Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics, a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

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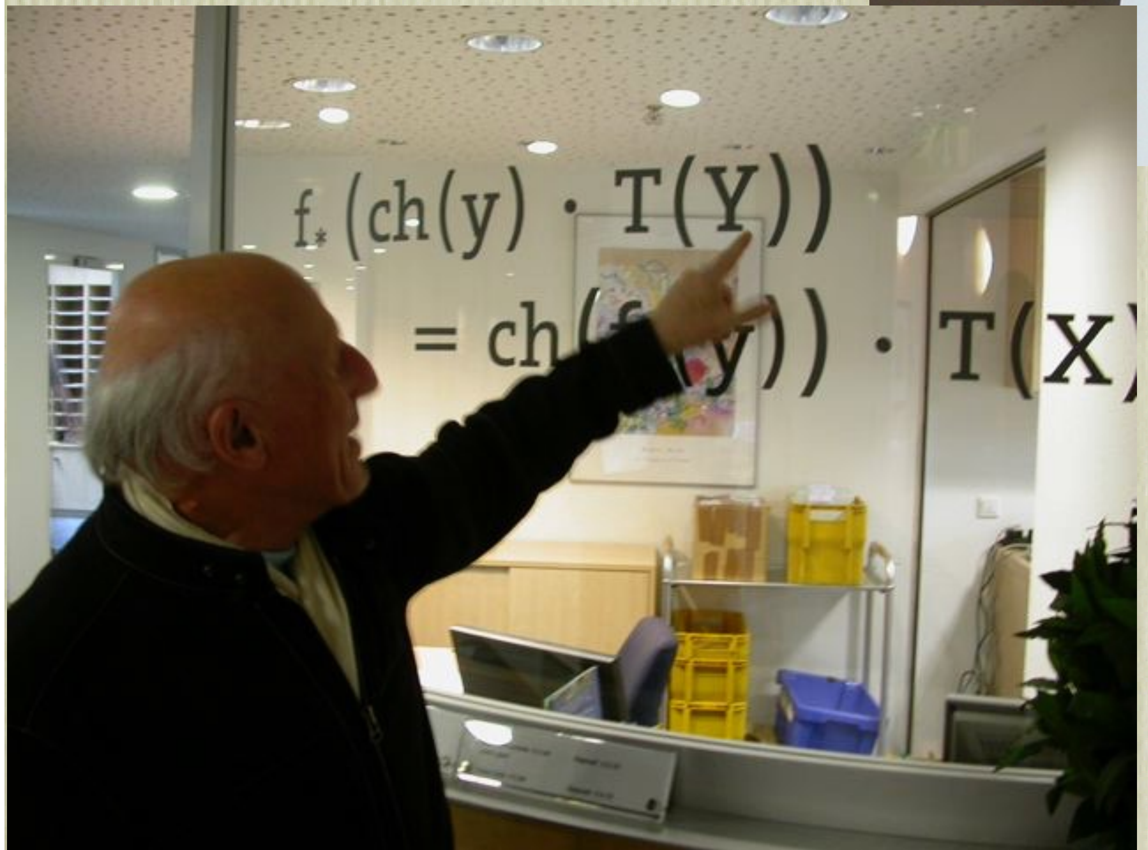
Chapter 3
Heaps and Paths, Flows and Rearrangements monoids
(1)

IMSc, Chennai

30 January 2017

"Commutation and Rearrangements" (1969)

Pierre Cartier



Dominique Foata

rearrangements monoid
$R(X)$

flows monoid
$F(X)$

The flows monoid

- X set
- $\underset{\text{basic pieces}}{P} = \underset{\text{alphabet}}{A} = \{(i,j)\}_{\substack{i \in X \\ j \in X}}$

flow monoid
(on X)

$F(X)$

- X set
- $\underset{\text{basic pieces}}{P} = \underset{\text{alphabet}}{A} = \{(i,j)\}_{\substack{i \in X \\ j \in X}}$
- $\mathcal{C}$ dependency relation: (or concurrency)

$$(i,j)\,\mathcal{C}\,(i',j') \iff i = i'$$

C commutations

$$(i,j)(i',j') = (i',j')(i,j) \text{ iff } i \neq i'$$

flow monoid (on X)

$F(X)$

$A = X \times X$

$\binom{i}{j}$

biword

$$w = \begin{pmatrix} 1 & 3 & 2 & 3 & 1 & 3 & 2 & 1 & 3 \\ 3 & 3 & 3 & 3 & 1 & 1 & 1 & 2 & 2 \end{pmatrix}$$

flow

$[w]$ equivalence class of biwords for C

$A = X \times X$

$\binom{i}{j}$

biword

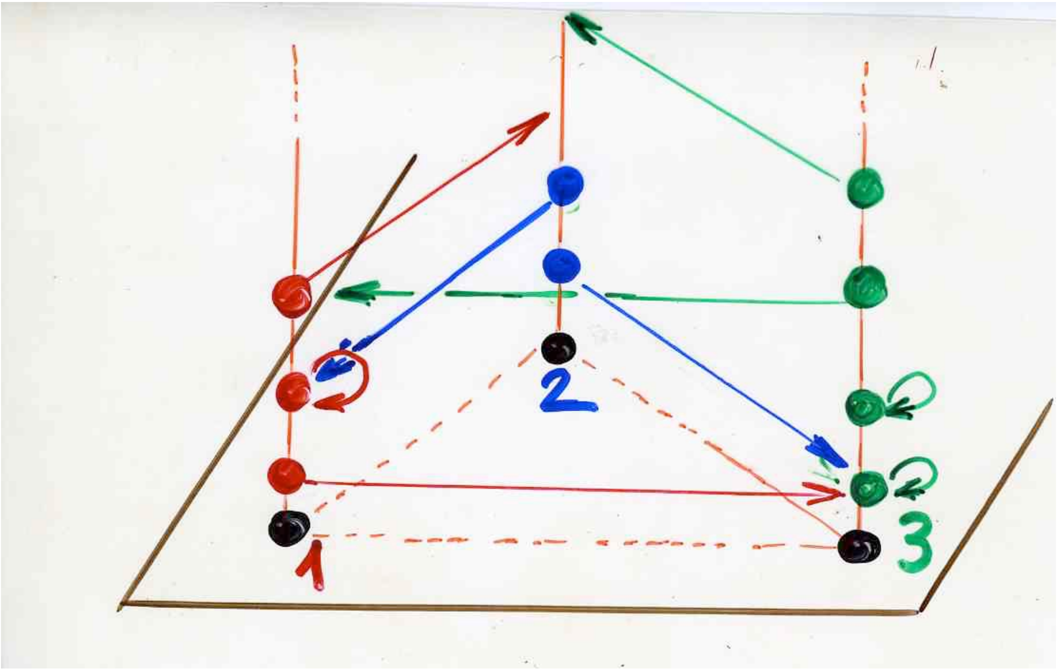
$$w = \begin{pmatrix} 1 & 3 & 2 & 3 & 1 & 3 & 2 & 1 & 3 \\ 3 & 3 & 3 & 3 & 1 & 1 & 1 & 2 & 2 \end{pmatrix}$$

total order on X

$$w \equiv_{c} \overrightarrow{w}$$

$$\underset{\text{biword}}{\overrightarrow{w}} = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\ 3 & 1 & 2 & 3 & 1 & 3 & 3 & 1 & 2 \end{pmatrix}$$

$X = \{1, 2, 3\}$



$A = X \times X$

heap of "half-edges"
(i, j) for $\mathcal{E}$

$$\vec{w} = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\ 3 & 1 & 2 & 3 & 1 & 3 & 3 & 1 & 2 \end{pmatrix}$$

biword

flow monoid
(on X)

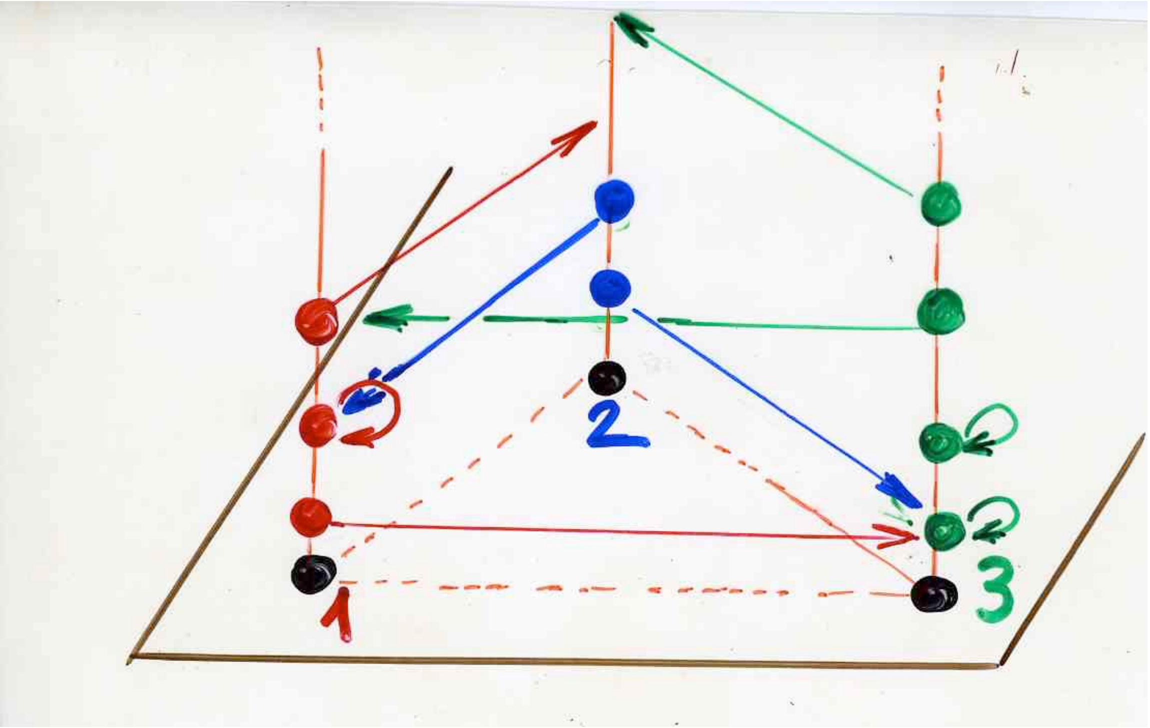
$$F(X) \simeq \prod_{s \in X} X_s^{*}$$

direct product

free monoid

$$X_s = \{ (s, t) \}_{t \in X}$$

$$F(\{1,2,3\}) = X_1^* \times X_2^* \times X_3^*$$



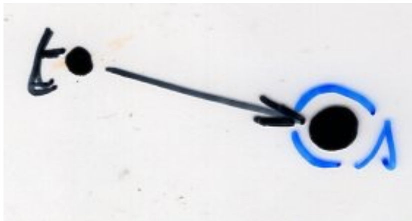
$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 1 & 2 \end{pmatrix}$$

Φ flow $F(X)$

for $s \in X$

$$\deg^{+}_{\Phi}(s) = \left\{ \begin{array}{l} \text{number of edges } \binom{s}{t} \\ t \in X, \ \text{in } \Phi \end{array} \right\}$$

$$\deg^{-}_{\Phi}(s) = \left\{ \begin{array}{l} \text{number of edges } \binom{t}{s} \\ t \in X \ \text{ in } \Phi \end{array} \right\}$$



Φ flow

$\Phi \in F(X)$

<u>definition</u>

Φ rearrangement iff
for any $s \in X$
$\deg^{+}_{\Phi}(s) = \deg^{-}_{\Phi}(s)$

$R(X)$ submonoid
of $F(X)$

$\mathbf{R}(X) \subseteq \mathbf{F}(X)$

"permutation"
with repetition of letters

word

3 1 2 3 1 3 3 1 2

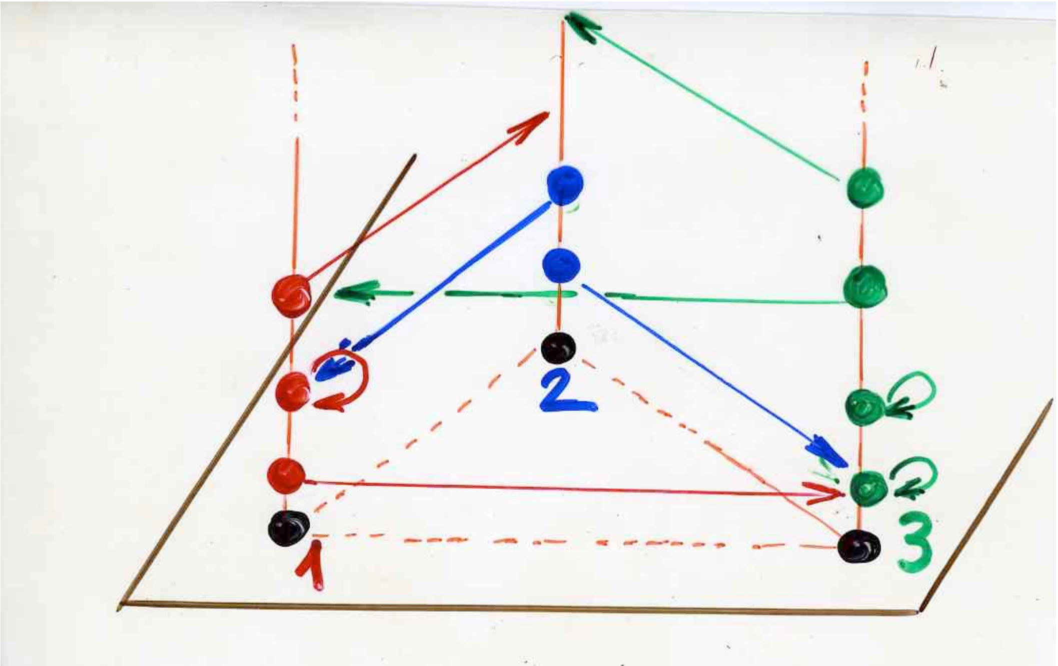
total order
on X

$$\begin{matrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\ 3 & 1 & 2 & 3 & 1 & 3 & 3 & 1 & 2 \end{matrix}$$

rearrangement

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\ 3 & 1 & 2 & 3 & 1 & 3 & 3 & 1 & 2 \end{pmatrix}$$

$$X = \{1, 2, 3\}$$



$$A = X \times X$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\ 3 & 1 & 2 & 3 & 1 & 3 & 3 & 1 & 2 \end{pmatrix}$$

paths and flows monoid

path on X

$$\omega = (s_0, \dots, s_i, s_{i+1}, \dots, s_n)$$

$$s_i \in X \quad i=0,\dots,n$$

ω goes from s_0 to s_n

path on a graph G (oriented or not)

(s_i, s_{i+1}) edge of G

notation $\quad s_0 \xrightarrow{\omega} s_n$

s_0 starting vertex

s_n ending vertex

(s_i, s_{i+1}) elementary step

length $|\omega| = n$

(number of elementary steps)

$n+1$ vertices

weight

$$v(\omega) = \prod_{0 \le i \le n-1} v(s_i, s_{i+1})$$

$$v : \quad X \times X \longrightarrow \mathbb{K}[z]$$

"formal variables"

