Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics, a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

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Chapter 1
Commutation monoids
and
heaps of pieces:

basic definitions
(3)

IMSc, Chennai
13 January 2017

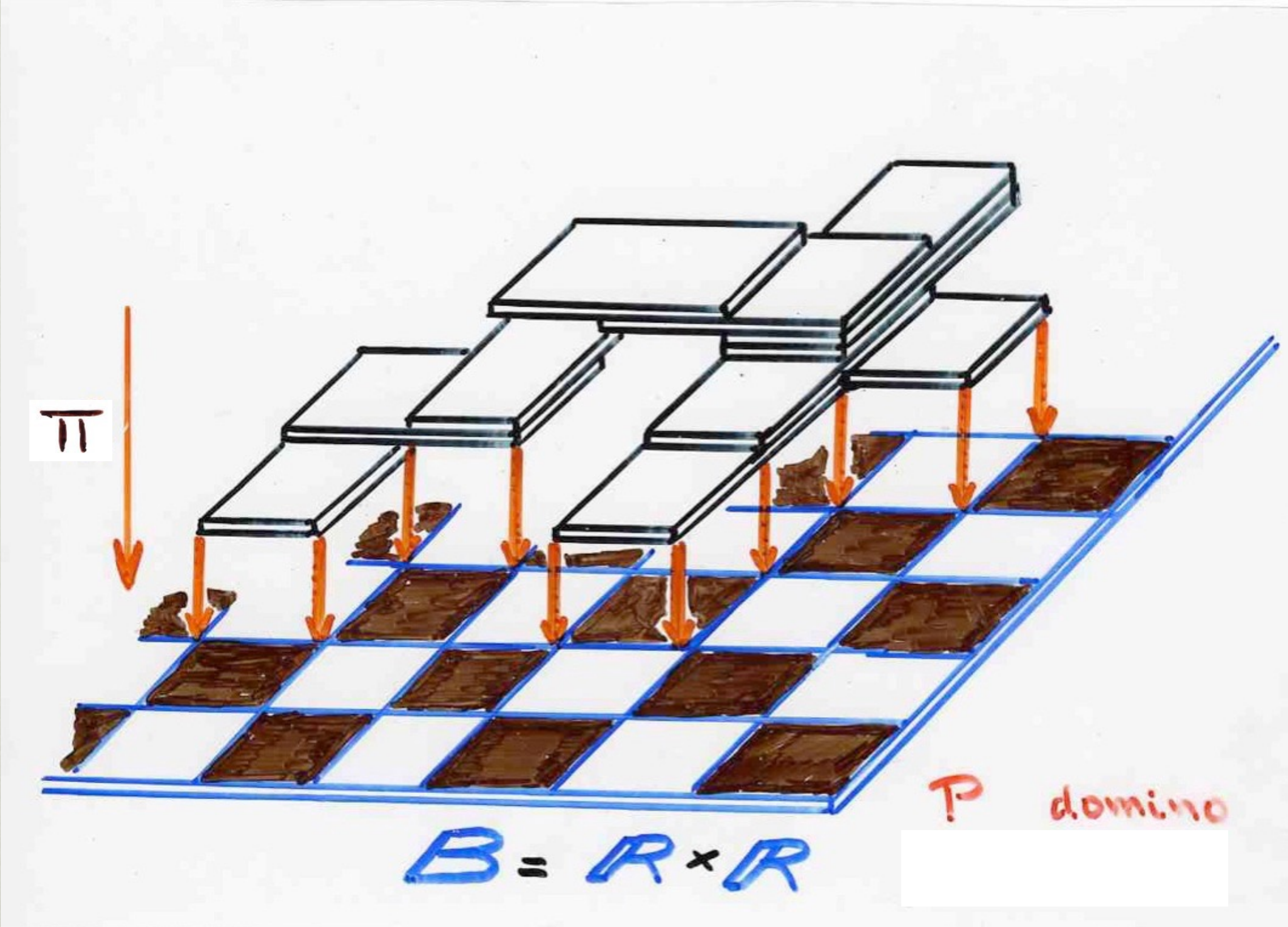
from the previous lecture

commutation monoid $A^*/\equiv_C$

commutation relation C antireflexive symmetric

$\equiv_C$ congruence of A^* generated by the commutations

$ab \equiv ba$ iff aCb


π
P domino
B = ℝ × ℝ

heap   definition

- $\mathbf{P}$ set (of basic pieces)
- $\mathcal{C}$ binary relation on $\mathbf{P}$ $\begin{cases} \text{symmetric} \\ \text{reflexive} \end{cases}$

  (**dependency** relation)

