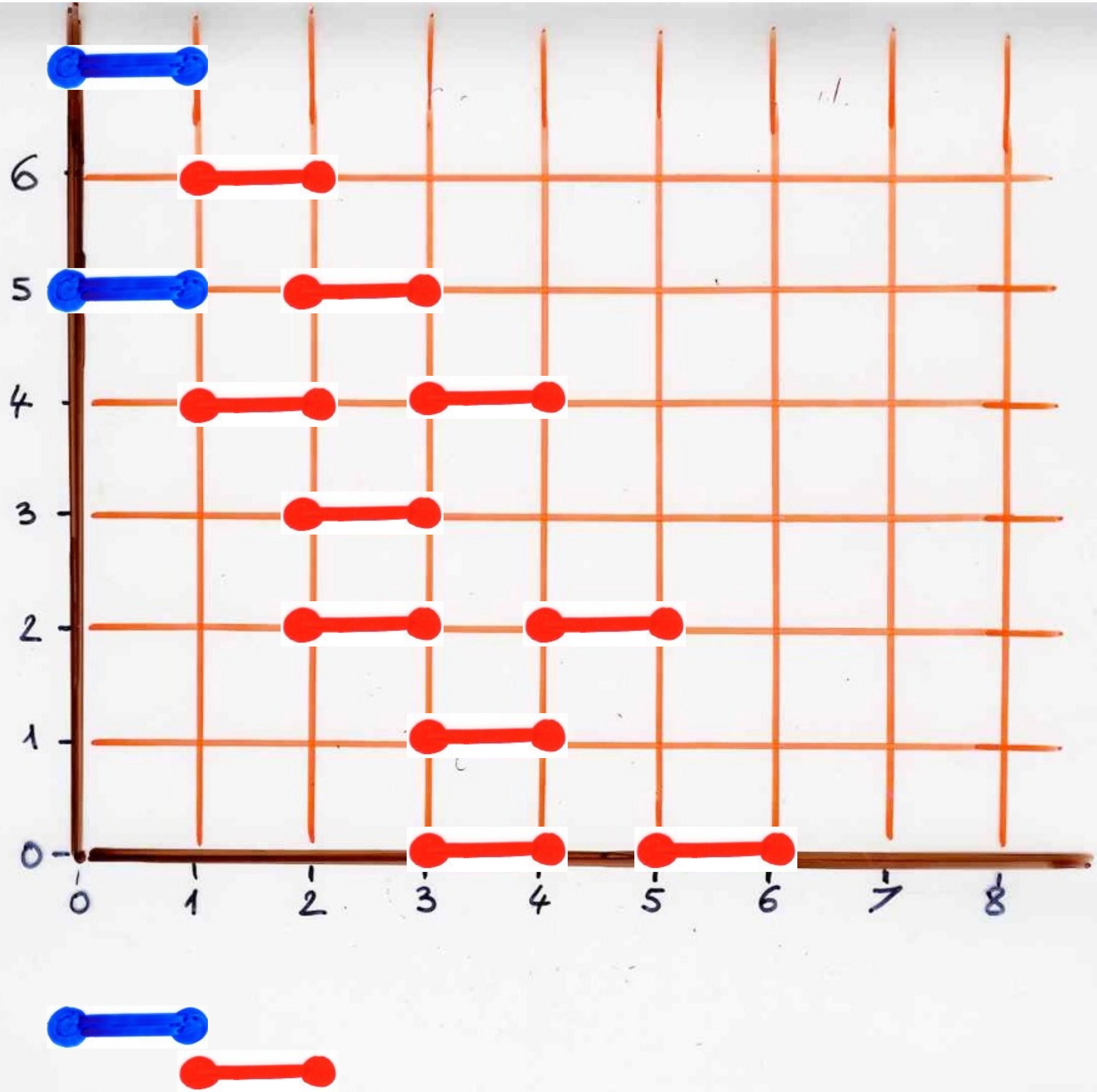
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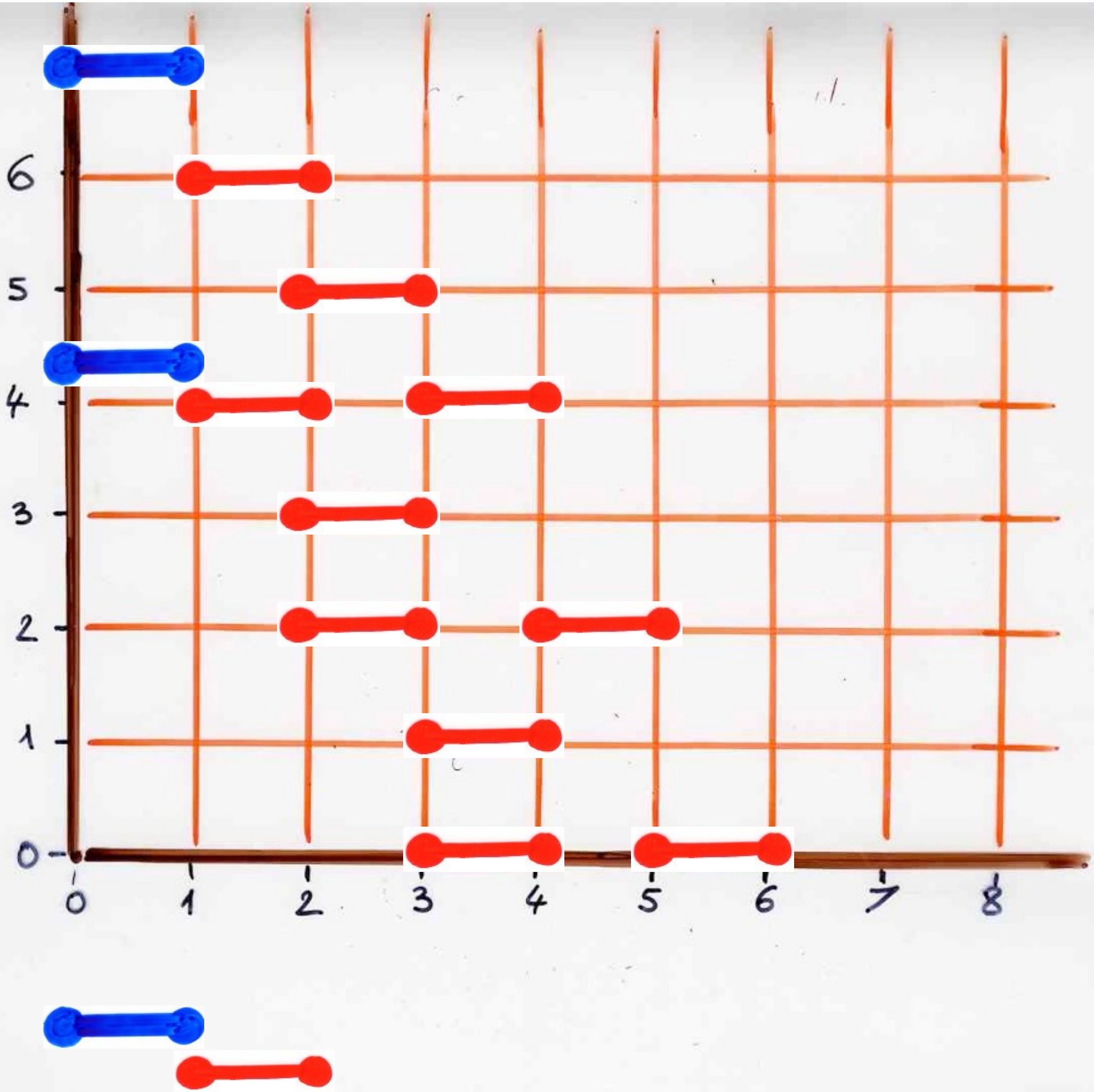
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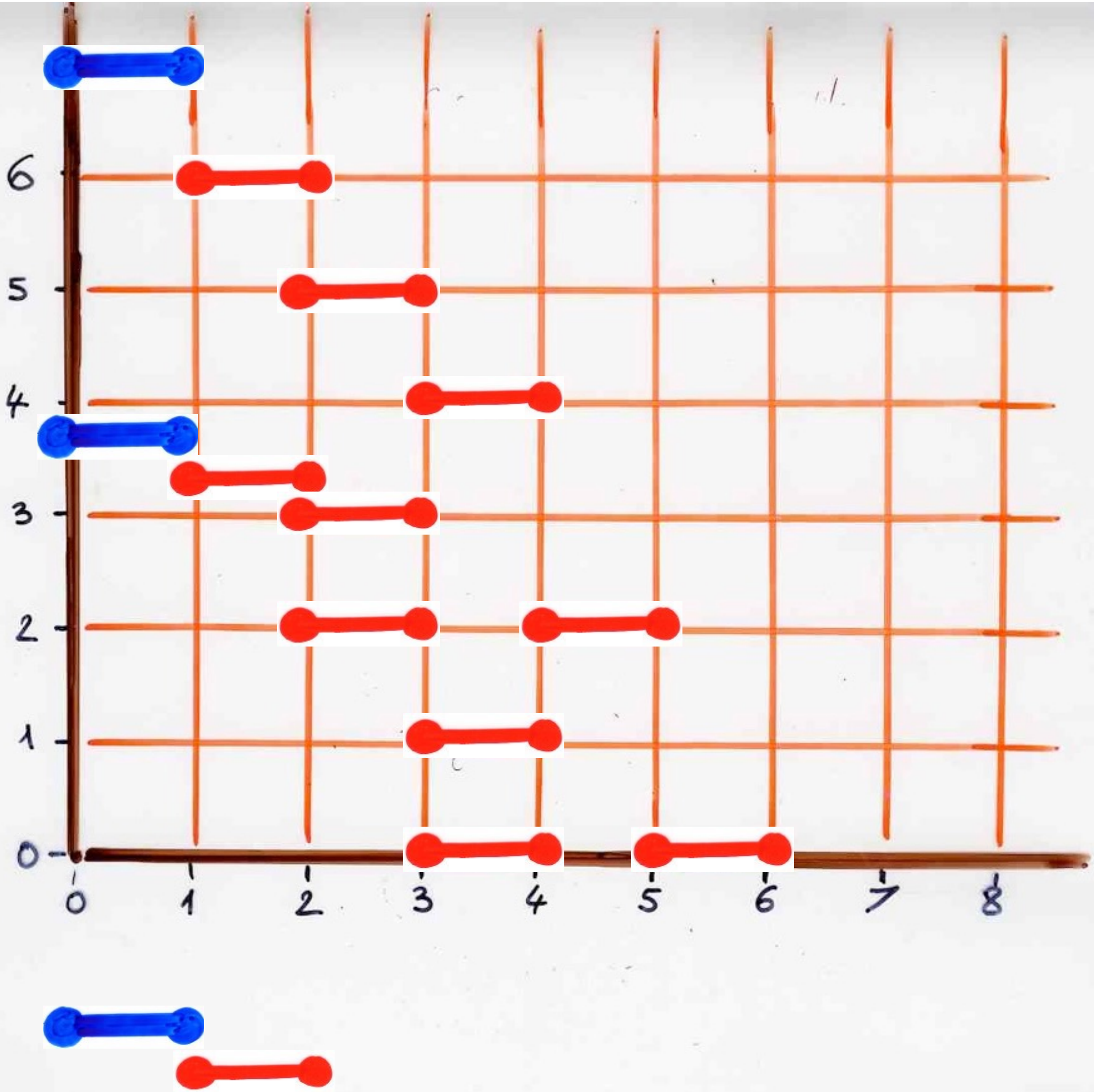