$\mathbb{K}$ ring

$$X = [1, k]$$

$$a_{i,j} = v(i, j)$$

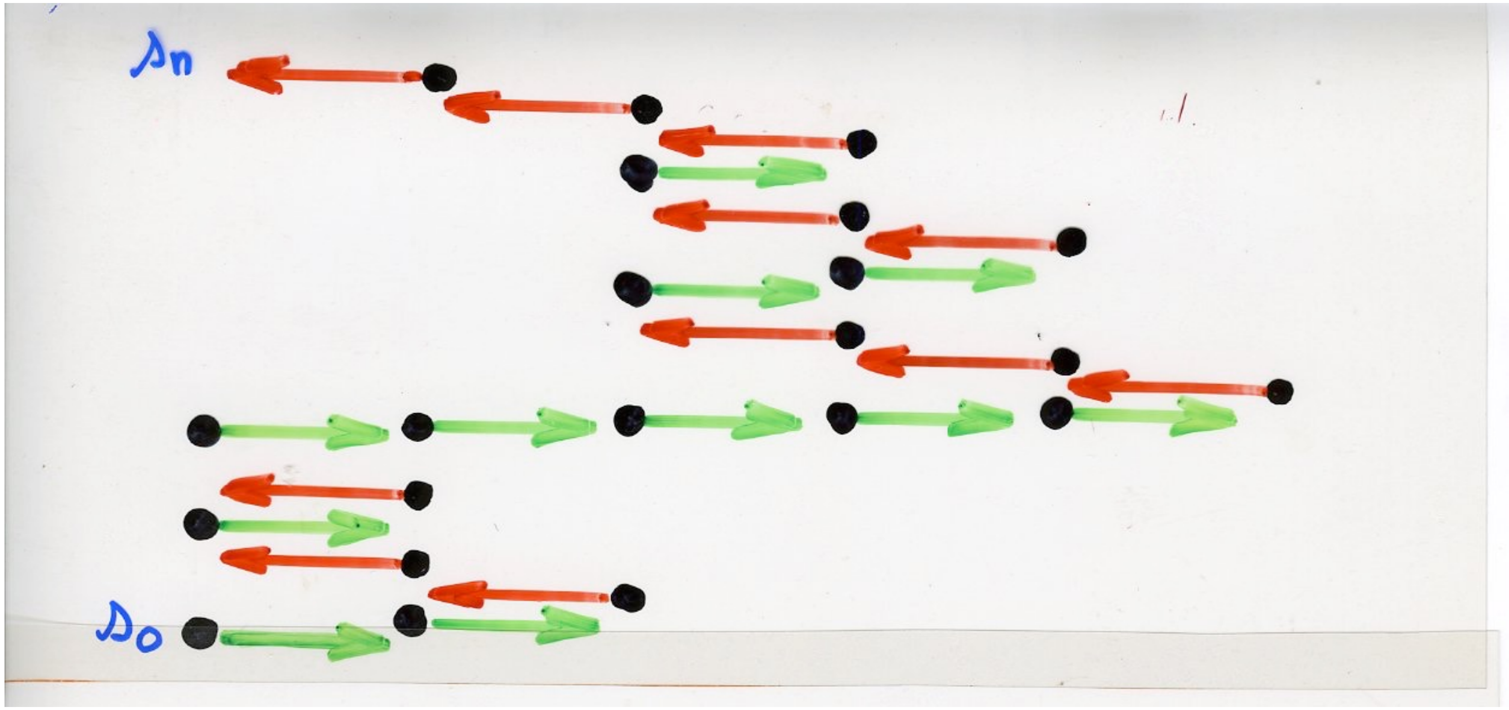
$$A = (a_{i,j})_{1 \le i,j \le k}$$

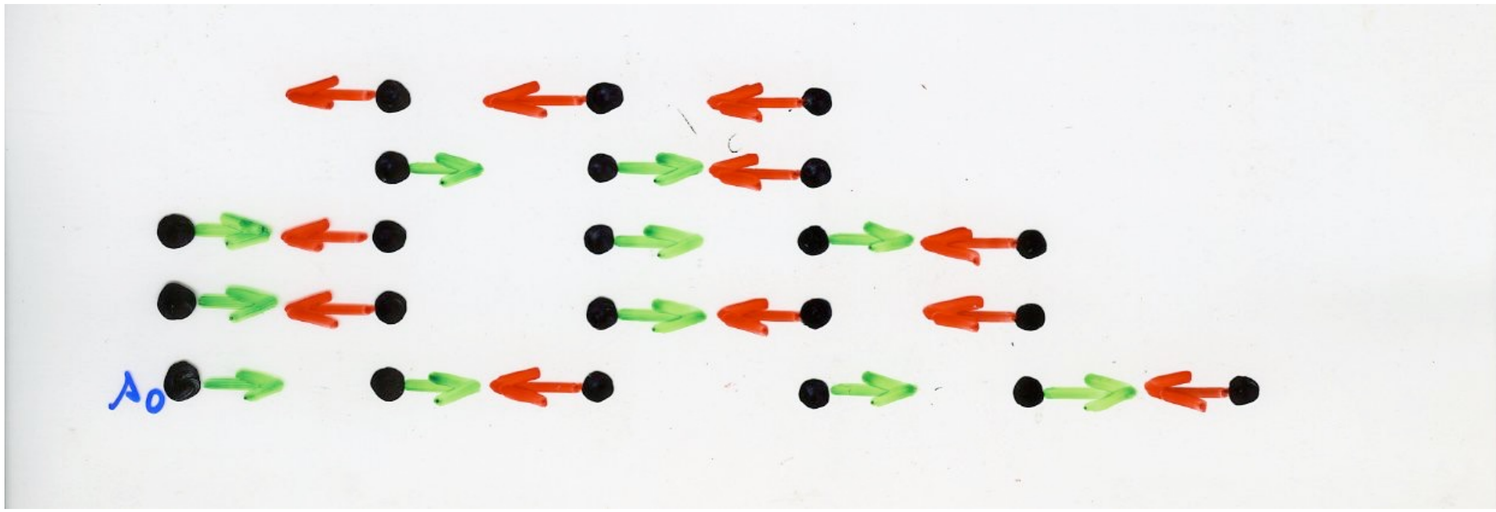
Path ω on X

$$\omega \longrightarrow f(\omega) \in F(X)$$

$$f(\omega) = \begin{bmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \end{bmatrix}$$

commutation class


Δn
Δo


Δo

can we reconstruct the path ω from the flow $\Phi = f(\omega)$?

Yes, knowing s_0 the starting point

algorithm "following" a flow $\Phi \in F(X)$

$(s, \Phi) \xrightarrow{h} \omega$ path on X

algorithm
« following a flow »

algorithm "following"
a flow $\Phi \in F(X)$

$(s, \Phi) \xrightarrow{h} \omega$ path on X

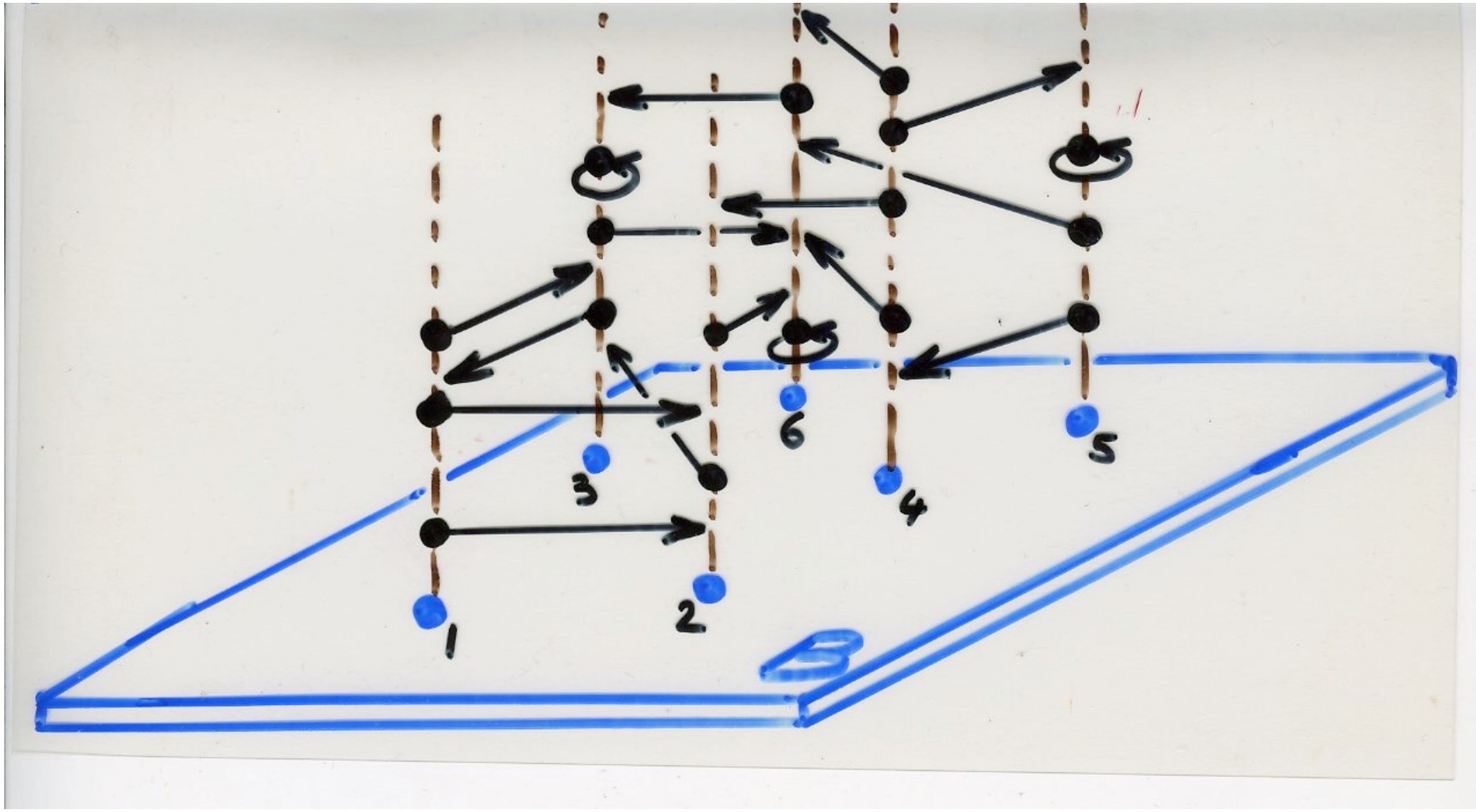
$$\omega = (s_0, \dots, s_n)$$

- starting at $s_0 = s$
- ending at s_n with $\deg^{+}(s_n) = 0$
- the flow $f(\omega)$ is a left divisor of Φ, i.e. $\Phi = f(\omega)\Psi$ in $F(X)$

for a certain Ψ in $F(X)$

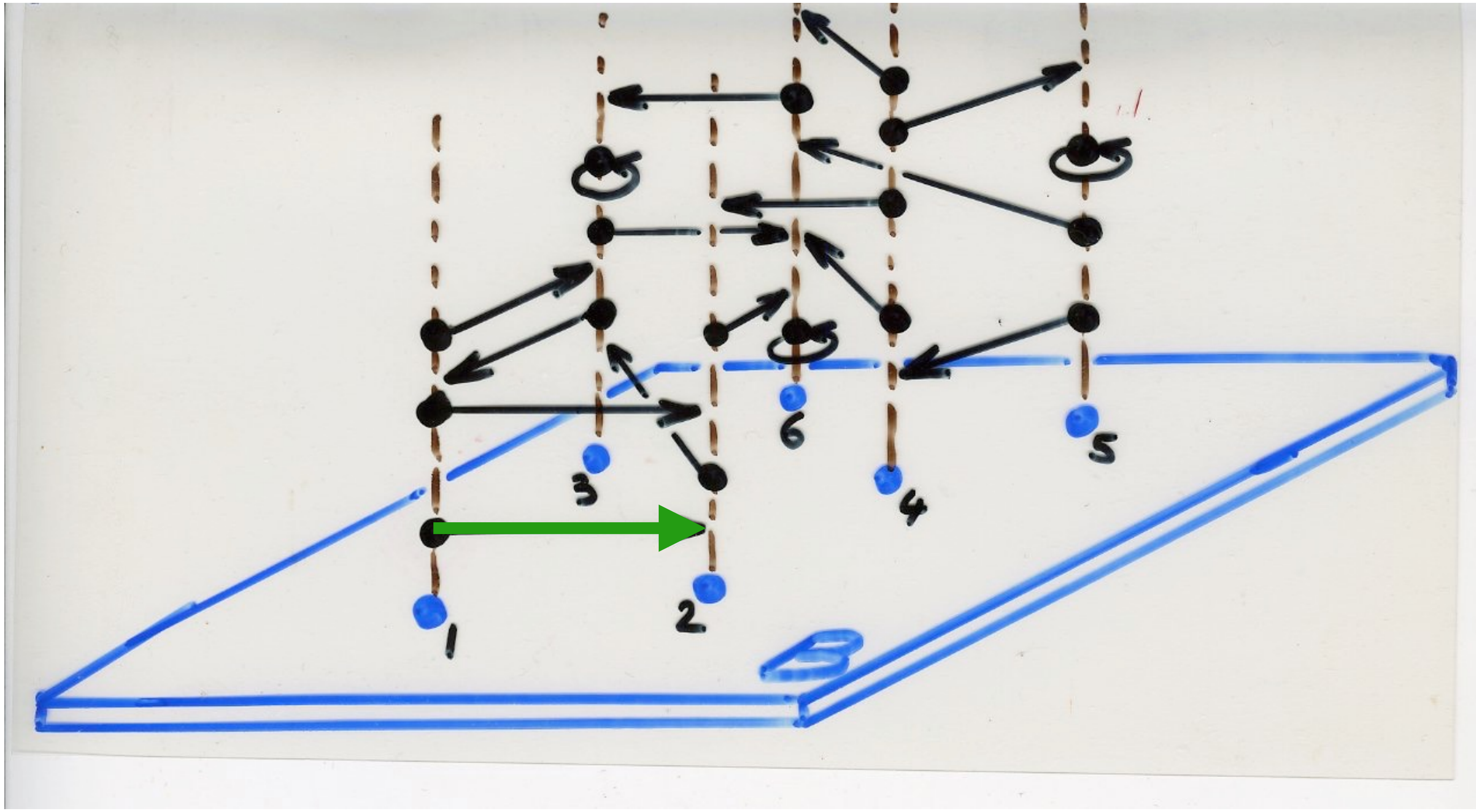
algorithm "following"

