Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics, a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

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Chapter 2
Heaps generating functions

(2)

IMSc, Chennai

16 January 2017

from the previous lecture

$$(a_0, a_1, a_2, \cdots, a_n, \cdots)$$

$$a_0 + a_1 t + a_2 t + \ldots + a_n t^n + \ldots$$

$\mathbb{K}[[t]]$ formal power series
algebra
(in one variable t and
coefficients in $\mathbb{K}$)

generating power series
of the coefficients (numbers a_n)

$$\sum_{n \geq 0} a_n t^n = f(t)$$

(ordinary generating function)

- sum
- product
- sequence

} "$\mathcal{L}$ combinatorial objects"

symbolic method

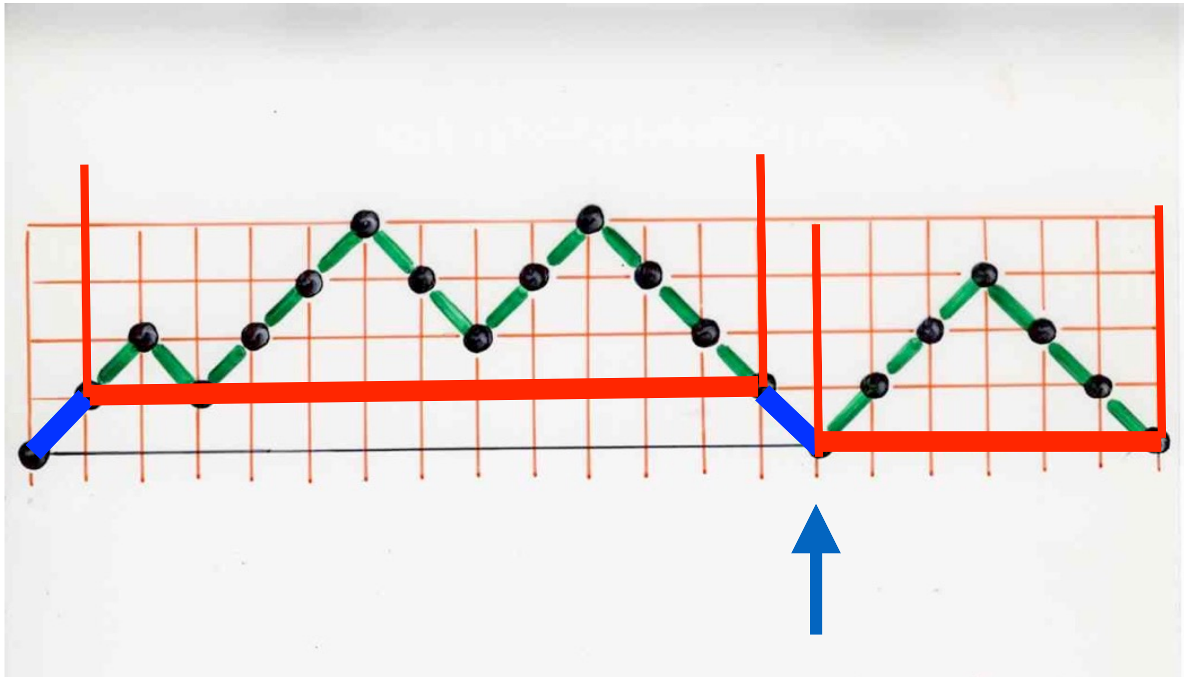
Philippe Flajolet (1948 – 2011)

(with Robert Sedgewick)

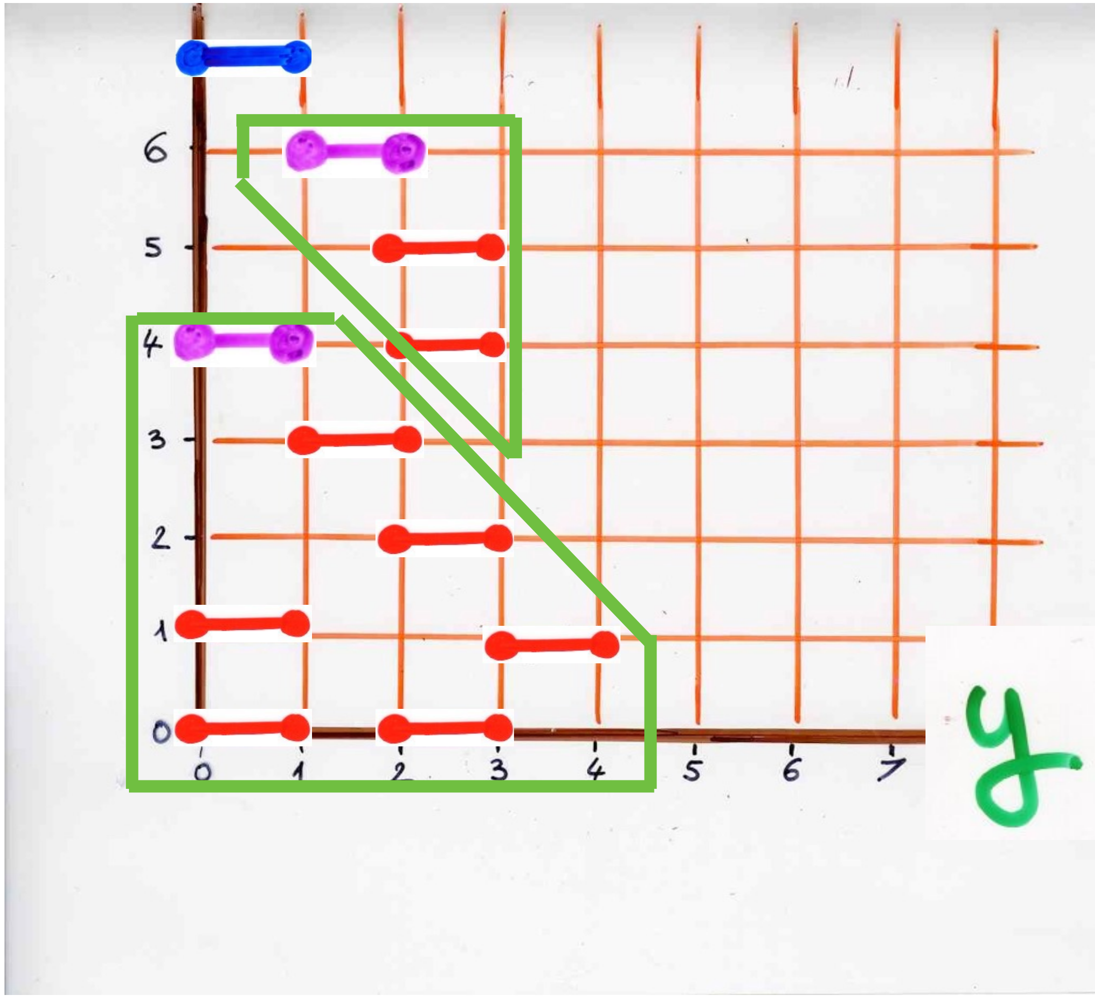
Analytic Combinatorics

(Cambridge Univ. Press, 2008)

Dyck path



$$y = 1 + t\,y^2$$



$$y = 1 + t\,y^2$$

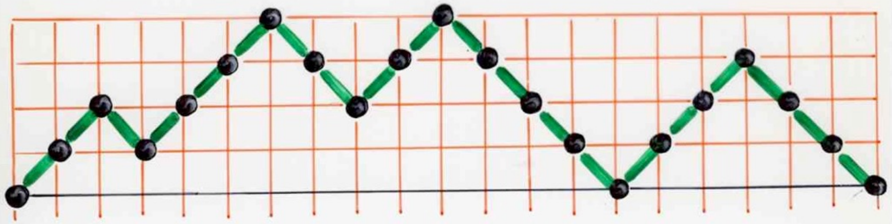
semi-pyramid of dimers on $\mathbb{N}$
the unique maximal piece
has projection [0,1]

Catalan number

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

"philosophy" underlying this course

Dyck path



$$y = 1 + t\,y^2$$

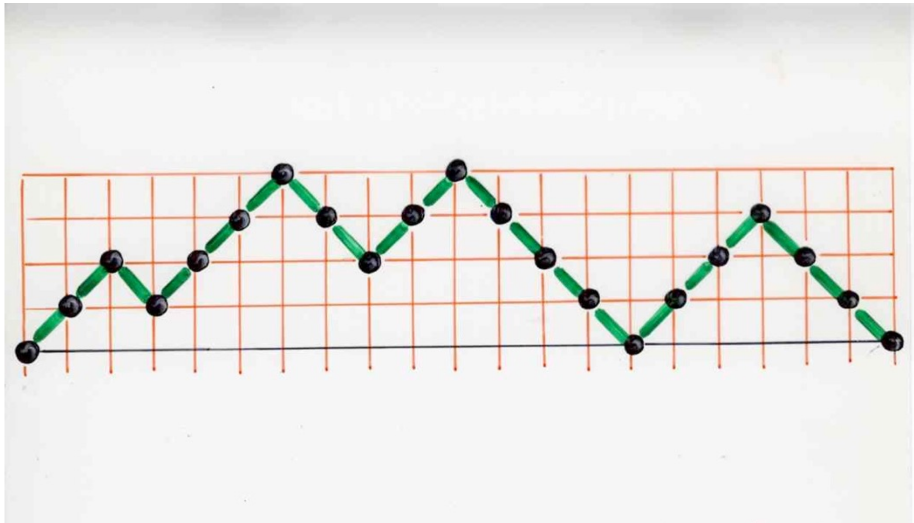
bijective proof

?

Catalan number

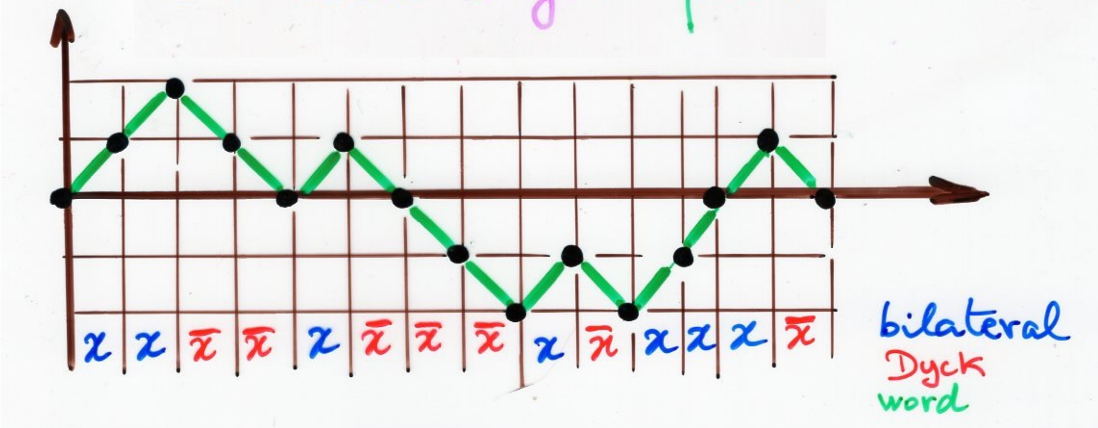
$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Dyck path



$$(n+1)\, C_n = \binom{2n}{n}$$

bilateral Dyck path



bijective proof

<u>exercise</u>

difficult !

bijective proof of an identity

symbolic method

Philippe Flajolet (1948-2011)

(with Robert Sedgewick)

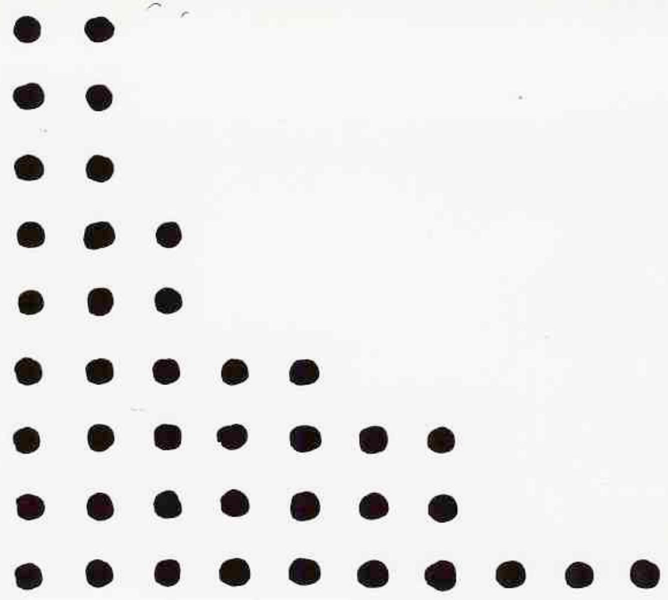
Analytic Combinatorics

(Cambridge Univ. Press, 2008)

$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

right handside

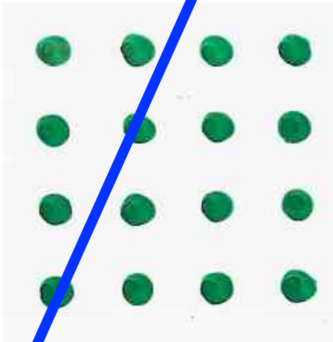


$$= \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

Ferrers diagram (= partition of an integer) left handside

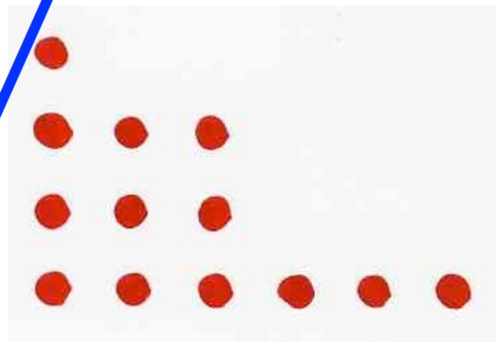
$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1}\frac{1}{(1-q^i)}$$

$$q^{m^2}$$



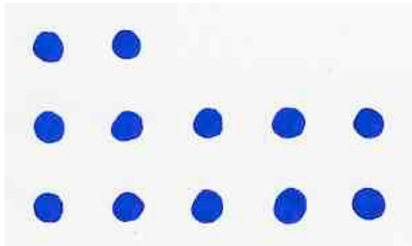
m × m square

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$



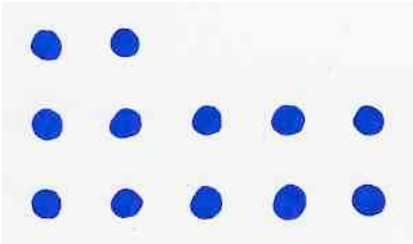
} at most m rows

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$

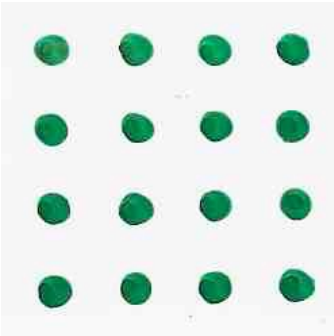


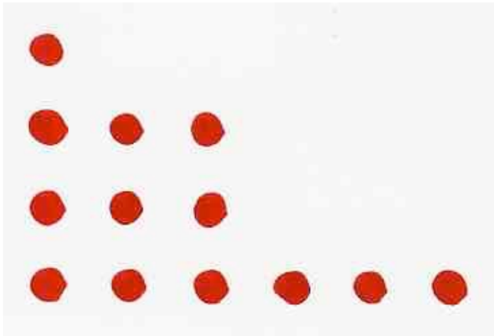
} at most m rows

$m \times m$
square



at most m rows



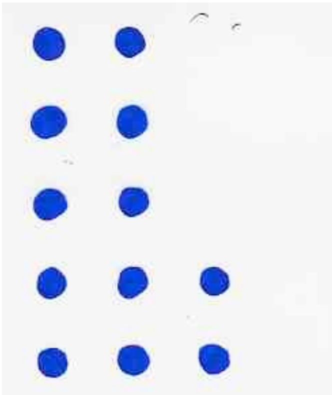


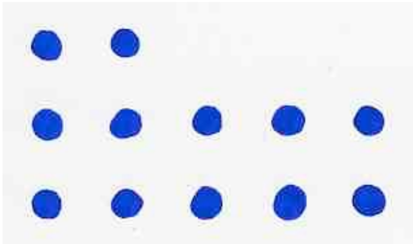
at most m rows

at most
m
columns

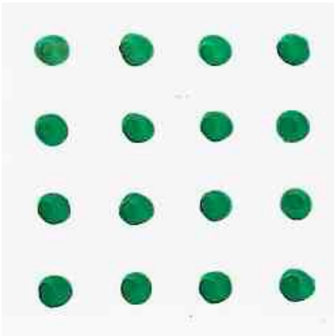
symmetry

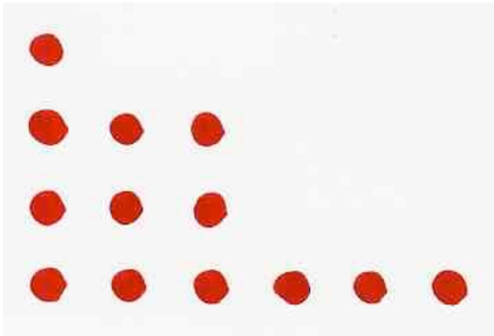
diagonal





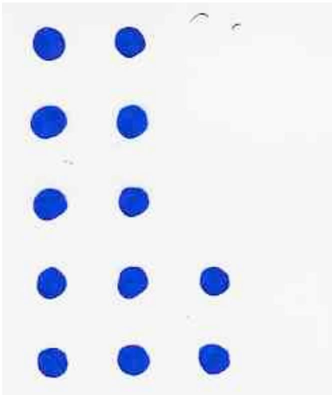
at most
m
rows

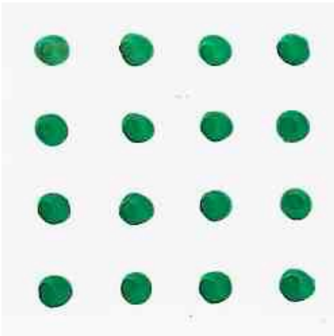


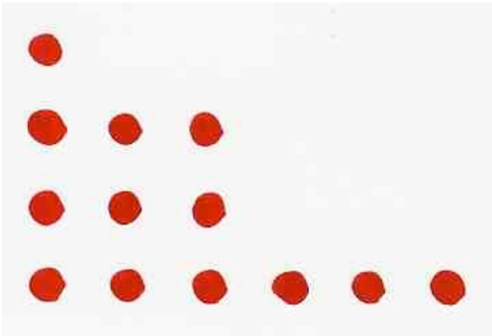


at most
m
rows

at most
m
columns



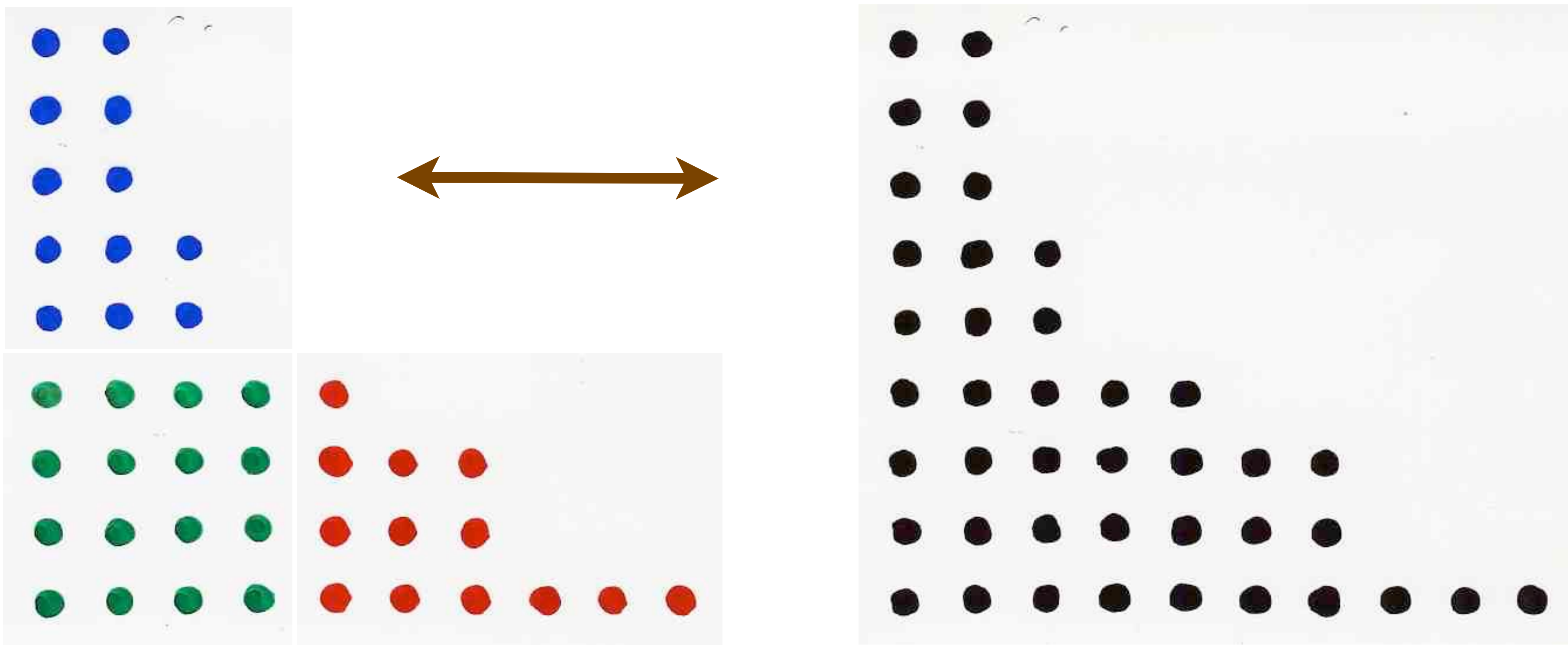




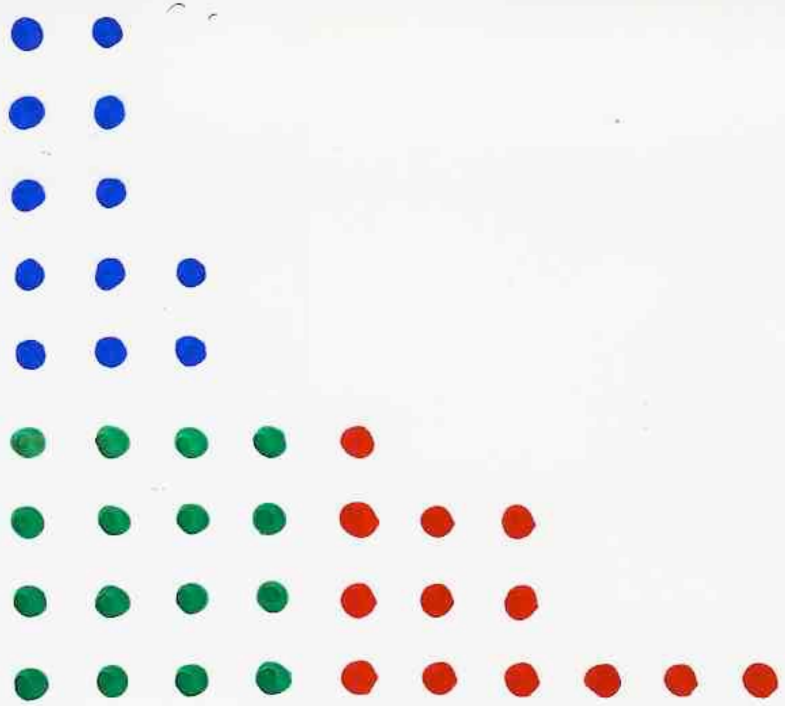
at most
m
rows

left handside

right handside



The identity means:



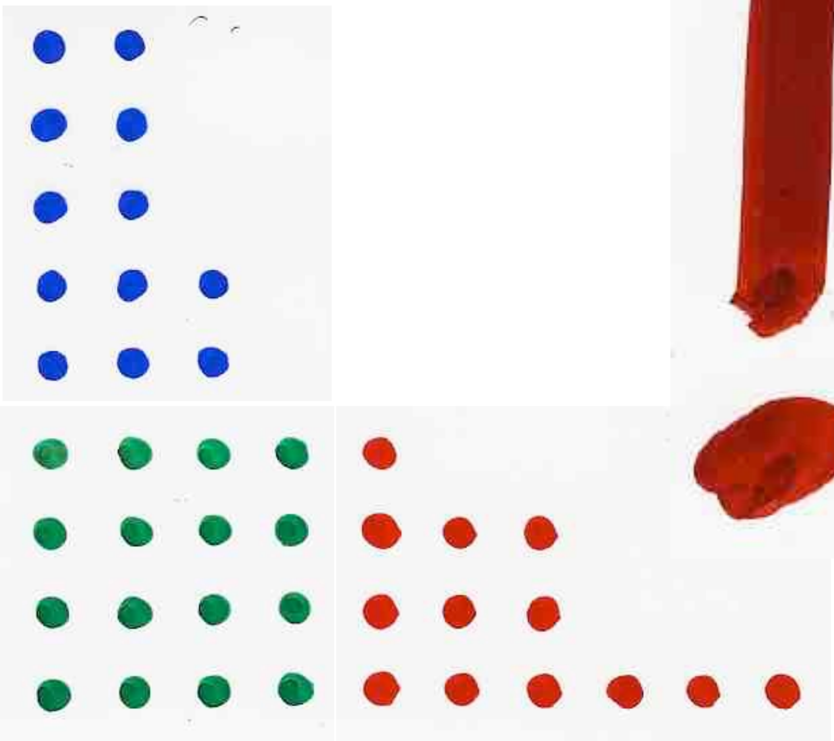
extract the biggest square ⊆ Ferrers diagram

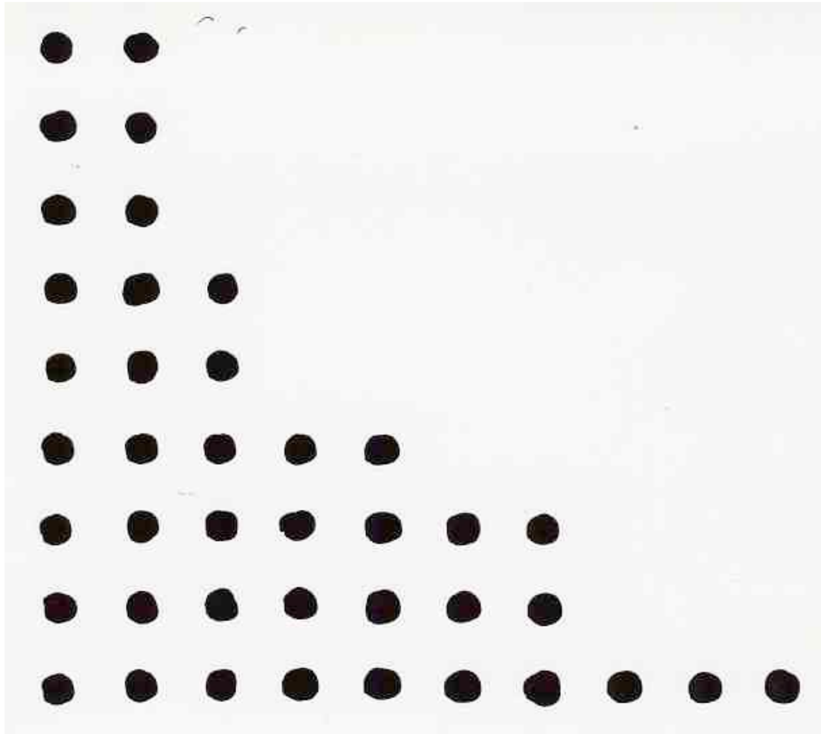
What remains
- diagram having at most m rows
- diagram having at most m columns

m size of the square

$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

!