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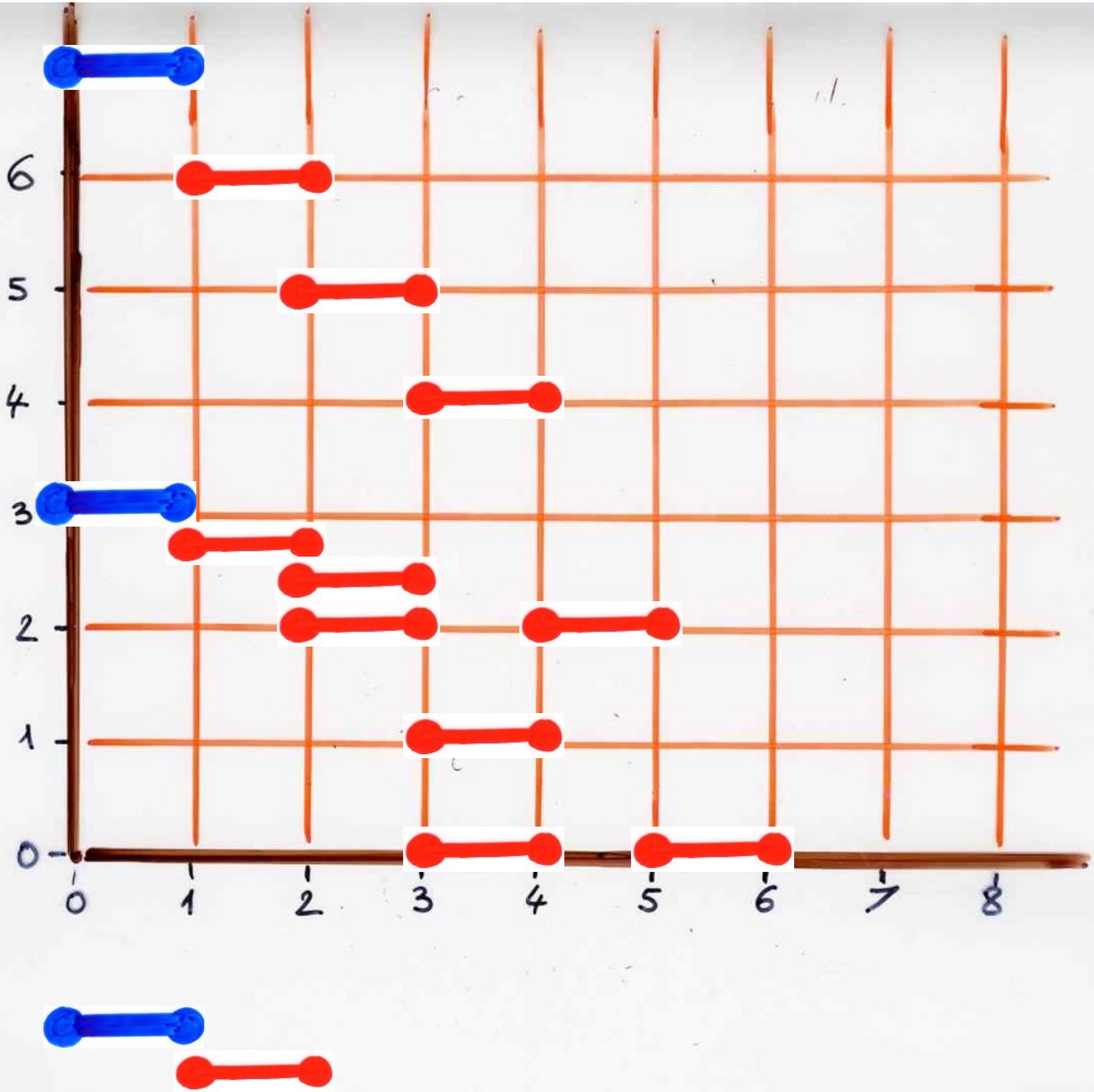
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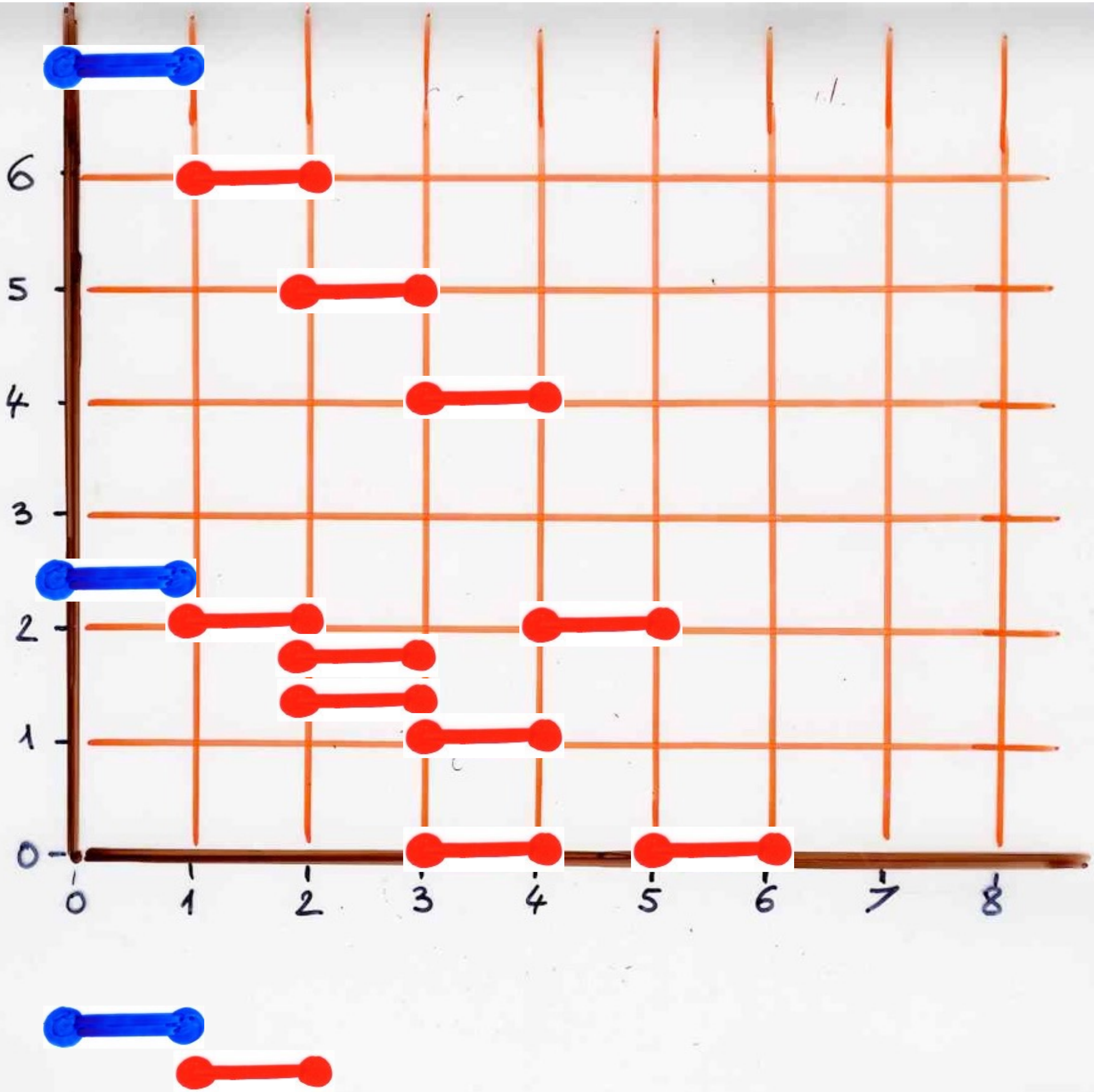