- **heap** E, finite set of pairs (α, i) $\alpha \in P$, $i \in \mathbb{N}$ (called pieces)

  α ↑ projection, i ↑ level

(i) $(\alpha, i), (\beta, j) \in E$, $\alpha \, \mathcal{C} \, \beta \implies i \neq j$

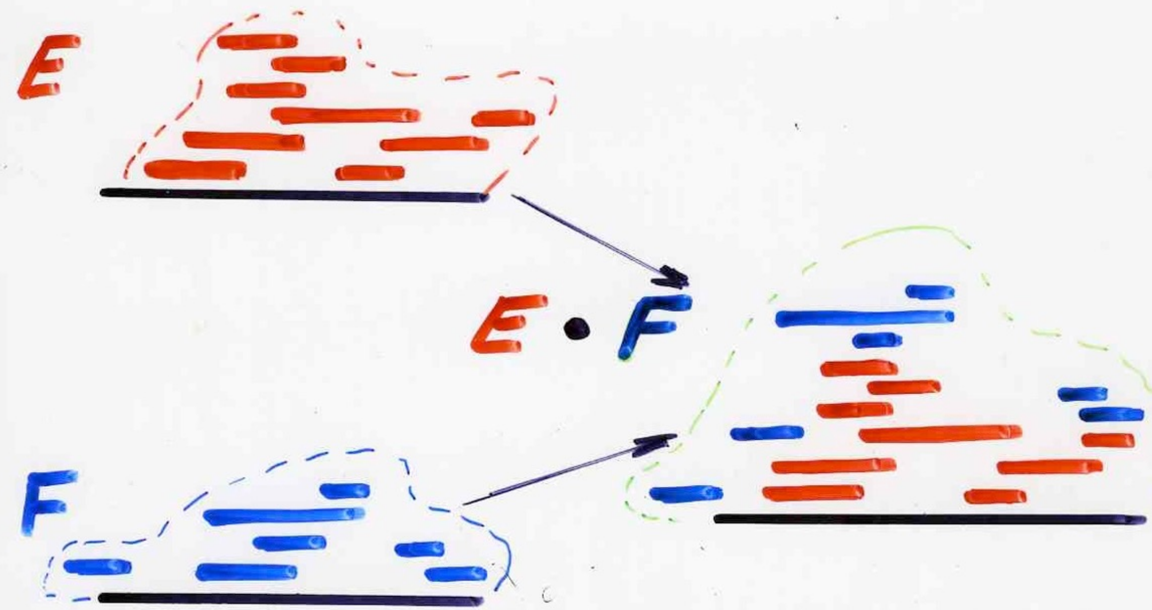
(ii) $(\alpha, i) \in E$, $i > 0 \implies \exists \beta \in P$, $\alpha \, \mathcal{C} \, \beta$, $(\beta, i-1) \in E$