"drawing" calculus ...

computing with "drawings"

(figures)

二

$$=$$

$$\longleftrightarrow$$

=

better
understanding

$\longleftrightarrow$

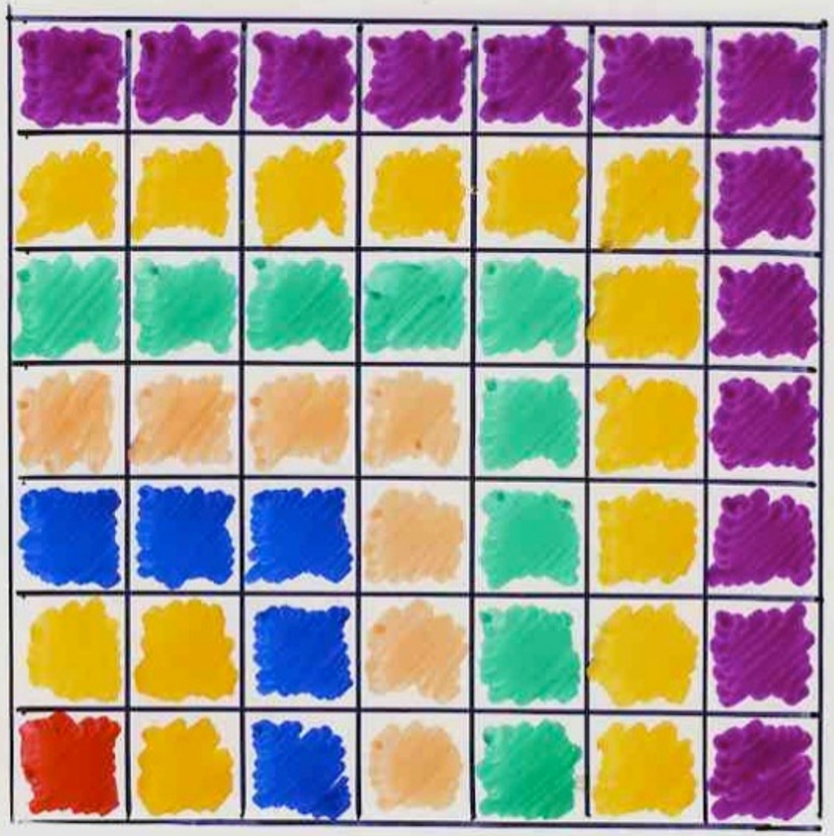
"philosophy" underlying this course

bijective proof

"visual" proof

proof without words

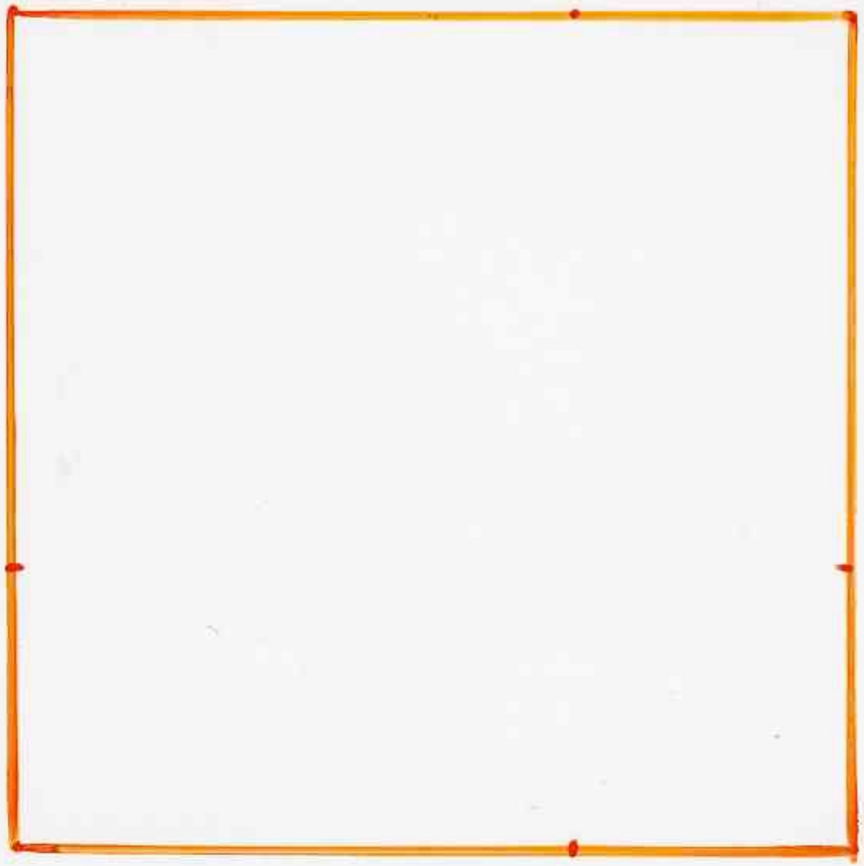
« proof without words »

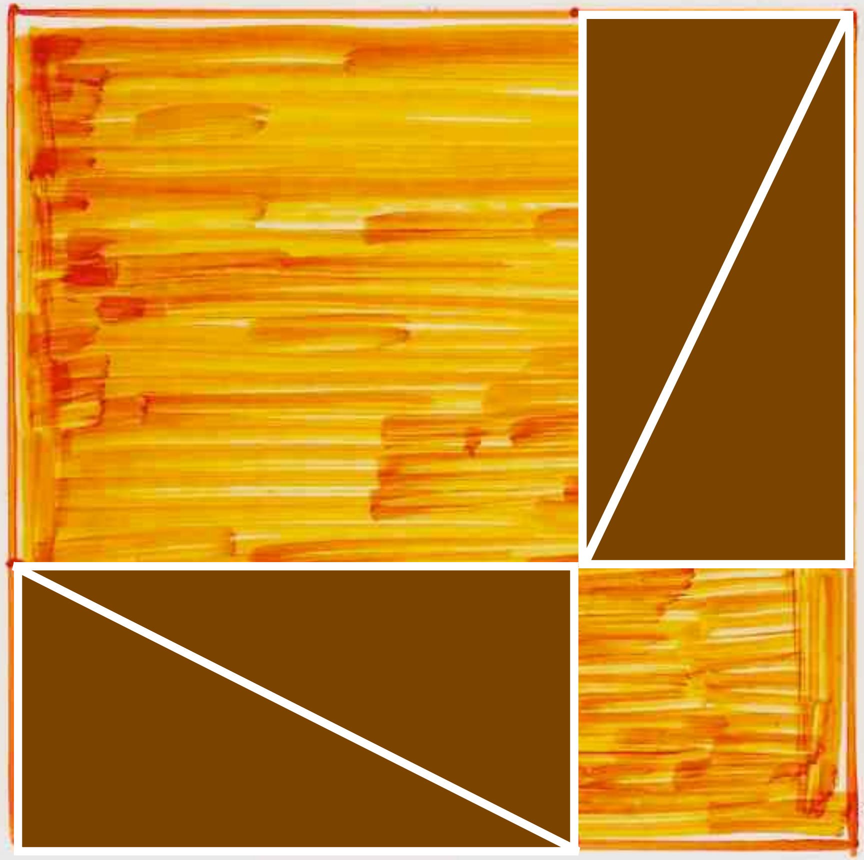


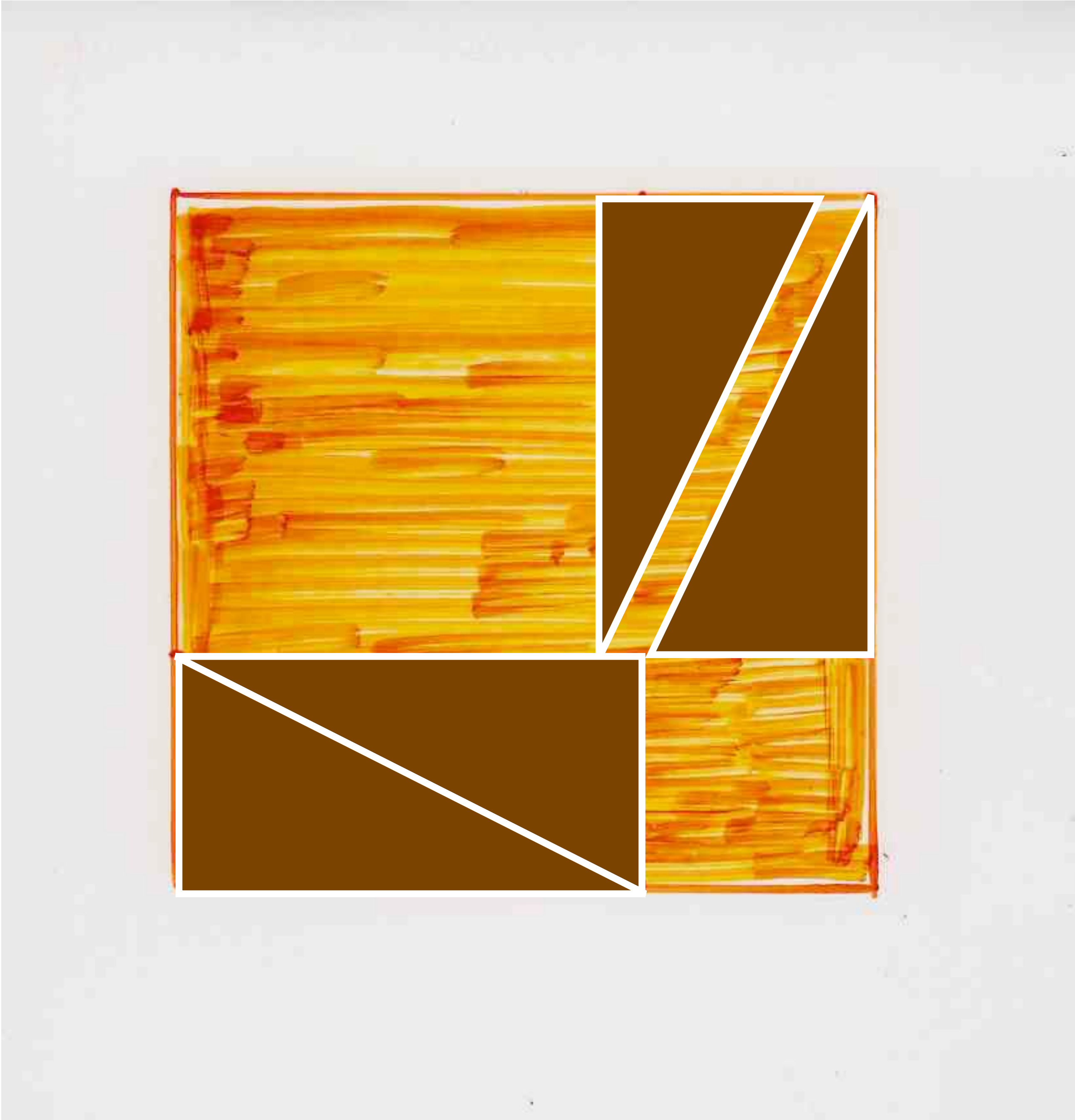
$$n^2 = 1 + 3 + \dots + (2n-1)$$

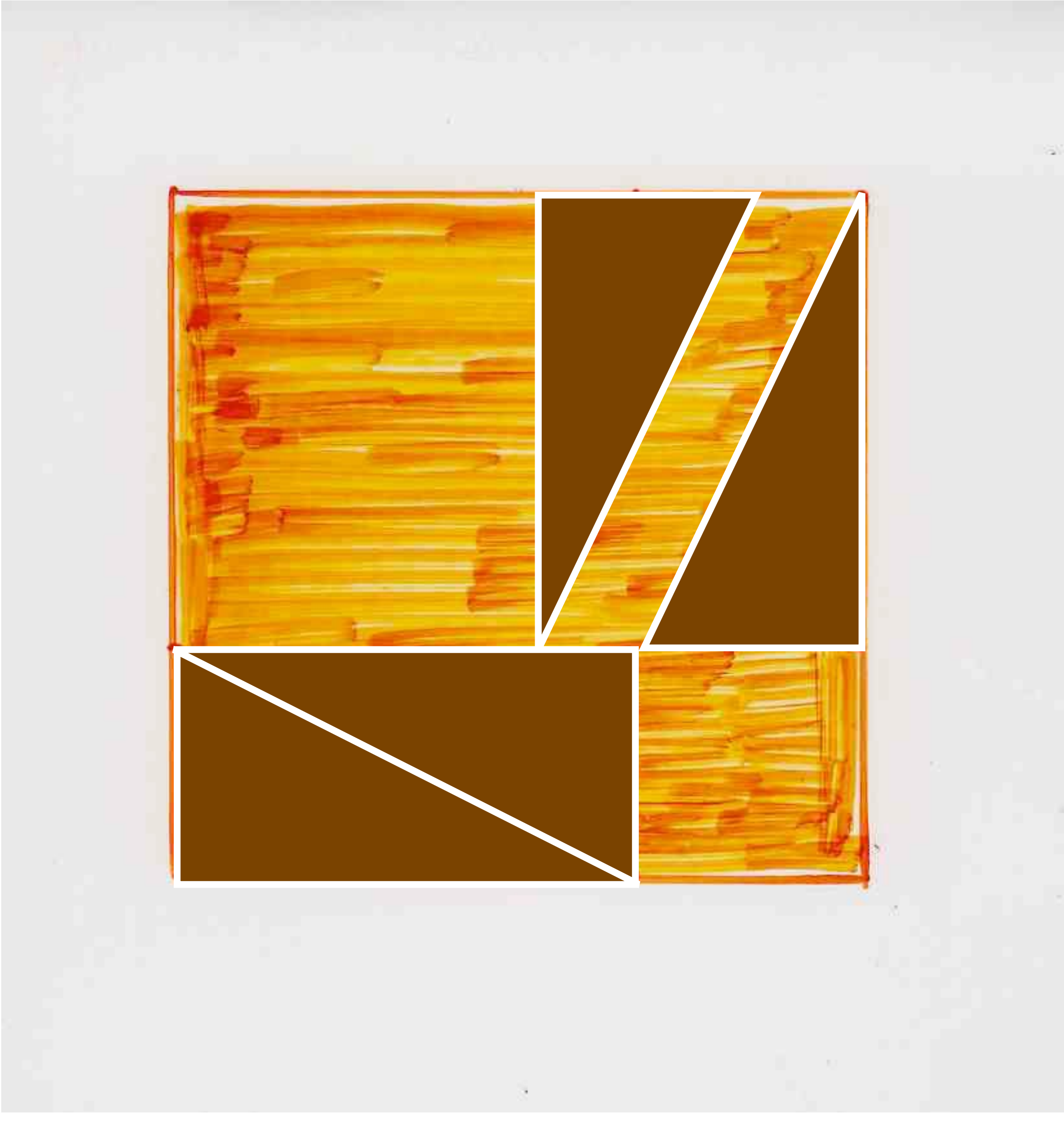
« proof without words »

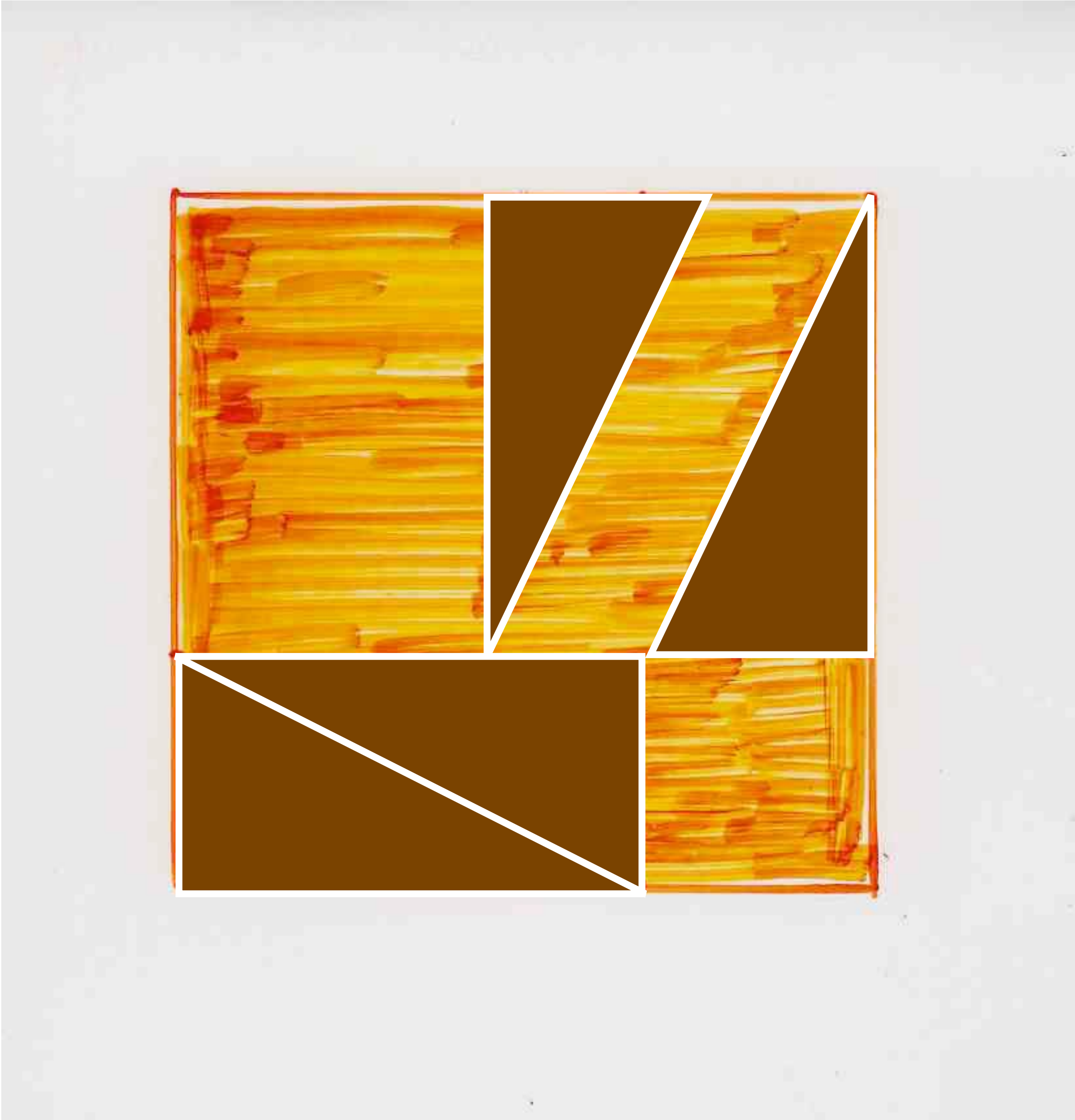
Pythagoras ....