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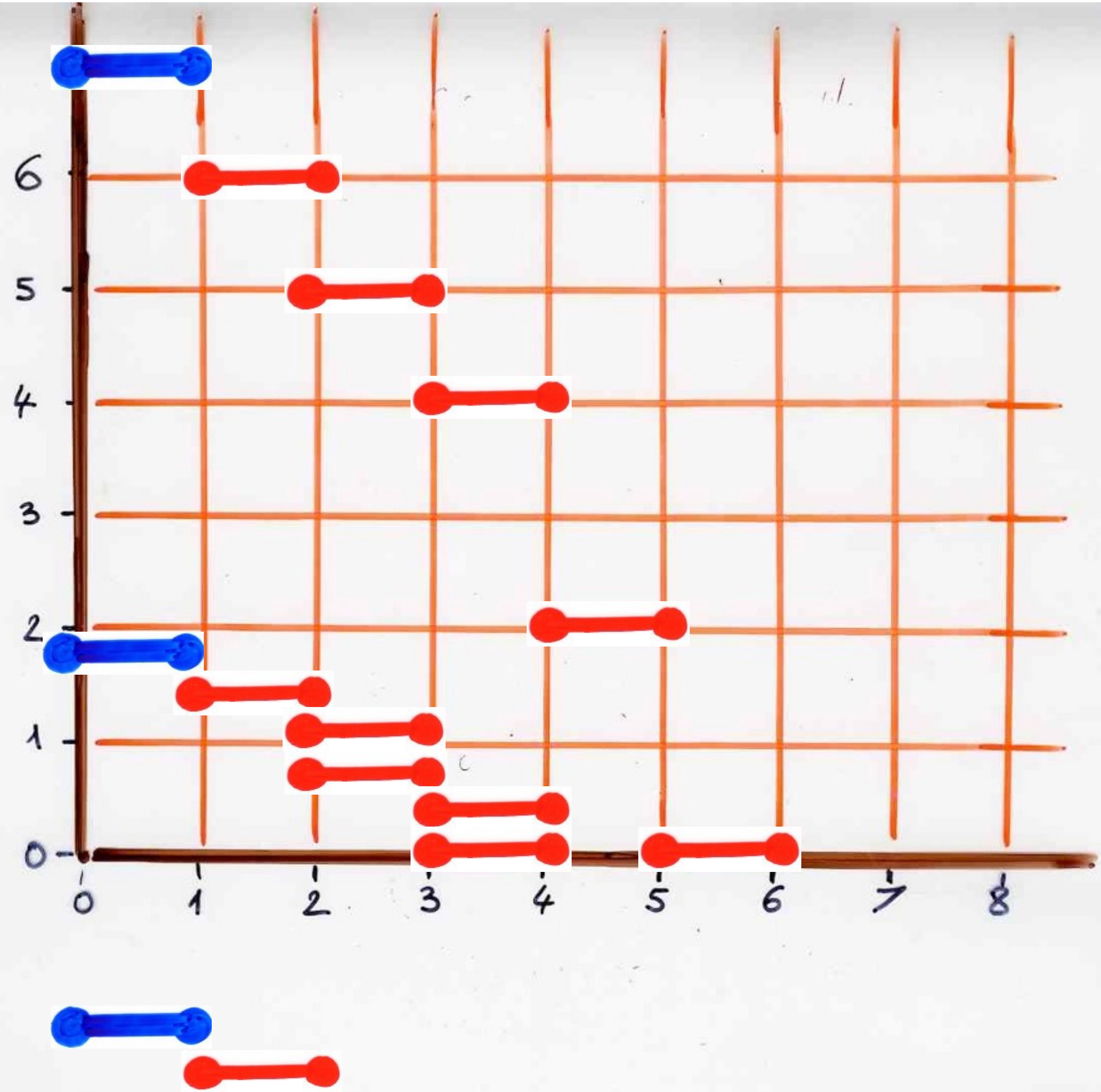
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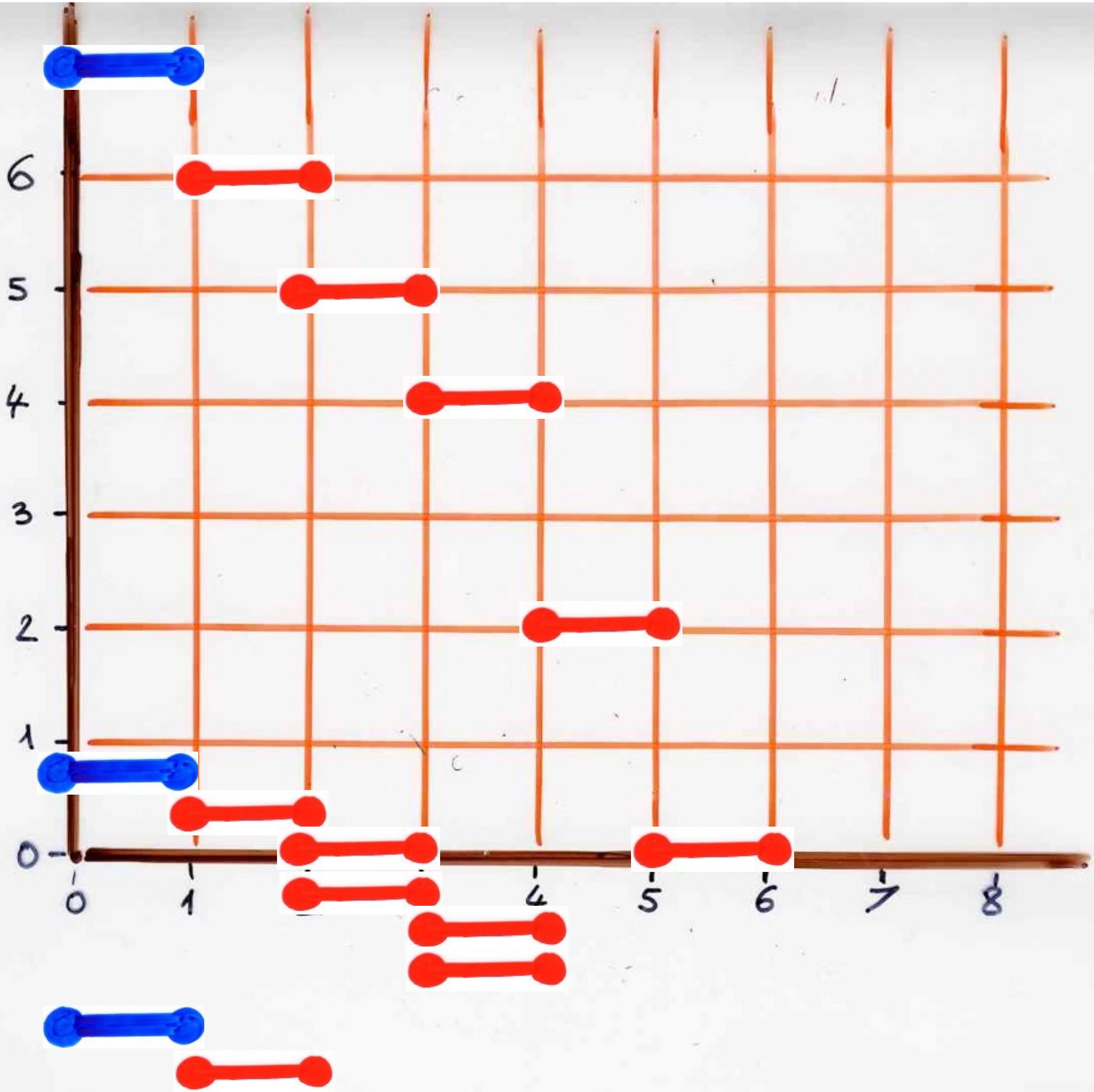
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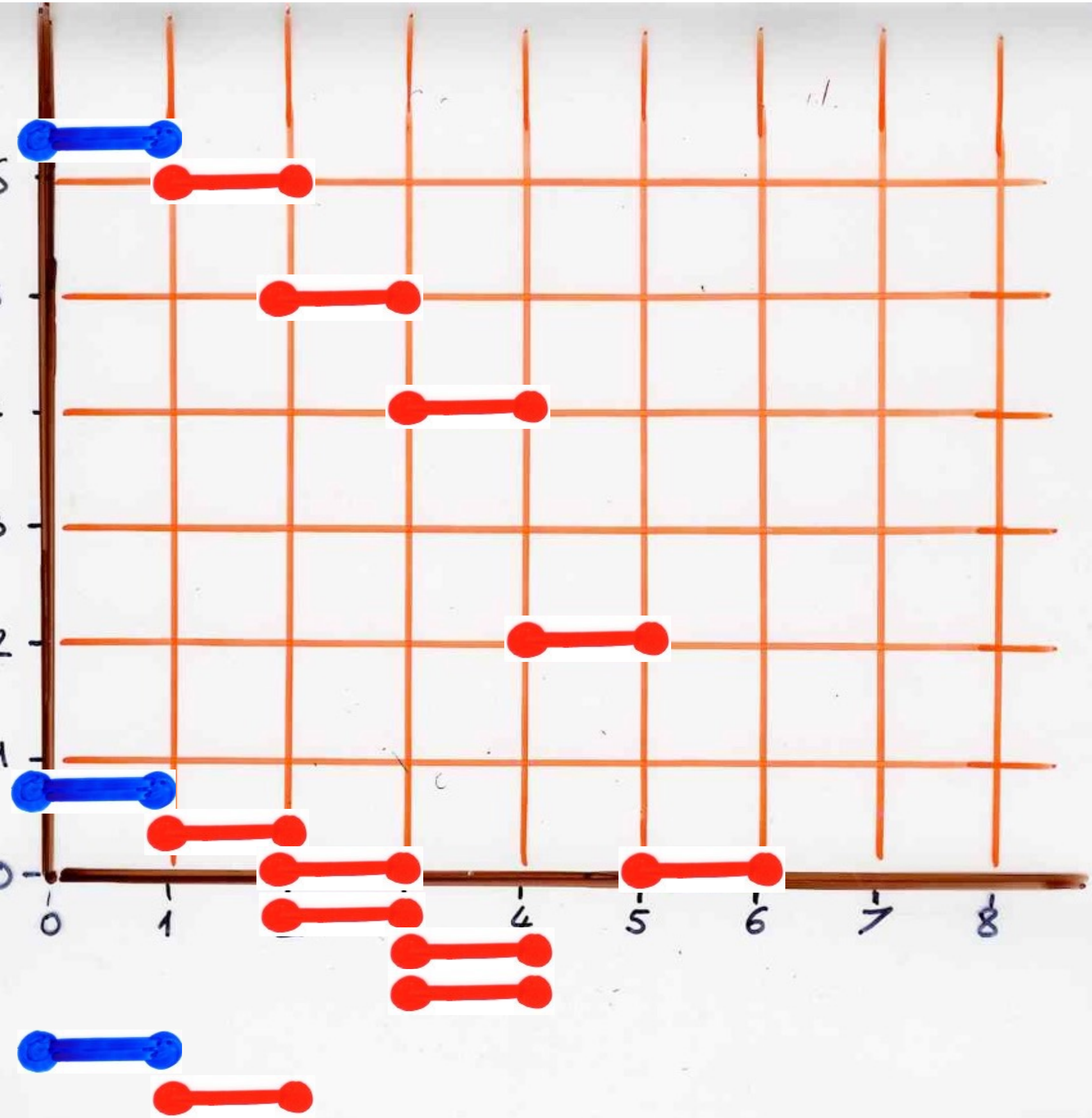