Heaps monoid

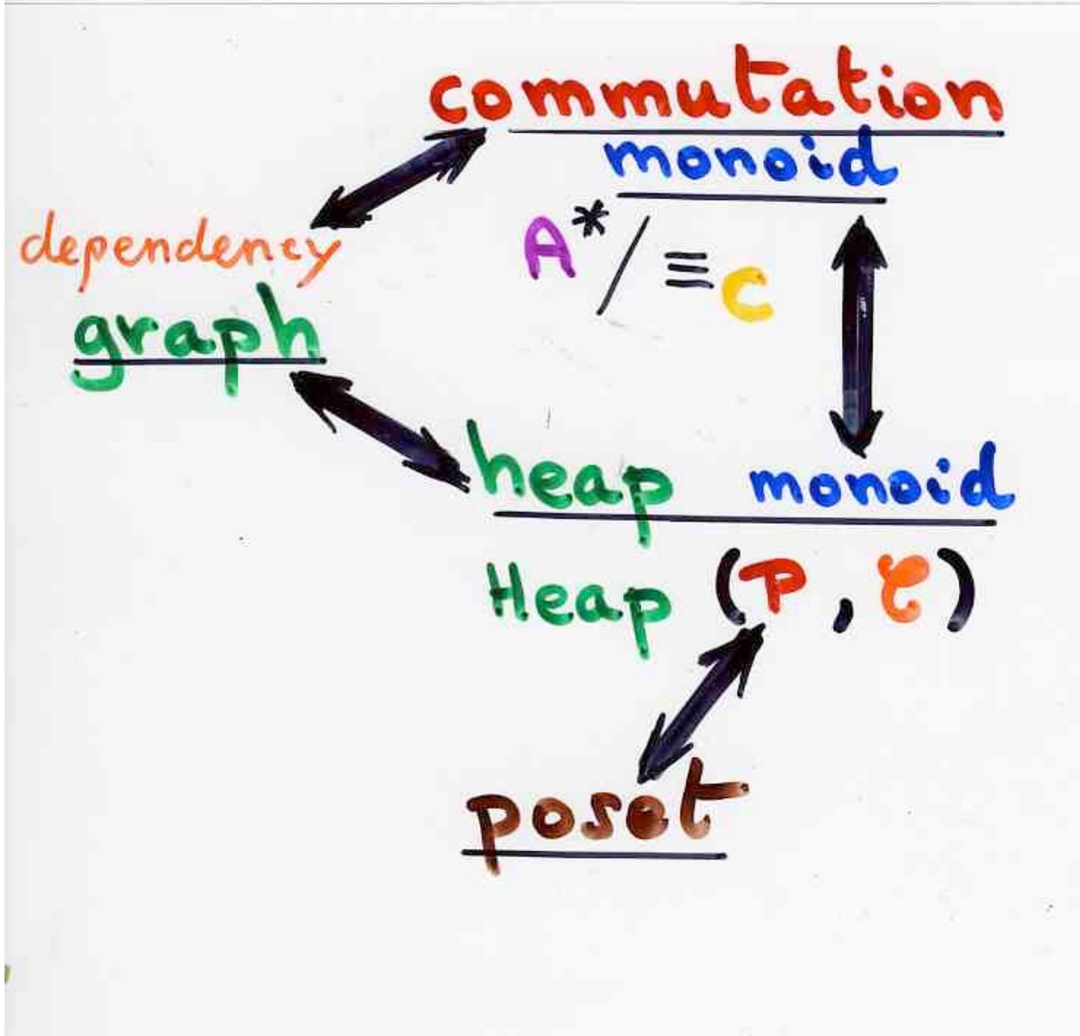
$H(P, \mathcal{C})$

product of two heaps

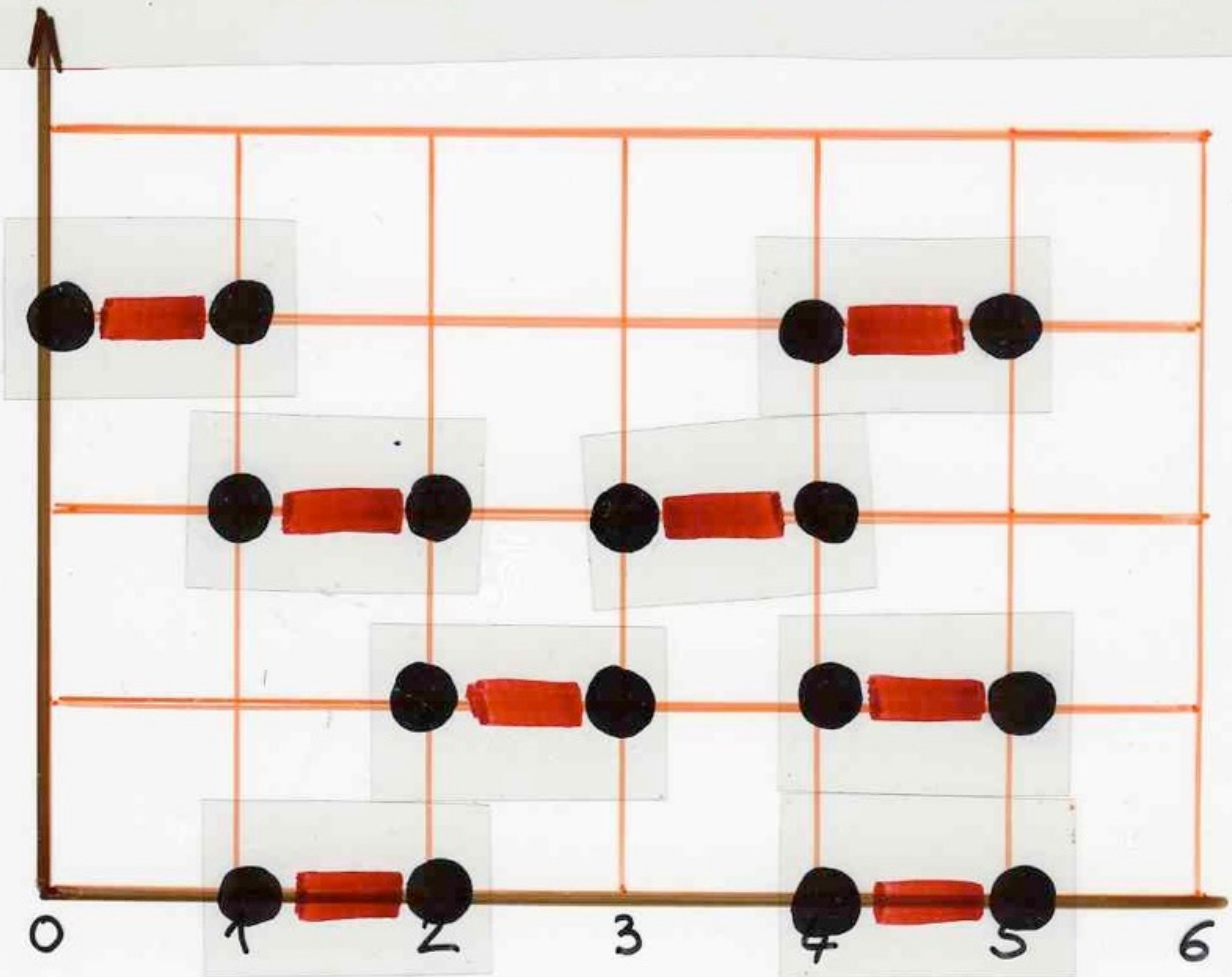