a flow $\Phi \in F(X)$



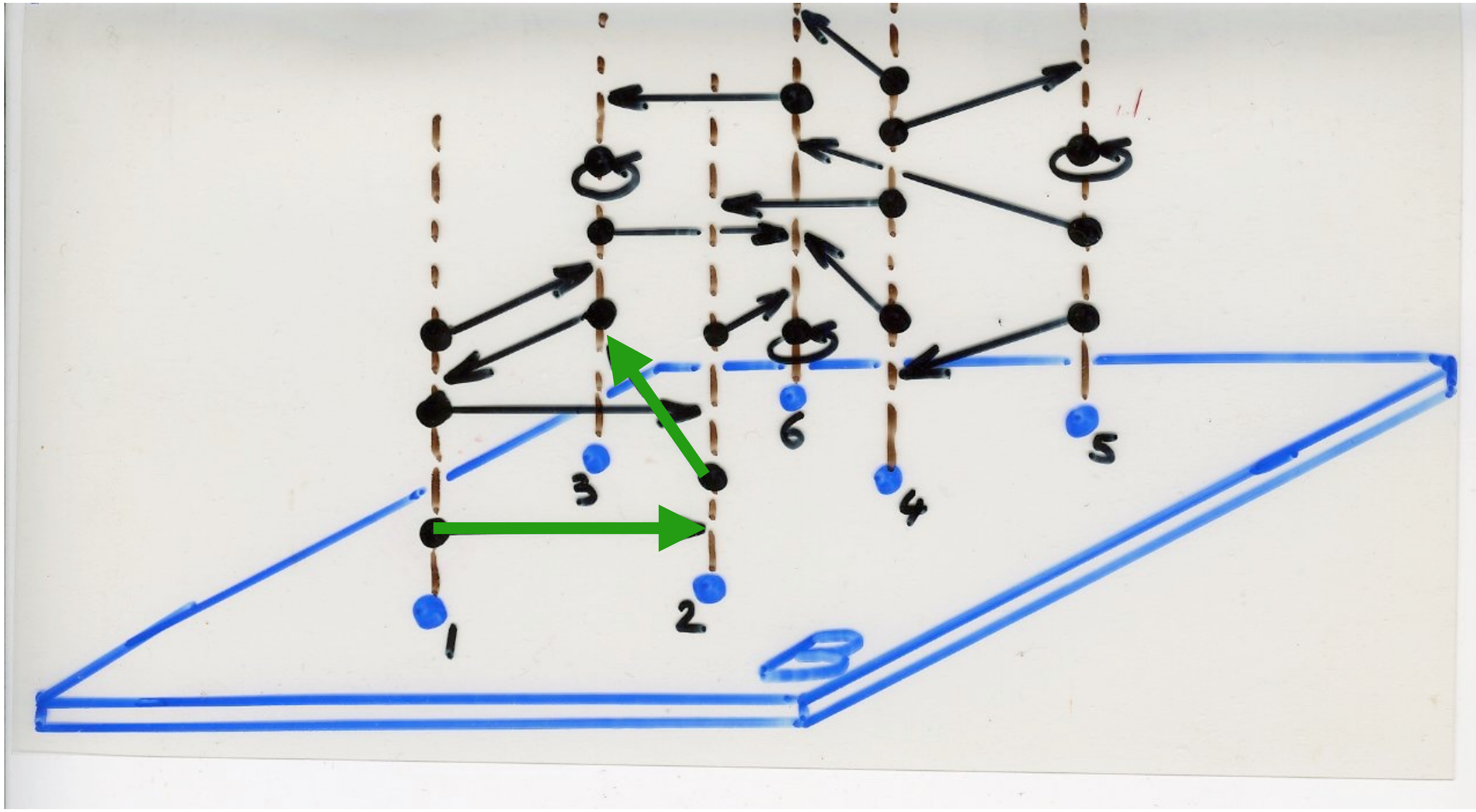
$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 3 & 3 & 3 \\ 1 & 6 & 3 \end{pmatrix}\begin{pmatrix} 4 & 4 & 4 & 4 \\ 6 & 2 & 5 & 6 \end{pmatrix}\begin{pmatrix} 5 & 5 & 5 \\ 4 & 6 & 5 \end{pmatrix}\begin{pmatrix} 6 & 6 \\ 6 & 3 \end{pmatrix}$$

algorithm "following"
a flow $\Phi \in F(X)$



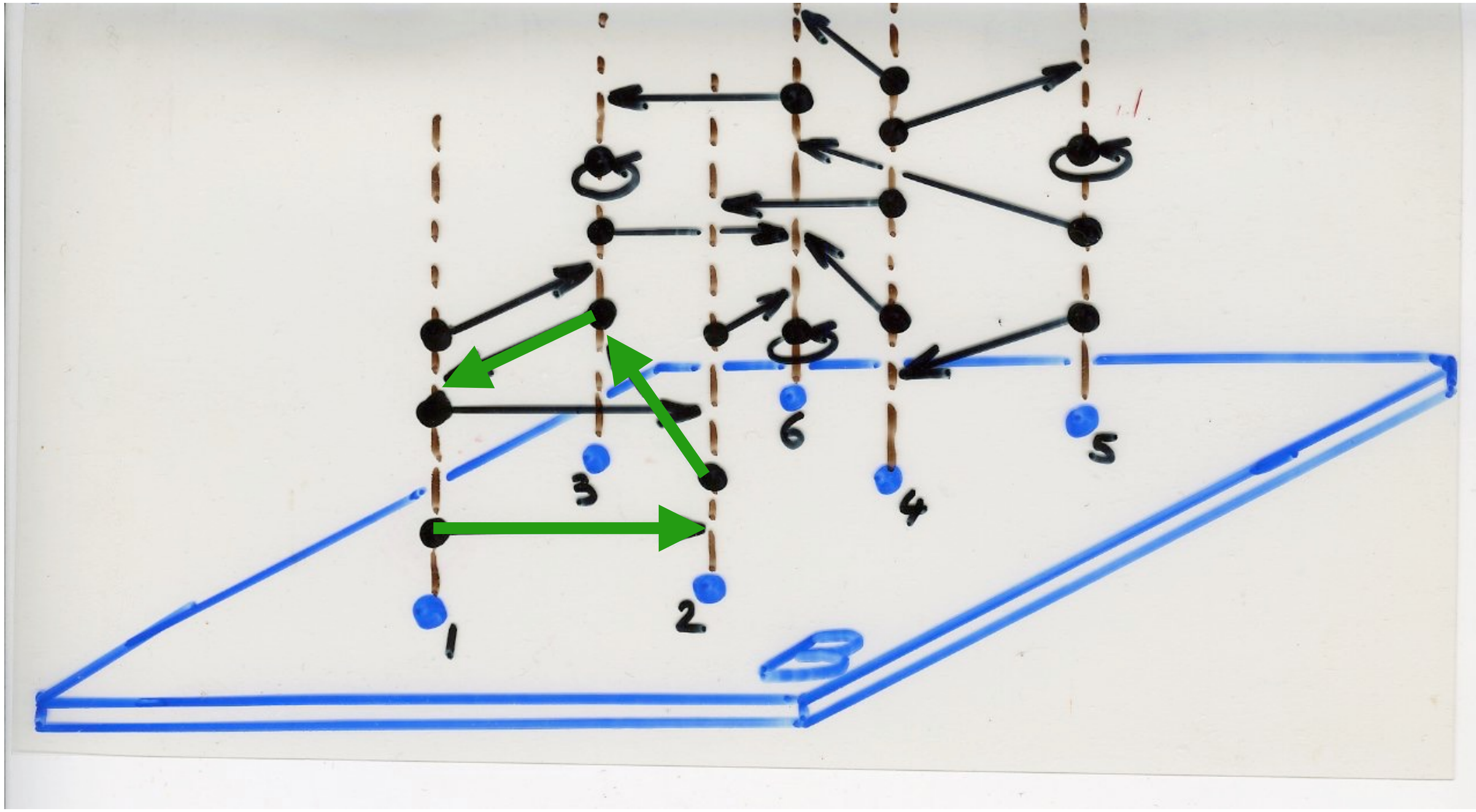
$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 3 & 3 & 3 \\ 1 & 6 & 3 \end{pmatrix}\begin{pmatrix} 4 & 4 & 4 & 4 \\ 6 & 2 & 5 & 6 \end{pmatrix}\begin{pmatrix} 5 & 5 & 5 \\ 4 & 6 & 5 \end{pmatrix}\begin{pmatrix} 6 & 6 \\ 6 & 3 \end{pmatrix}$$

algorithm "following"
a flow $\Phi \in F(X)$



$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 3 & 3 & 3 \\ 1 & 6 & 3 \end{pmatrix}\begin{pmatrix} 4 & 4 & 4 & 4 \\ 6 & 2 & 5 & 6 \end{pmatrix}\begin{pmatrix} 5 & 5 & 5 \\ 4 & 6 & 5 \end{pmatrix}\begin{pmatrix} 6 & 6 \\ 6 & 3 \end{pmatrix}$$

algorithm "following" a flow $\Phi \in F(X)$



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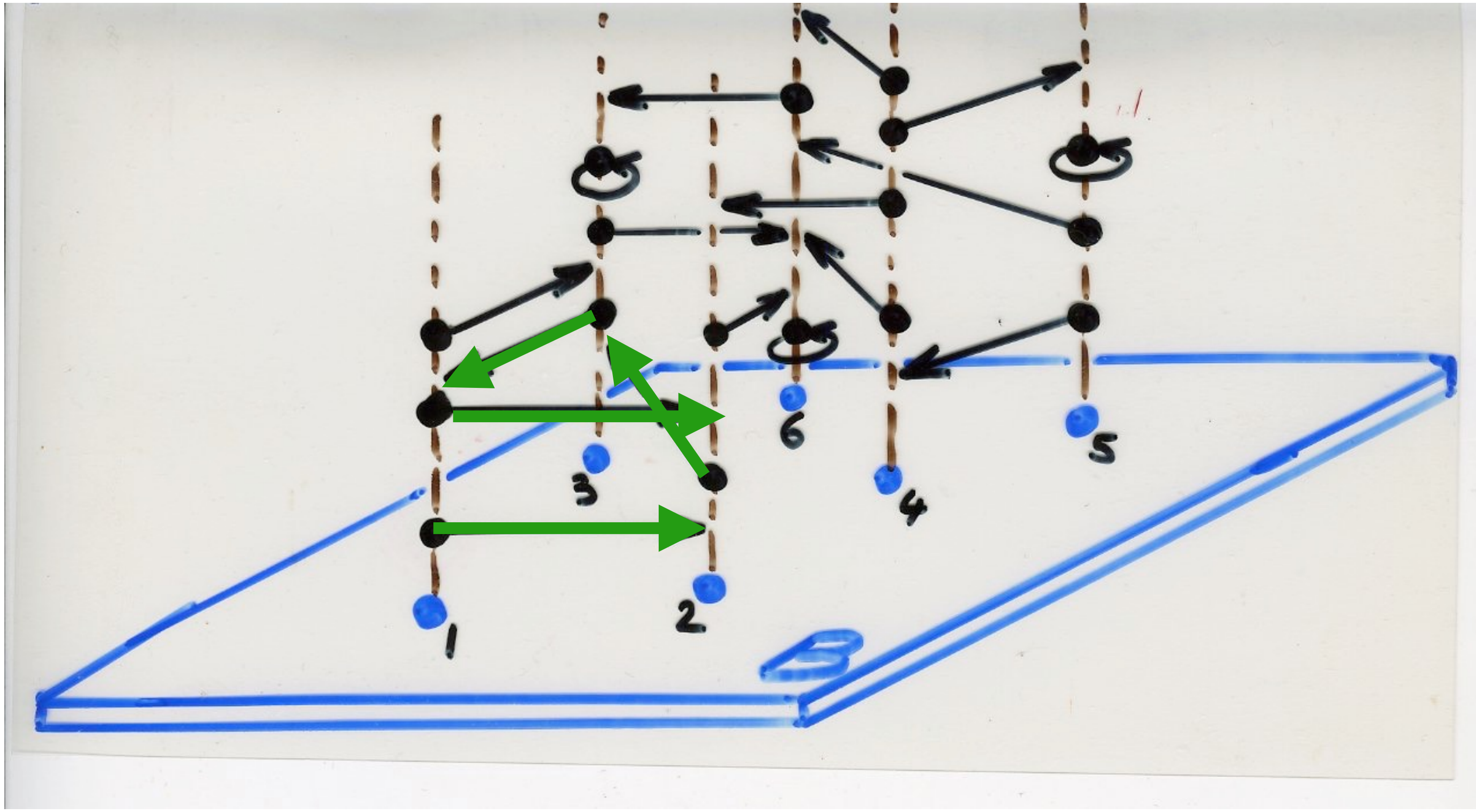
algorithm "following"
a flow $\Phi \in F(X)$