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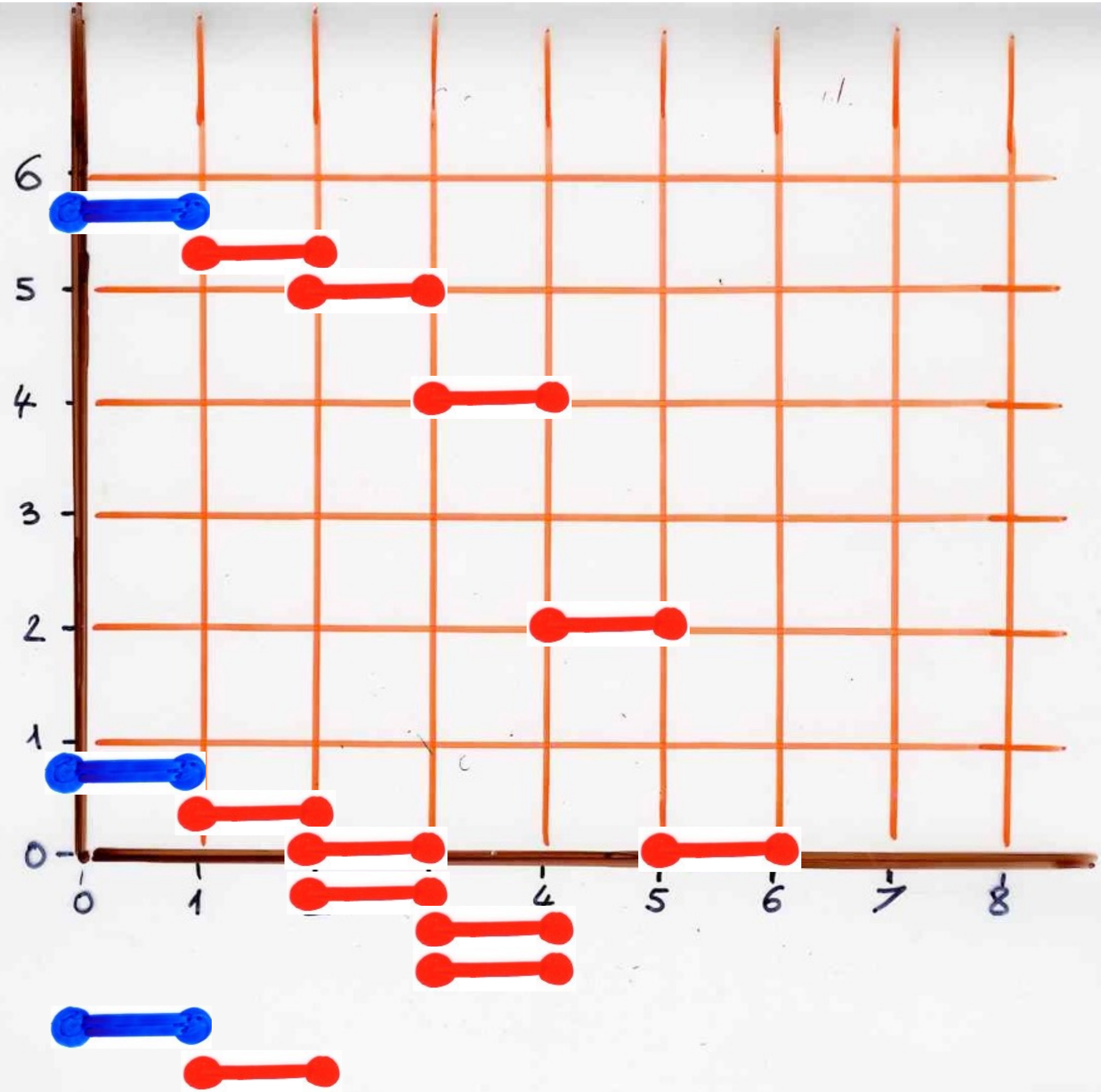