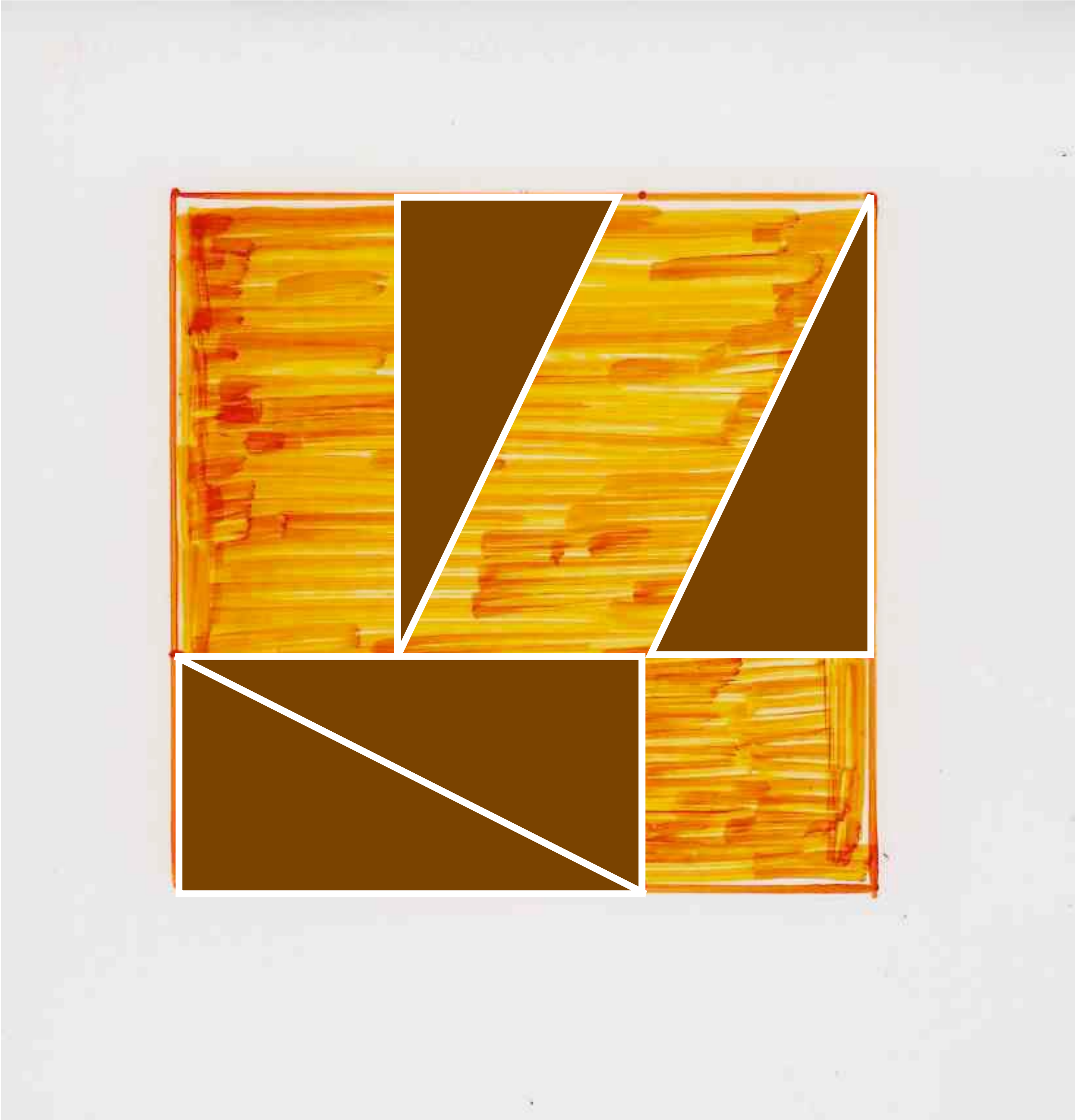


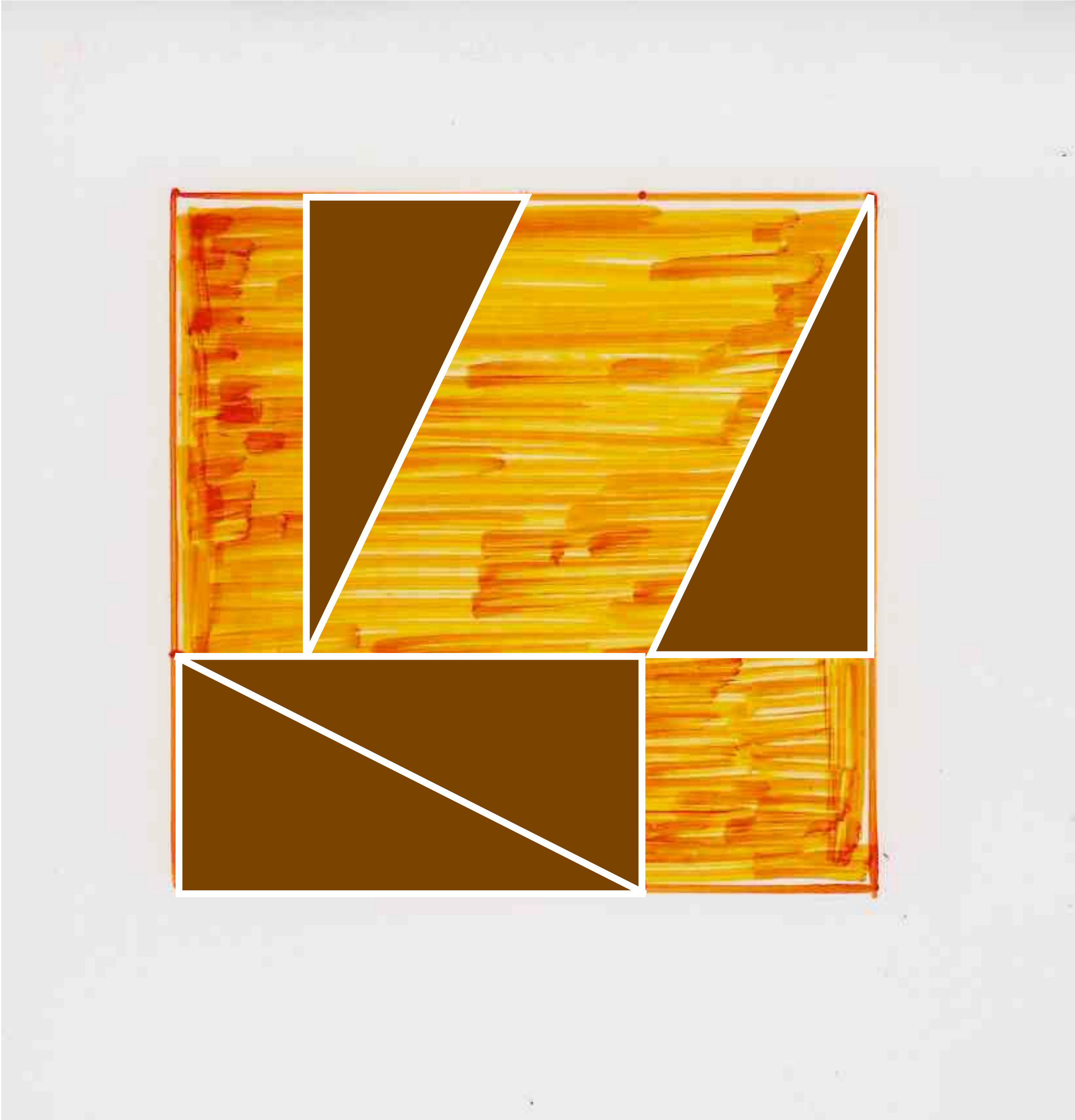


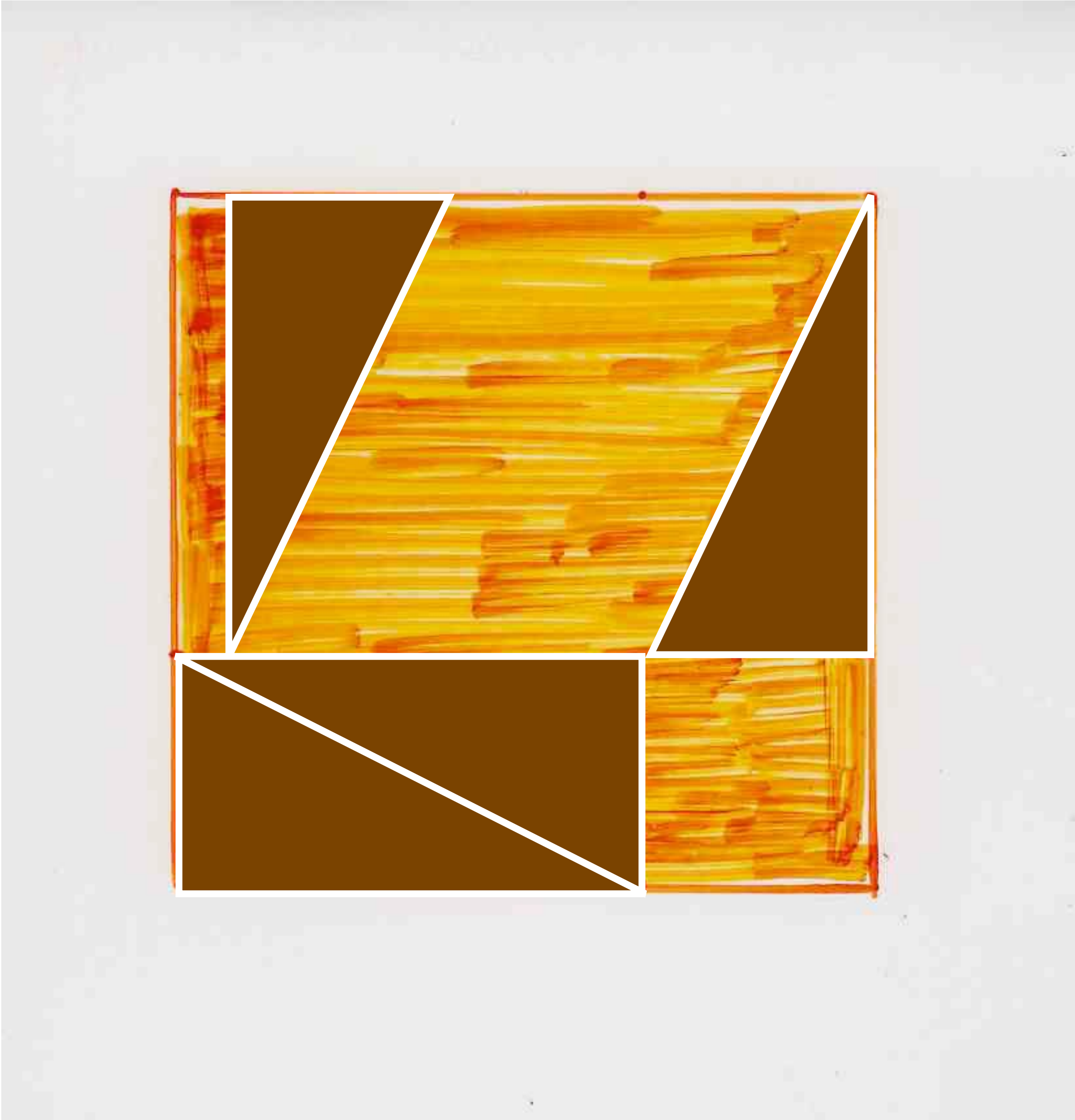


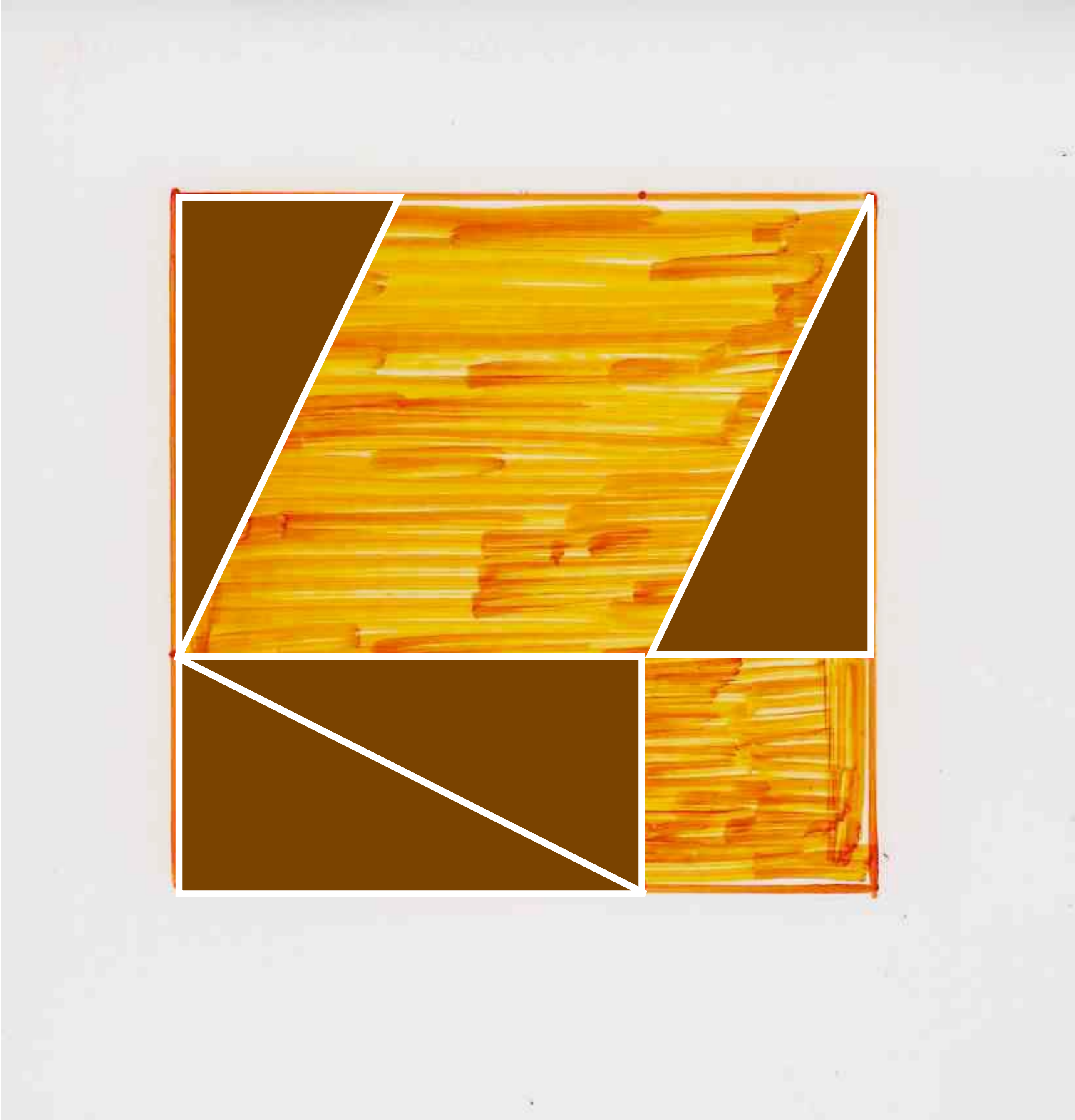


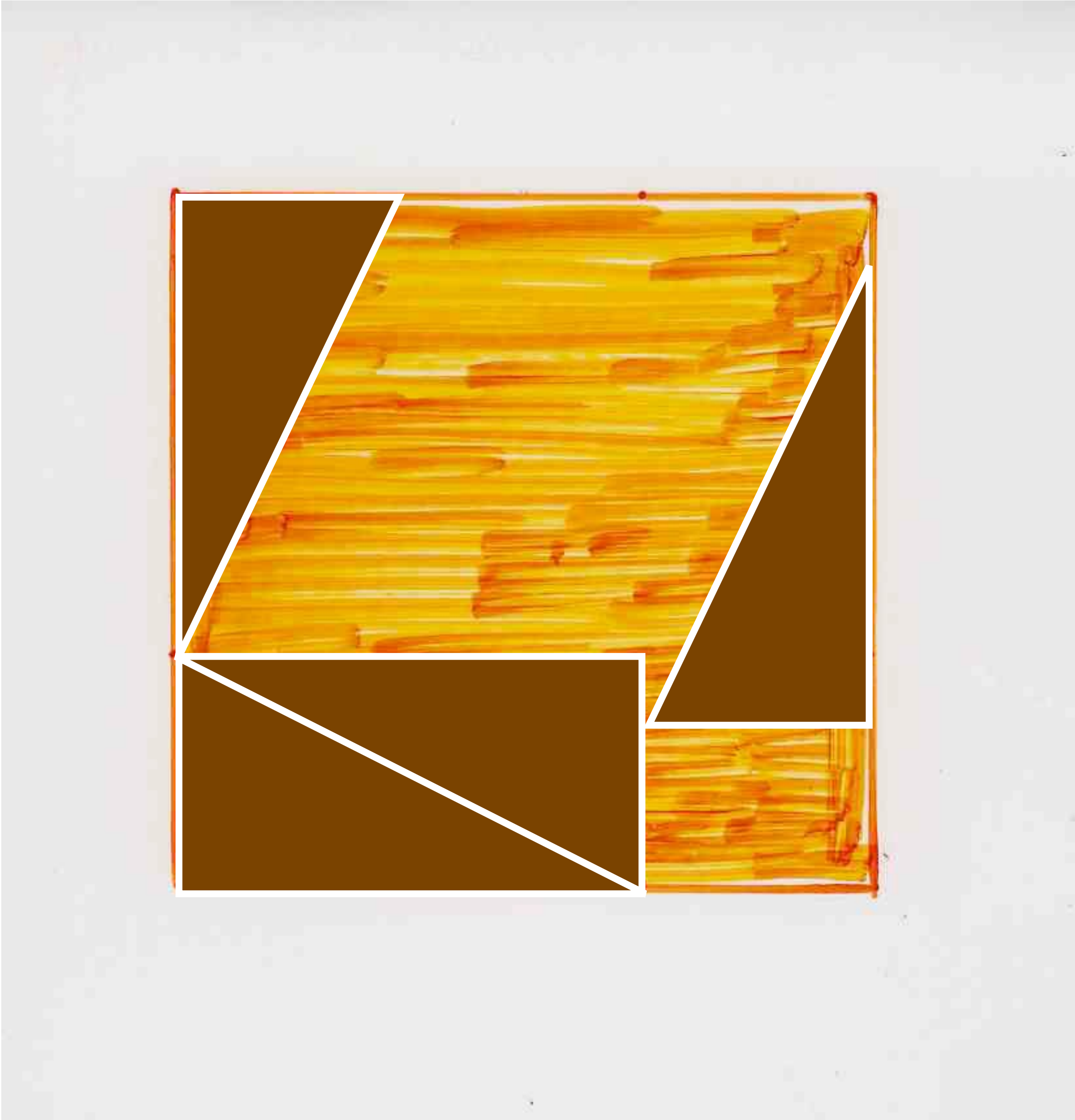


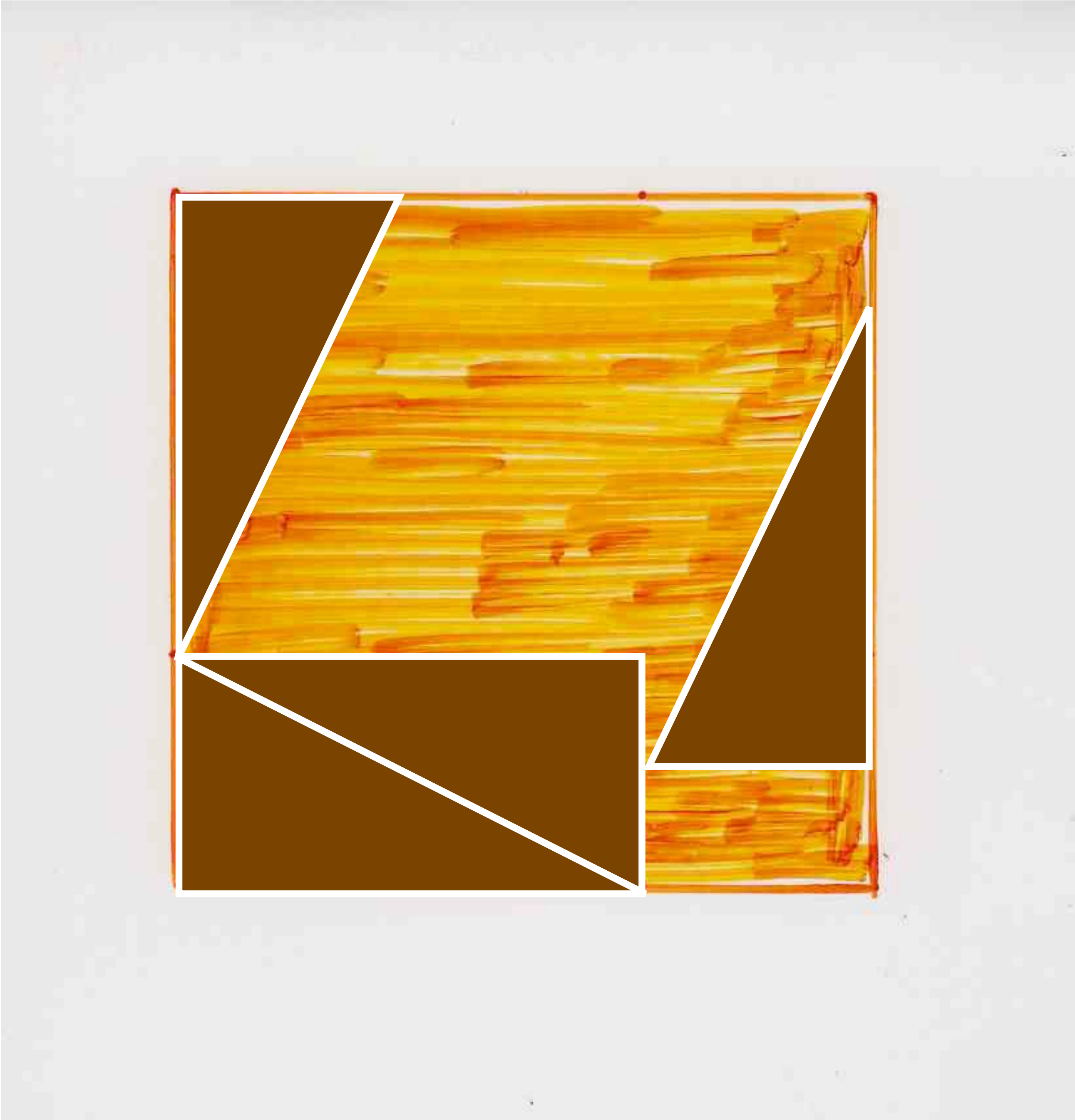


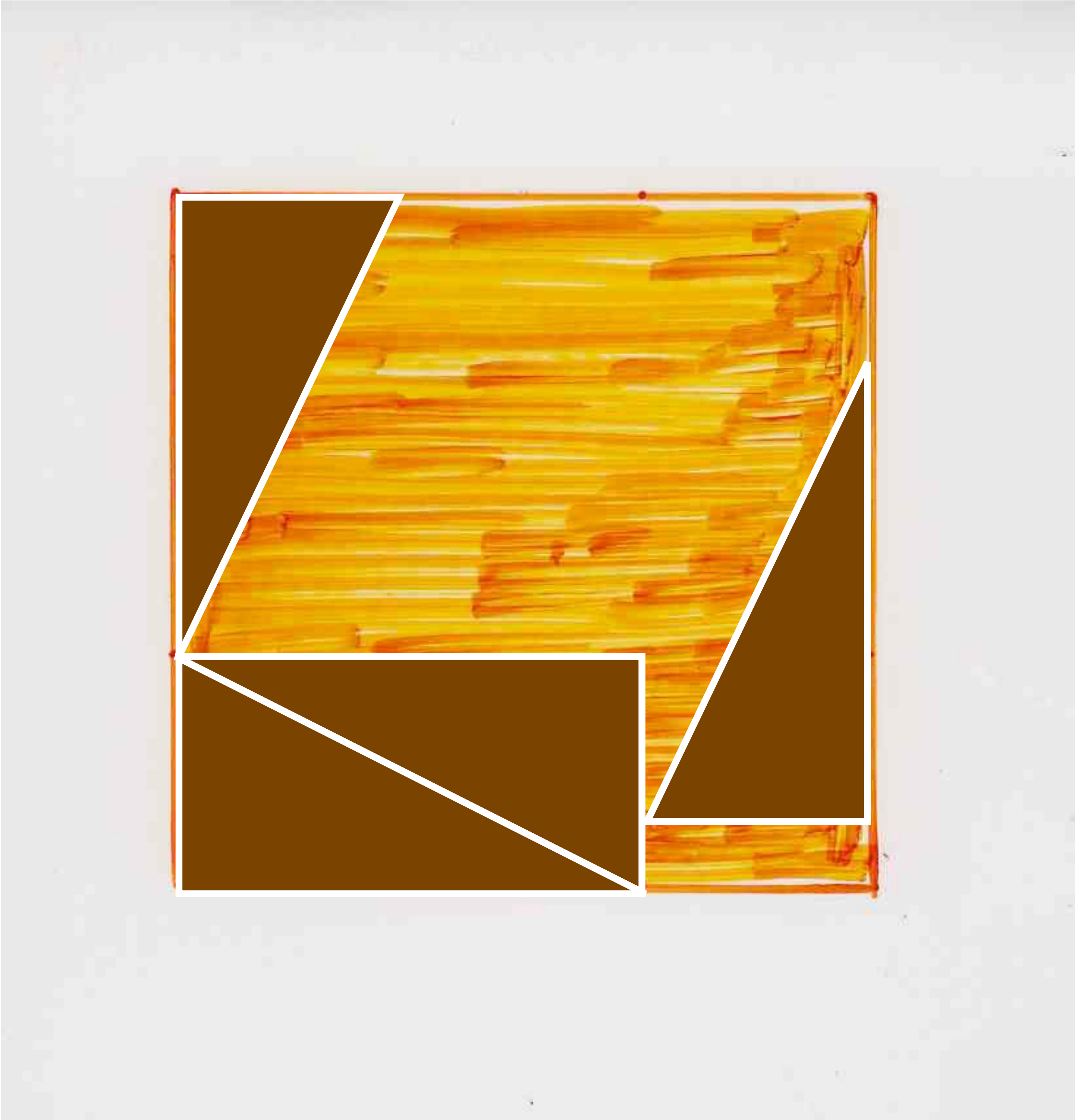


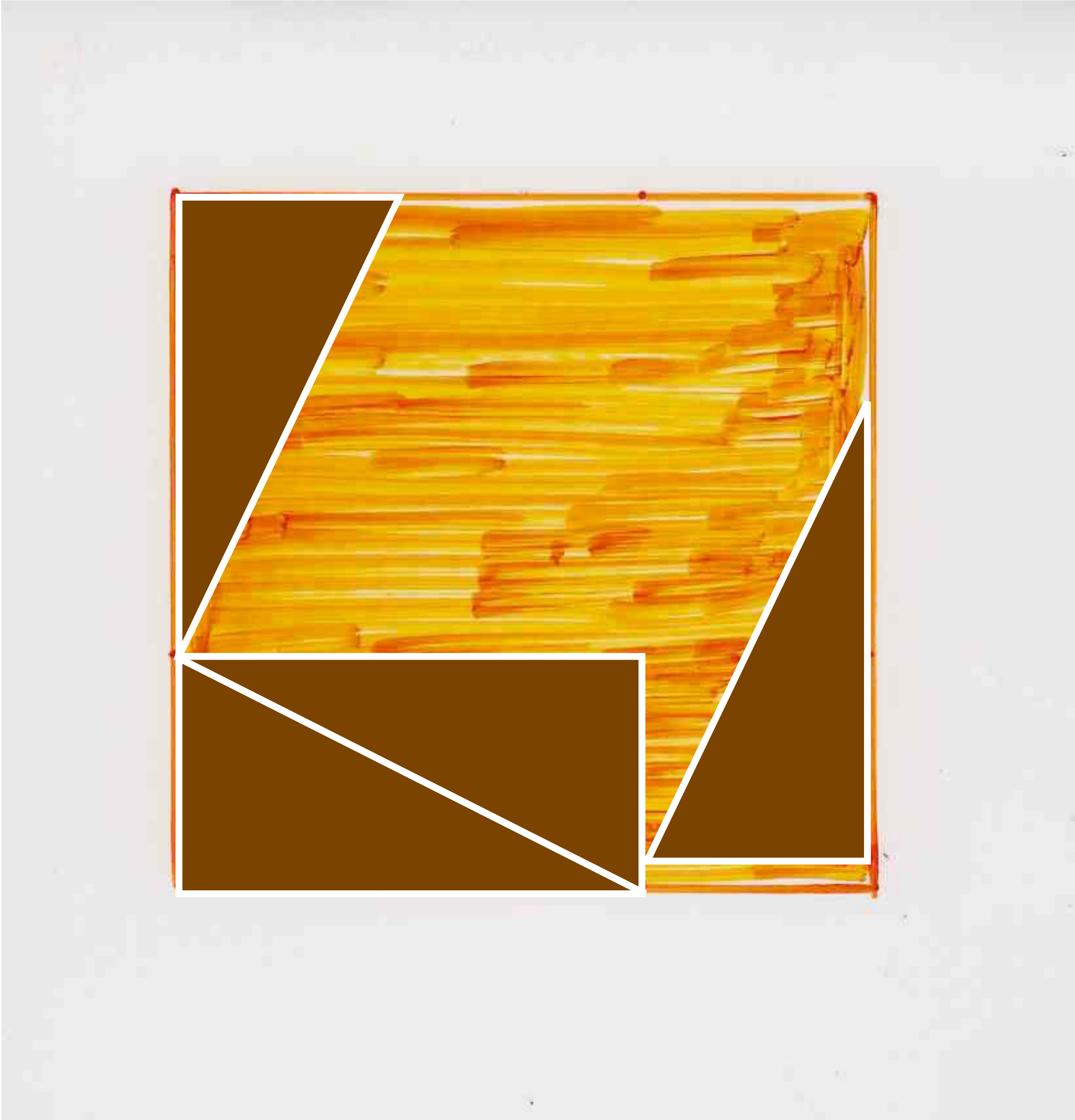


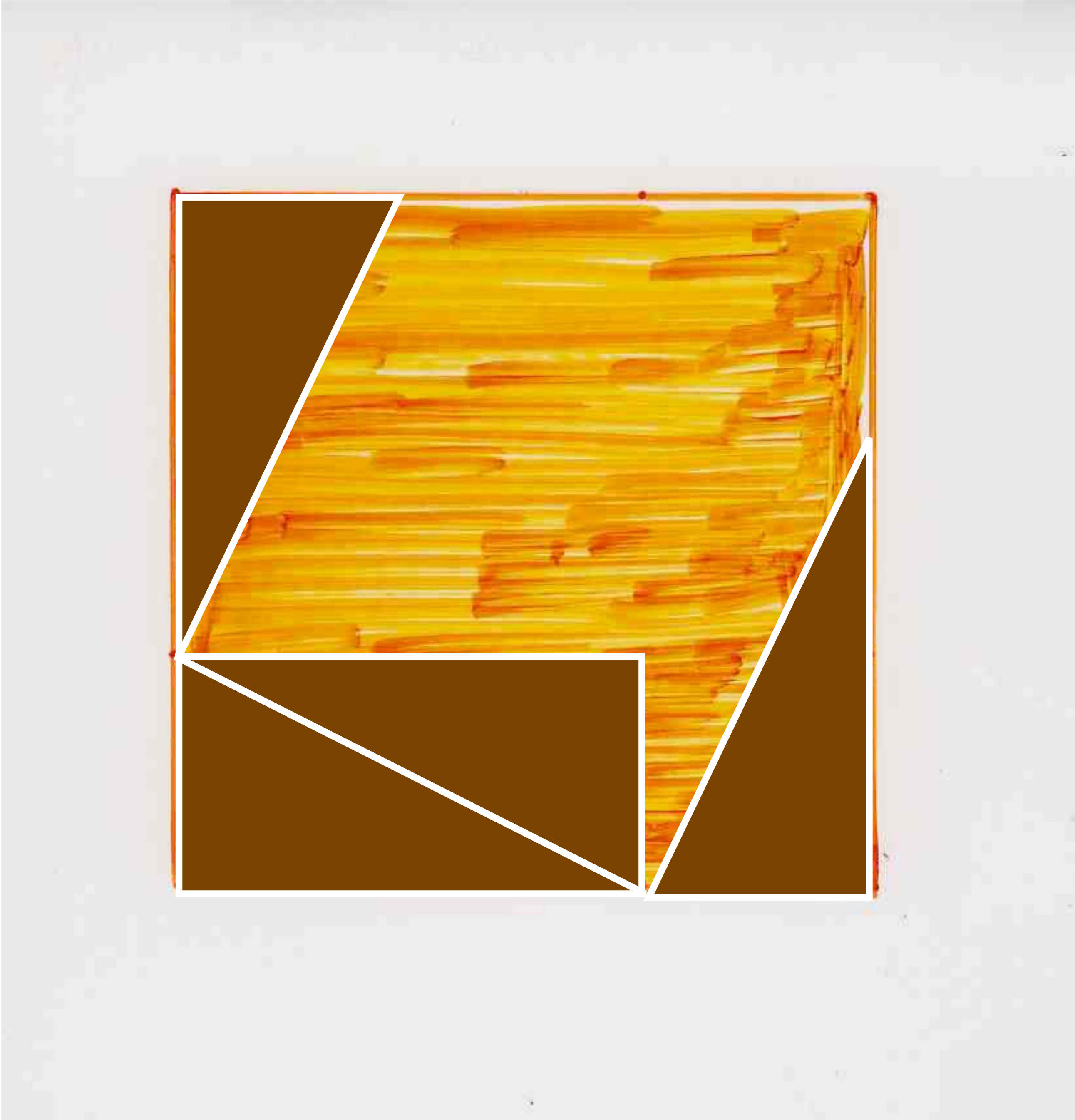


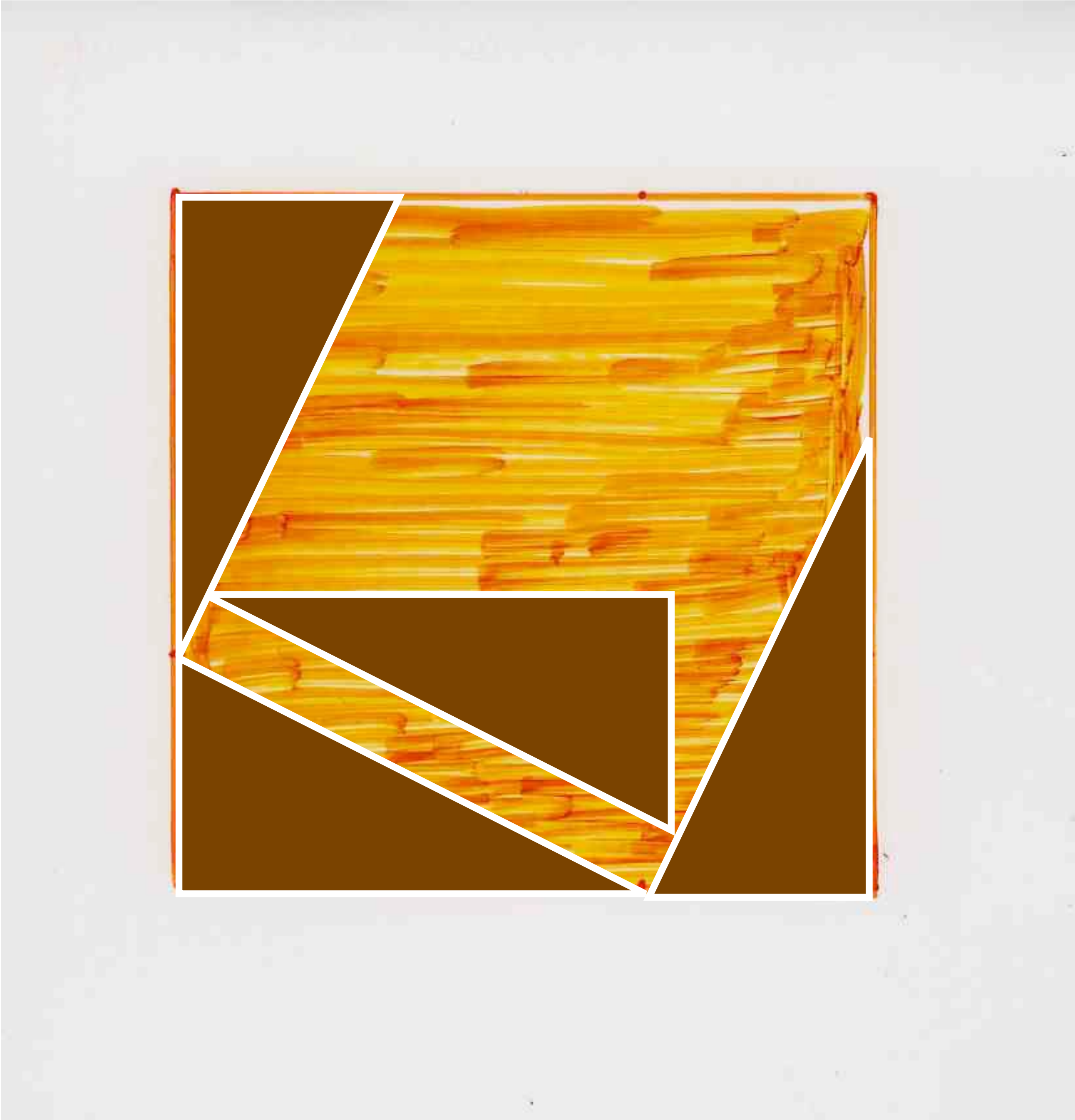


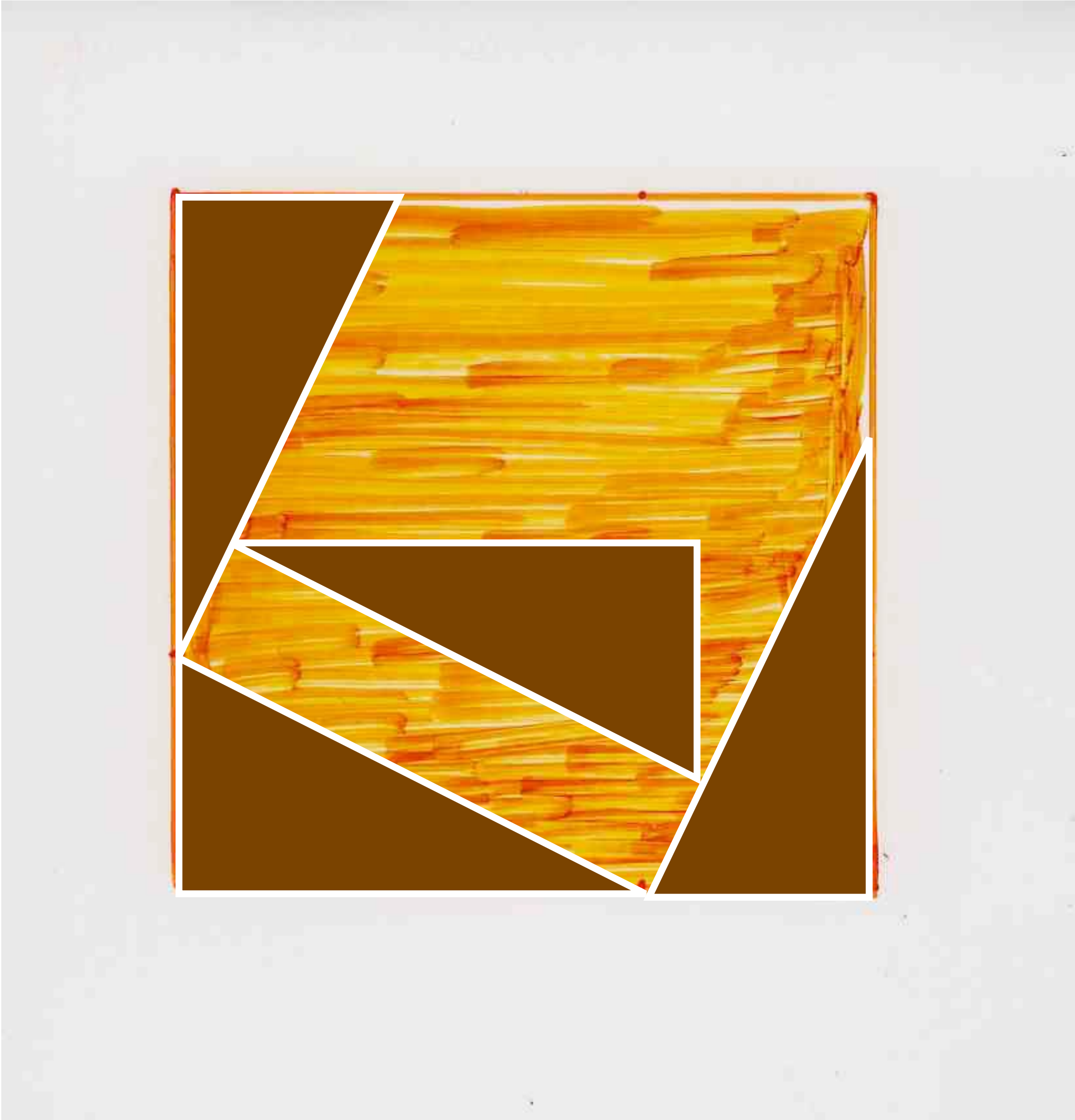


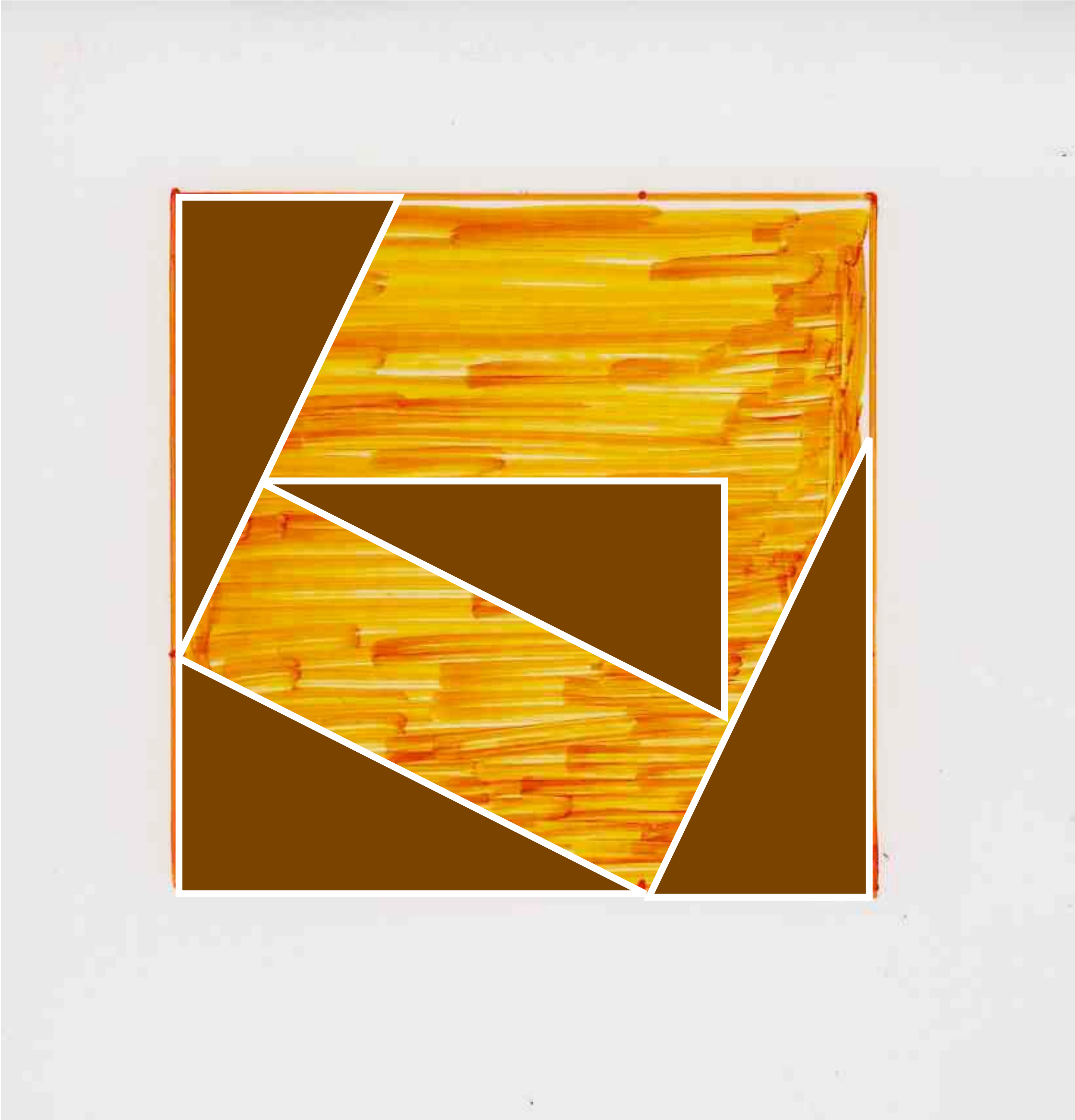


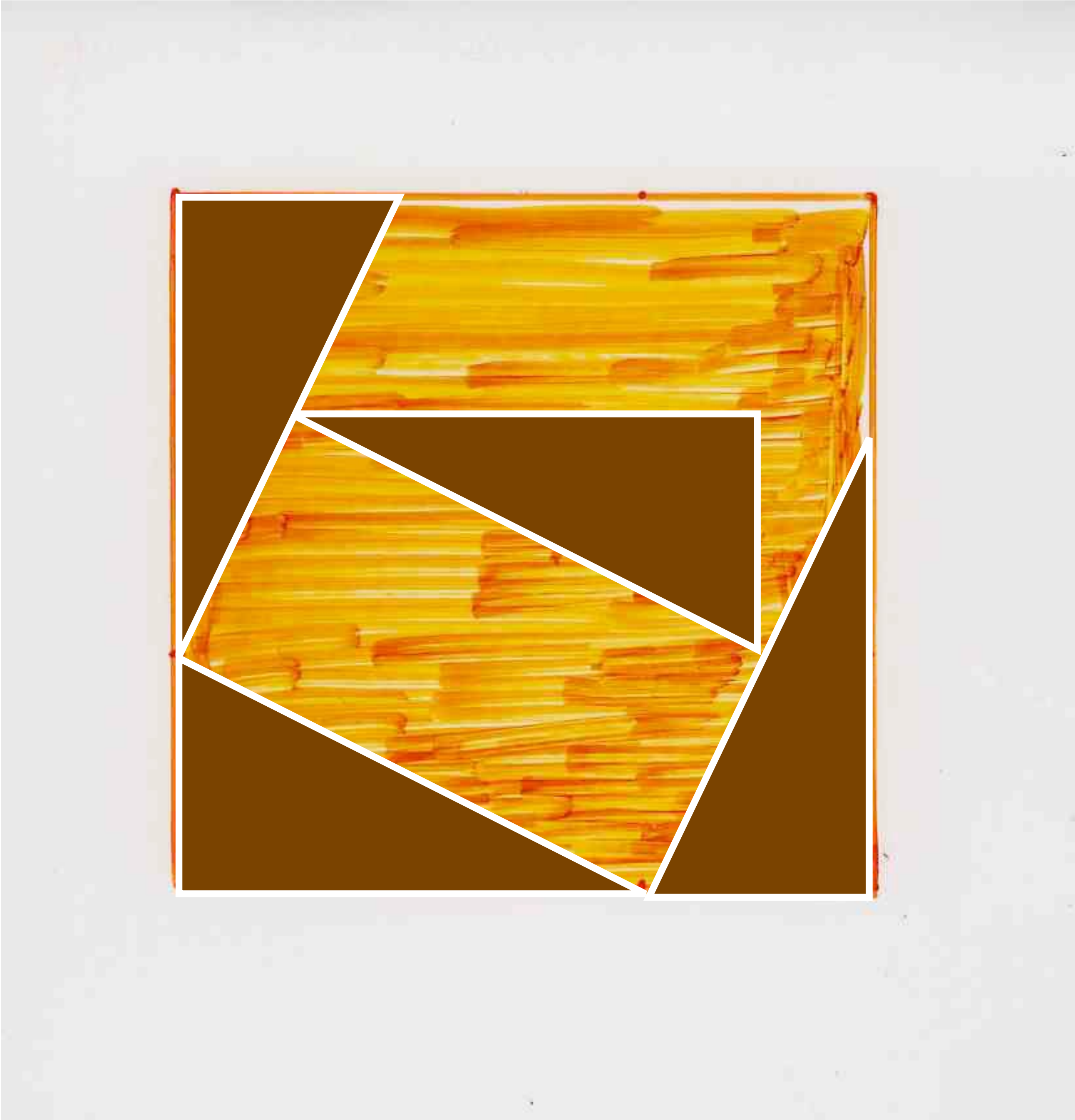


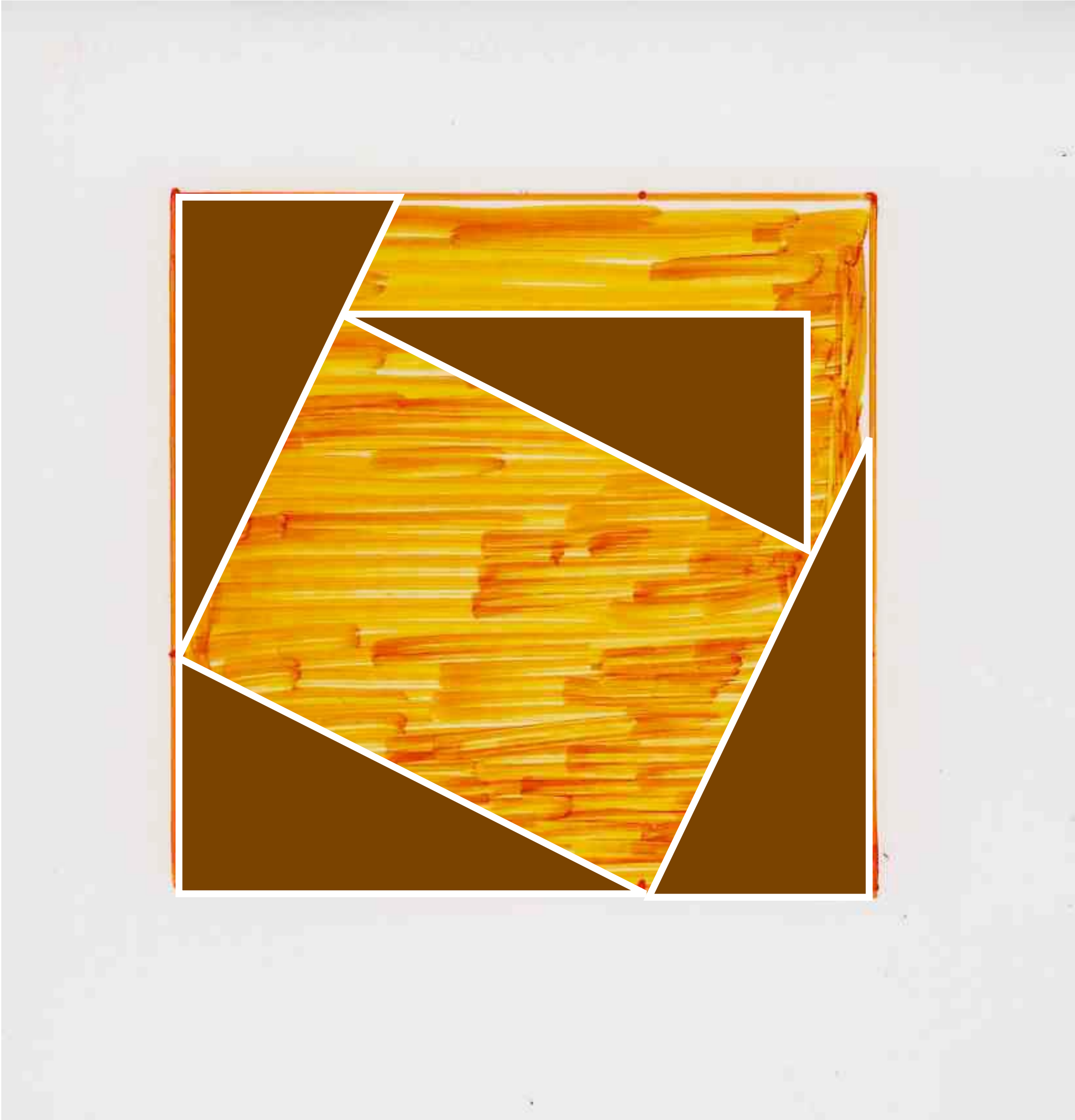


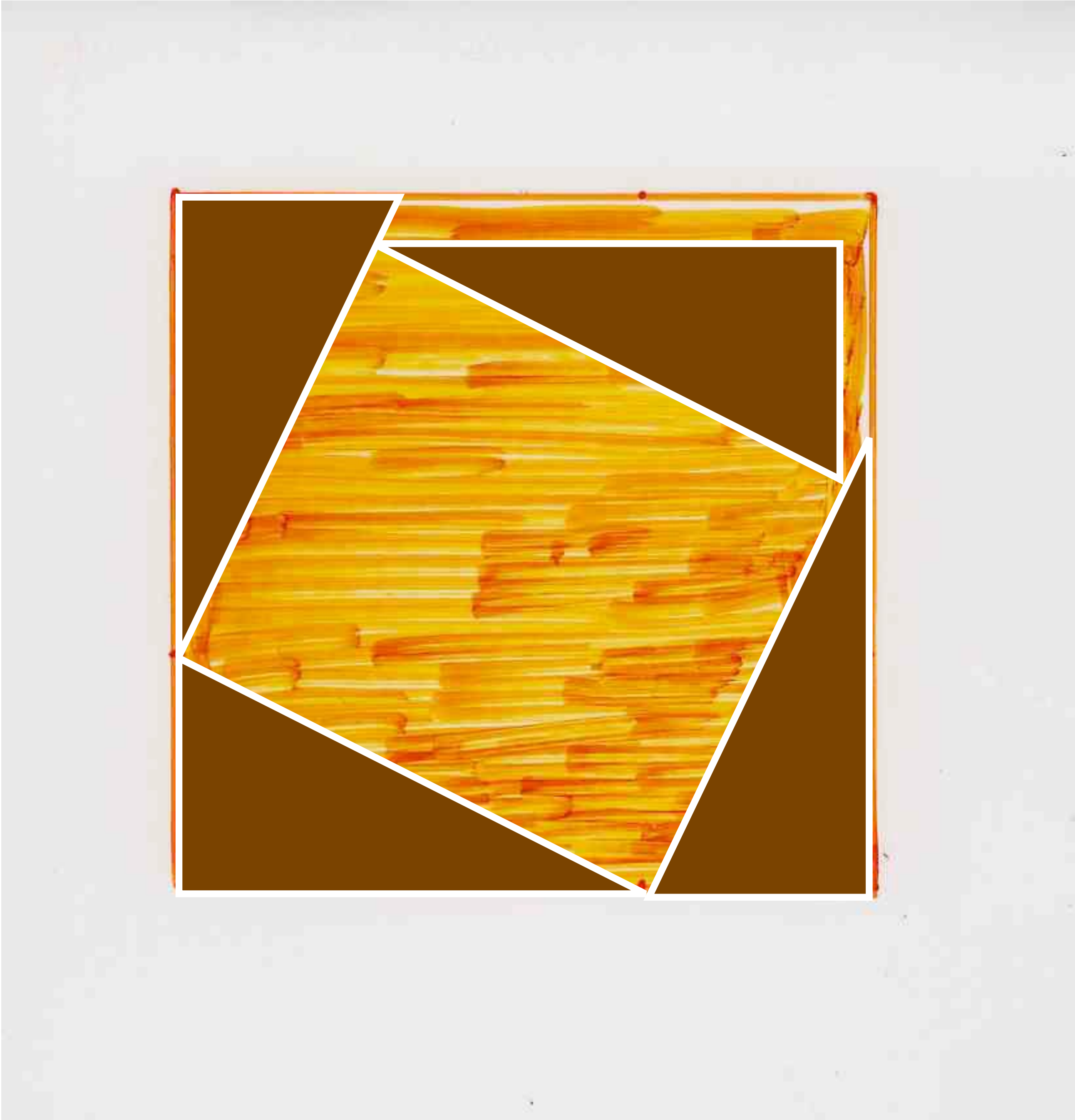


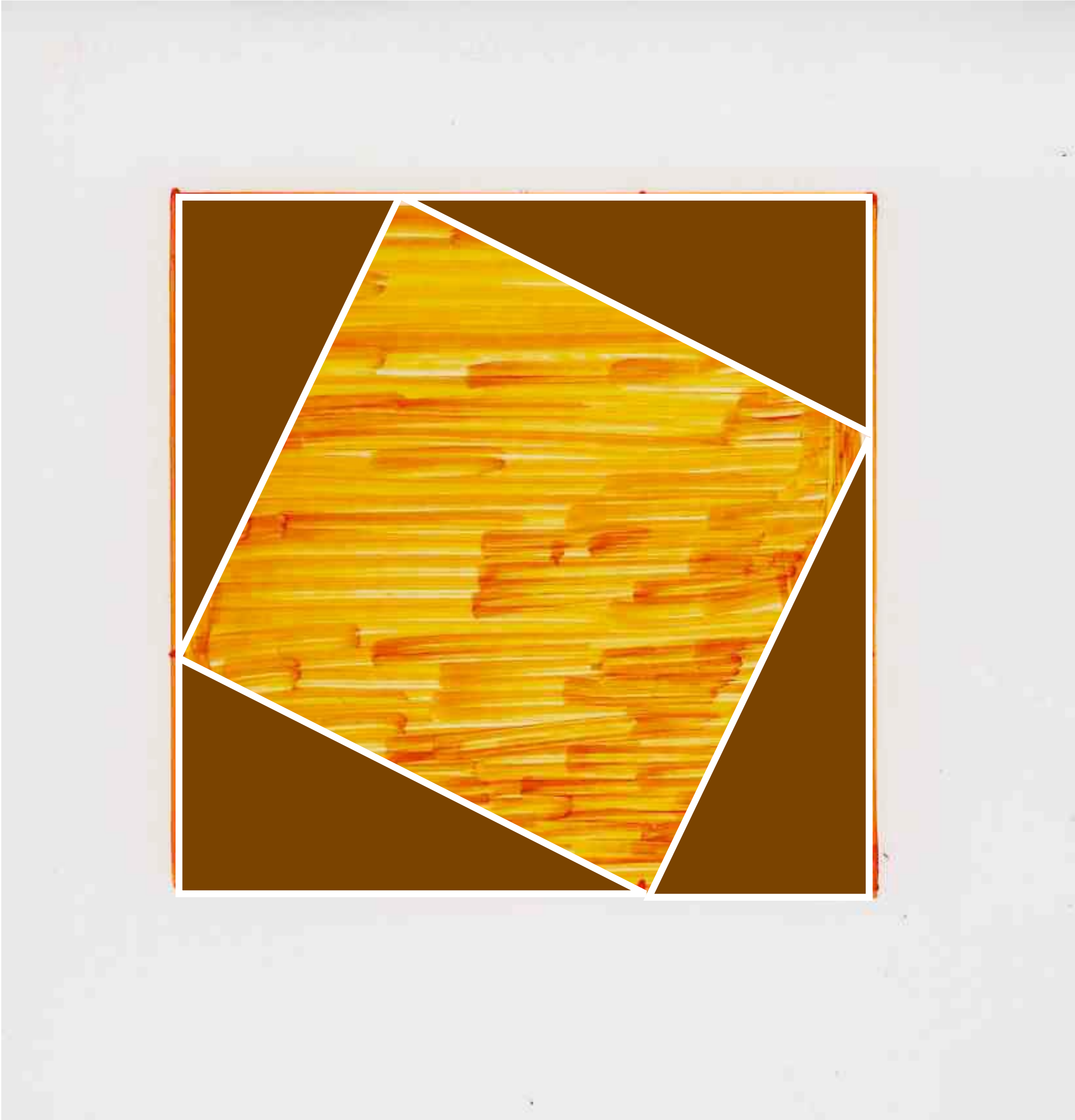


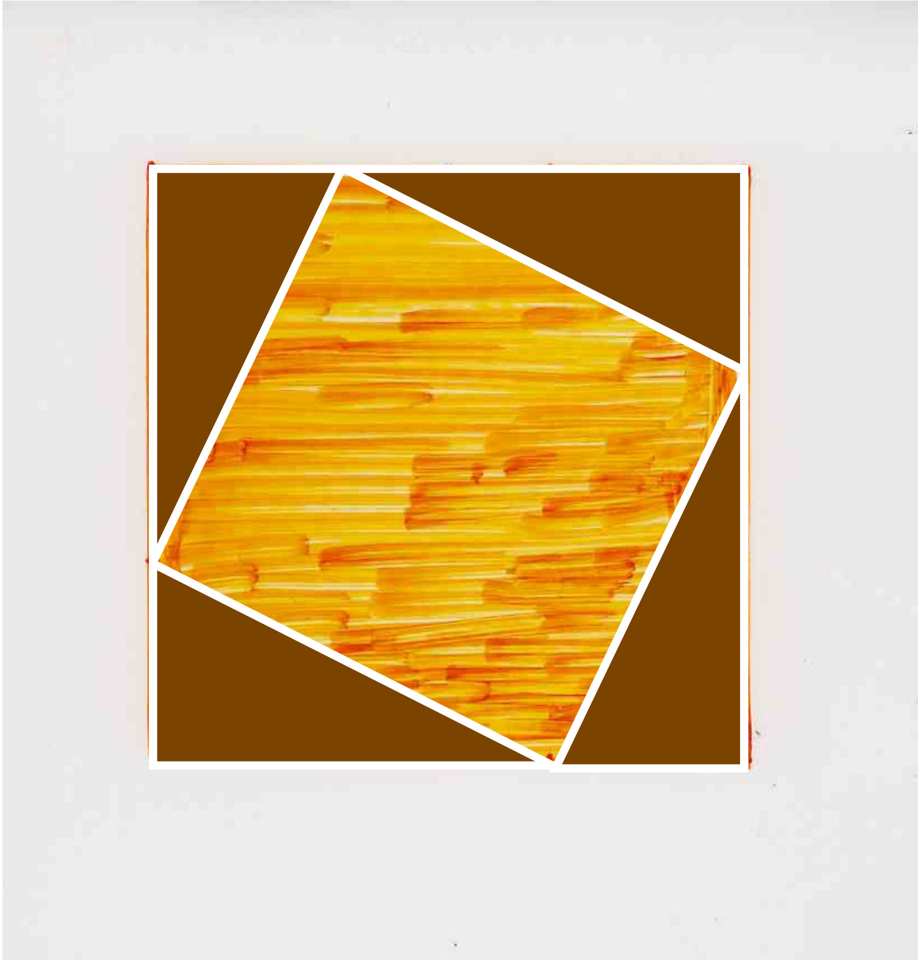




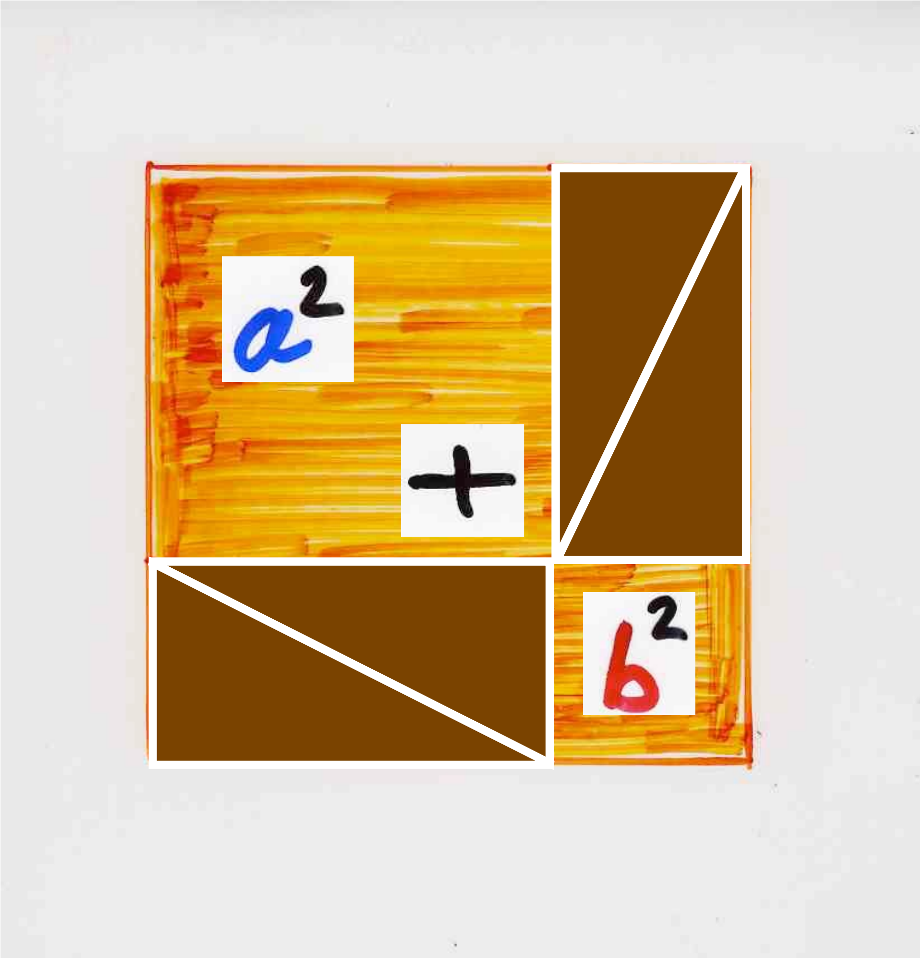


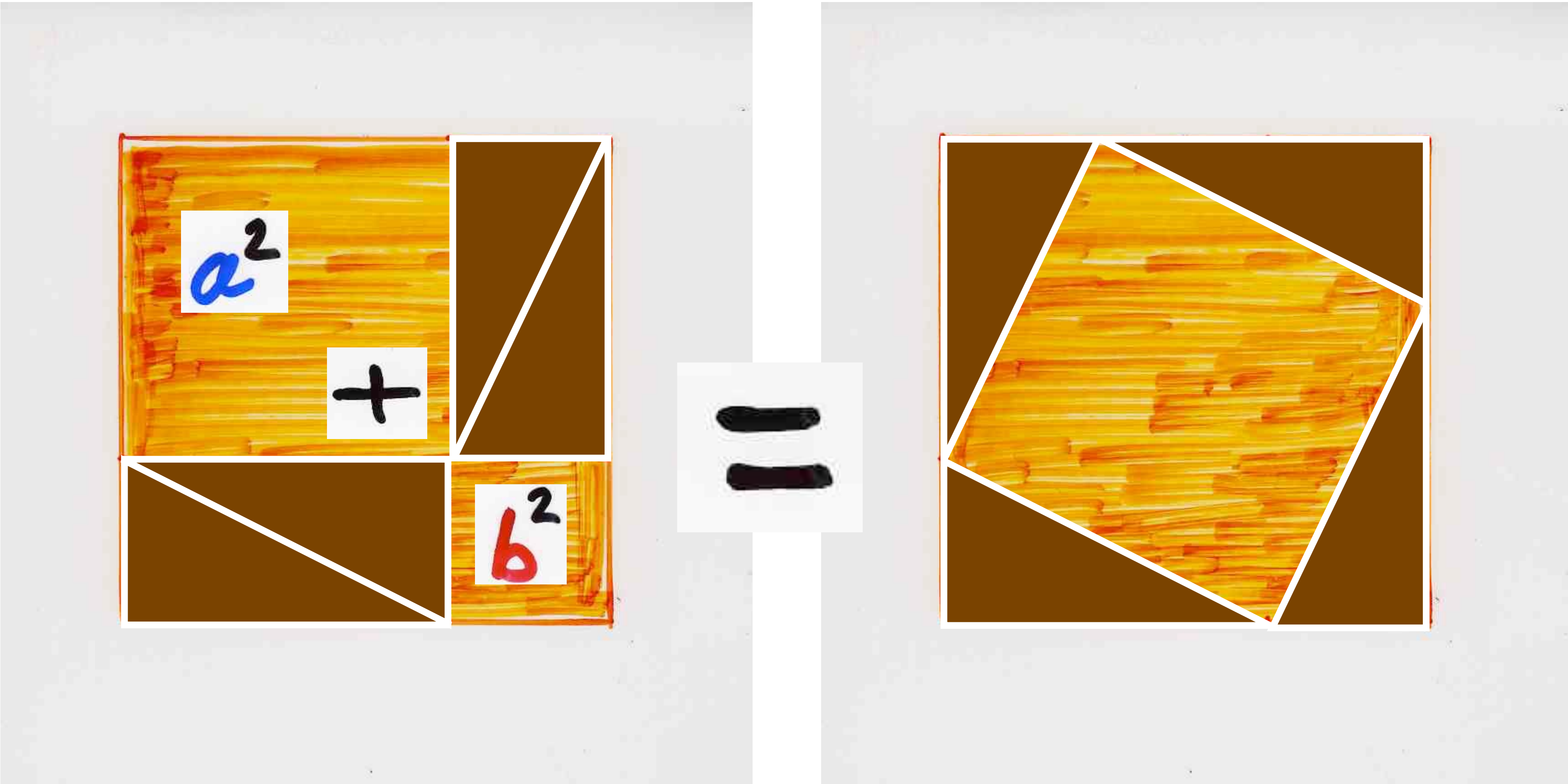


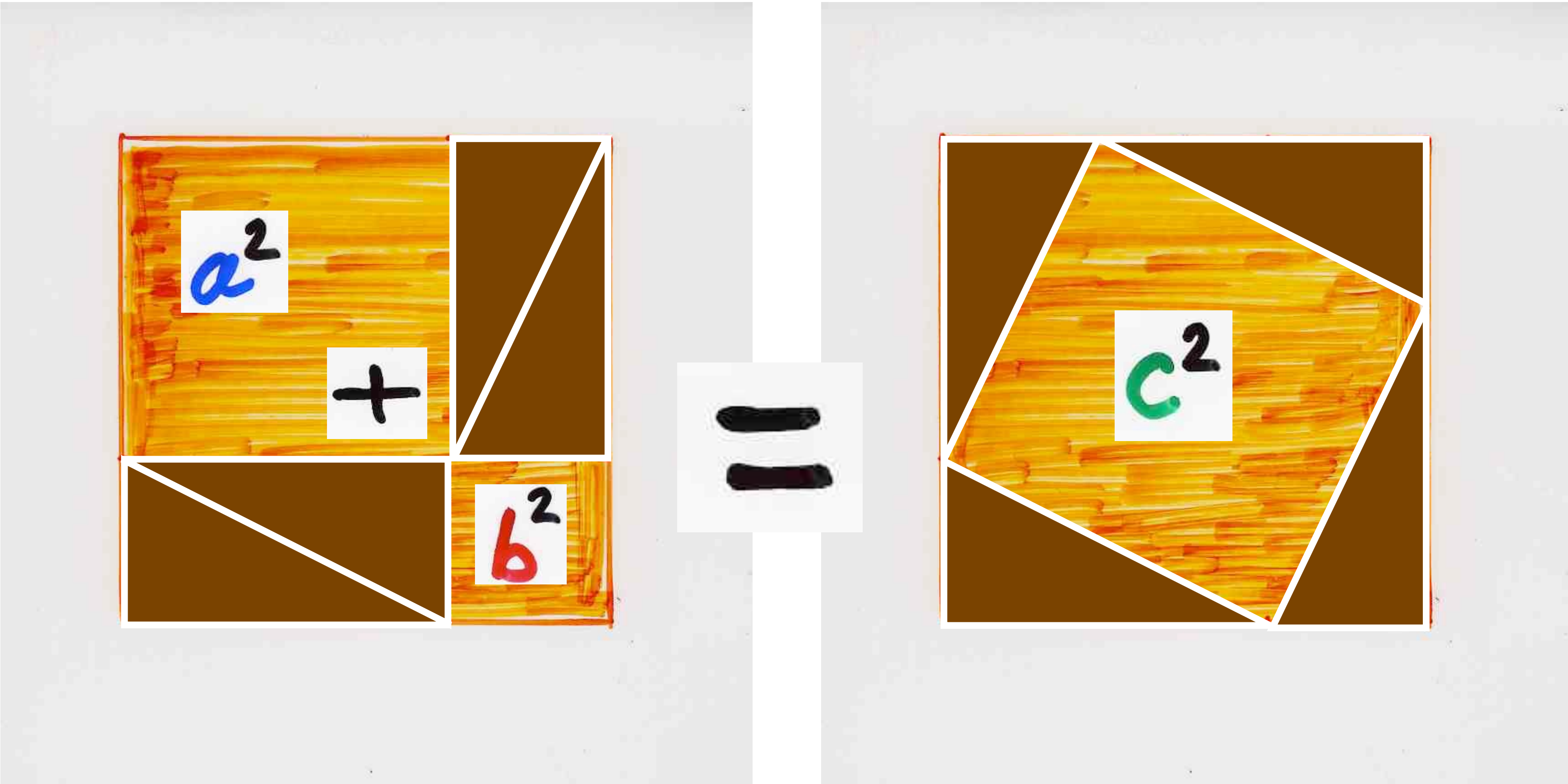





a^2


a^2
+
b^2


a^2
+
b^2
=

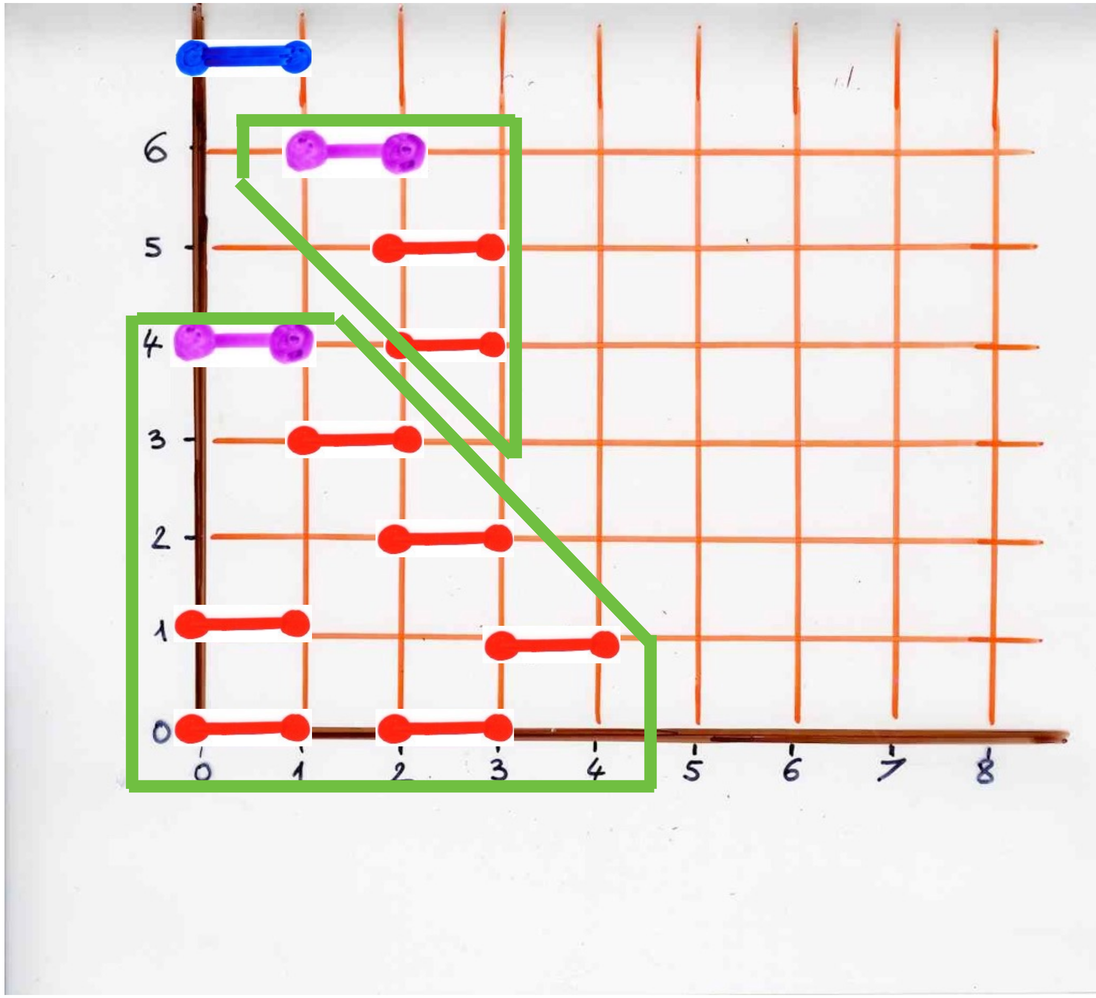

a^2
+
b^2
=
c^2

generating functions

rational

algebraic

D-finite



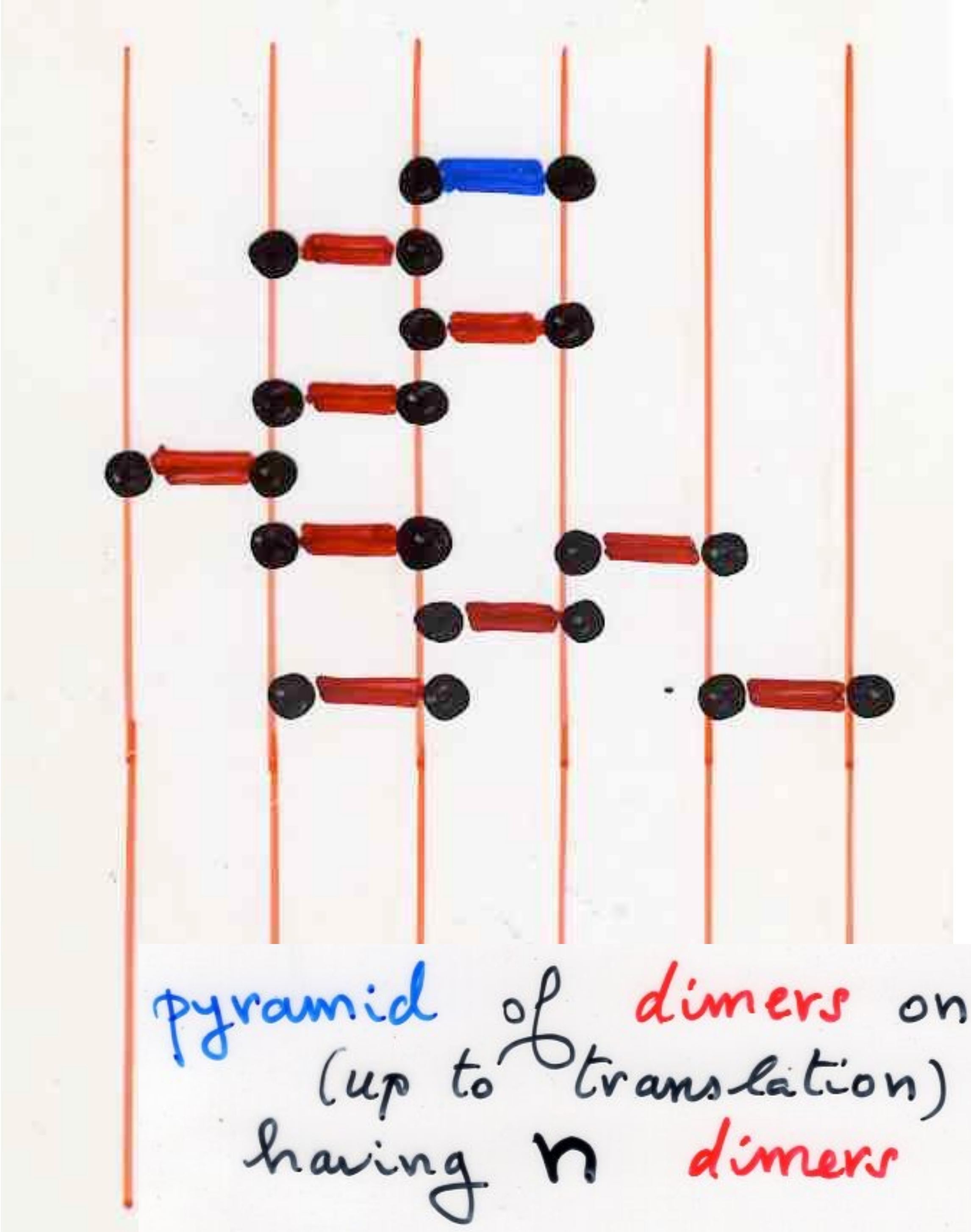
algebraic
generating
function

$$y = 1 + t\,y^2$$

semi-pyramid of dimers on $\mathbb{N}$
the unique maximal piece
has projection [0,1]

Catalan number

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$



pyramid of dimers on $\mathbb{Z}$
(up to translation)
having n dimers

algebraic generating function

$$\frac{1}{2}\binom{2n}{n}$$