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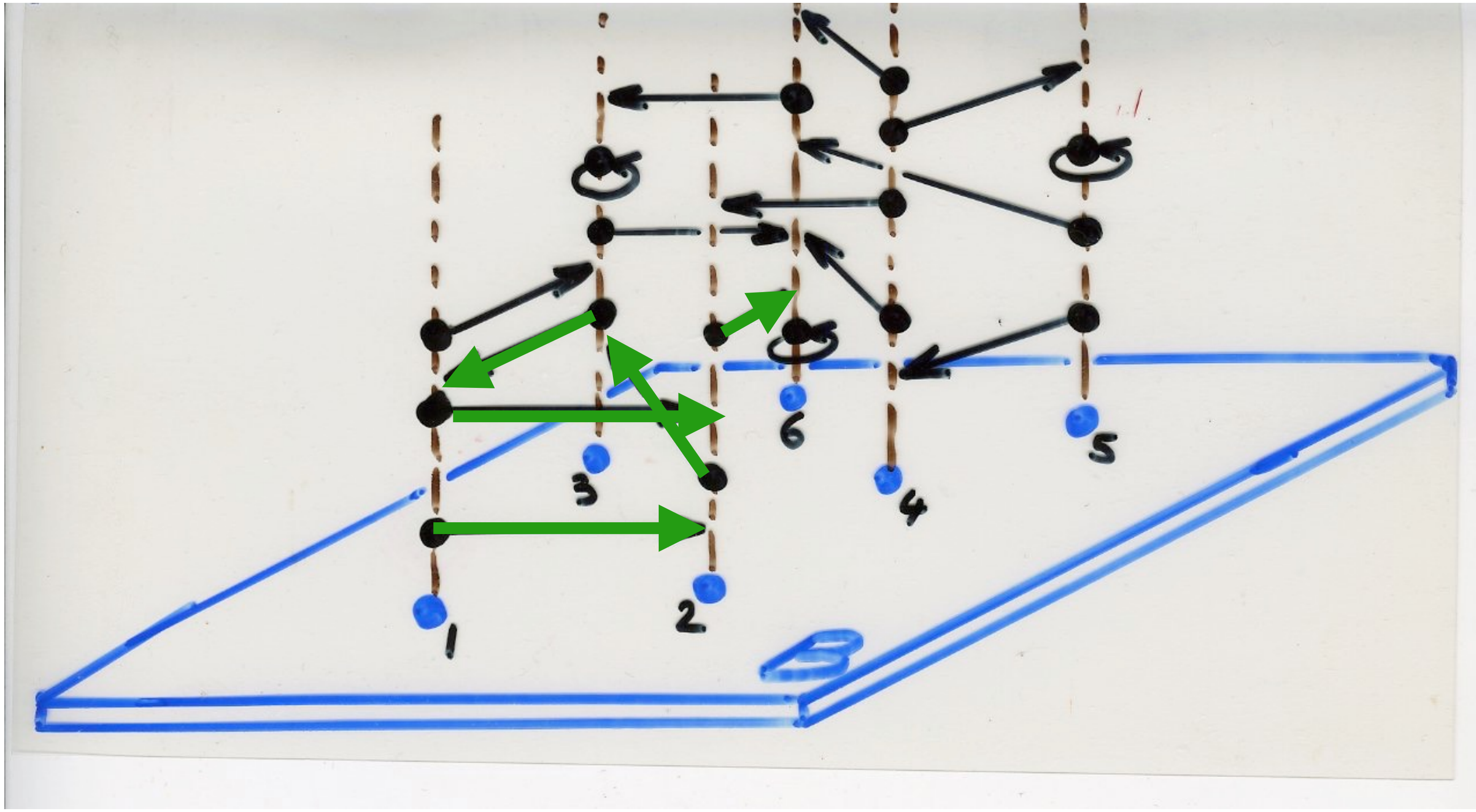
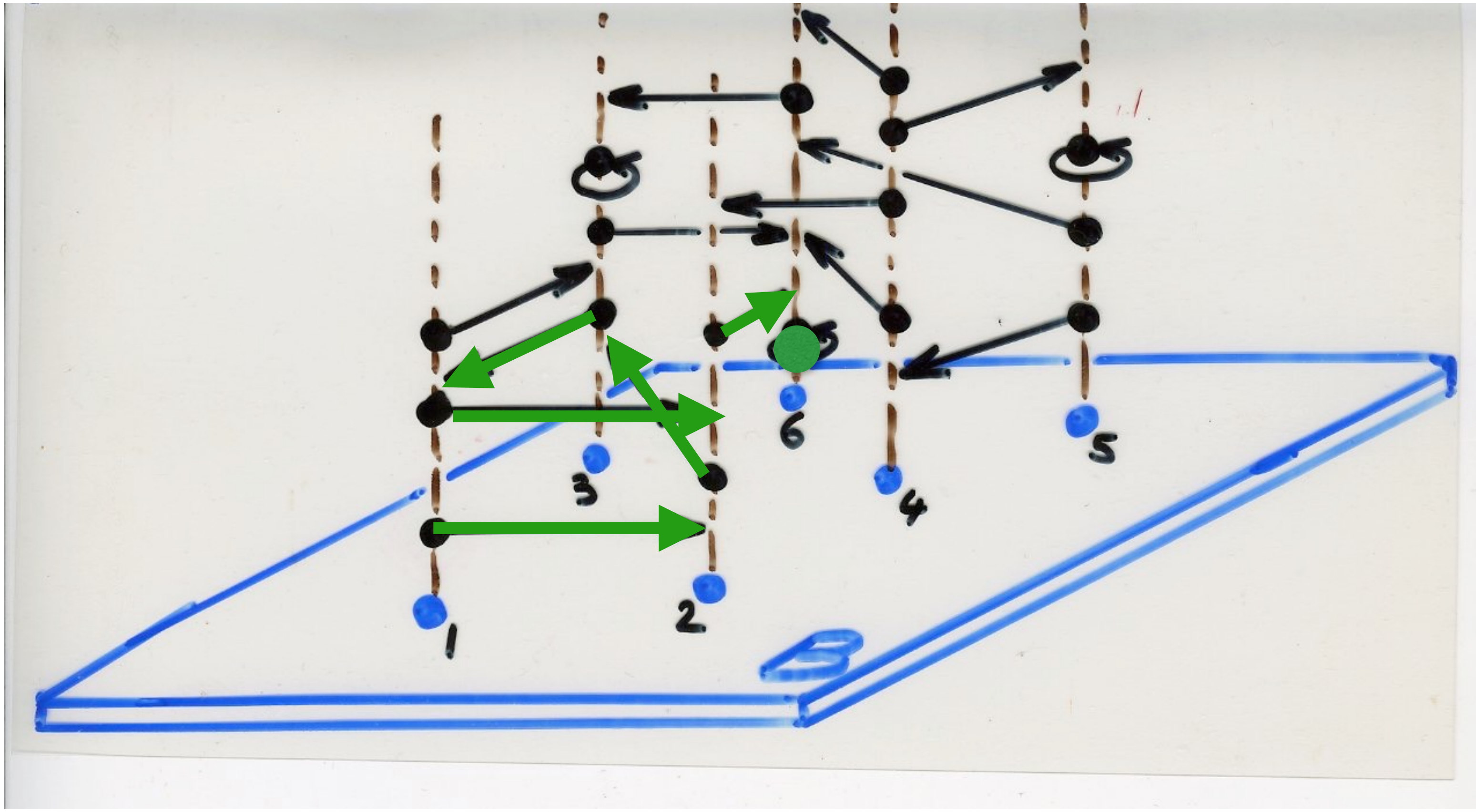
algorithm "following"
a flow $\Phi \in F(X)$



$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 3 & 3 & 3 \\ 1 & 6 & 3 \end{pmatrix}\begin{pmatrix} 4 & 4 & 4 & 4 \\ 6 & 2 & 5 & 6 \end{pmatrix}\begin{pmatrix} 5 & 5 & 5 \\ 4 & 6 & 5 \end{pmatrix}\begin{pmatrix} 6 & 6 \\ 6 & 3 \end{pmatrix}$$

algorithm "following" a flow $\Phi \in F(X)$



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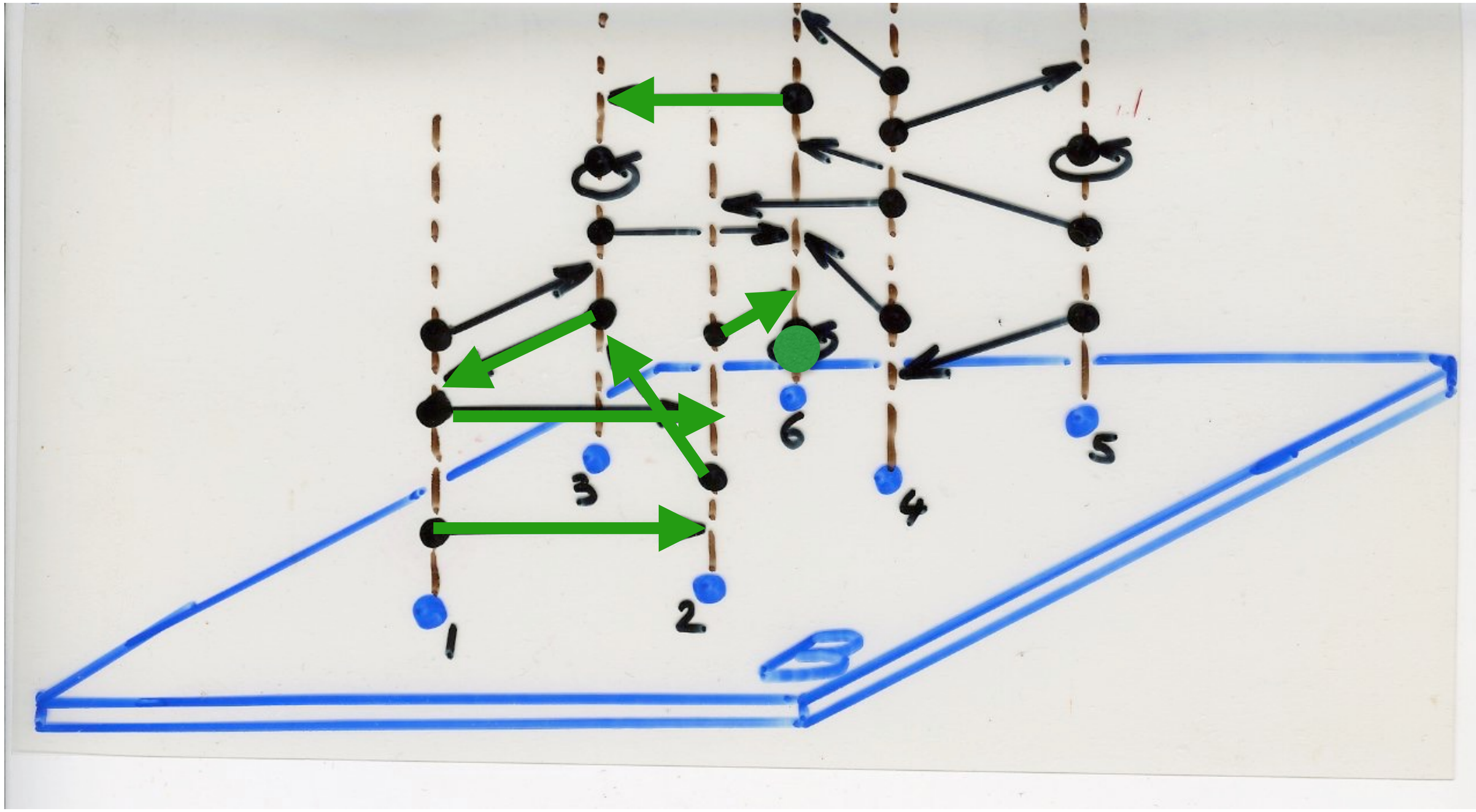
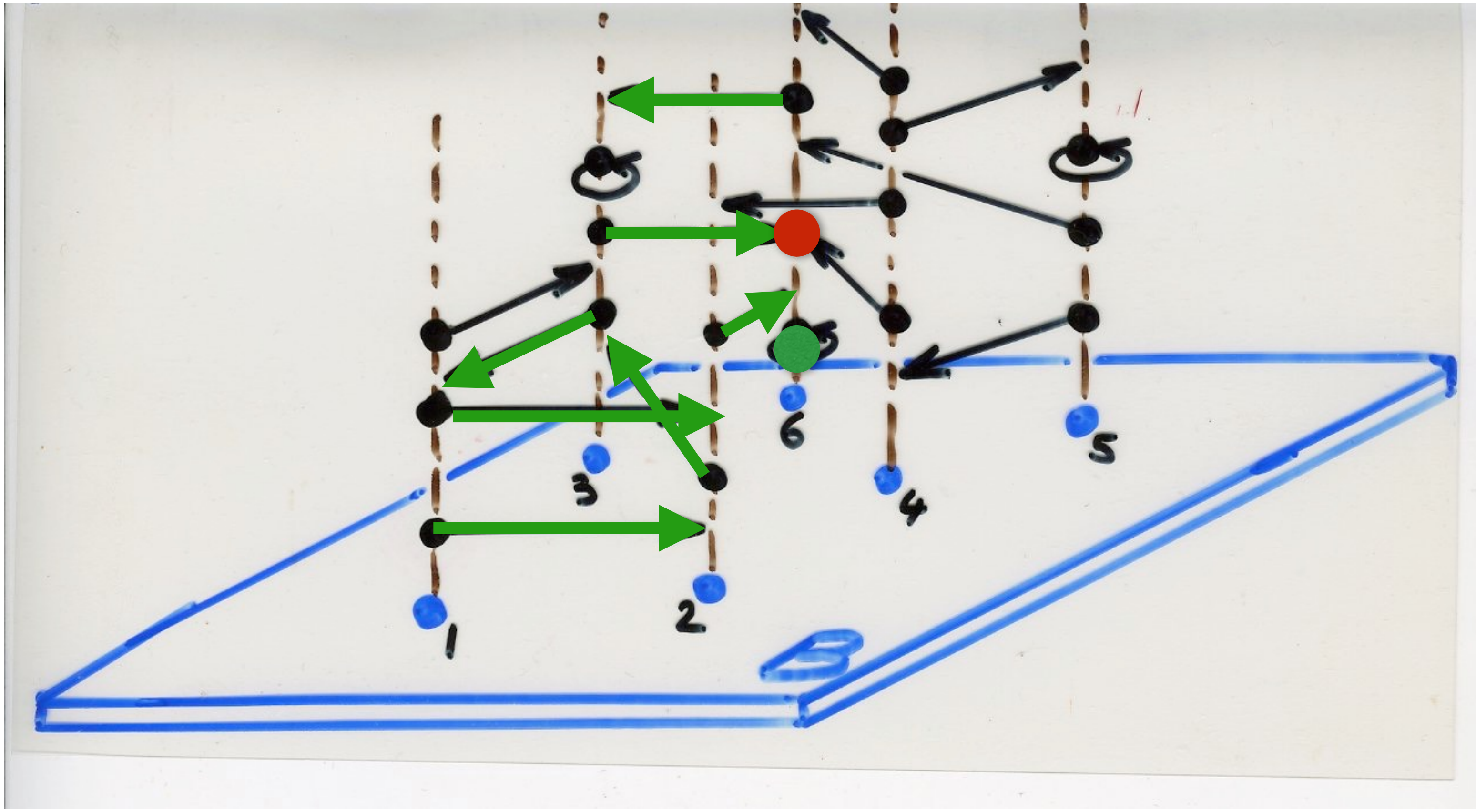
algorithm "following" a flow $\Phi \in F(X)$



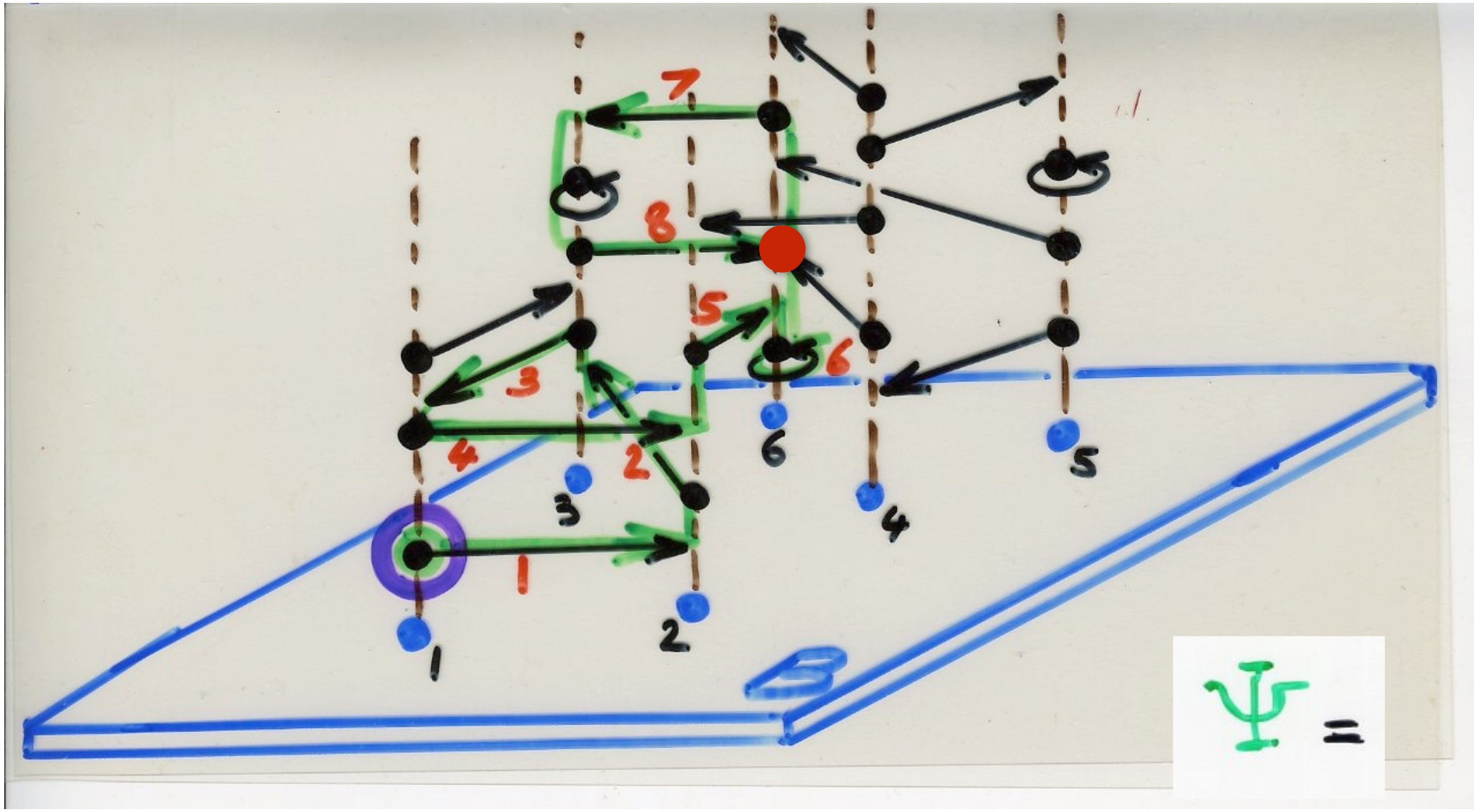
$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 3 & 3 & 3 \\ 1 & 6 & 3 \end{pmatrix}\begin{pmatrix} 4 & 4 & 4 & 4 \\ 6 & 2 & 5 & 6 \end{pmatrix}\begin{pmatrix} 5 & 5 & 5 \\ 4 & 6 & 5 \end{pmatrix}\begin{pmatrix} 6 & \boxed{\begin{matrix} 6 \\ 3 \end{matrix}} \\ 6 & \end{pmatrix}$$