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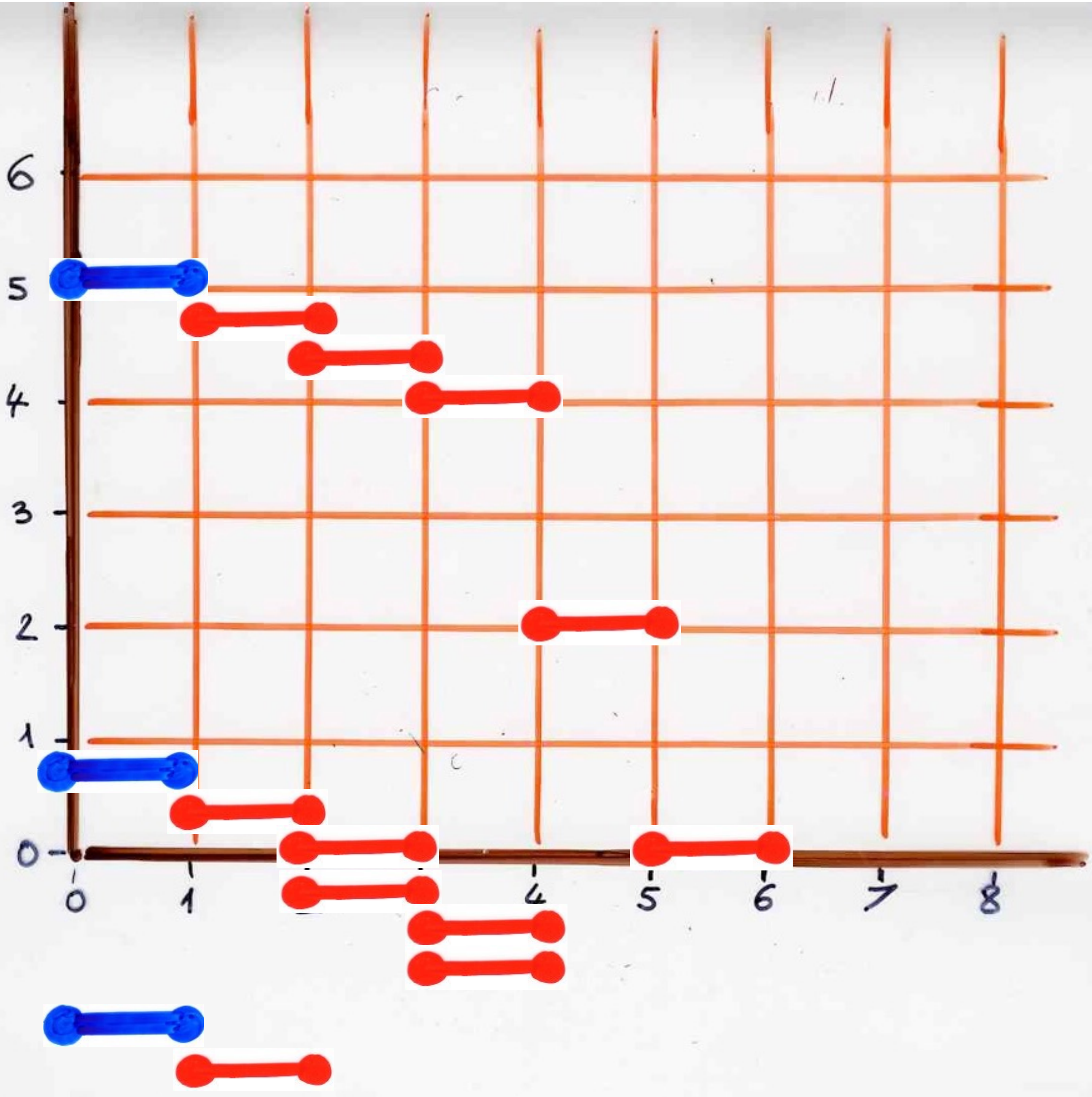
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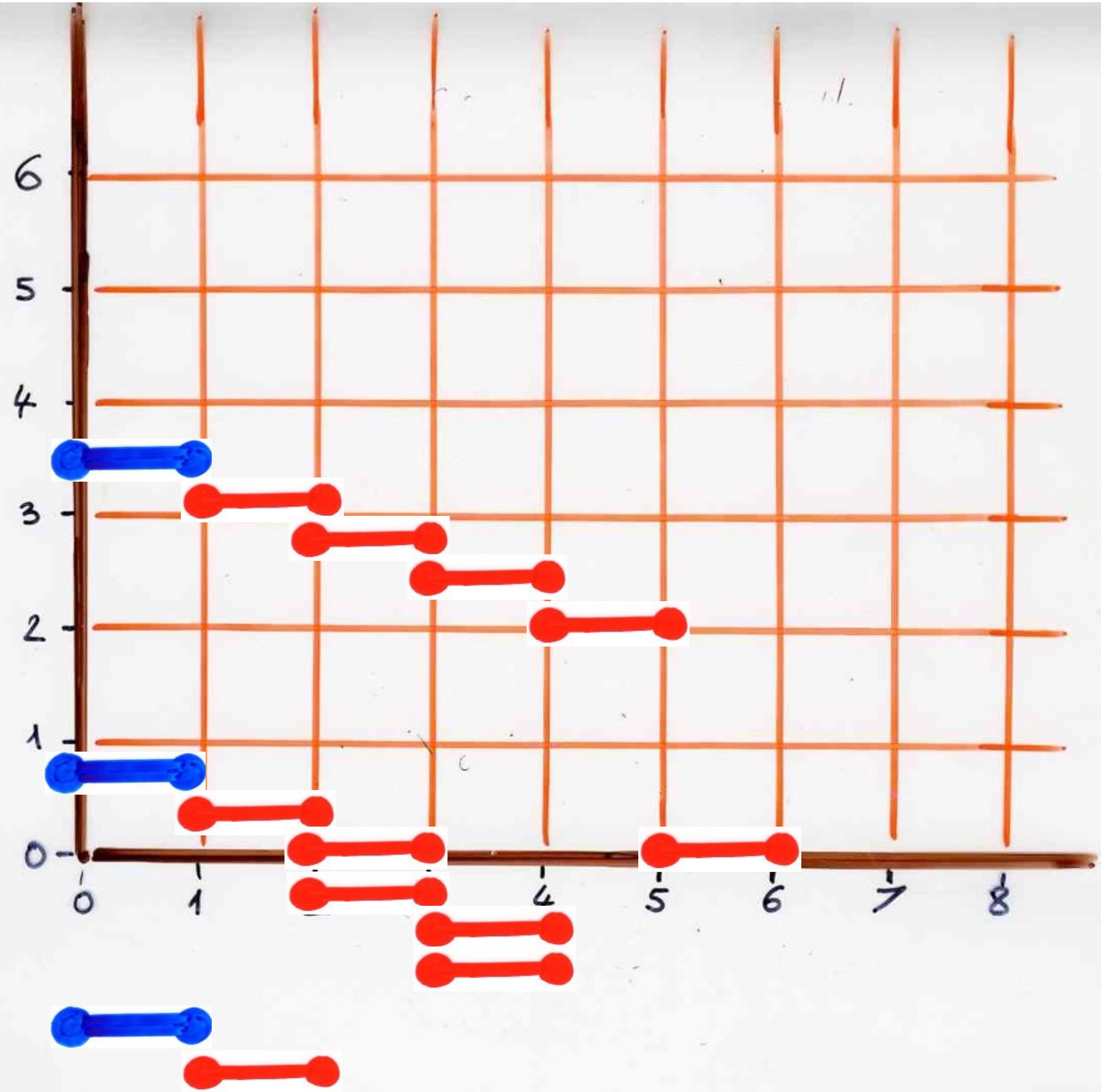