system of algebraic equations

rational power series

$\cap|$

algebraic power series

$\cap|$

P-recursive (D-finite) power series

$$\sum_{n\geqslant 0} a_n t^n = \frac{N(t)}{D(t)}$$

$$P(y,t)=0$$

$$P_k(n)a_{n+k} + P_{k-1}(n)a_{n+k-1} + \cdots + P_0(n)a_n = 0$$

rational power series $\Leftrightarrow$ recurrence relation with $P_0, \ldots, P_k$ constants

- Rat

$$\sum_{n\geq 0} F_n t^n = \frac{1}{1-t-t^2}$$

Fibonacci numbers

$$F_{n+1} = F_n + F_{n-1}$$

- Alg

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

Catalan numbers

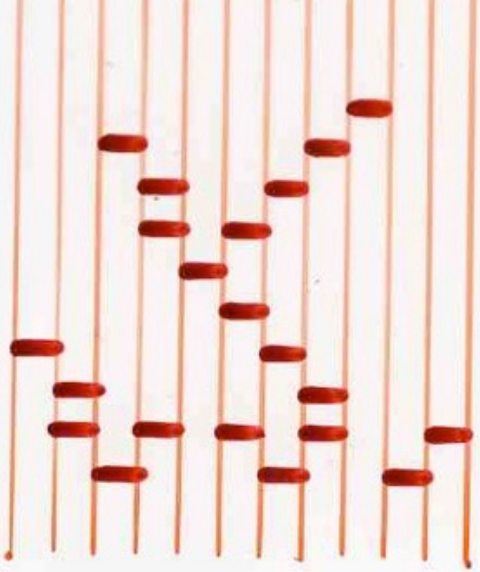
$$y = 1 + t y^2$$

$$2(2n+1)C_n = (n+2)C_{n+1}$$

- D-finite

$$a_n = n!$$

$$a_n = n\, a_{n-1}$$



connected heap of dimers = no empty column

$C(t)$ g.f. connected heap

(Bousquet-Mélou, Rechnitzer) 2002

not D-finite

$$C(t) = \frac{Q}{(1-Q)\left[1-\sum_{k\geq 1}\frac{Q^{k+1}}{1-Q^{k}(1+Q)}\right]}$$

$$Q = \sum_{n\geq 1} C_n t^n$$

Catalan number

First basic lemma on heaps:
the inversion lemma

$1/D$

the inversion lemma

$$(\text{Heaps}) = \frac{1}{(\text{Trivial heaps})}$$

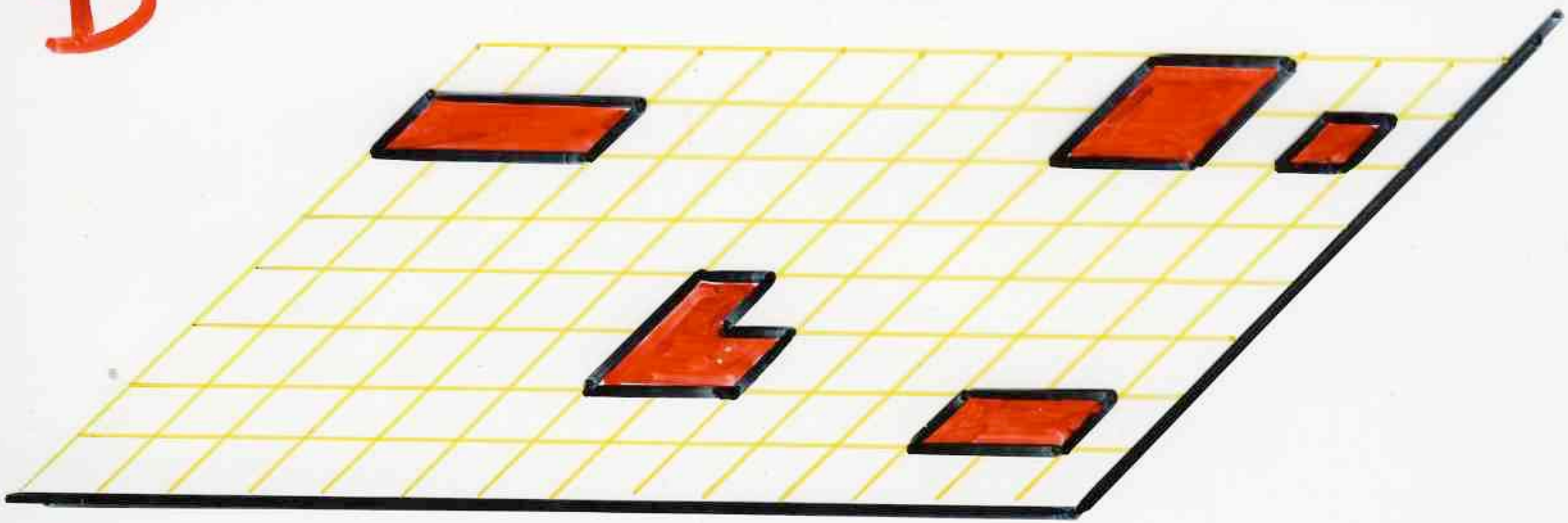
all pieces (α, i)
at level 0

trivial heap

F

all pieces (α, i) at level 0

D



weight
valuation

$v(E)$

- $v : \underset{\text{basic piece}}{P} \longrightarrow \mathbb{K}[x, y, \ldots]$
- $v(\underset{\text{piece}}{\alpha, i}) = v(\alpha)$