algorithm "following"
a flow $\Phi \in F(X)$



$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 3 & \boxed{\begin{matrix} 3 \\ 6 \end{matrix}} & 3 \\ 1 & & 3 \end{pmatrix}\begin{pmatrix} 4 & 4 & 4 & 4 \\ 6 & 2 & 5 & 6 \end{pmatrix}\begin{pmatrix} 5 & 5 & 5 \\ 4 & 6 & 5 \end{pmatrix}\begin{pmatrix} 6 & 6 \\ 6 & 3 \end{pmatrix}$$

$$h(1, \Phi) = (1, 2, 3, 1, 2, 6, 6, 3, 6)$$



$$\Psi = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 3 & 3 & 3 \\ 1 & 6 & 3 \end{pmatrix} \begin{pmatrix} 4 & 4 & 4 & 4 \\ 6 & 2 & 5 & 6 \end{pmatrix} \begin{pmatrix} 5 & 5 & 5 \\ 4 & 6 & 5 \end{pmatrix} \begin{pmatrix} 6 & 6 \\ 6 & 3 \end{pmatrix}$$

Path ω on X

$$\omega \longrightarrow f(\omega) \in F(X)$$

$$f(\omega) = \begin{bmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \end{bmatrix}$$

commutation class

can we reconstruct the path ω from the flow $\Phi = f(\omega)$?

Yes, knowing s_0 the starting point

$$h(s_0, f(\omega)) = \omega$$

let $\omega = (s_0, \dots, s_i, \dots, s_n)$ path on X with $s_0 \neq s_n$

can we reconstruct the path ω from the flow $\underline{\Phi} = f(\omega)$?

Yes !

(i) • s_0 is the unique element $s \in X$ such that $\deg^+_\Phi(s) - \deg^-_\Phi(s) = 1$

(ii) • s_n is the unique element $s \in X$ such that $\deg^+_\Phi(s) - \deg^-_\Phi(s) = -1$

(iii) • for all other $s \in X$
$\deg^+_\Phi(s) = \deg^-_\Phi(s)$

$$h(s_0, f(\omega)) = \omega$$

if $s_0 = s_n$ (ω is a circuit)

we need to know $s_0 = s_n$

$f(\omega)$ is a rearrangement

(i) • s_0 is the unique element $s \in X$ such that $\deg^+_\Phi(s) - \deg^-_\Phi(s) = 1$

(ii) • s_n is the unique element $s \in X$ such that $\deg^+_\Phi(s) - \deg^-_\Phi(s) = -1$

(iii) • for all other $s \in X$
$$\deg^+_\Phi(s) = \deg^-_\Phi(s)$$

Conversely, any flow Φ satisfying (i)(ii)(iii) can be uniquely factorized as

$$\Phi = f(\omega)\,\Psi$$

(unique) path   rearrangement

obvious!

called "open hike"   (Giscard, Rochet, 2016)

Rearrangements and heaps of cycles

definition $\quad \Phi$ flow $F(X)$

Φ rearrangement iff for any $s \in X$

$$\deg^{+}_{\Phi}(s) = \deg^{-}_{\Phi}(s)$$

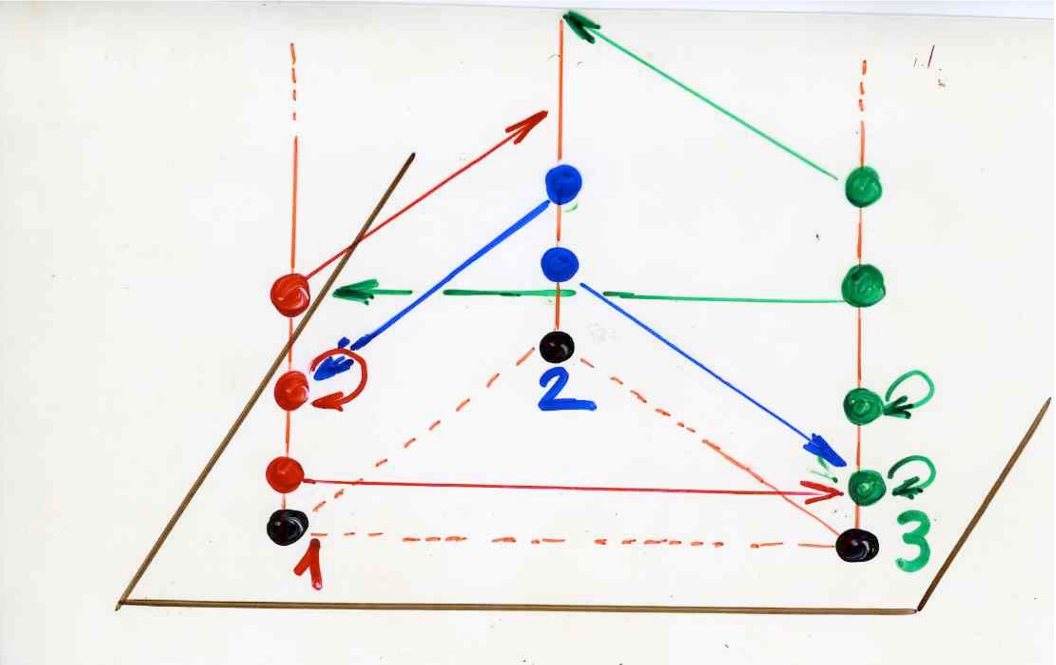
$$\deg^{+}_{\Phi}(s) = \left\{ \begin{array}{l} \text{number of edges } \binom{s}{t} \\ t \in X, \text{ in } \Phi \end{array} \right\}$$

$$\deg^{-}_{\Phi}(s) = \left\{ \begin{array}{l} \text{number of edges } \binom{t}{s} \\ t \in X \text{ in } \Phi \end{array} \right\}$$

$$\mathbf{R}(X) \subseteq \mathbf{F}(X)$$

$R(X)$ submonoid of $F(X)$

$X = \{1, 2, 3\}$



$A = X \times X$

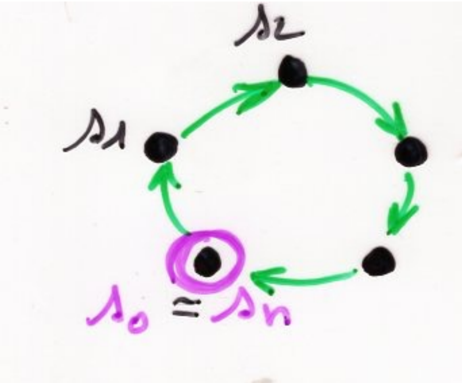
$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\ 3 & 1 & 2 & 3 & 1 & 3 & 3 & 1 & 2 \end{pmatrix}$$

<u>notation</u> $HC(X)$

heaps of cycles on X

monoid

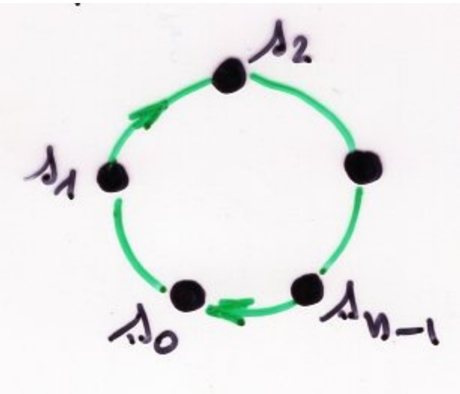
basic pieces : cycles on X

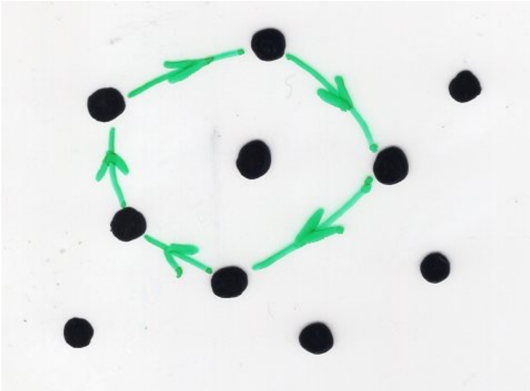


(from Chapter 2d)

elementary circuit $\omega = (s_0, \ldots, s_n)$
with $s_0 = s_n$, all vertices are disjoint
except $s_0 = s_n$.

Cycle = elementary circuit up to a
circular permutation of the vertices

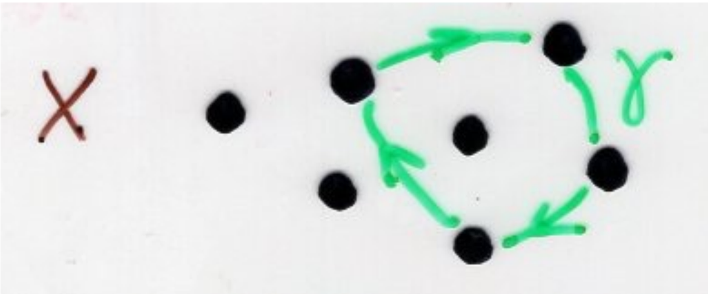




<u>notation</u> **HC**(X)