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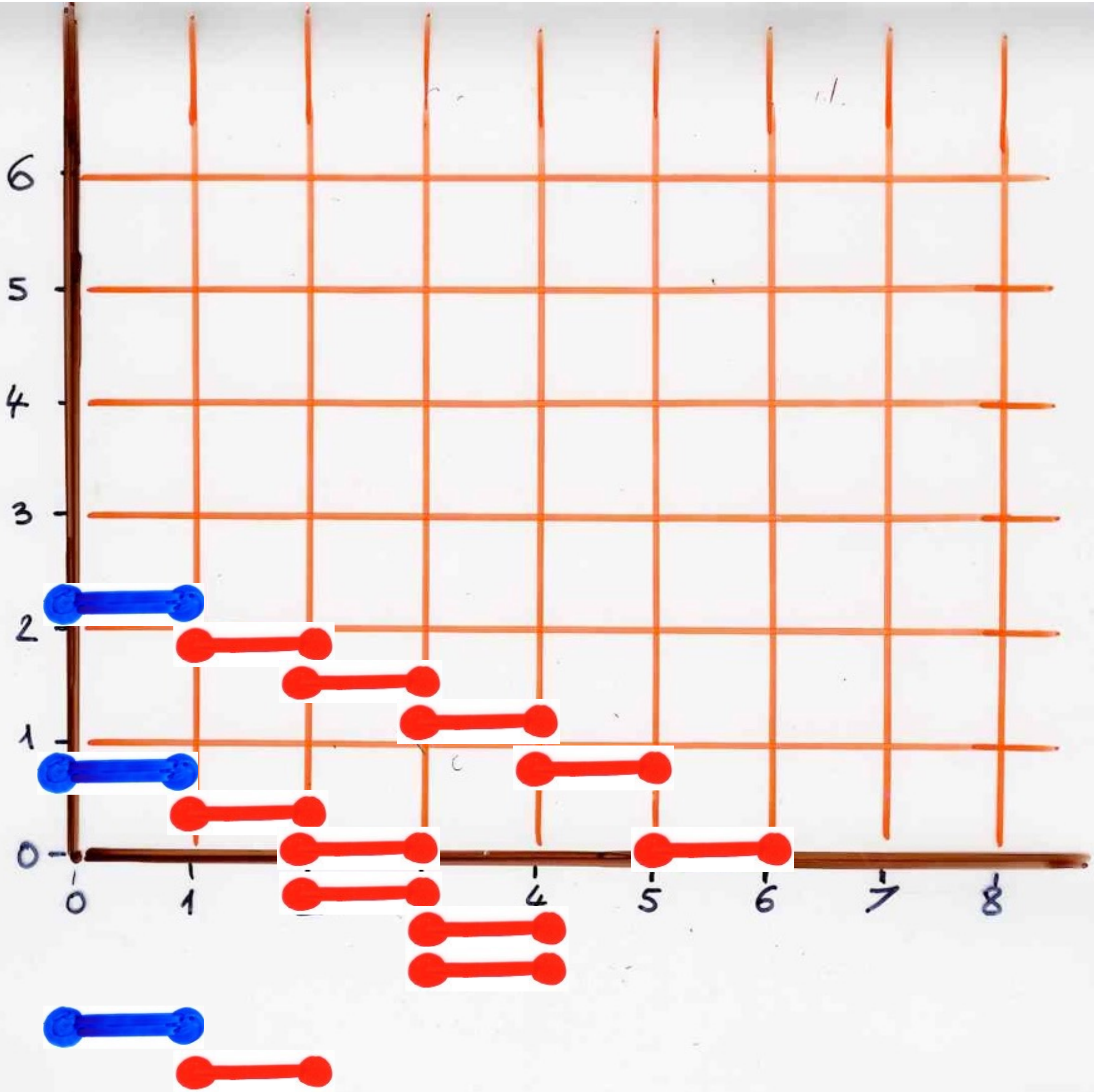

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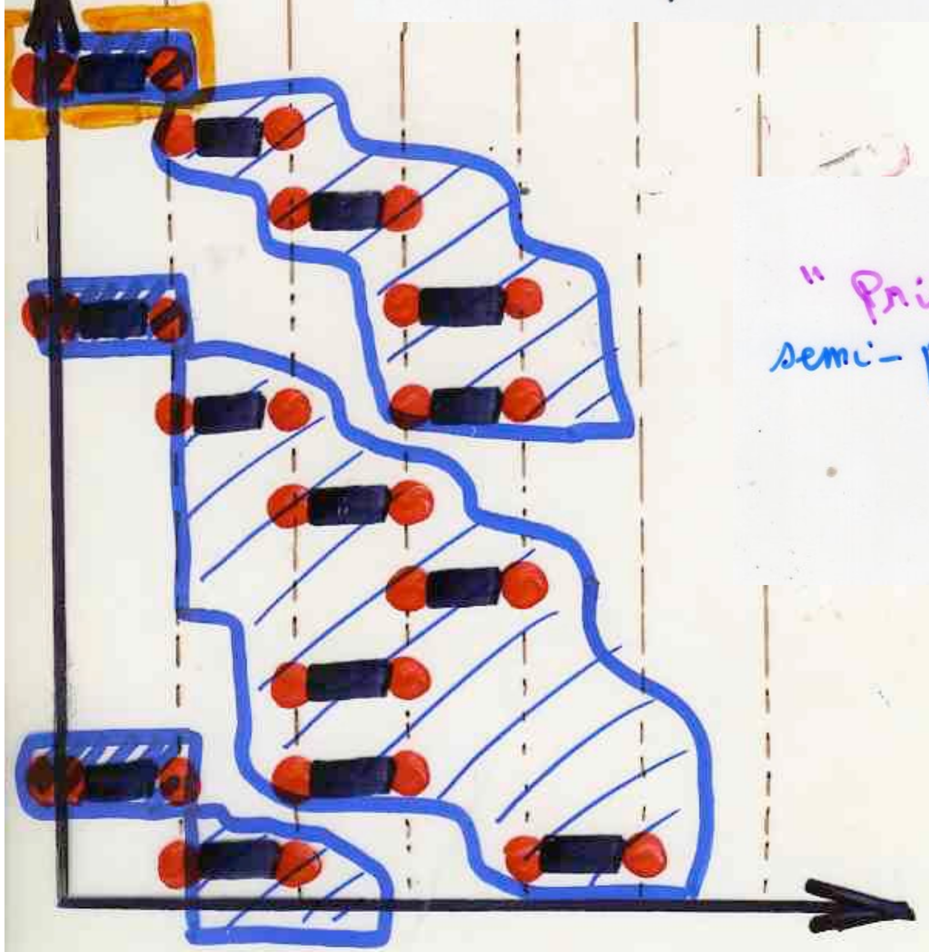
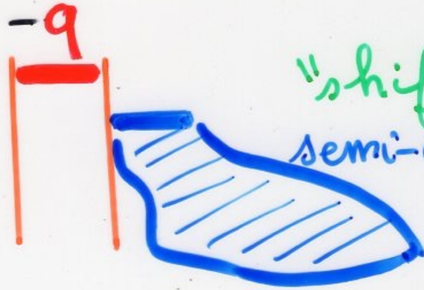
Semi-pyramid

= sequence of "primitive" semi-pyramids



"Primitive" semi-pyramid =



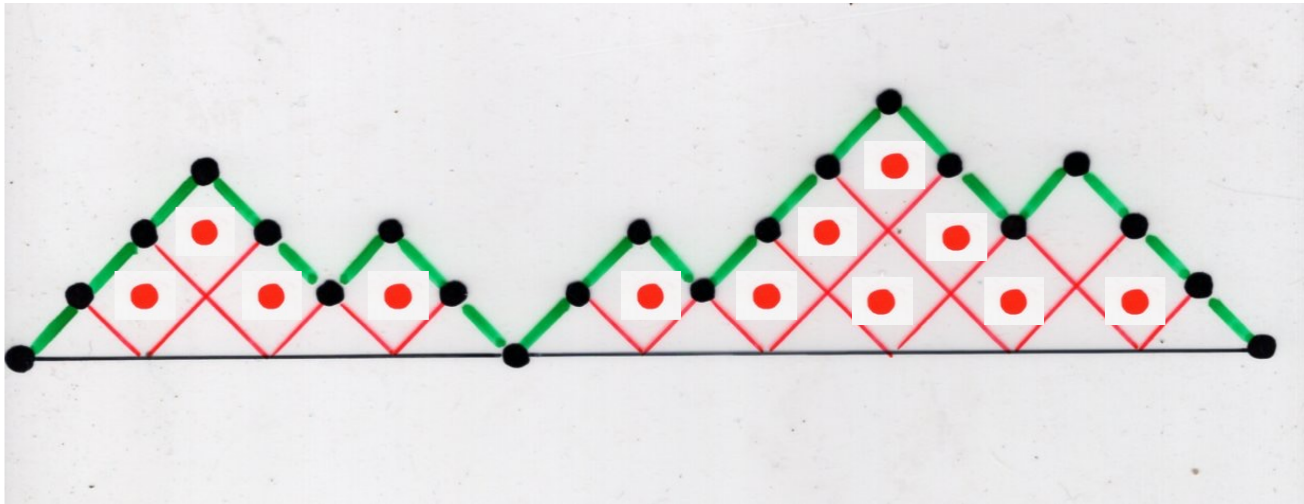
"shifted" semi-pyramid

weight: $q^{i} \xrightarrow{\delta} q^{i+1}$

$$\sum_{\substack{E \\ \text{semi-pyramids}}} v(E) = \frac{1}{1-(-q)\displaystyle\sum_{\substack{E \\ \text{semi-pyramids}}} \delta v(E)}$$

$$= \frac{1}{1+\dfrac{q}{1+q^2\displaystyle\sum_{\substack{E \\ \text{semi-pyramids}}} \delta^2 v(E)}}$$