- $v(\underset{\text{heap}}{E}) = \prod_{(\alpha, i) \in E} v(\alpha\ i)$

the inversion lemma

$$\left(\sum_{\substack{E \\ \text{heaps}}} v(E)\right) = \frac{1}{\left(\sum_{\substack{F \\ \text{trivial} \\ \text{heaps}}} (-1)^{|F|} v(F)\right)}$$

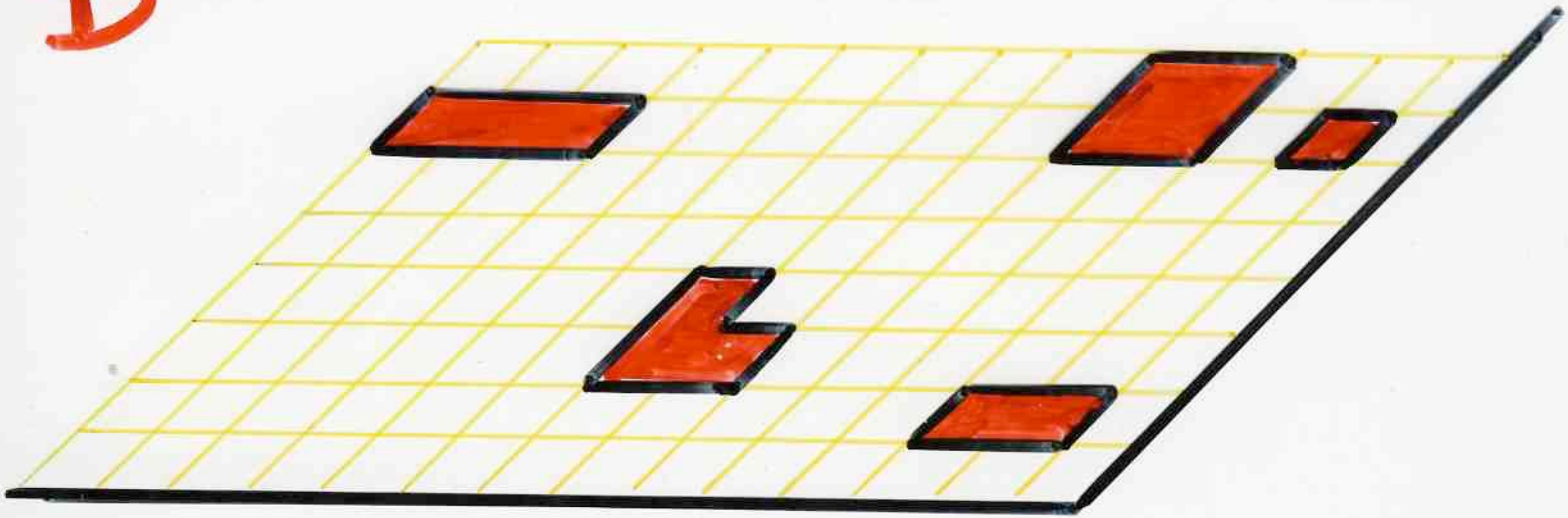
the inversion lemma

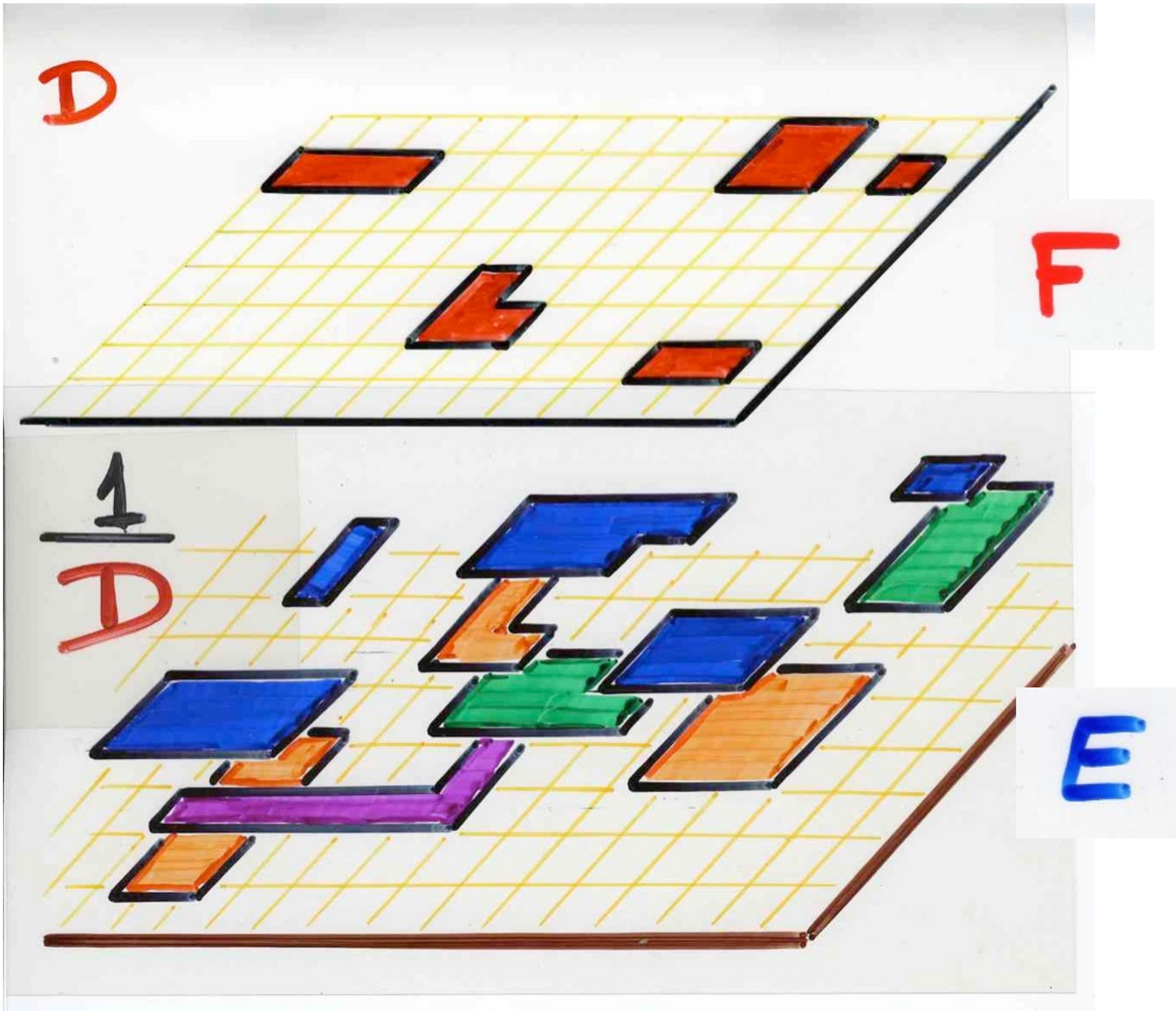
$$\left(\sum_{\substack{E \\ \text{heaps}}} v(E)\right) = \frac{1}{\boxed{\left(\sum_{\substack{F \\ \text{trivial} \\ \text{heaps}}} (-1)^{|F|} v(F)\right)}}$$

D

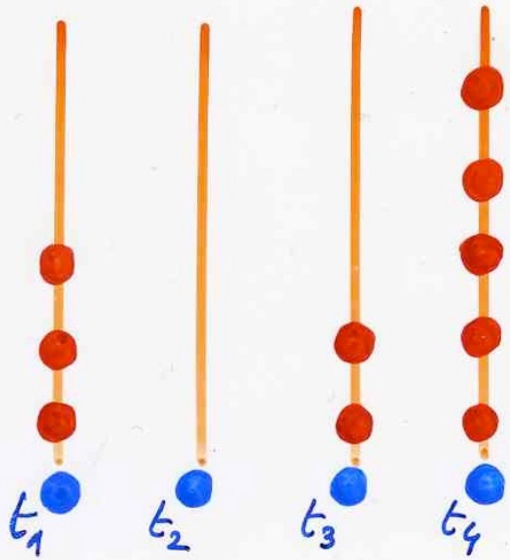
F

D

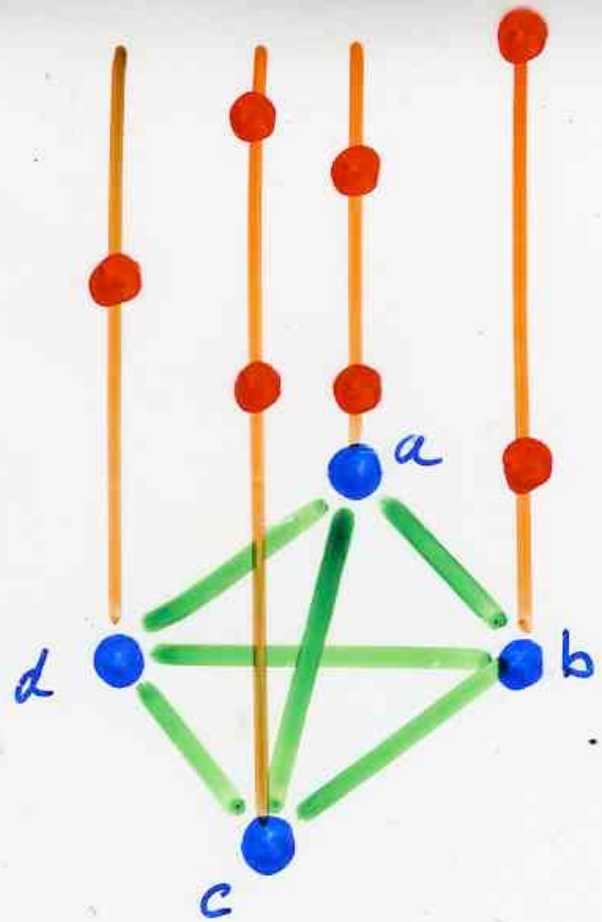



D
F
1
D
E

$$\frac{1}{1-t} = 1 + t + t^2 + \dots + t^n \dots$$



$$\frac{1}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)} = \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4 \geq 0} t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} t_4^{\alpha_4}$$



$$X = \{a, b, c, d\}$$

$$\frac{1}{1-X} = \underline{\underline{X^*}}$$

$$\left(= \sum_{w \in X^*} w \right)$$

Proof of

the inversion lemma

$$\left(\sum_{\substack{E \\ \text{heaps}}} v(E)\right)\left(\sum_{\substack{F \\ \text{trivial} \\ \text{heaps}}} (-1)^{|F|} v(F)\right) = 1$$

define an involution φ

$\varphi(E, F) = (E', F')$

E: heap

F: trivial heap

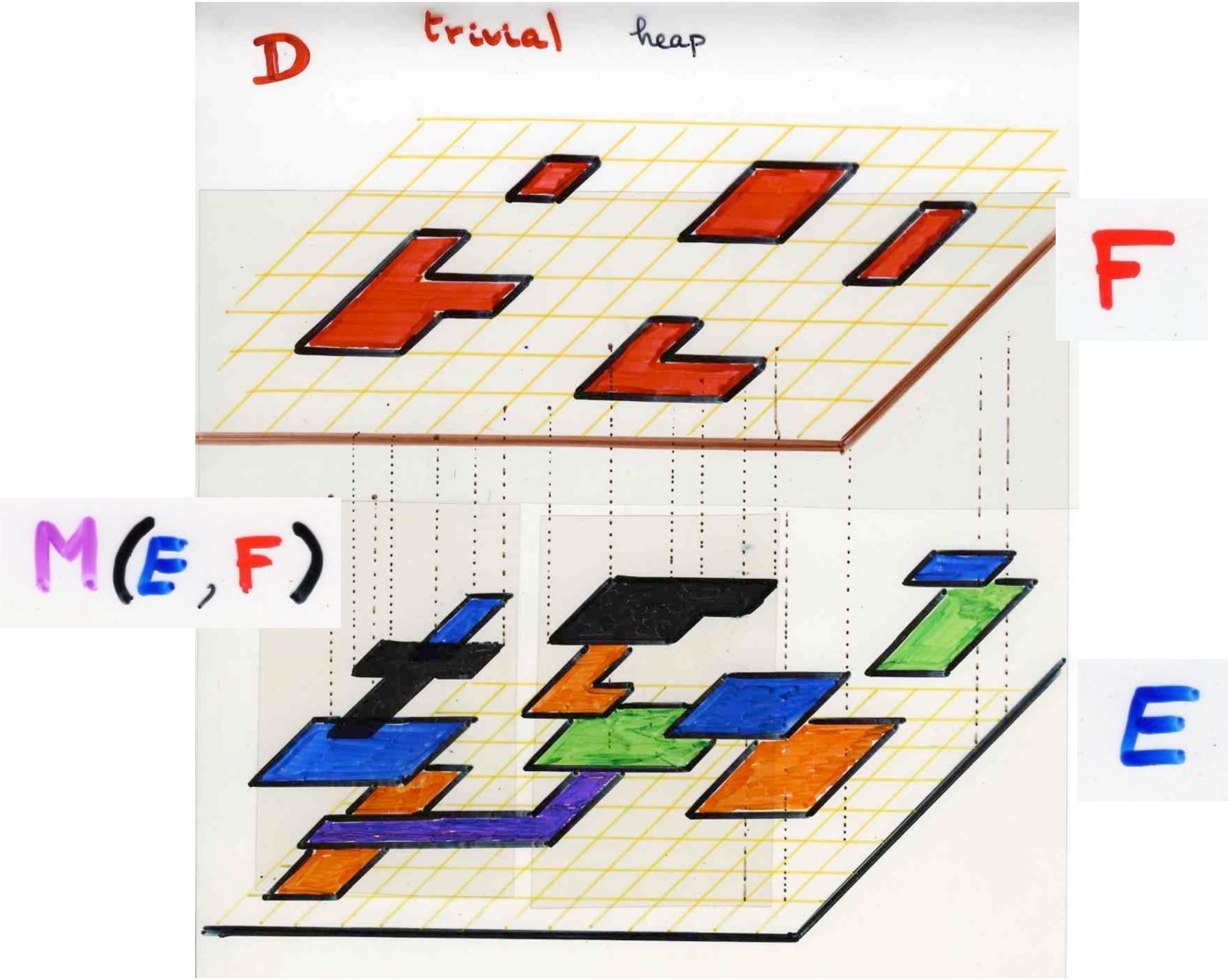
$$\begin{cases} \bullet\ v(E)\,v(F) = v(E')\,v(F') \\ \bullet\ (-1)^{|F|} = -(-1)^{|F'|} \end{cases}$$