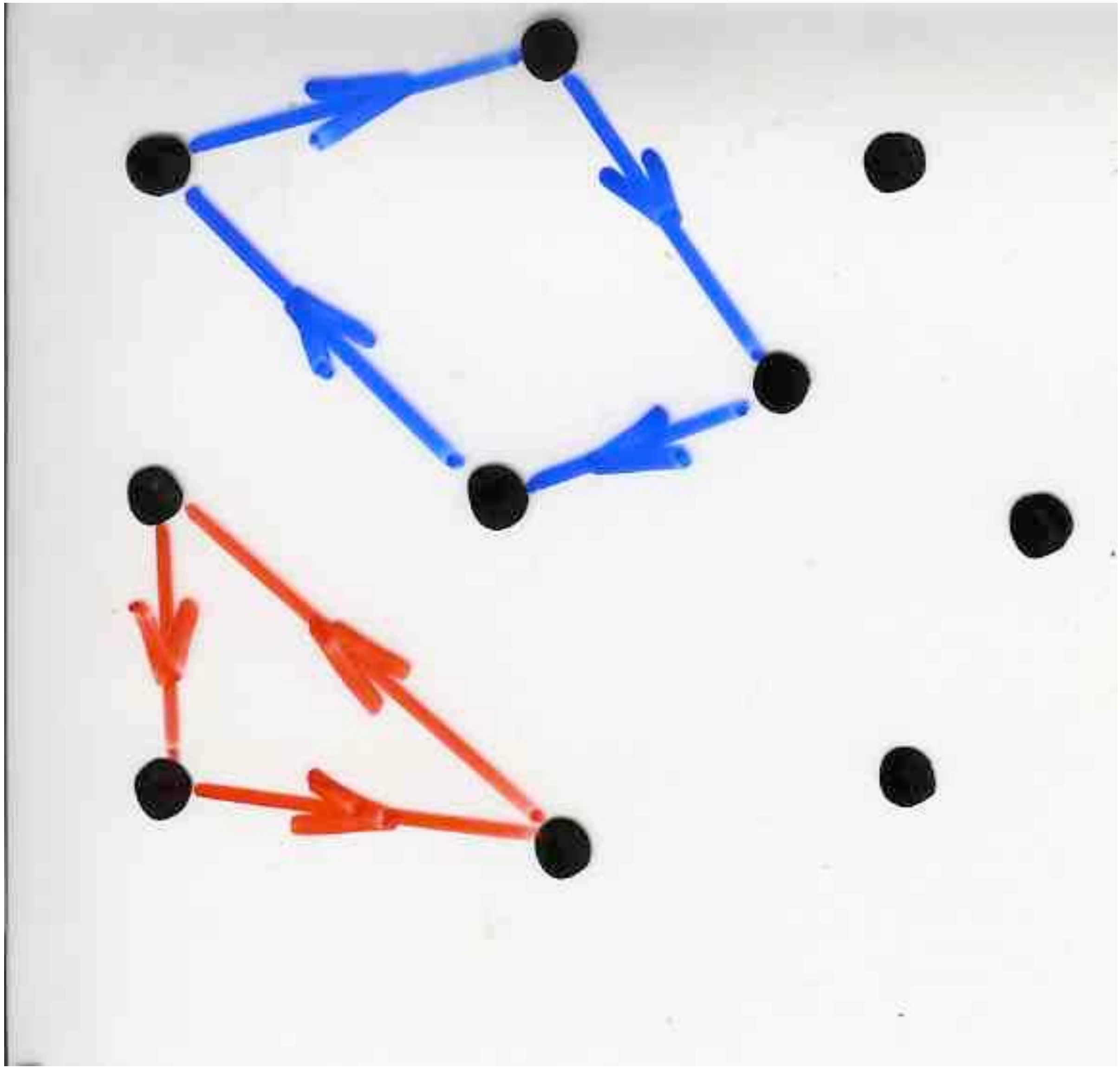
heaps of cycles on X
monoid

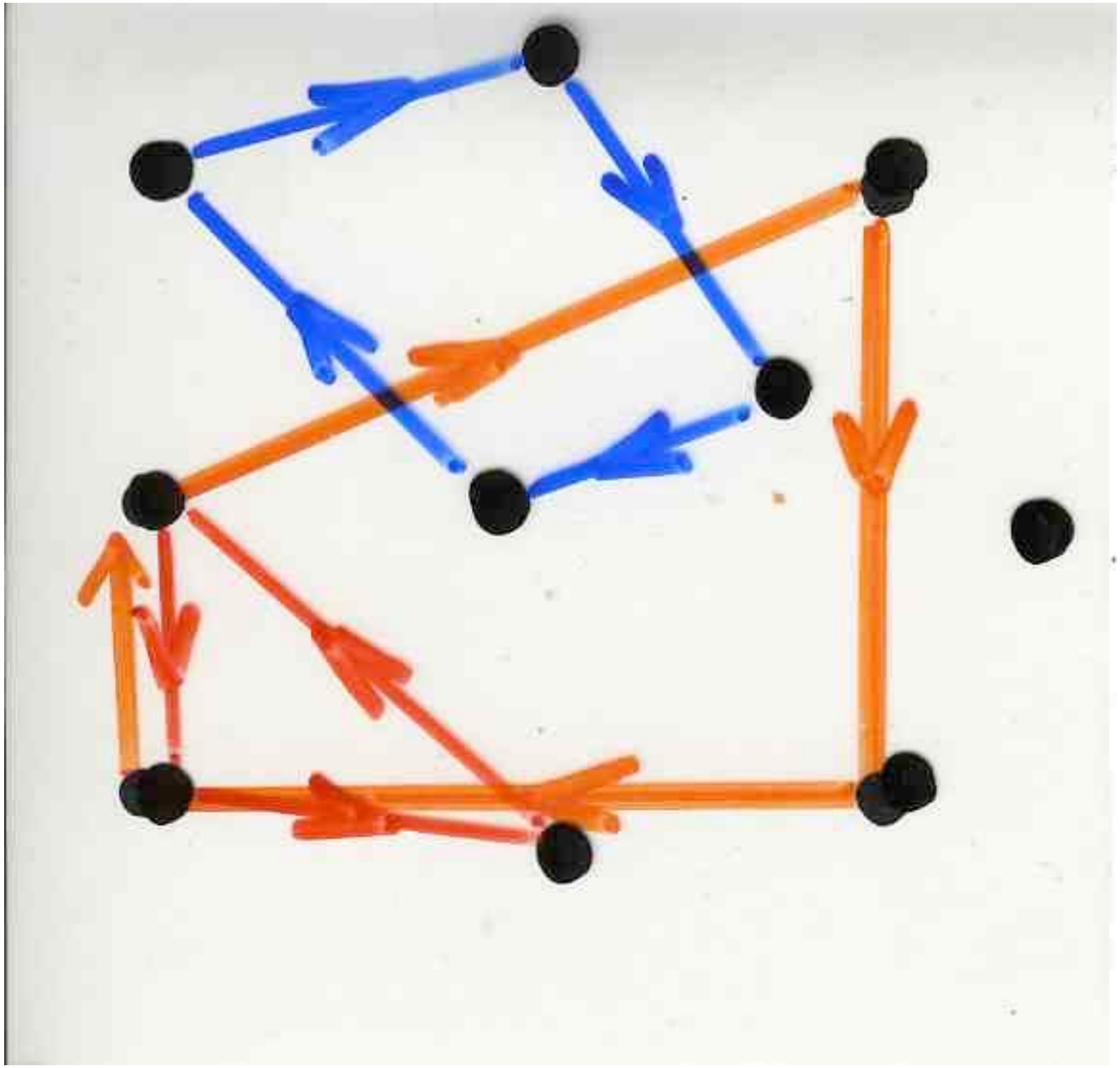
basic pieces : cycles on X

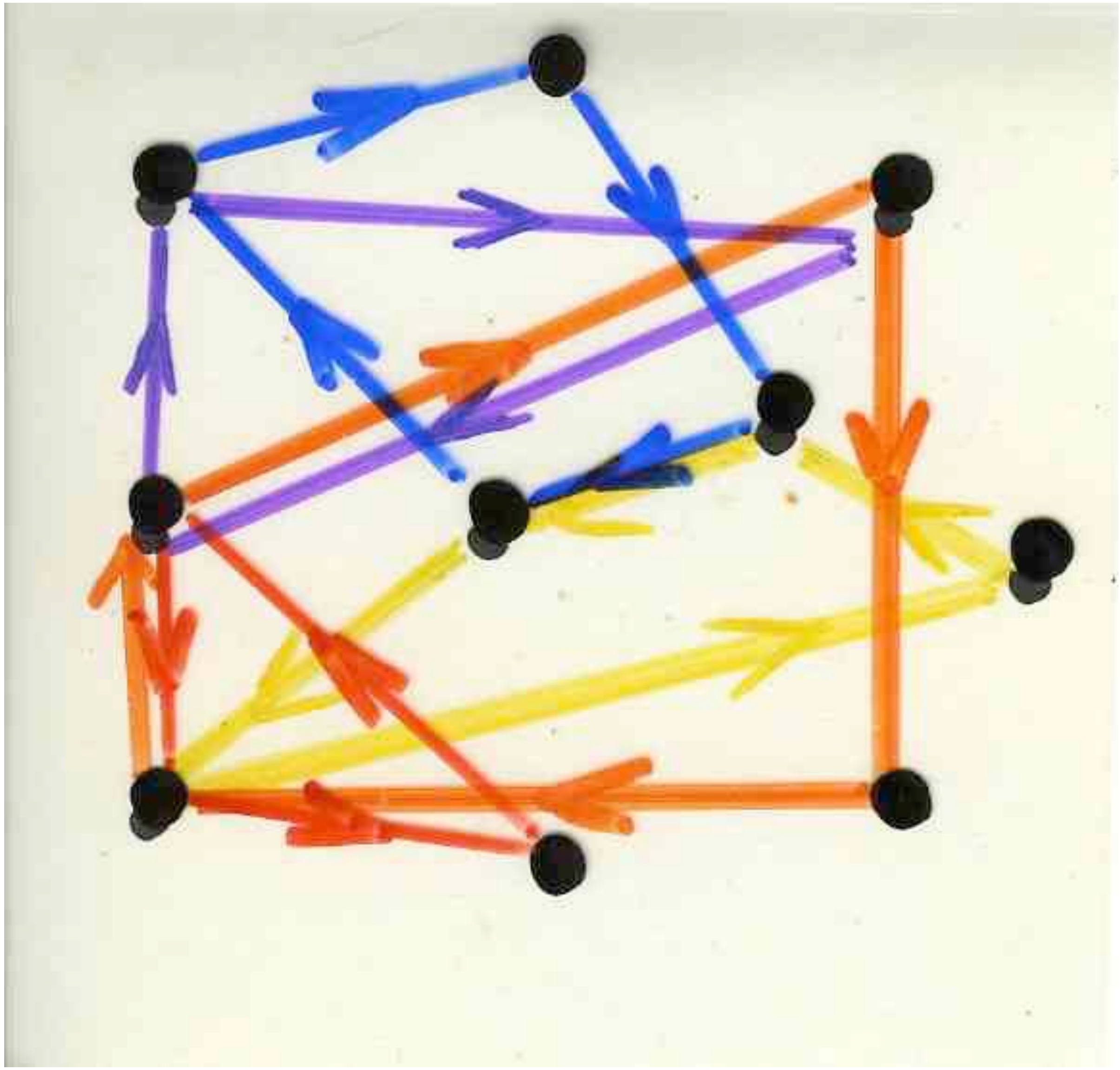


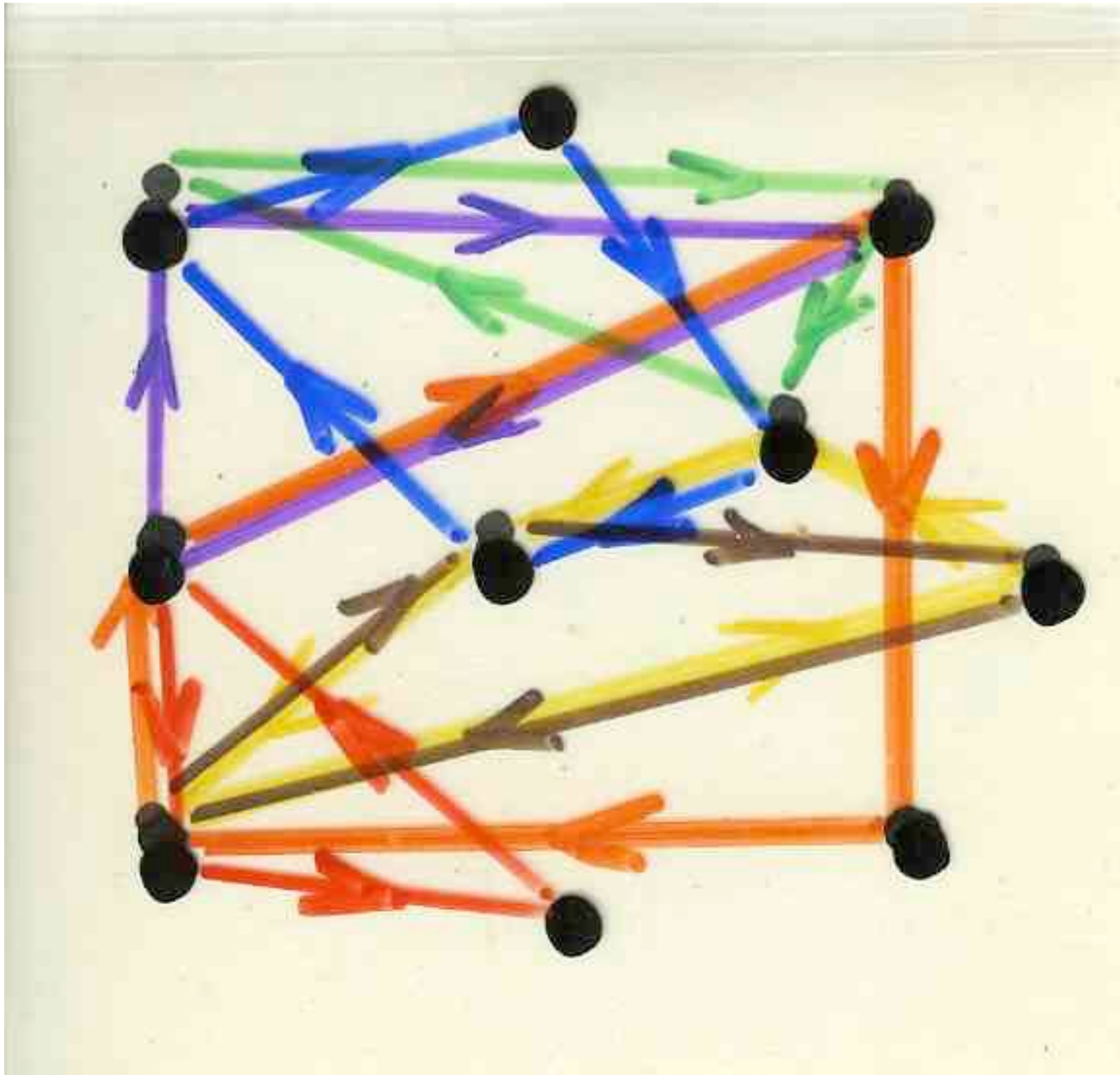
dependancy relation
$\gamma \,\mathcal{C}\, \gamma'$
iff $\mathrm{supp}(\gamma) \cap \mathrm{supp}(\gamma') \neq \emptyset$

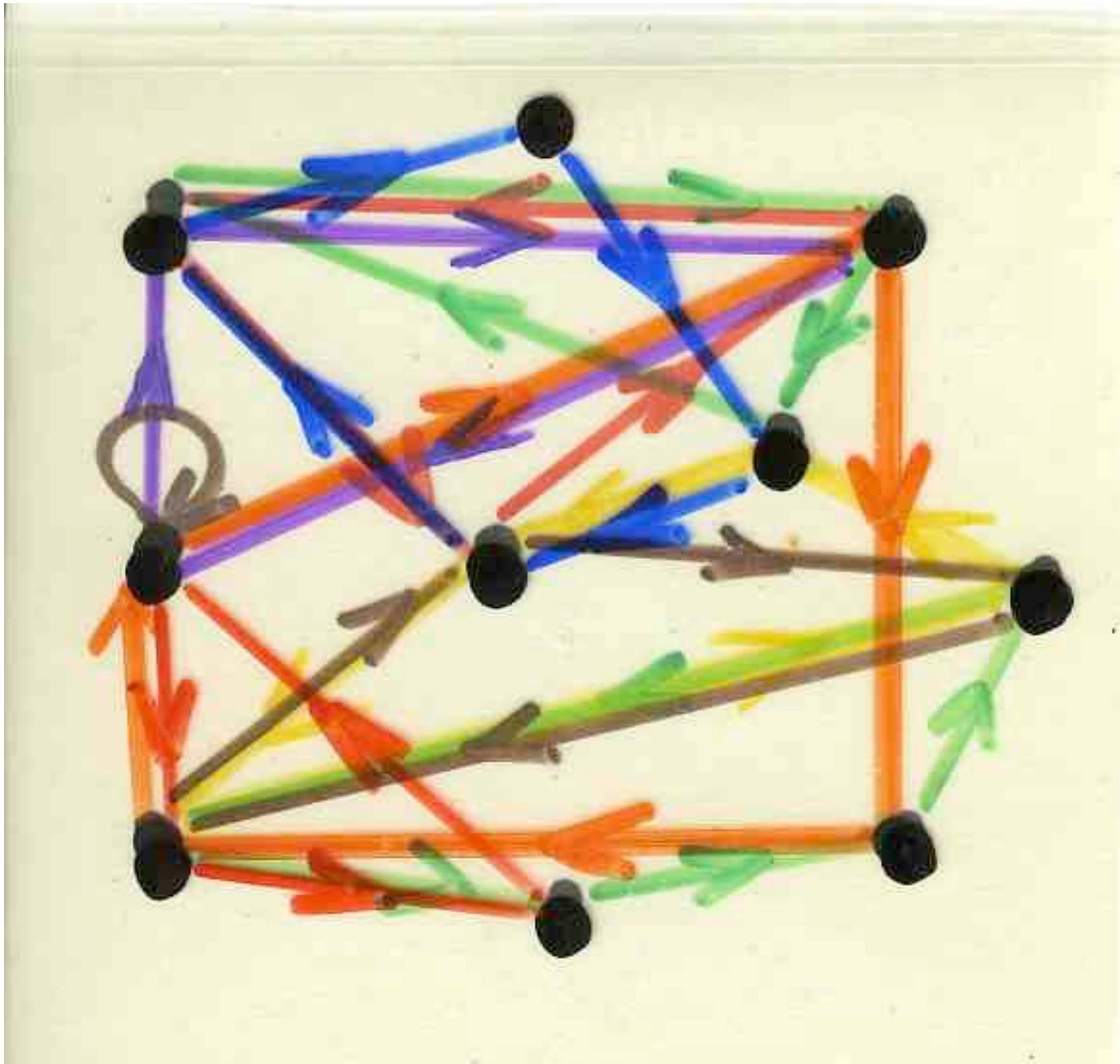
$\mathrm{supp}(\gamma)$ = underlying set of vertices $\subseteq X$