$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E) = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots}}}}}$$



$$\sum_{\substack{\omega \\ \text{Dyck paths}}} q^{\mathrm{area}(\omega)}\, t^{|\omega|/2} = \cfrac{1}{1-\cfrac{t}{1-\cfrac{tq}{1-\cfrac{tq^2}{\cdots\cfrac{}{1-\cfrac{tq^k}{\cdots}}}}}}$$

$$\sum_{\substack{E \\ \text{semi-pyramid}}} v(E) = \frac{N}{D}$$

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\cfrac{\cdots\cdots}{1+\cfrac{q^k}{\cdots\cdots}}}}}} = \frac{\displaystyle\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}}{\displaystyle\sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}}$$

Hard
Hexagons
gas model

$$R(q) = \prod_{n \geq 0} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+3})(1-q^{5n+2})} = \frac{R_{II}}{R_{I}}$$

Baxter
(1980)

$$t = -q \left[R(q) \right]^5$$

$$Y(q) = \prod_{n \geq 0} \frac{(1-q^{6n+2})(1-q^{6n+3})^2(1-q^{6n+4})(1-q^{5n+1})^2(1-q^{5n+4})^2(1-q^{5n})^2}{(1-q^{6n+1})(1-q^{6n+5})(1-q^{6n})^2(1-q^{5n+2})^3(1-q^{5n+3})^3}$$

$Z(t)$

partition function

$$Z(t) = Y(q(t))$$

Andrews interpretation of the «reciprocal» of Ramanujan continued fraction

quasi-partitions of n

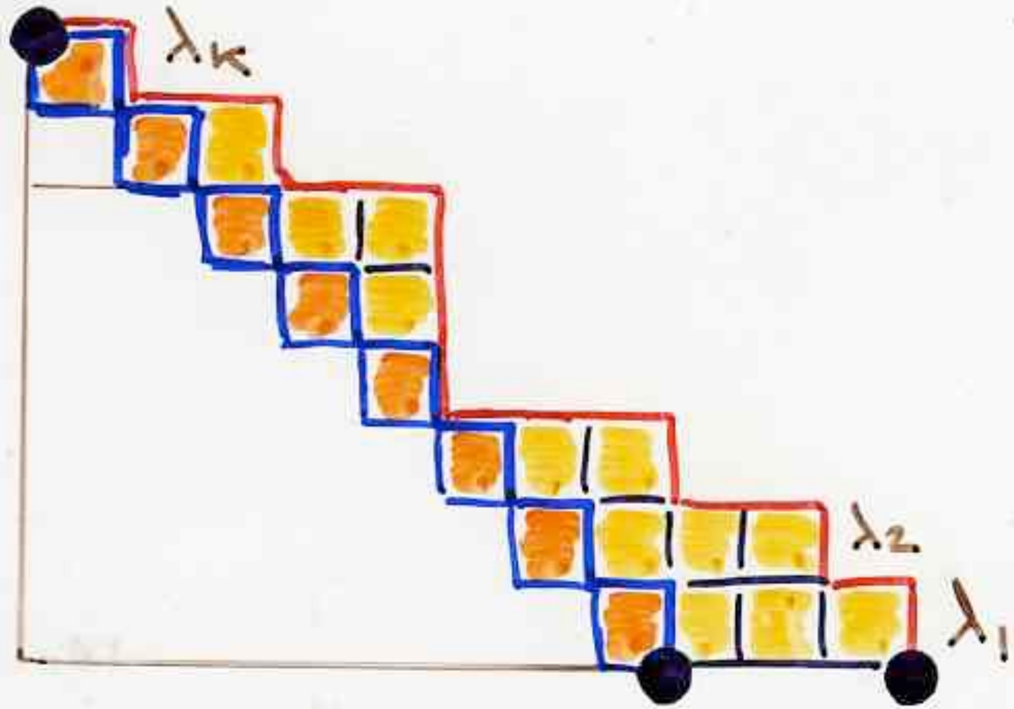
G. Andrews (1981) reciprocal of Rogers-Ramanujan identities