φ not defined for $(E, F) = (\emptyset, \emptyset)$

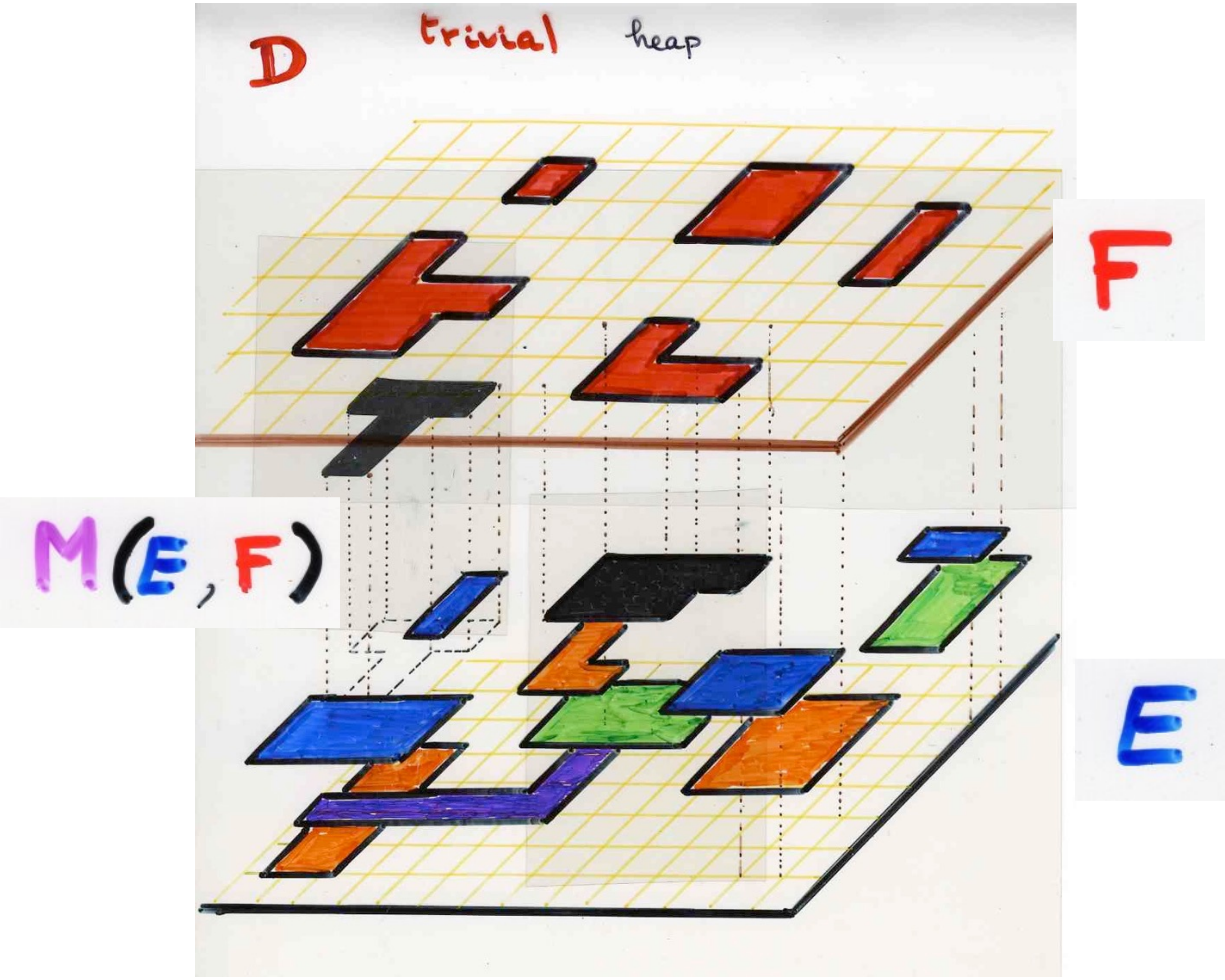
$$M(E,F) = \left\{ \begin{array}{l} m = (\beta, i) \\ \text{maximal piece} \\ \text{of } E \end{array} \right\}$$

such that $\alpha \not\mathcal{C} \beta$ for all $\alpha \in F$

i.e $F \cup \{\beta\}$ is a trivial heap



D
trivial heap
F
M(E, F)
E


D
trivial heap
F
M(E, F)
E

$$M(E,F) = \left\{ \begin{array}{l} m = (\beta, i) \\ \text{maximal piece of } E \end{array} \right\}$$

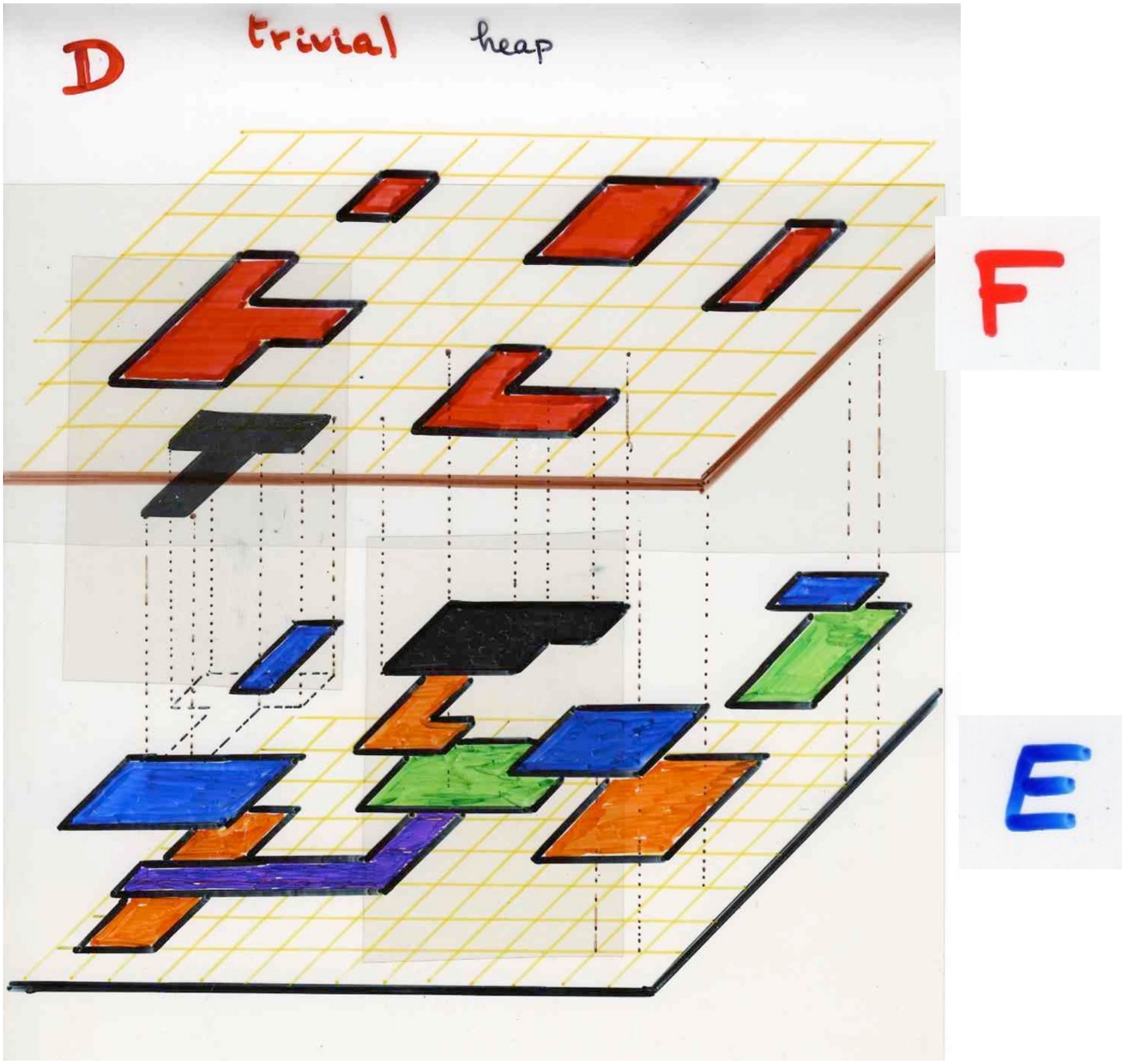
such that $\alpha \not\mathcal{C} \beta$ for all $\alpha \in F$

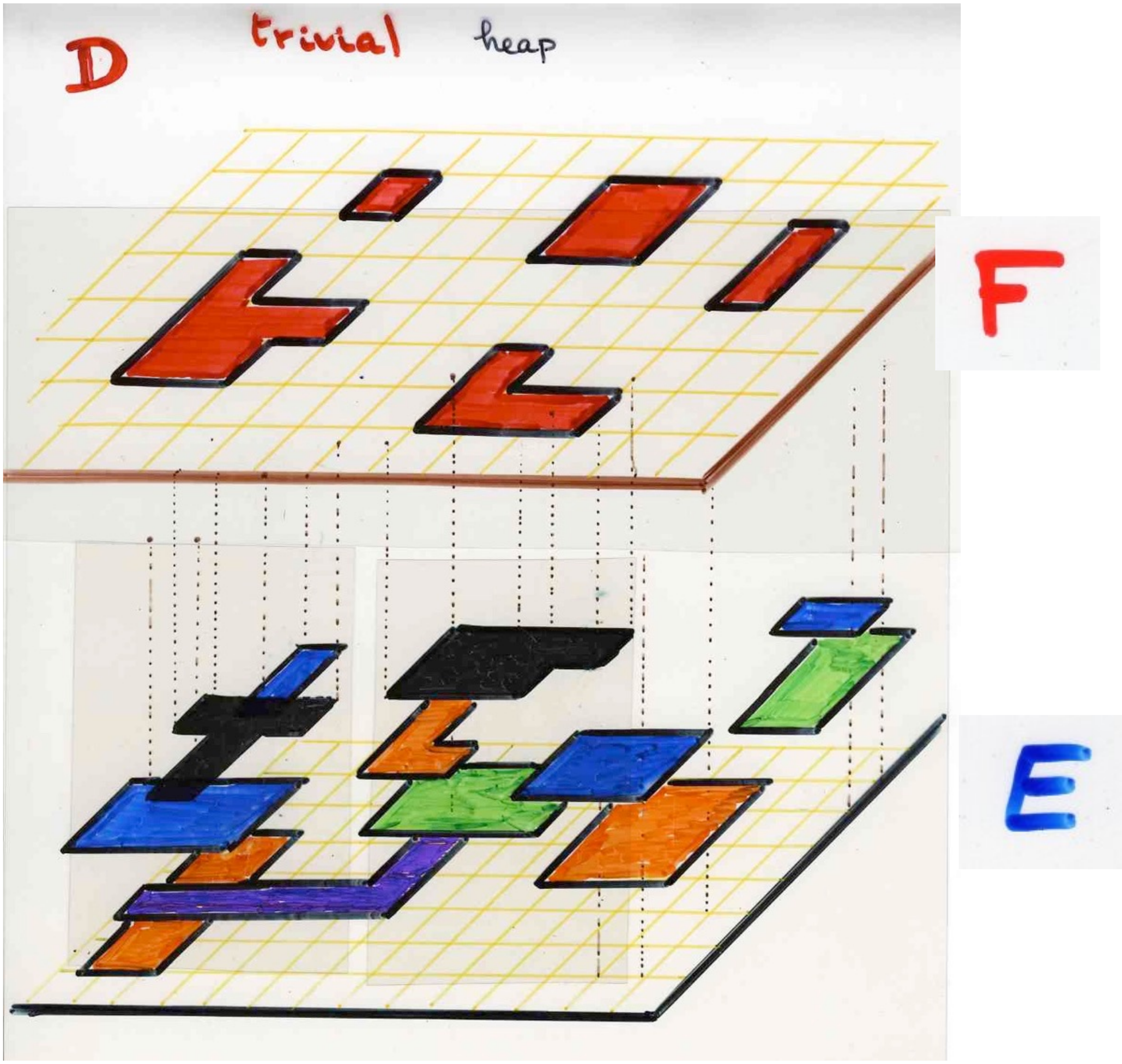
i.e $F \cup \{\beta\}$ is a trivial heap

$$\mathrm{Trans}(E,F) = F \cup M(E,F)$$

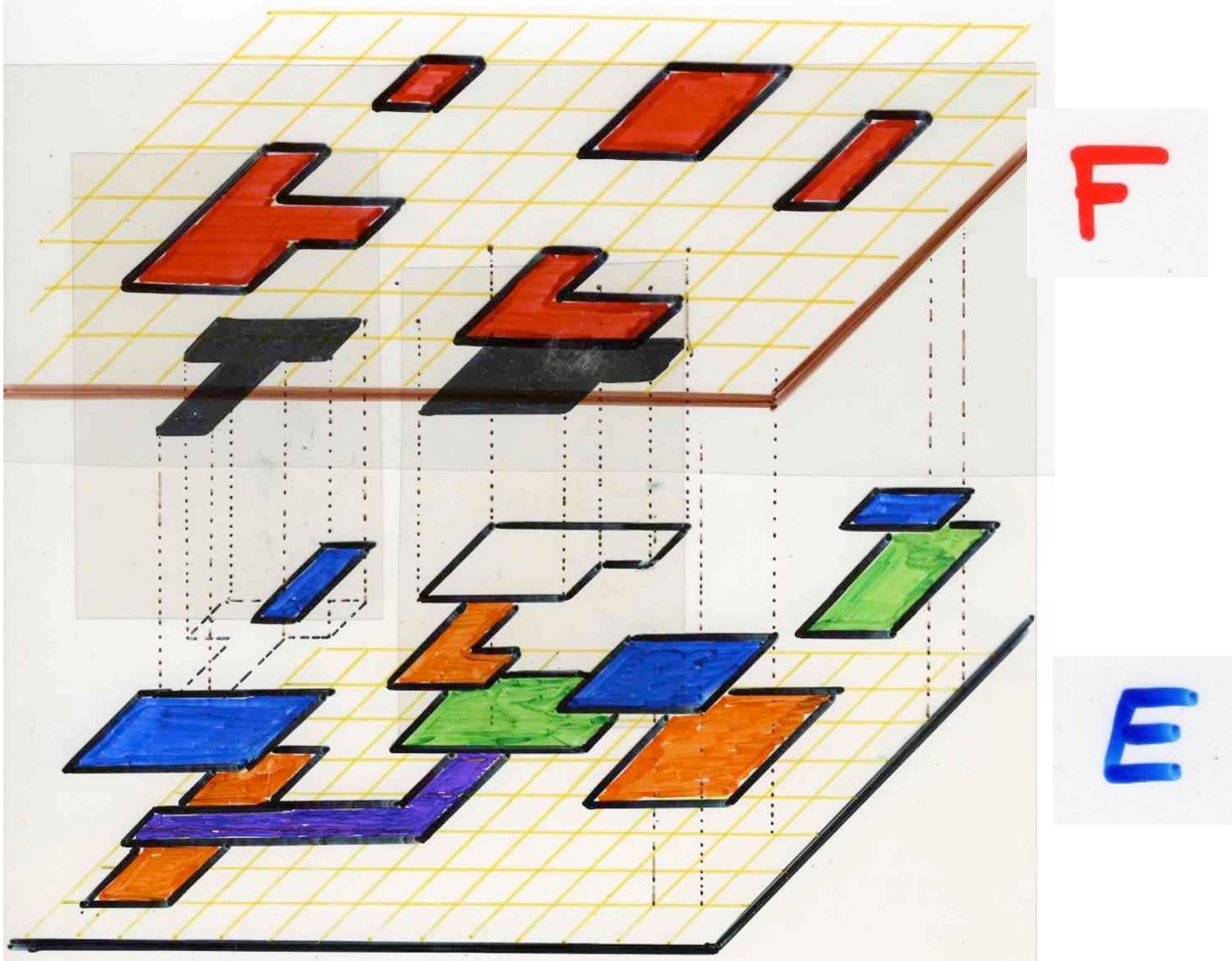
$$F \cap \pi(M(E,F)) = \emptyset$$

π projection $(\alpha, i) \longrightarrow \alpha$


D
trivial
heap
F
E


D
trivial
heap
F
E

D trivial heap



define a total order on the set $\mathcal{P}$ of basic pieces

Let $\gamma \in \mathcal{P}$ be the smallest basic piece such that

$(\gamma, i) \in \mathrm{Trans}(E, F)$

(for a certain $i \geq 0$)

$$F \cap \pi(M(E,F)) = \emptyset$$

- if $\gamma \in F$, then $F' = F \setminus \{\gamma\}$

  $[\gamma \simeq (\gamma, 0)]$ $\quad E' = E \odot \{\gamma\}$

  (adding γ on the top of E)

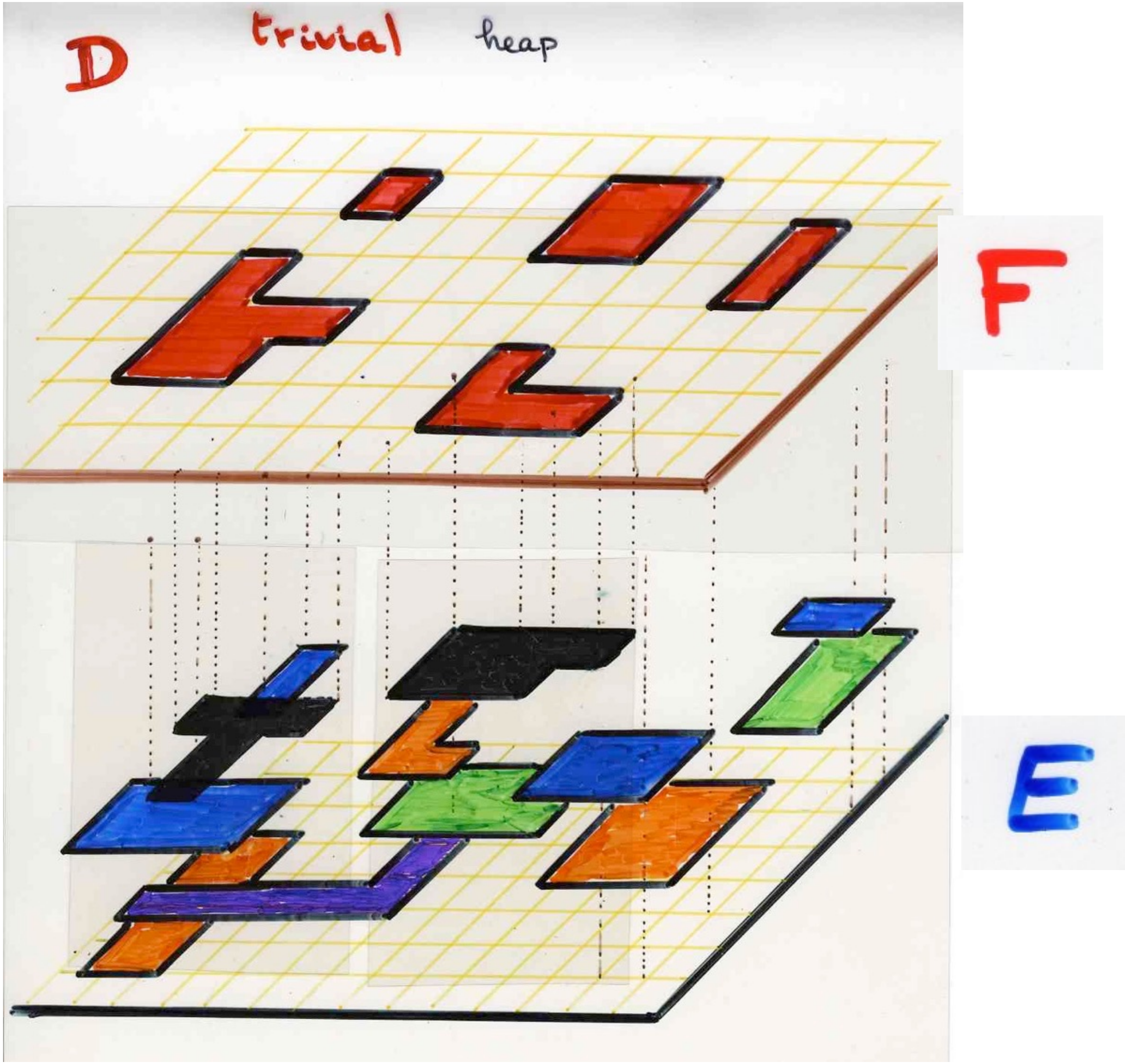
  $\odot$ product of heaps $\gamma \simeq (\gamma, 0)$

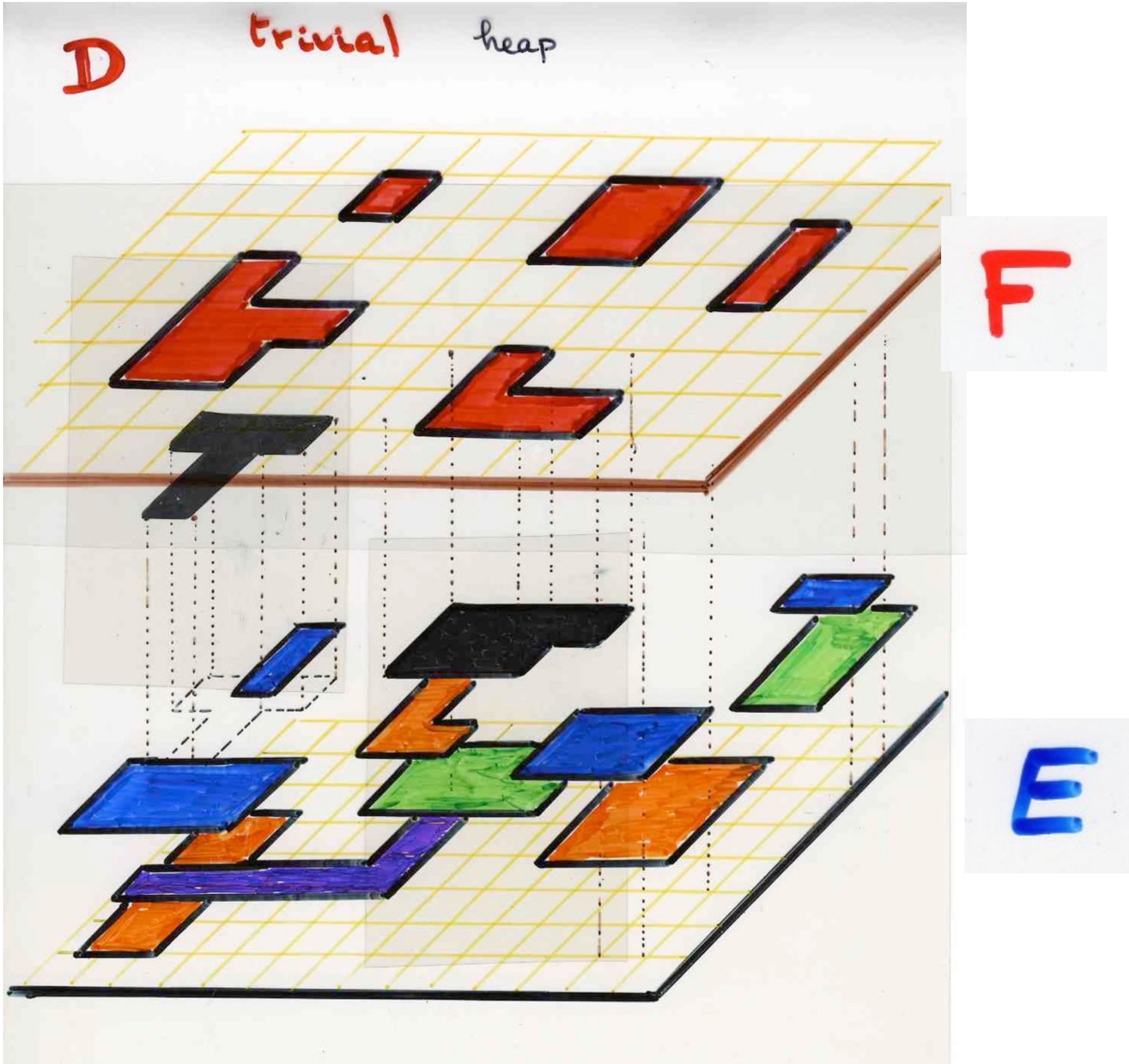
- if $\gamma \in \pi(M(E,F))$, then $F' = F \cup \{\gamma\}$

  $E' = E - \{\gamma\}$

deleting a maximal piece from a heap

E' is the unique heap such that
$E' \odot (\gamma, 0) = E$


D
trivial heap
F
E



D
trivial heap
F
E

$$\mathrm{Trans}(E, F) = \emptyset \quad \Rightarrow F = \emptyset$$
$$\Rightarrow E = \emptyset$$

$$(E, F) \xrightarrow{\varphi} (E', F')$$

φ is an involution $\qquad \varphi^2 = \mathrm{Id}$

$$\begin{cases} \bullet \; v(E)\,v(F) = v(E')\,v(F') \\ \bullet \; (-1)^{|F|} = -(-1)^{|F'|} \end{cases}$$

$$\Big(\sum_{\substack{E \\ \text{heaps}}} v(E)\Big)\Big(\sum_{\substack{F \\ \text{trivial heaps}}} (-1)^{|F|} v(F)\Big) = 1$$

extension of the inversion lemma

N/D

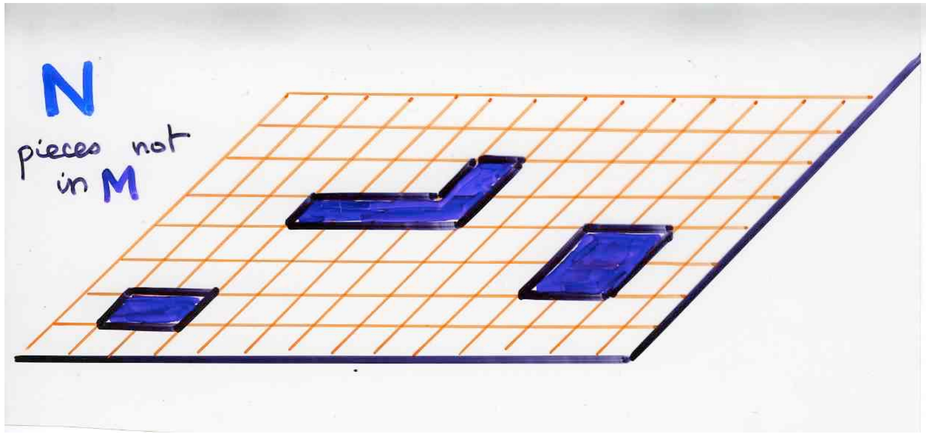
extension of the inversion lemma

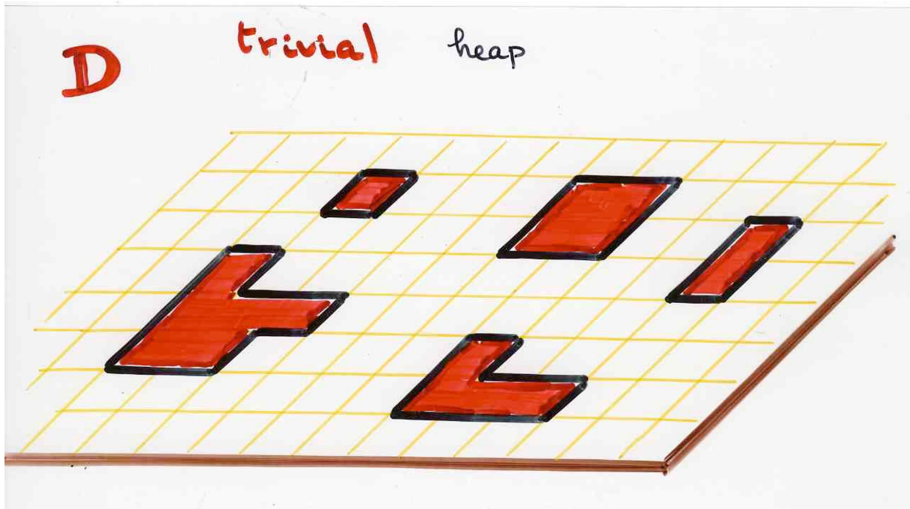
$M \subseteq P$

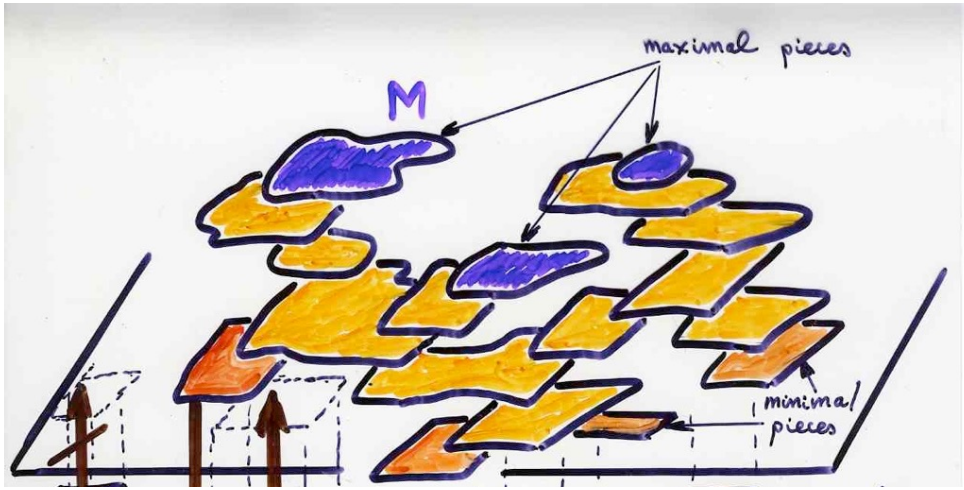
$$\sum_{\substack{E \\ \pi(\text{maximal pieces}) \in M}} v(E) = \frac{N}{D}$$

$$D = \sum_{\substack{F \\ \text{trivial heaps}}} (-1)^{|F|} v(F)$$

$$N = \sum_{\substack{F \\ \text{trivial heaps} \\ \text{pieces} \notin M}} (-1)^{|F|} v(F)$$