γ cycle $\quad \dot{\gamma} = (s_0, s_1, \dots, s_n = s_0)$

$$f(\gamma) = f(\dot{\gamma}) \qquad \dot{\gamma} \text{ is a path}$$

$$= \binom{s_0}{s_1}\binom{s_1}{s_2}\cdots\binom{s_{n-1}}{s_n}$$

E heap of cycles

$$E = \gamma_1 \odot \cdots \odot \gamma_k$$

(product in the heaps of cycles on X monoid

$$f(E) = f(\gamma_1) \odot \cdots \odot f(\gamma_k)$$

(product in the rearrangement monoid $\mathbf{R}(X)$)

"breaking" paths and heap of cycles

Proposition The map $f : \mathbf{HC}(X) \longrightarrow \mathbf{R}(X)$ is an isomorphism from the heaps of cycles monoid to the rearrangements monoid

for any $s, t \in X$

the numbers of occurrences of the edge (s, t) in Φ and E are the same.

$$\Rightarrow \qquad v(\Phi) = v(E)$$

Construction of the reciprocal isomorphism

$$g = f^{-1}$$

$$\Phi \xrightarrow{g} E$$

rearrangement $\in R(X)$ heap of cycles on X

for Φ rearrangement of $R(X)$
from any vertex $s \in X$
"follow" the flow Φ

at the end we are back in s
giving a sequence of cycles

$(\gamma_1, \ldots, \gamma_k)$

if all edges of Φ has been used

$g(\Phi) = \gamma_1 \odot \cdots \odot \gamma_k$

else we have

$$\Phi = \gamma_1 \odot \cdots \odot \gamma_k \odot \Psi$$

heap of cycles

rearrangement

$$\deg^{+}_{\Psi}(s) = \deg^{-}_{\Psi}(s) = 0$$

choosing another vertex $t \in X$ with $\deg^{+}(t) = \deg^{-}(t) \neq 0$ we repeat the process for Ψ

Recursively we get a sequence of cycles $(\gamma_1, \ldots, \gamma_r)$ such that

$$\Phi = f(\gamma_1) \cdots f(\gamma_r) \in R(X)$$

The heap $E = \gamma_1 \odot \cdots \odot \gamma_r$ is independant of the successive choices of the vertices $s, t, \ldots$

define $g(\Phi) = E$

$$g = f^{-1}$$

"gluing" bijections

Paths and heaps of cycles

path on X

$$\omega = (s_0, \dots, s_i, s_{i+1}, \dots, s_n)$$

$$s_i \in X \quad i = 0, \dots, n$$

ω goes from s_0 to s_n

path on a graph G (oriented or not)

notation $s_0 \overset{\omega}{\rightsquigarrow} s_n$

(s_i, s_{i+1}) edge of G

s_0 starting vertex

s_n ending vertex

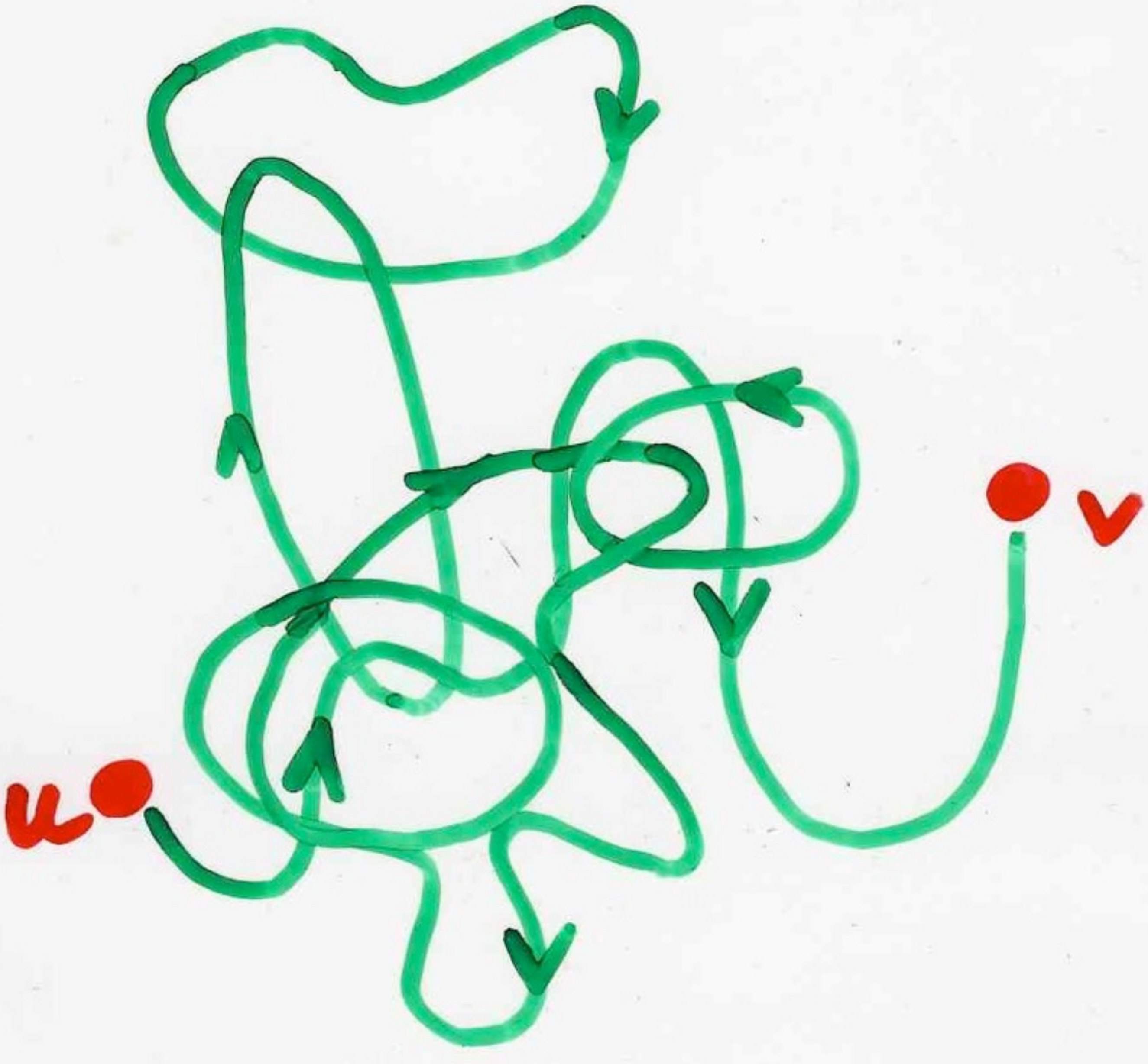
(s_i, s_{i+1}) elementary step

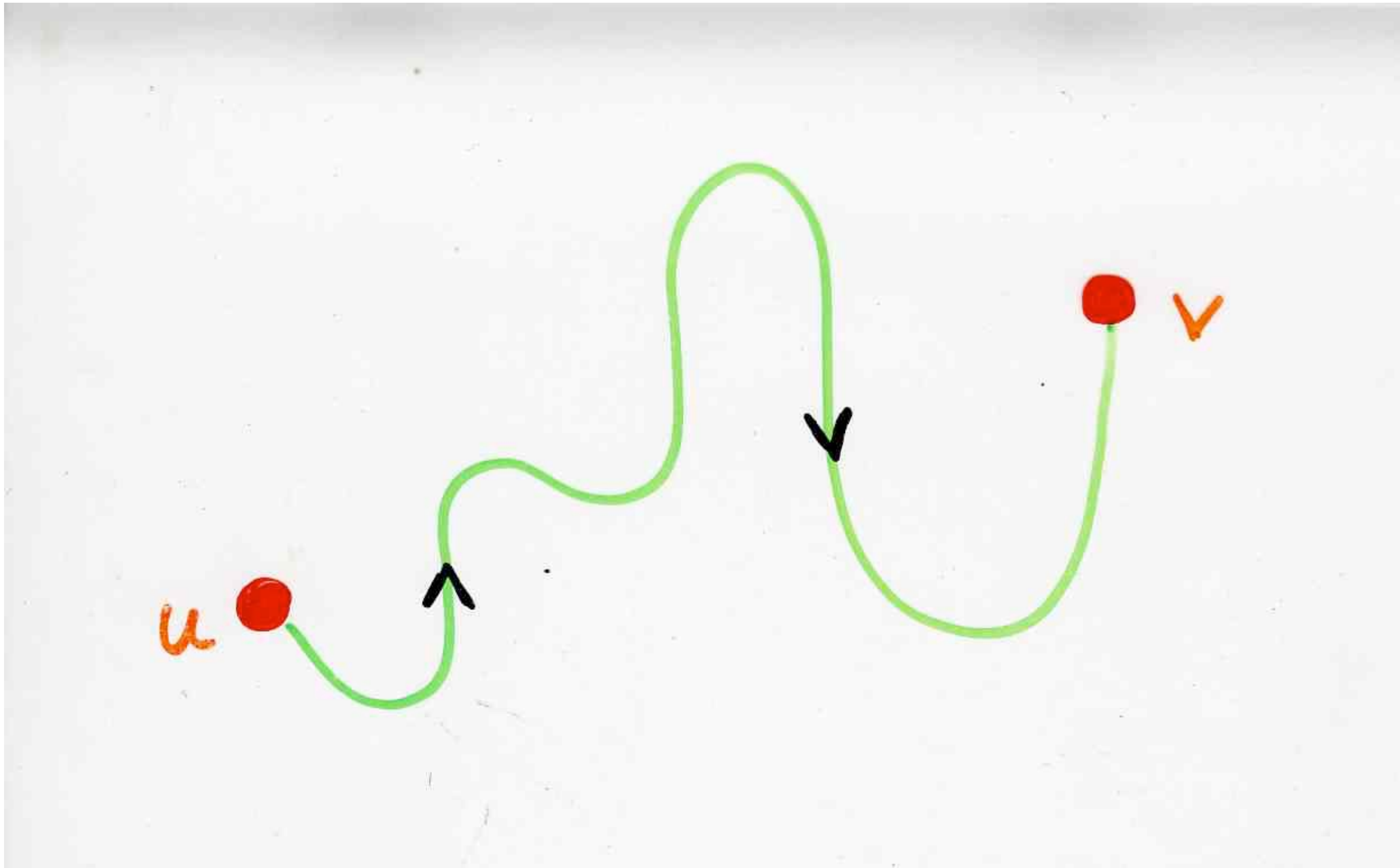
length $|\omega| = n$

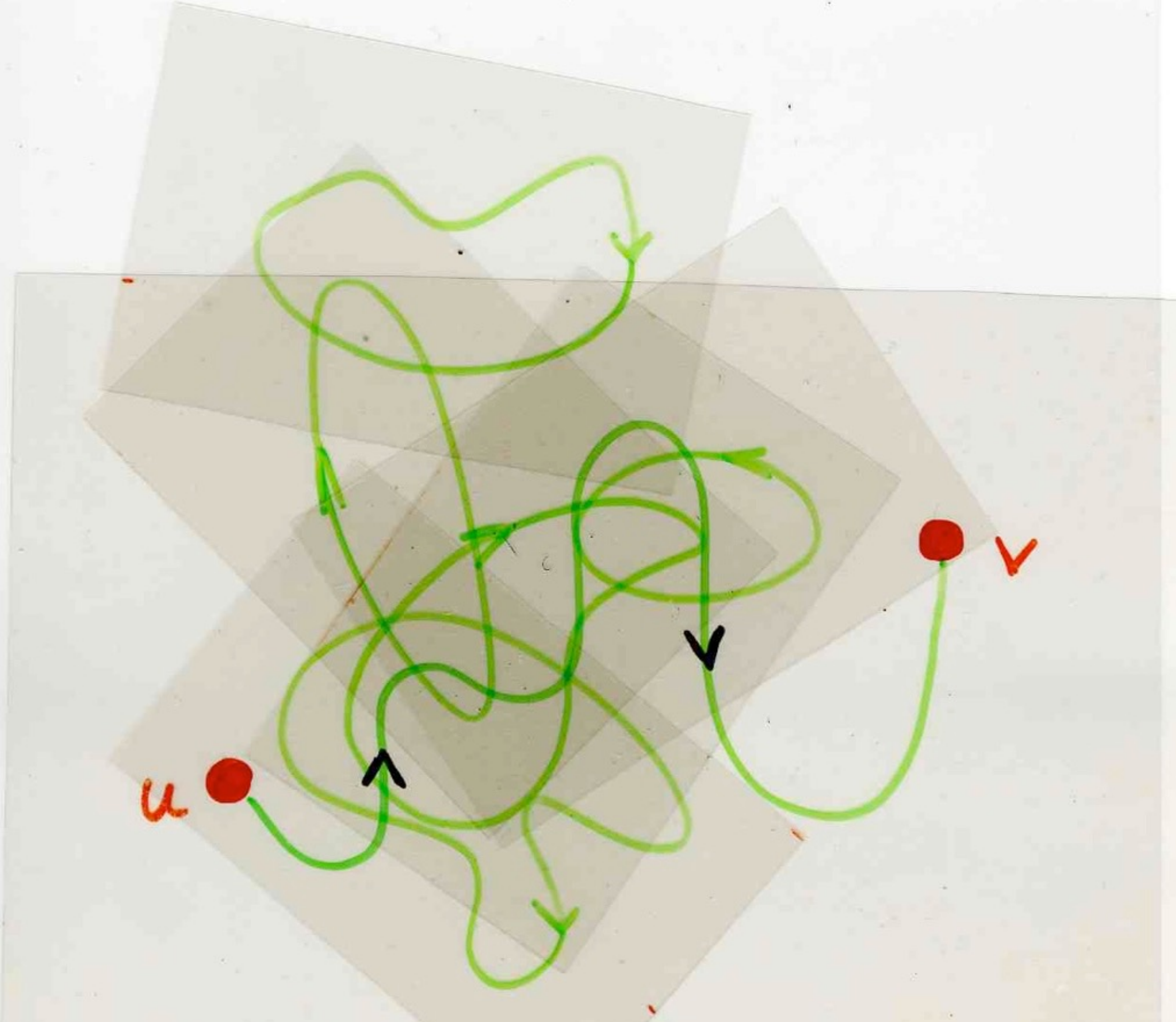
(number of elementary steps)