$E \bullet F$



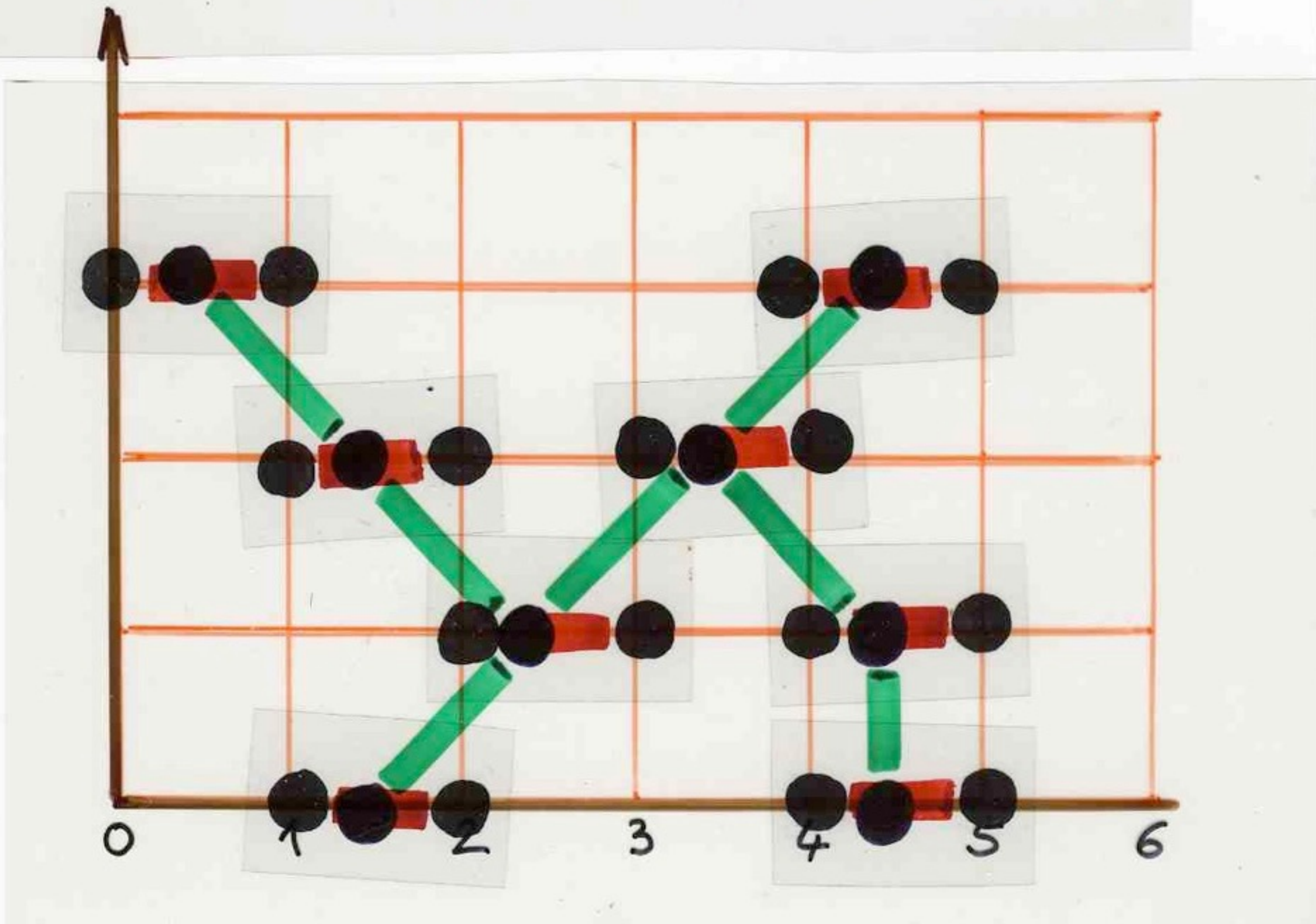
$$E \cdot F = E_{\sim}(E \cup T(F))$$


commutation
monoid
$A^*/\equiv_C$
dependency