N
pieces not in M


D
trivial heap

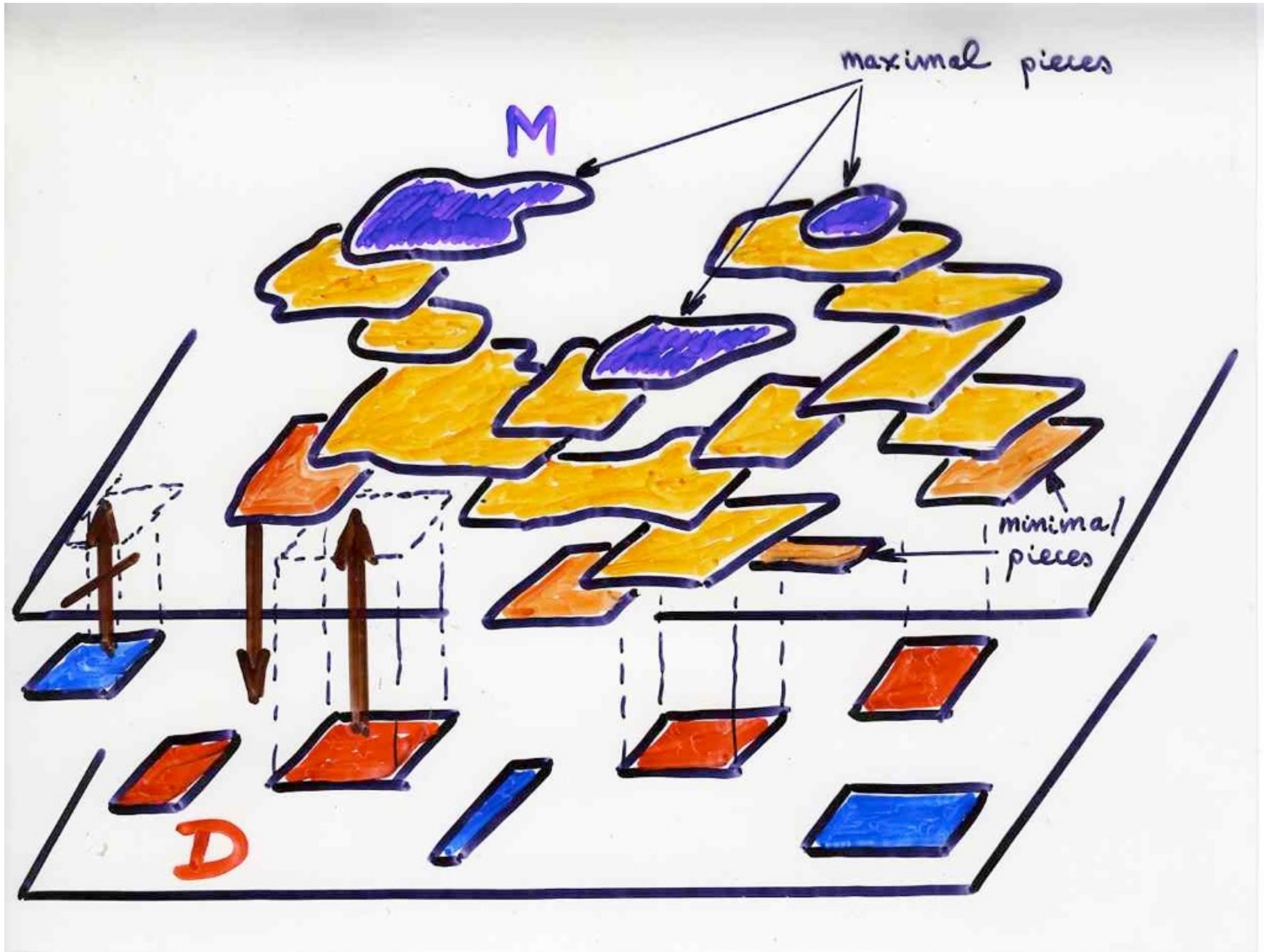

maximal pieces
M
minimal pieces

$$\left(\sum_{\substack{E\\ \text{heaps}}} v(E)\right)\left(\sum_{\substack{F\\ \text{trivial}\\ \text{heaps}}} (-1)^{|F|} v(F)\right) = N$$

$\prod \begin{pmatrix}\text{maximal}\\ \text{pieces}\end{pmatrix} \in M$

D

Proof by involution



define an involution φ

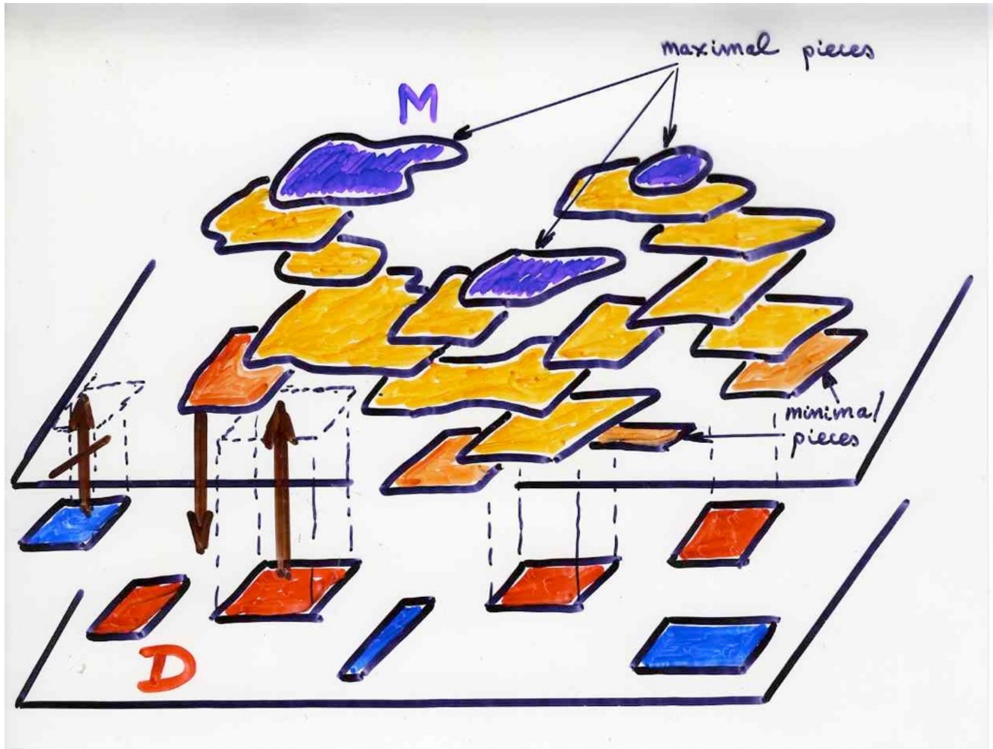
$$\varphi(E, F) = (E', F')$$

heap trivial heap

$$\prod \left(\begin{smallmatrix}\text{maximal}\\ \text{pieces}\end{smallmatrix}\right) \in M$$

$$\begin{cases} \bullet\ v(E)\,v(F) = v(E')\,v(F') \\ \bullet\ (-1)^{|F|} = -(-1)^{|F'|} \end{cases}$$

φ not defined for (E, F) with $E = \emptyset$, $F \subseteq P - M$



$$\mathrm{Trans}(E,F) = \left\{ \alpha \in F,\ (\alpha, 0) \odot E \text{ is a heap with } \pi\left(\begin{matrix}\text{maximal}\\ \text{pieces}\end{matrix}\right) \in M \right\}$$

$$\cup$$

$$m(E,F) = \left\{ \begin{matrix} m = (\beta, 0) \text{ minimal piece of } E \\ \text{such that } \alpha \not\mathcal{C}\, \beta \text{ for all } \alpha \in F \end{matrix} \right\}$$

define a total order on the set $\mathcal{P}$ of basic pieces

Let $\gamma \in \mathcal{P}$ be the smallest basic piece such that

$(\gamma, i) \in \mathrm{Trans}(E, F)$

(for a certain $i \geq 0$)

(i) • if $\gamma \in \left\{ \alpha \in F \,,\ (\alpha, 0) \odot E \text{ is a heap with } \Pi\left(\begin{smallmatrix}\text{maximal}\\ \text{pieces}\end{smallmatrix}\right) \in M \right\}$

$[\gamma = (\gamma, 0)]$

then $F' = F \setminus \{\gamma\}$

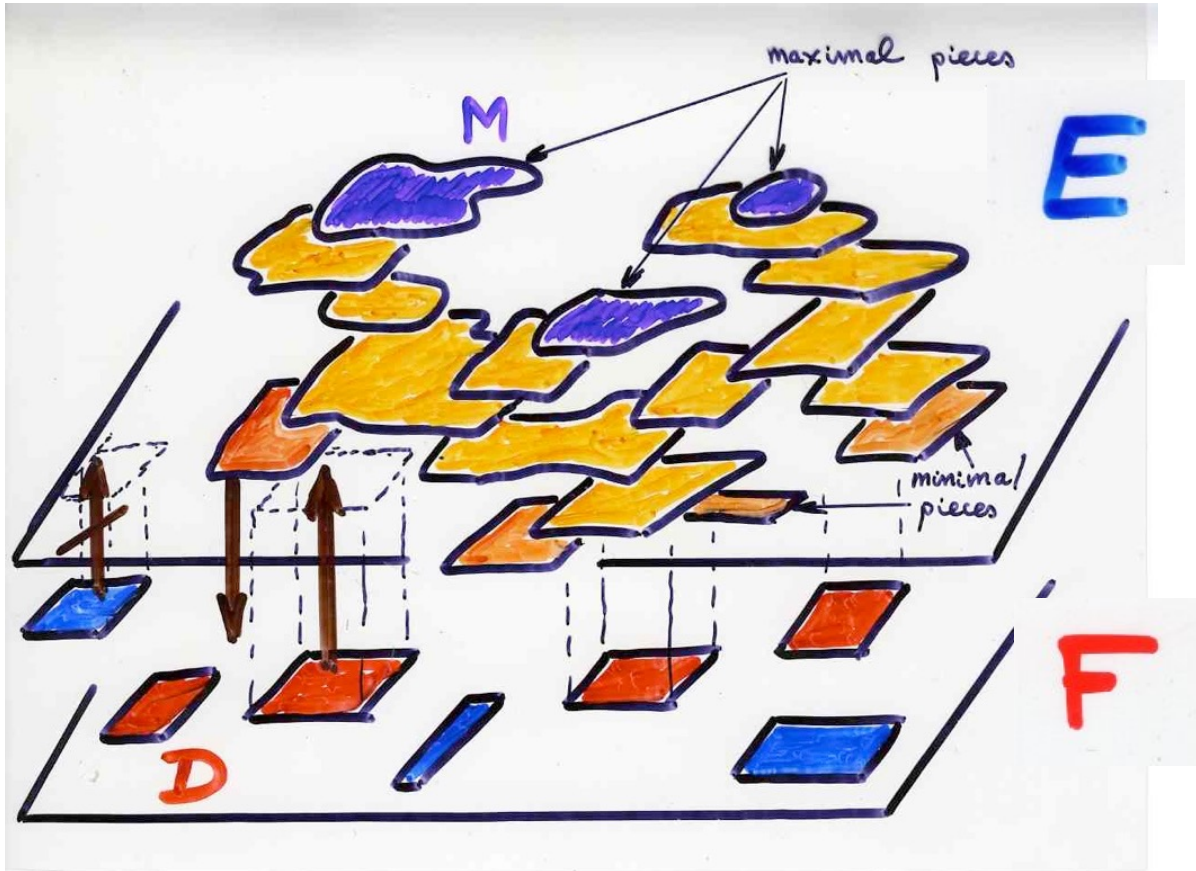
$E' = \{\gamma\} \odot E$

(ii) • if $\gamma \in \Pi(m(E, F))$

$F' = F \cup \{\gamma\}$

E' unique heap such that $\{\gamma\} \odot E' = E$

deleting a minimal piece from a heap



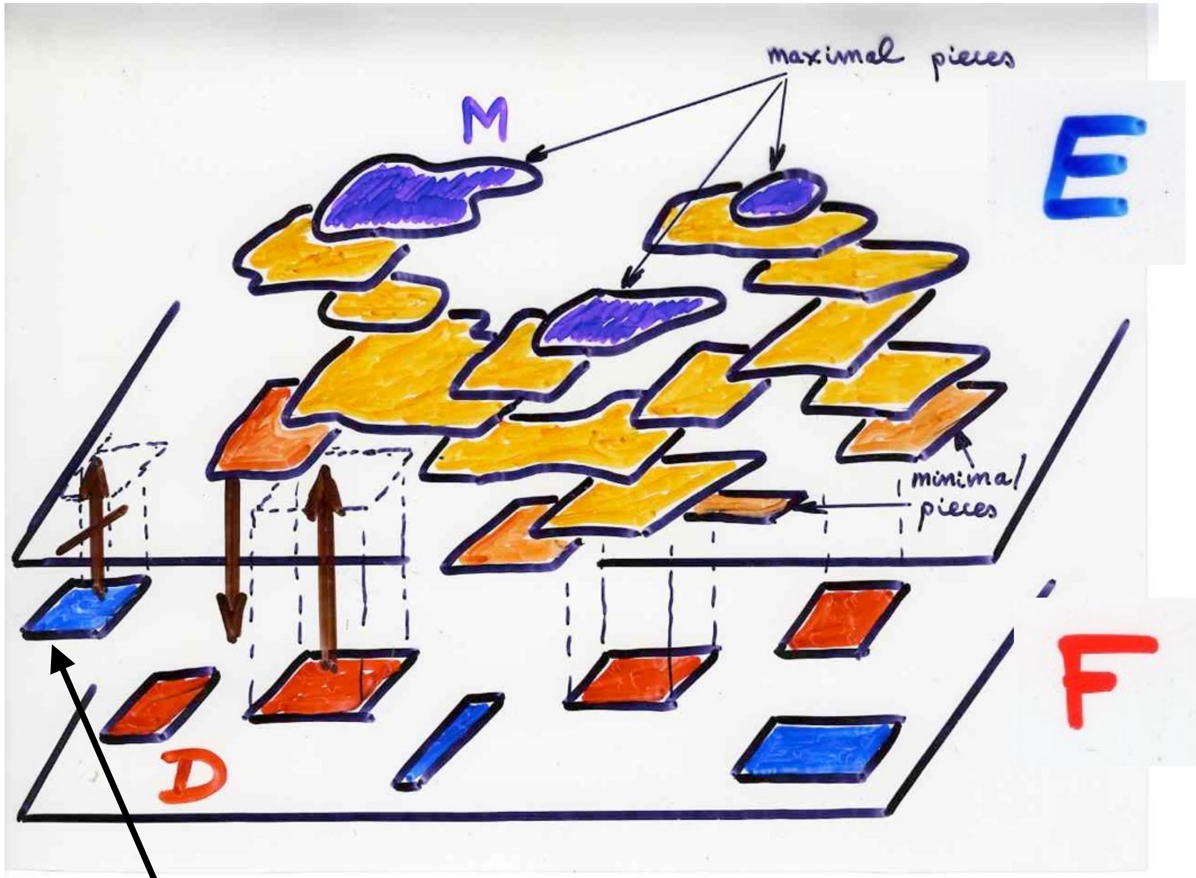
(i) • if $\gamma \in \left\{ \alpha \in F,\ (\alpha, 0) \odot E \text{ is a heap with } \pi\left(\begin{smallmatrix}\text{maximal}\\ \text{pieces}\end{smallmatrix}\right) \in M \right\}$

$[\gamma = (\gamma, 0)]$

$\uparrow \gamma$

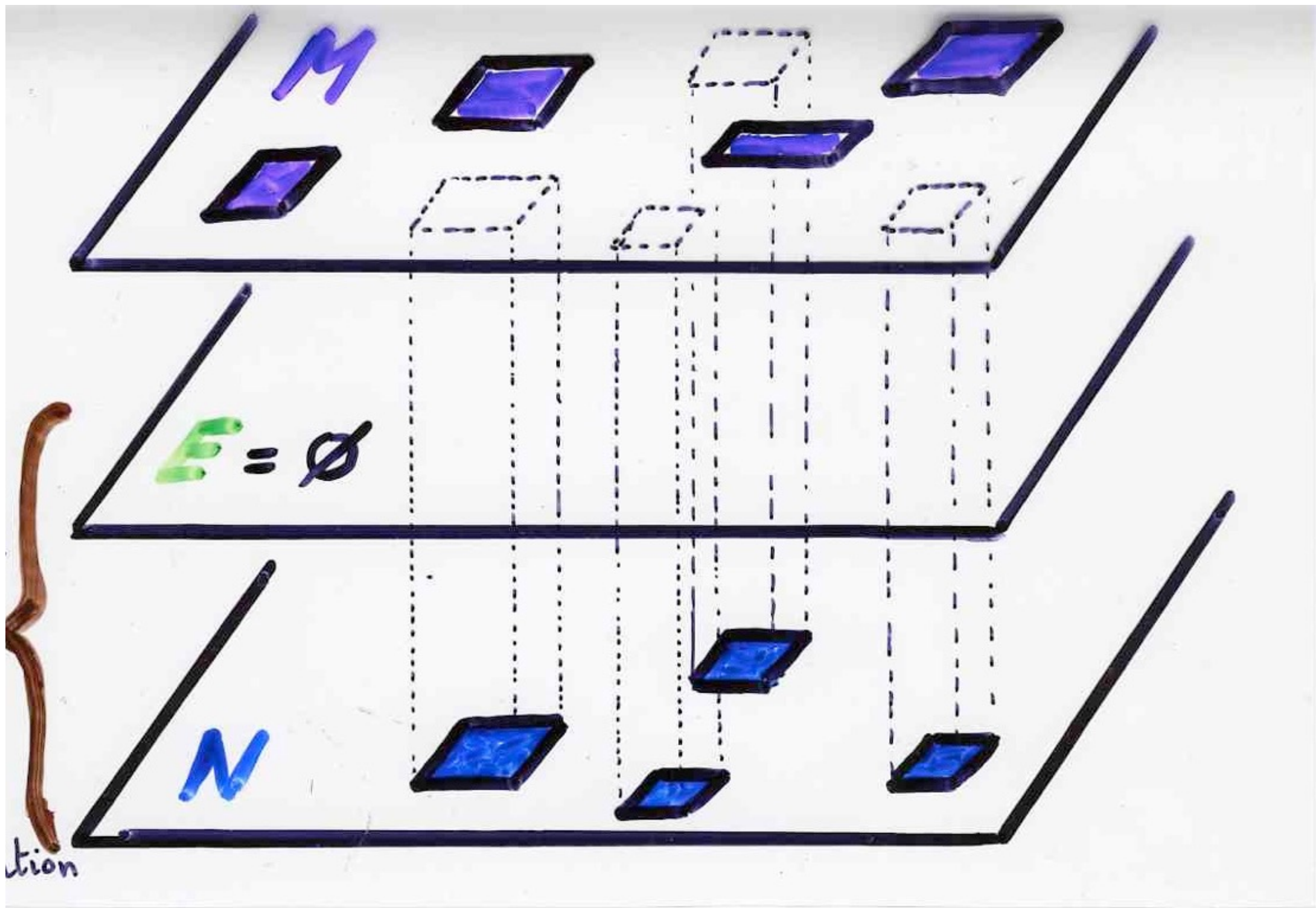
$\downarrow \gamma$

(ii) • if $\gamma \in \pi(m(E, F)) = \left\{ \begin{array}{l} m = (\beta, 0) \text{ minimal piece of } E \\ \text{such that } \alpha \not{\mathcal{C}} \beta \text{ for all } \alpha \in F \end{array} \right\}$



φ not defined for (E, F) with $E = \emptyset$, $F \subseteq P - M$

$$\left\{ \alpha \in F,\ \alpha \not\subset \beta,\ \text{for any } \beta \in E \text{ and } \alpha \notin M \right\}$$



φ not defined for (E, F) with $E = \emptyset$, $F \subseteq P - M$

φ is an involution $\qquad \varphi^2 = Id$

$$\begin{cases} \bullet \ v(E)v(F) = v(E')v(F') \\ \bullet \ (-1)^{|F|} = -(-1)^{|F'|} \end{cases}$$

$$\left(\sum_{\substack{E \\ \text{heaps}}} v(E)\right) D = N$$

$\prod$ (maximal pieces) $\in M$

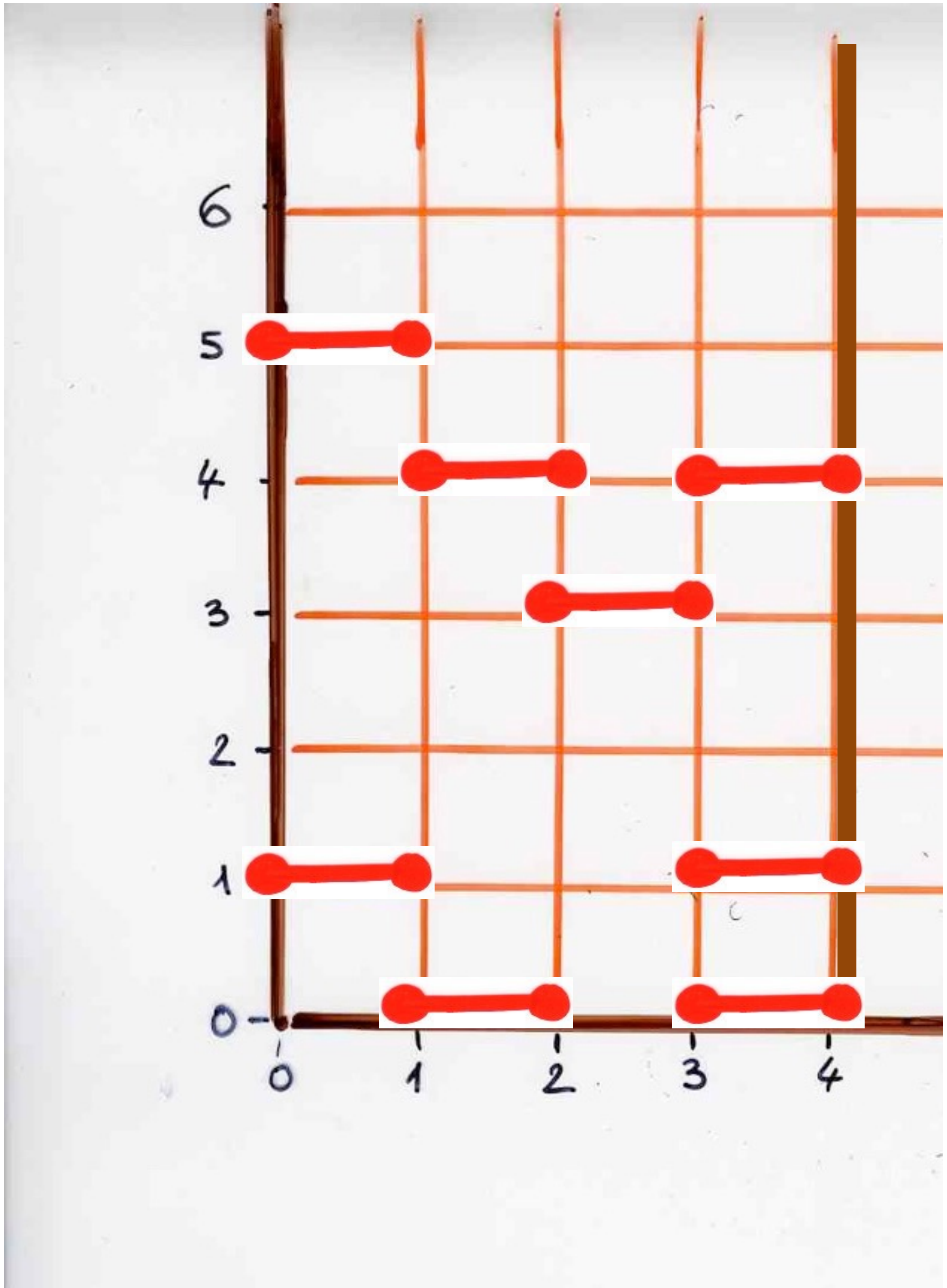
φ not defined for (E, F) with $E = \emptyset$, $F \subseteq P - M$

$$D = \sum_{\substack{F \\ \text{trivial heaps}}} (-1)^{|F|} v(F)$$

$$N = \sum_{\substack{F \\ \text{trivial heaps} \\ \text{pieces} \notin M}} (-1)^{|F|} v(F)$$

examples:

heaps of dimers
on a segment



generating function of heaps of dimers on the segment $[0, k]$ (enumerated by the number of dimers)

$$\frac{1}{F_{k+1}(t)}$$

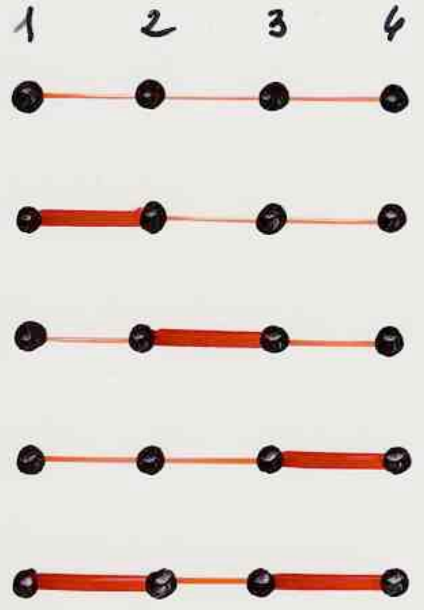
$$F_n(x) = \sum_{k \geq 0} (-1)^k a_{n,k} x^k$$

$$= \sum_{\substack{M \\ \text{matchings} \\ \text{of } \{1, \dots, n\}}} (-x)^{|M|}$$

Fibonacci polynomials

$= n$

$a_{n,k}$ = number of matchings of $\{1, 2, \dots, n\}$ with k dimers



1

$-x$

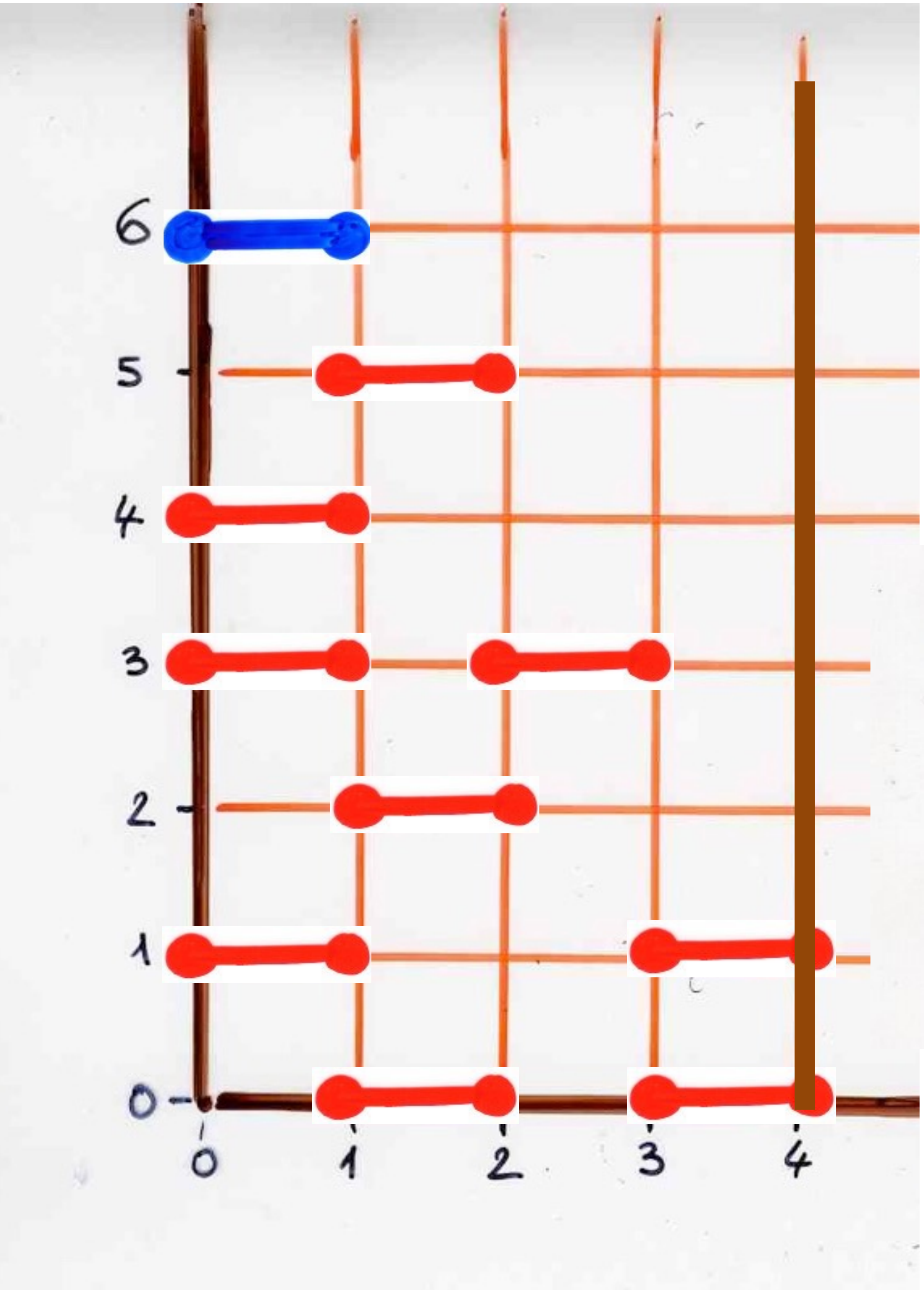
$-x$

$-x$

x^2

$$F_4(x) = 1 - 3x + x^2$$

Fibonacci polynomials



$$\frac{F_k(t)}{F_{k+1}(t)}$$

generating function of semi-pyramids of dimers on the segment $[0, k]$ (enumerated by the number of dimers)

exercise

$a_{n,k}$ = number of matchings of $\{1,2,\dots,n\}$ with k dimers

$$a_{n,k} = \binom{n-k}{k}$$

addition +

1

1

1 1

1

1 2 1

2

1 3 3 1

3

1 4 6 4 1

5

1 5 10 10 5 1

8

1 6 15 20 15 6 1

13

1 7 21 35 35 21 7 1

21

1 8 28 56 70 56 28 8 1

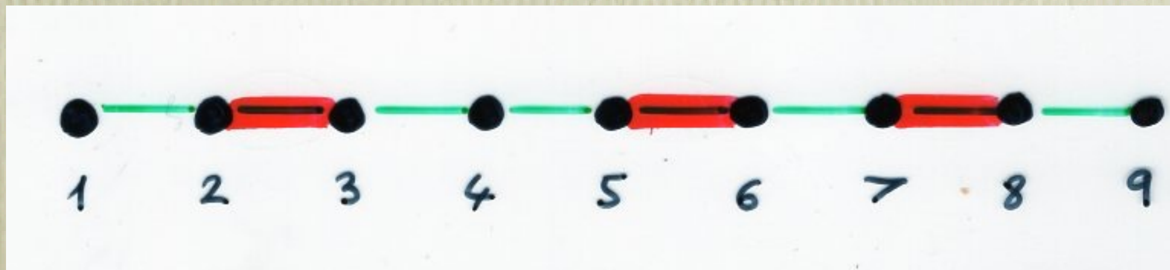
Pingala (2nd century B.C.)