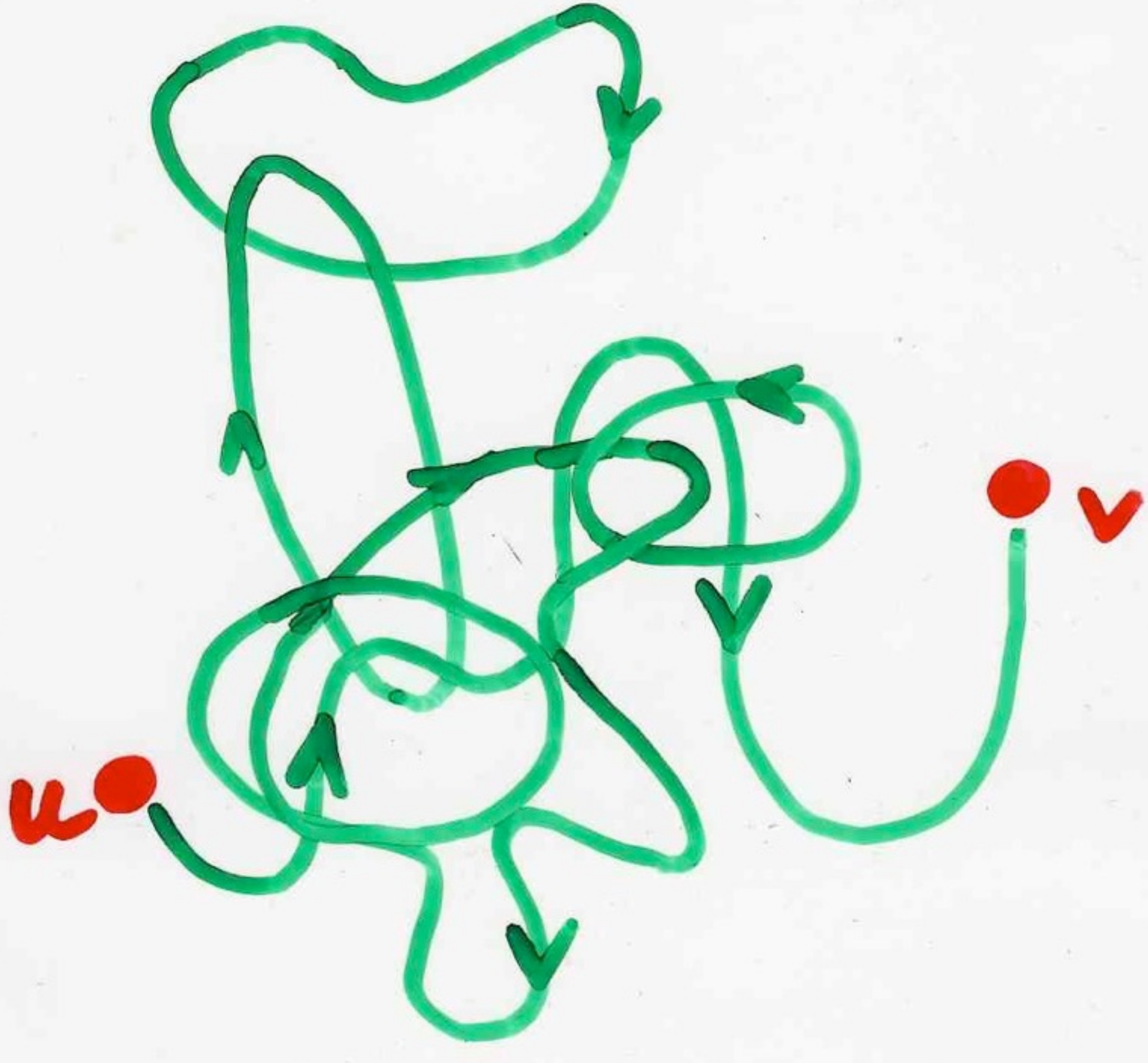
$n+1$ vertices

paths = $\left\{ \begin{array}{l} \text{heaps of cycles} \\ + \\ \text{self-avoiding path} \end{array} \right.$




u
v


u
v


u
v

Bijection

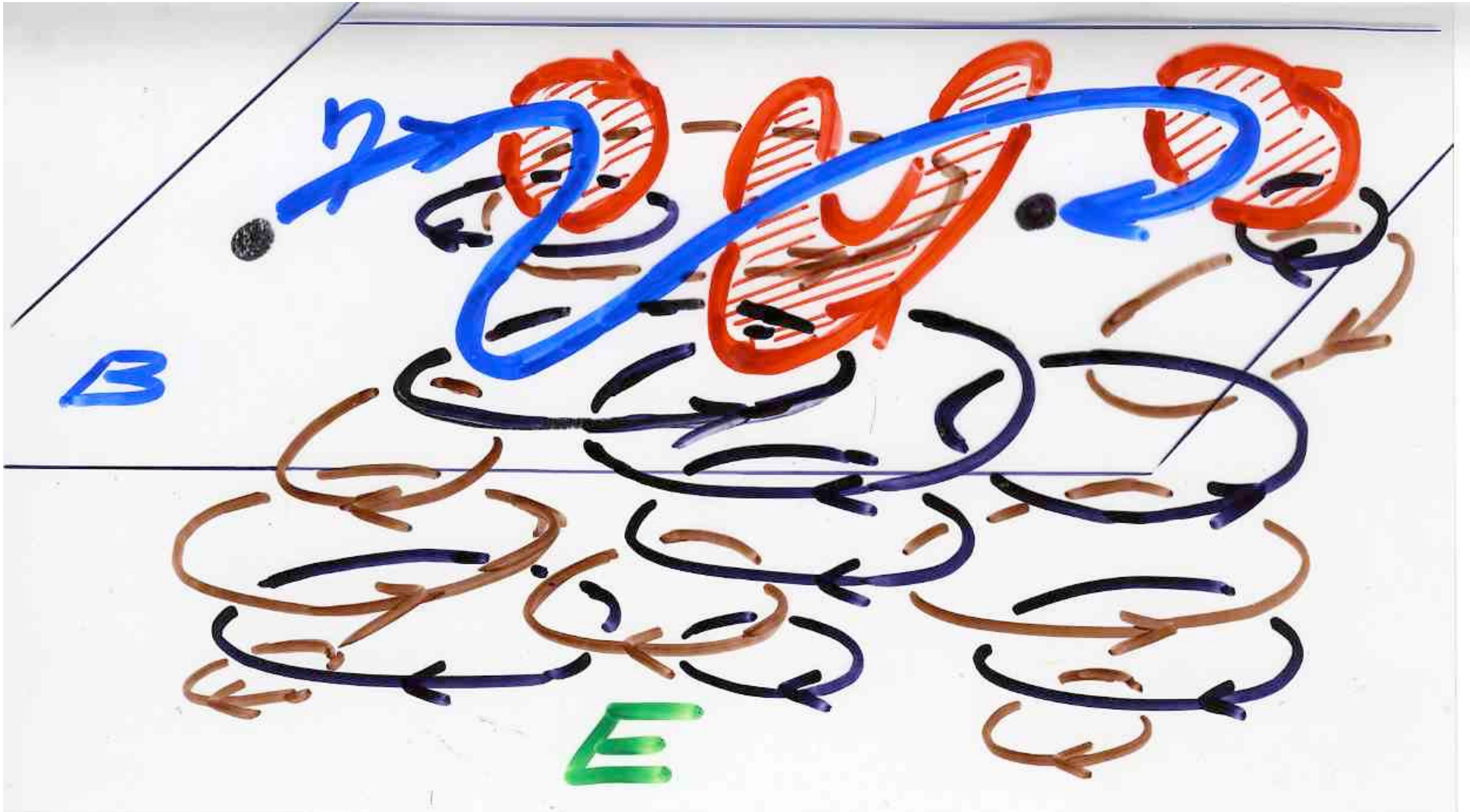
$u, v \in X$

path ω on X going from u to v $\longleftrightarrow$ (η, E)

- η self-avoiding path going from u to v
- E heap of cycles such that the projections $\alpha = \pi(m)$ of the maximal pieces intersect η (α and η has a common vertex)
  (cycle) (path)

for any $s, t \in X$ the numbers of occurrences of the edge (s,t) in ω and in (η, E) are the same.

$$\Rightarrow \quad v(\omega) = v(\eta)\, v(E)$$



B
E

algorithm "Cut and heap"

$\omega = (s_0 = u, \ldots, s_n = v)$

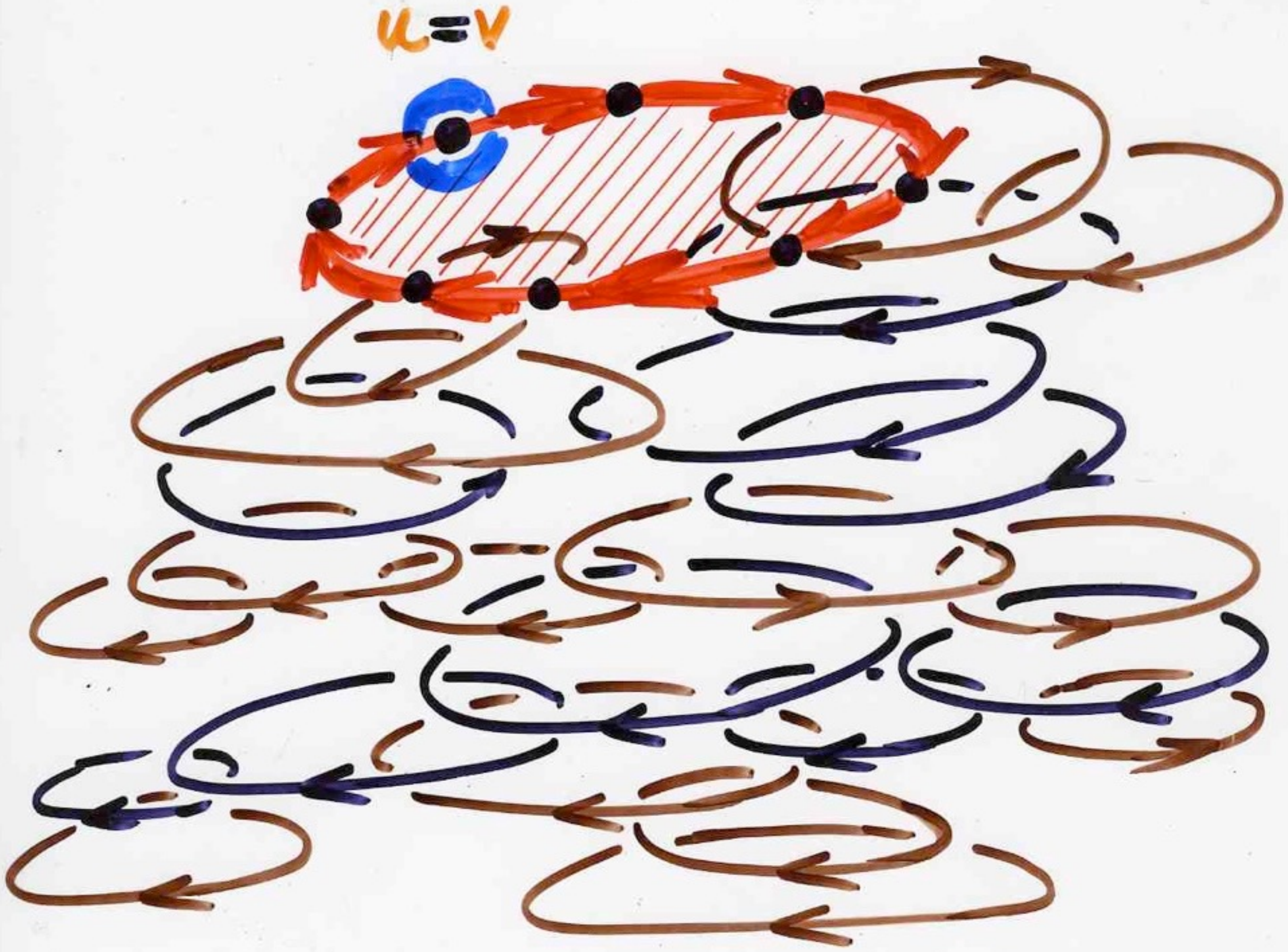
path on X

see next course
Chapter 3b

$\omega \longrightarrow (\eta, E)$

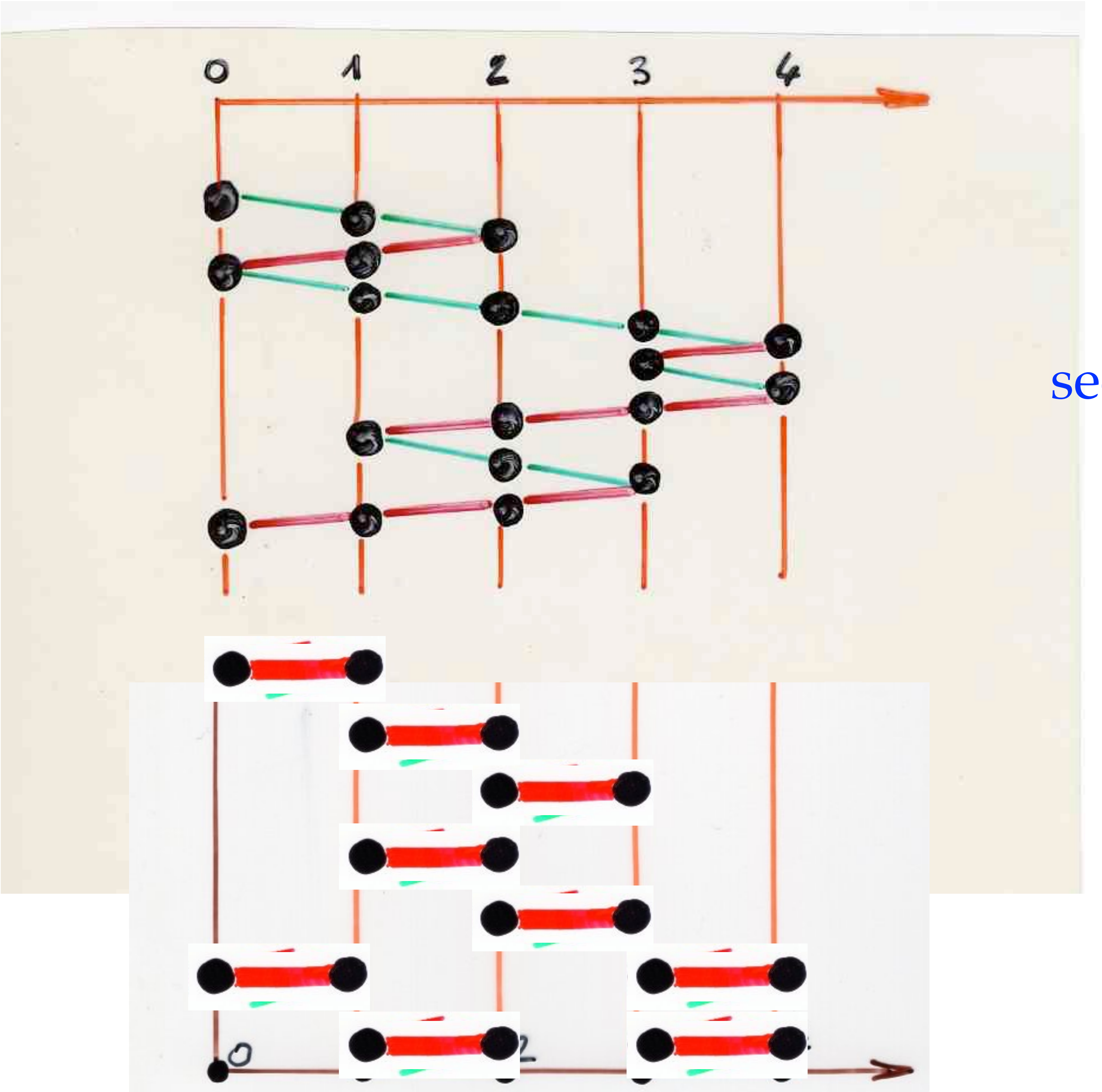
self-avoiding path $u \rightsquigarrow v$

heap of cycles



pointed cycles pyramid

an example with Dyck paths



see the animation
on the video

violin:
G. Duchamp