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$$

$$1 + \lambda_i \geq \lambda_{i+1}$$

$$i = 1, \ldots, k-1$$

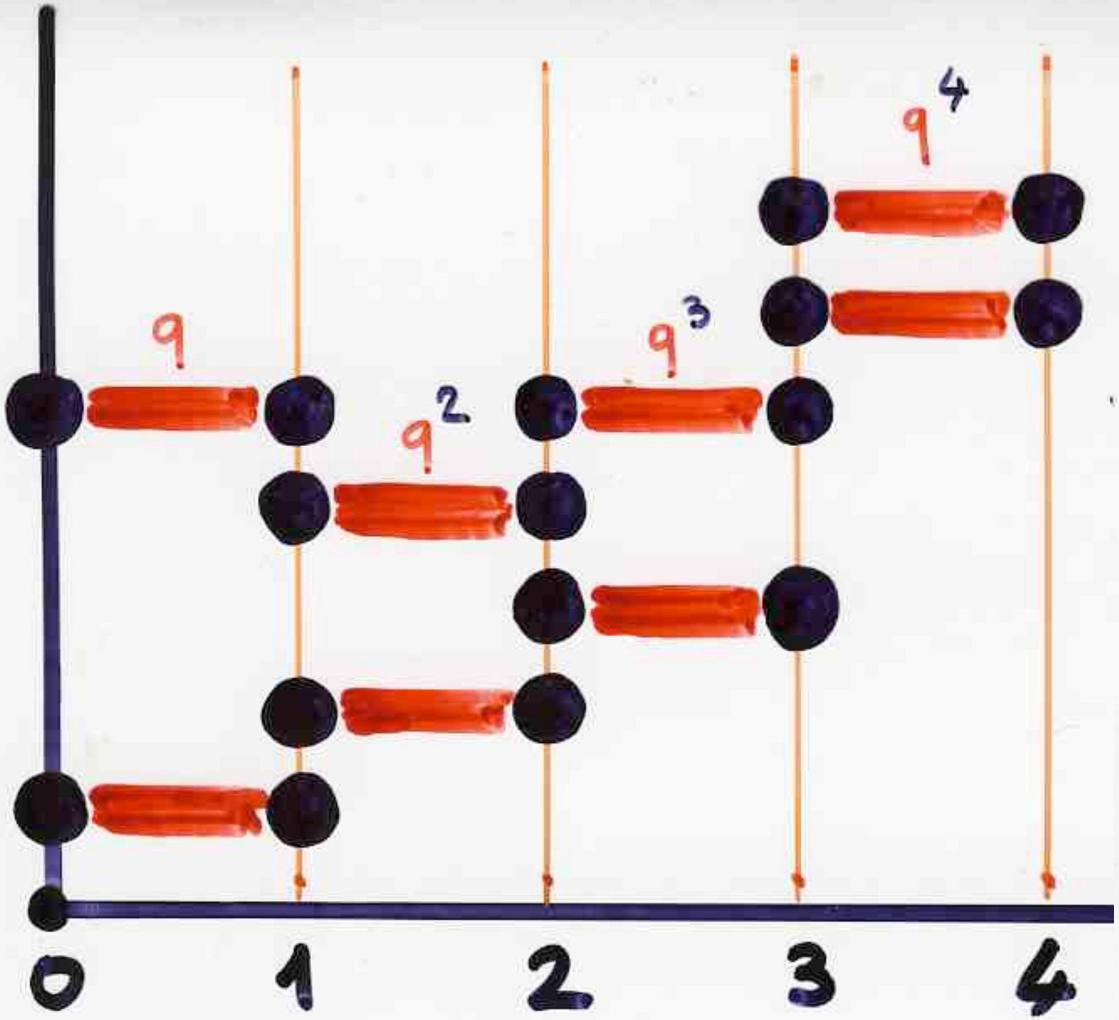
$\lambda = (4, 4, 3, 1, 2, 3, 2, 1)$



exercise
Ch1b, p71

weight-preserving
bijection

quasi-partitions
$\updownarrow$
heaps of dimers
on $\mathbb{N}$



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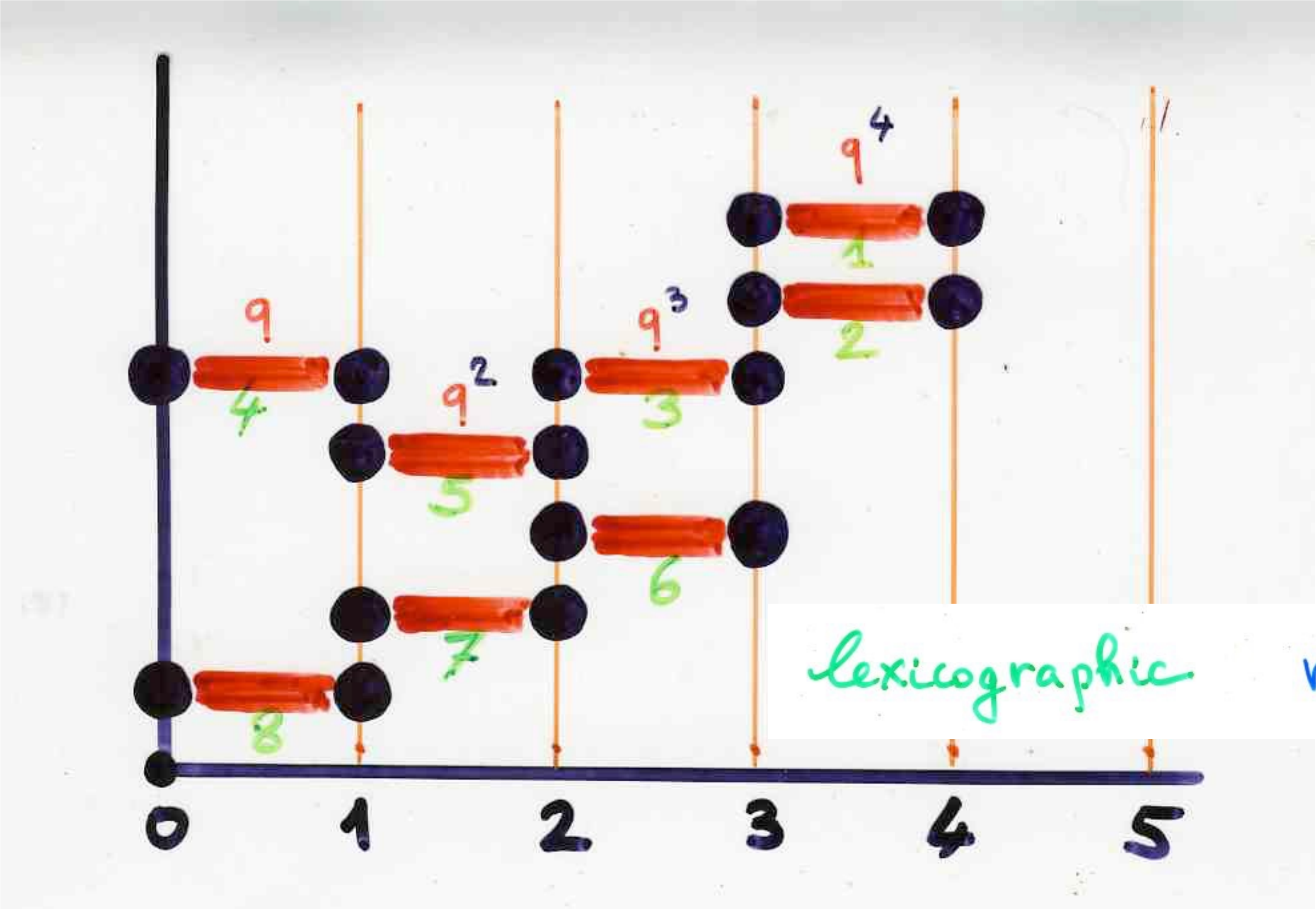


lexicographic normal form

$$H \longrightarrow \lambda = (\underset{1}{4}, \underset{2}{4}, \underset{3}{3}, \underset{4}{1}, \underset{5}{2}, \underset{6}{3}, \underset{7}{2}, \underset{8}{1})$$

quasi-partition

$$\frac{1}{D} = \sum_{\substack{E \\ \text{heaps of dimers}}} v(E)$$

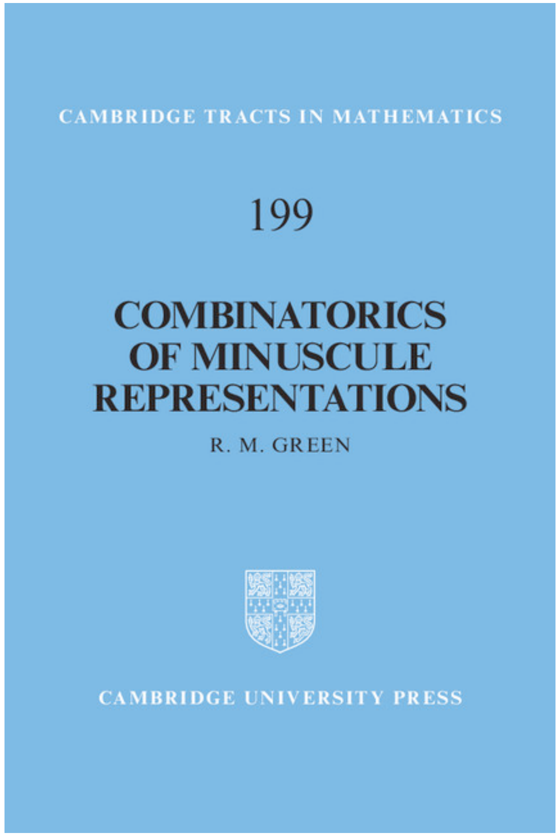
$$\frac{1}{R_I} = \sum_{\substack{\lambda \\ \text{quasi-partitions}}} (-1)^{\ell(\lambda)} q^{|\lambda|}$$

G. Andrews (1981) reciprocal of Rogers–Ramanujan identities

other future chapters

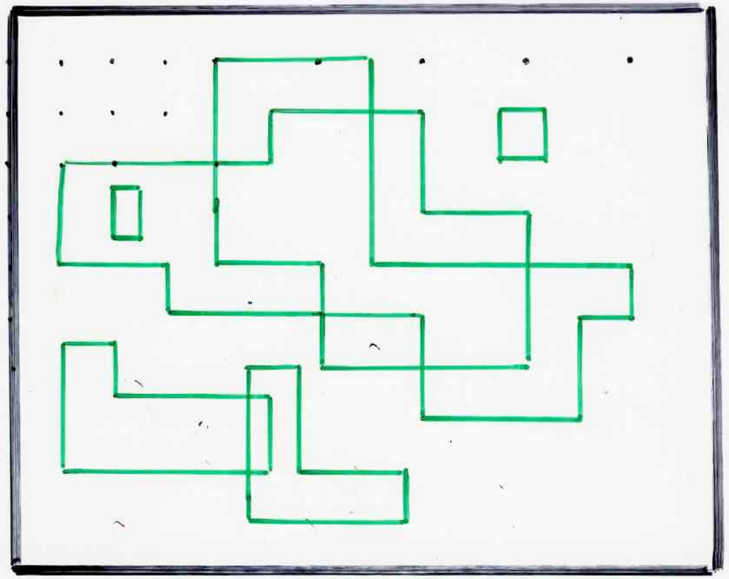
Complementary topics

- minuscule representations of Lie algebra (R. Green and students) book
- basis of free partially commutative Lie algebra (Lalonde, Duchamp-Krob, ...)



Lyndon words
Lyndon heaps

R. Green (2013)



"closed" graph

Ising model

- statistical physics : (T. Helmuth)
  Ising model revisited
- string theory and heaps
  gauge theory, quivers
  (Ramgoolam)

Q-systems, heaps, paths and clusters positivity

Di Francesco, Kedem (2008)

- computer science:
  the SAT problem revisited with heaps
  (D. Knuth, vol 4, Fascicle 6)

- computer science:
  Petri nets, asynchronous automata,
  Zielonka theorem