Pingala   Laghu (short syllabe)
Guru (long syllabe)

two classes of meters in Sanskrit

- Akṣarachandaḥ
  Chandaḥ number of syllables
  later 4 feet (pāda)
- number of mātrās (time measure)
  short syllabe : one mātrās
  long syllabe : two mātrās

relation with Fibonacci numbers ?



exercise

Fibonacci polynomials
and
generating function of Catalan numbers

notations $D = 1 + Q$

generating function of Catalan numbers

$$Q(t) = \frac{1-2t-\sqrt{1-4t}}{2t}$$

generating function for

half-pyramid $(\neq \emptyset)$

$$= \sum_{n\geq 1} C_n t^n$$

Catalan

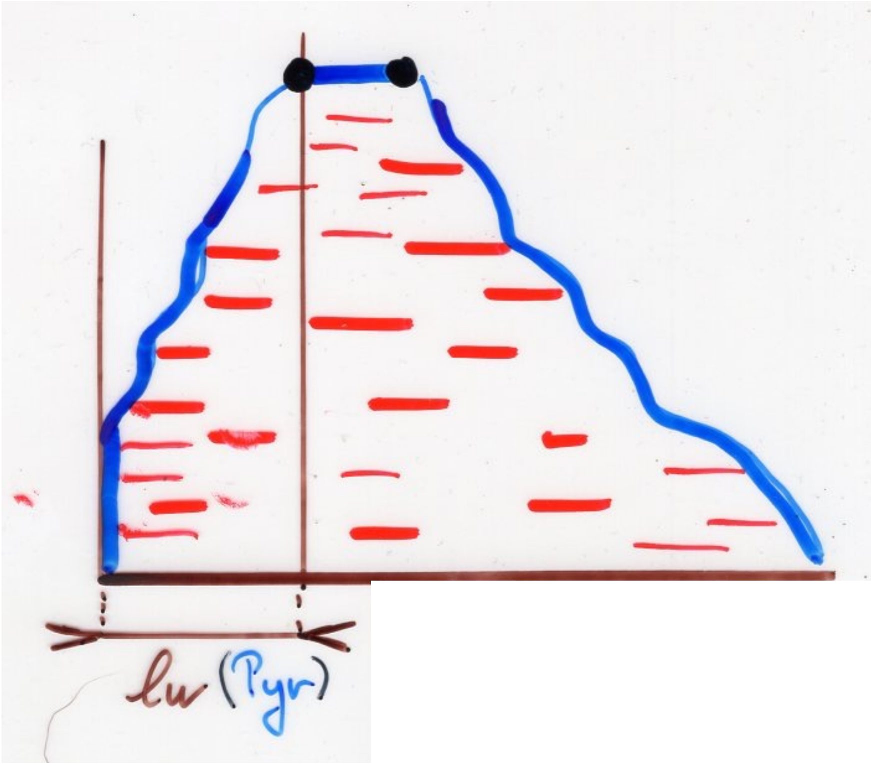
$F_n(t)$ n^{th} Fibonacci polynomial

we want to prove
the following identity.

$$F_n = \frac{(1-Q^{n+1})}{(1-Q)(1+Q)^n}$$

$$\Longleftrightarrow$$

$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n} \times (1+Q+\dots+Q^n)$$



semi-pyramid:
$lw(Pyr) = 0$

left-width of a pyramid of dimers $lw(Pyr)$

a) Prove that the generating function of (non-empty) pyramids of dimers Pyr with left-width $lw(Pyr) = k$, is equal to
$$Q^{k+1}$$

b)

Prove that both sides of the identity are the generating function of :

$$(1+Q)^n = \frac{1}{F_n} \times (1+Q+\ldots+Q^n)$$

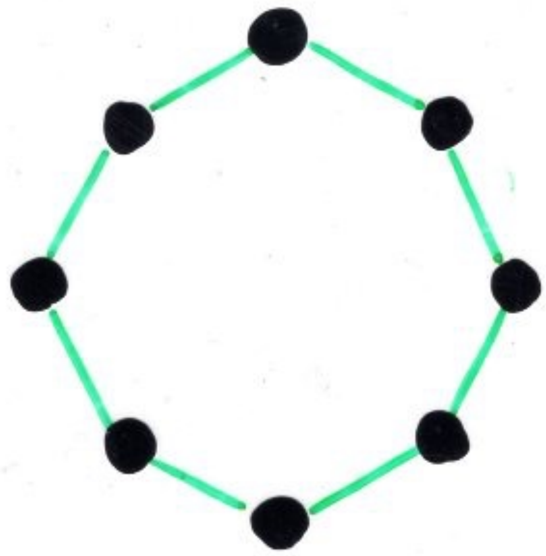
$$\underbrace{(1+Q)^n}_{D^n}$$

heaps of dimers on $[0,\infty[$

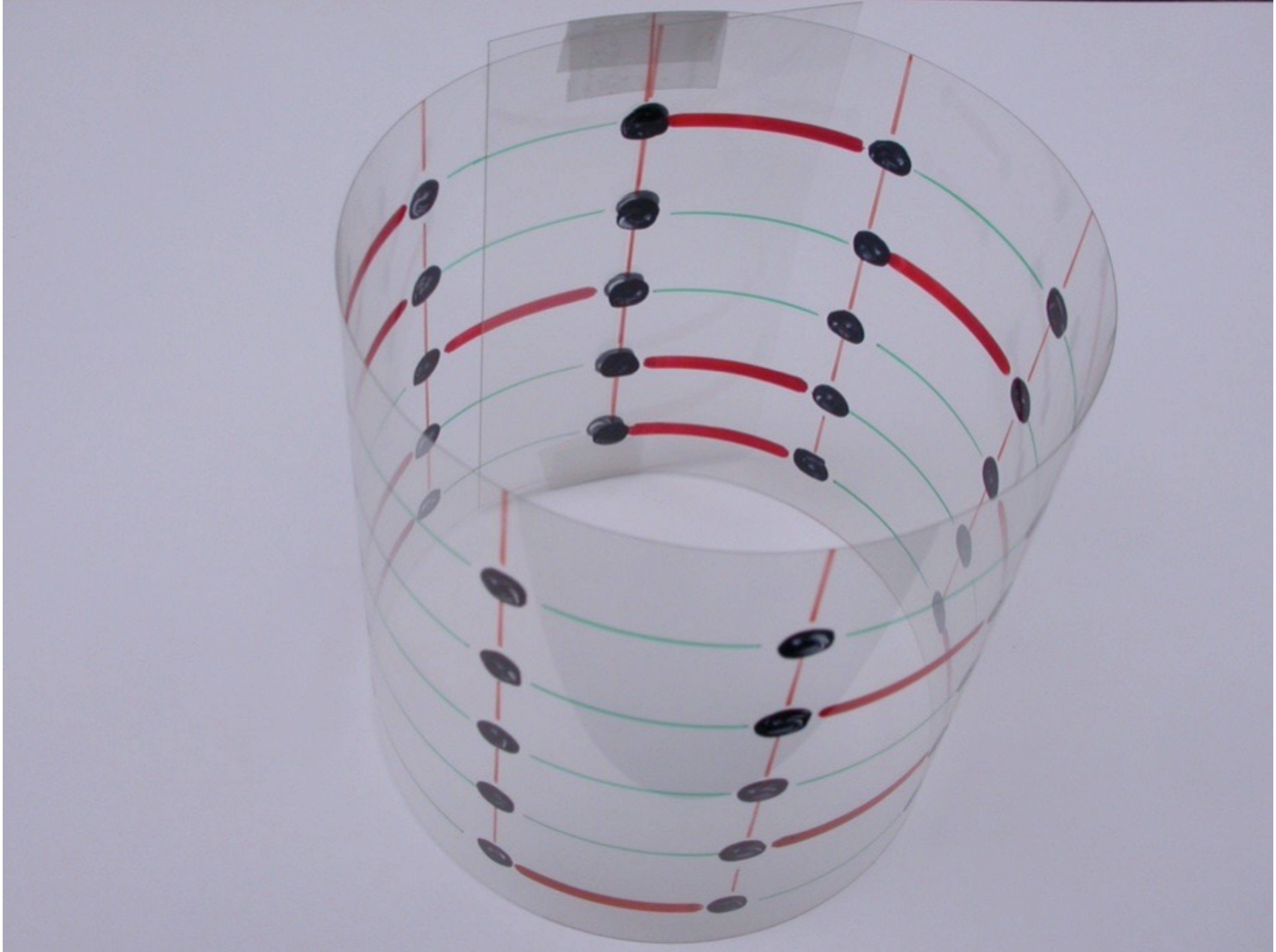
maximal pieces, projection $\subseteq [0,n]$

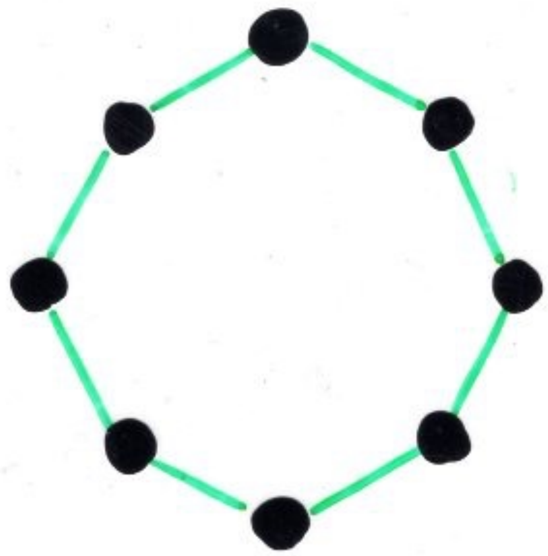
examples:

heaps of dimers
on a circle



heaps of dimers on the "cycle" γ_k of length k

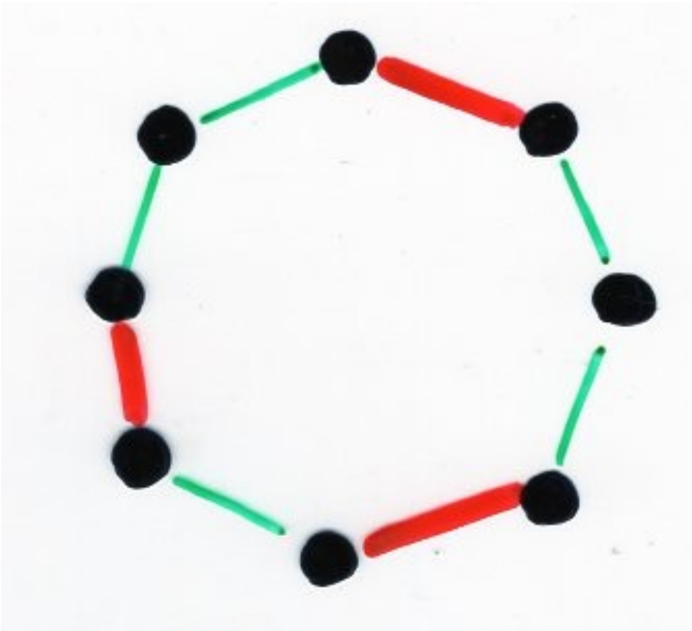




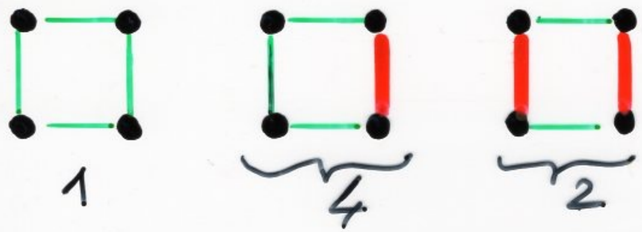
heaps of dimers
on the "cycle" γ_k
of length k

generating
function
(enumerated by the
number of dimers)

$$\frac{1}{L_k(t)}$$



Lucas polynomial



$$L_4(x) = 1 \quad - 4x \quad + 2x^2$$

$$L_n(x) = \sum_{\substack{\text{matchings } M \\ \text{of a cycle } \gamma \\ \text{length } n}} (-x)^{|M|}$$

exercise (very easy)

generating function for pyramids of dimers on the cycle γ_k ?

Fibonacci
and
Tchebychef polynomials

$$\sin\big((n+1)\theta\big) = \sin\theta \, U_n(\cos\theta)$$

$U_n(x)$ Tchebychef polynomial 2^{nd} kind

sequence of orthogonal polynomials

$$\frac{2}{\pi}\int_{-1}^{+1} U_n(x)\, U_m(x)\,(1-x^2)^{1/2}\,dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{else} \end{cases}$$

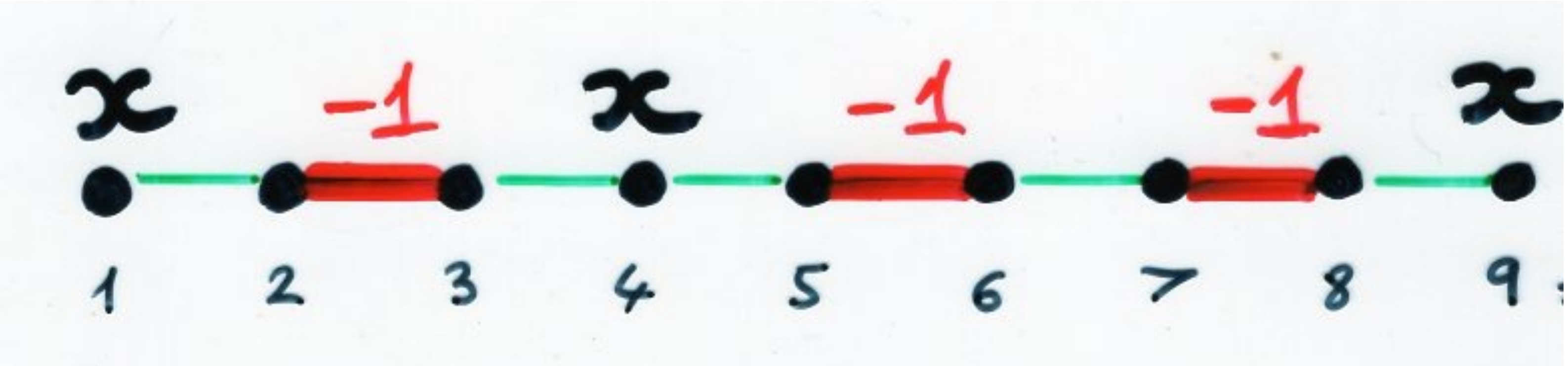
$$M_n(x) = \sum_{k \geq 0} (-1)^k a_{n,k} \, x^{n-2k}$$

matching polynomial of the segment graph

$$= \sum_{M} (-1)^{|M|} \, x^{ip(M)}$$

M matchings of $\{1, .., n\}$

$ip(M)$ is the number of isolated points of M



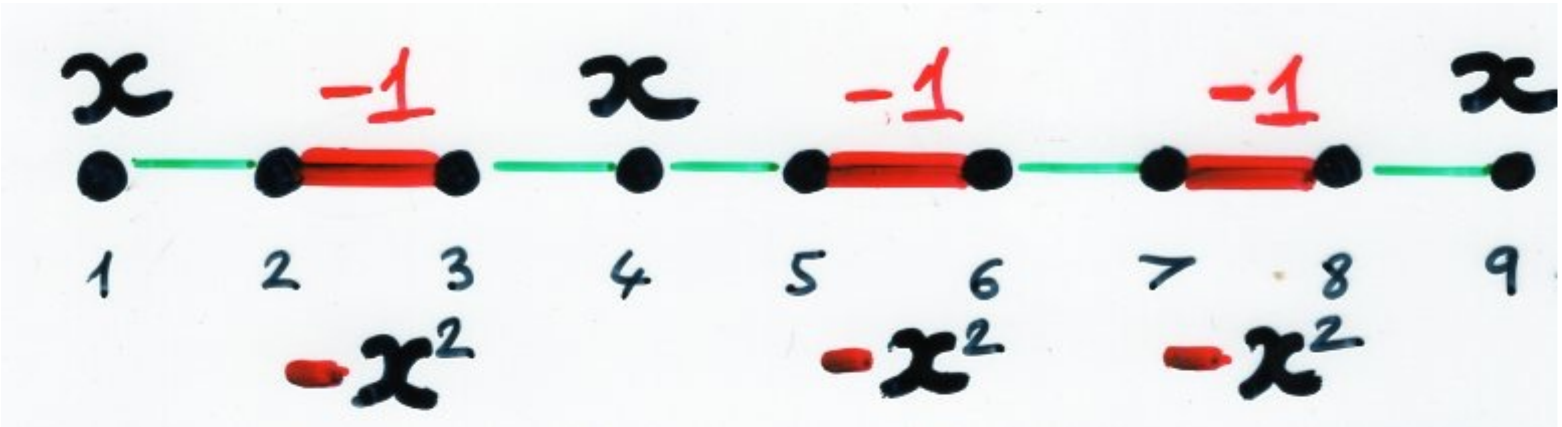
$a_{n,k}$ = number of matchings of $\{1, 2, \ldots, n\}$ with k dimers

$$M^{*}_{n}(x) = x^{n} M_{n}(1/x)$$

reciprocal

$$= \sum_{\substack{M \\ \text{matchings} \\ \text{of } \{1,..,n\}}} (-x^{2})^{|M|}$$

$$= F_{n}(x^{2})$$



$$\sin((n+1)\theta) = \sin\theta \, U_n(\cos\theta)$$

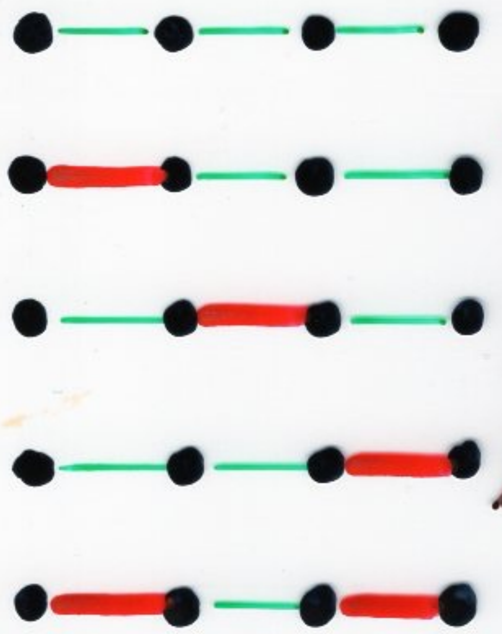
$U_n(x)$ Tchebychef polynomial 2nd kind

$$U_n(x) = M_n(2x)$$

same 3-terms recurrence relation

$$M_{n+1}(x) = x\,M_n(x) - M_{n-1}(x)$$

$$\begin{cases} M_0 = 1 \\ M_1 = x \end{cases}$$



$x^4 \qquad (2\cos\theta)^4$

$-3x^2 \qquad -3(2\cos\theta)^2$

$+1 \qquad 1$

$$\sin 5\theta = \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right)$$

Lucas
and
Tchebycheff polynomials

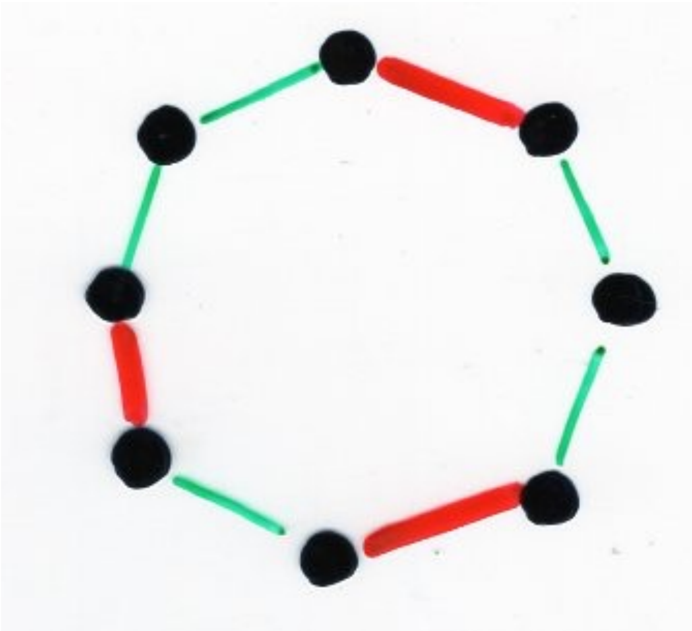
$$\cos(n\theta) = T_n(\cos\theta)$$

$T_n(x)$

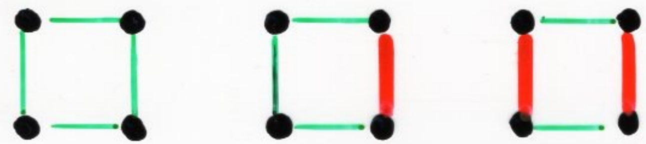
Tchebychef

polynomial

1st kind



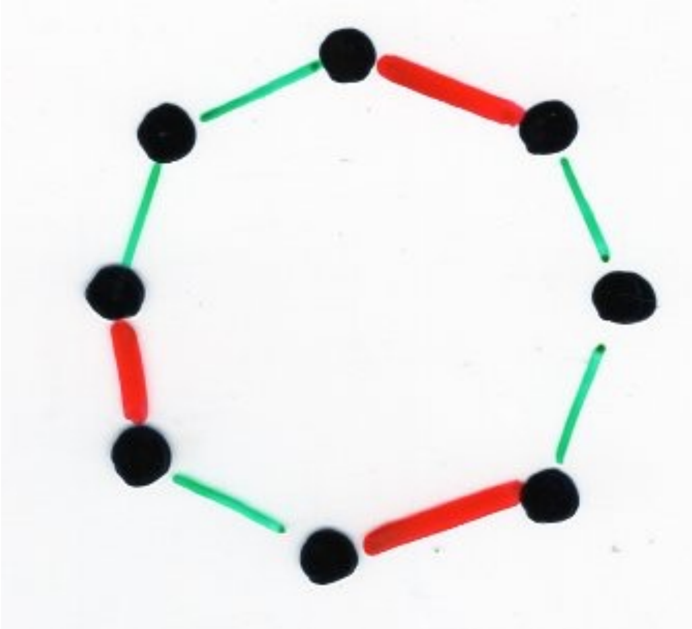
Lucas polynomial



$$\underbrace{\quad}_{1} \qquad \underbrace{\quad}_{4} \qquad \underbrace{\quad}_{2}$$

$$L_4(x) = 1 \quad - 4x \quad + 2x^2$$

$$L_n(x) = \sum_{\substack{\text{matchings } M \\ \text{of a cycle } \gamma \\ \text{length } n}} (-x)^{|M|}$$

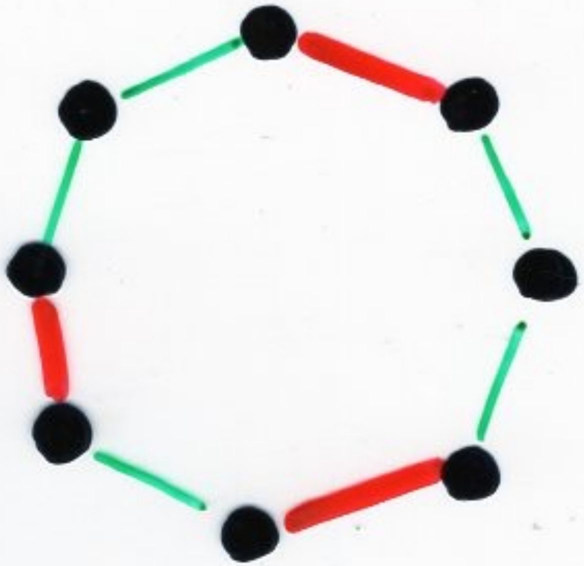


matching polynomial
of the cycle graph

reciprocal of $L_n(x^2)$ is

$$C_n(x) = \sum_{\substack{\text{matching } M \\ \text{of } \gamma}} (-1)^{|M|} \, x^{ip(M)}$$

number of isolated points



matching polynomial
of the cycle graph

$$T_n(x) = \frac{1}{2} C_n(2x)$$

$$C_{n+1}(x) = x\,C_n(x) - \lambda_n C_{n-1}(x)$$

same 3-terms
recurrence
relation

$$\begin{cases} C_0 = 1 \\ C_1 = x \end{cases}$$

$$\begin{cases} \lambda_1 = 2 \\ \lambda_n = 1 \\ (n \geq 2) \end{cases}$$

1 $\qquad$ 4 $\qquad$ 2

$$L_4(x) = 1 - 4x + 2x^2$$

$$L_4(x^2) = 1 - 4x^2 + 2x^4$$

$$L_4^*(x^2) = x^4 - 4x^2 + 2$$

$$C_4(x) = 16x^4 - 16x^2 + 2$$

$$\left(8\cos^4\theta - 8\cos^2\theta + 1\right) = \cos 4\theta$$

$$T_n(\cos\theta)$$

exercise

factorisation

$$F_{2n+1}(t) = F_n(t) \times L_{n+1}(t)$$