graph
heap monoid
Heap (P, C)
poset

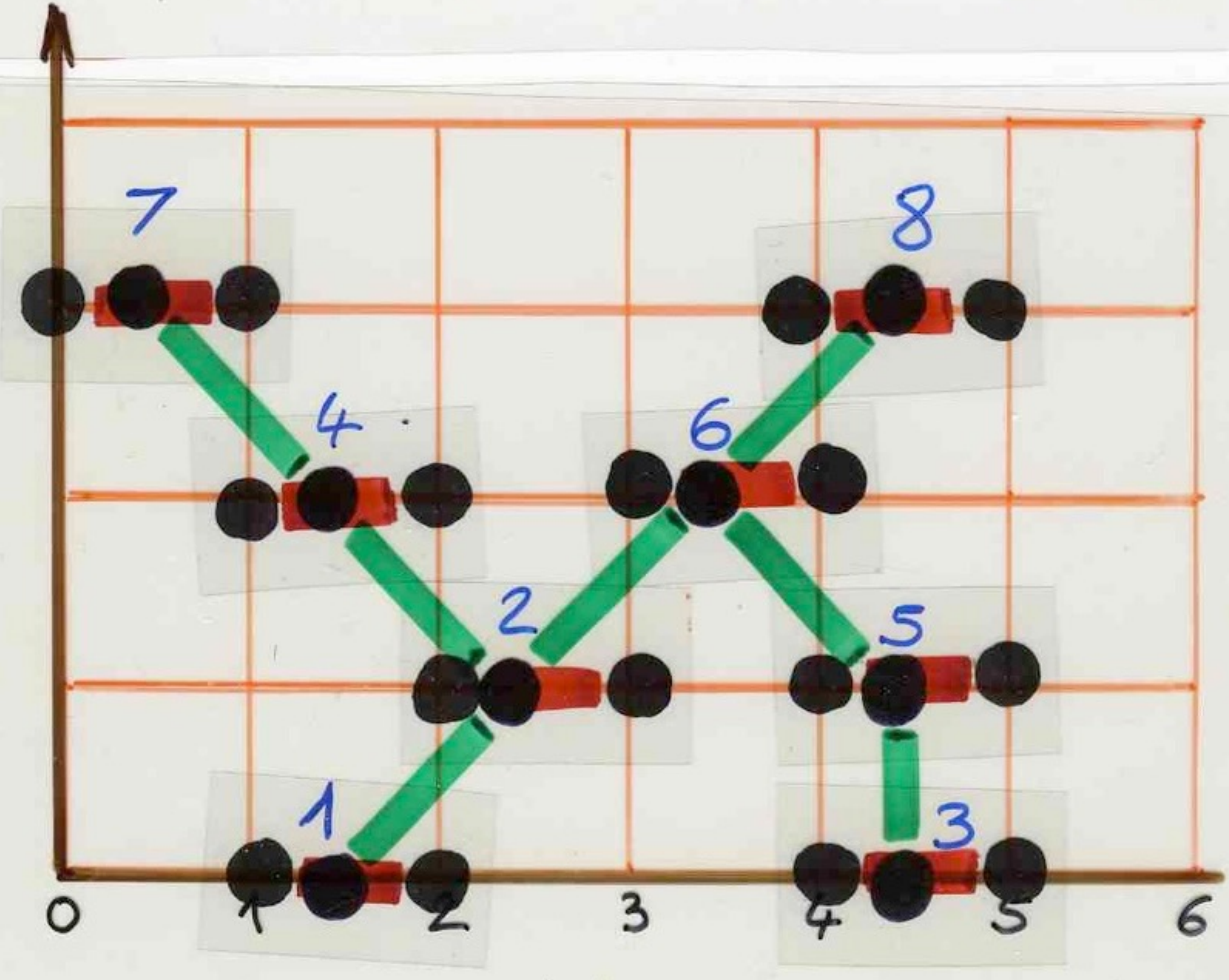
$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$

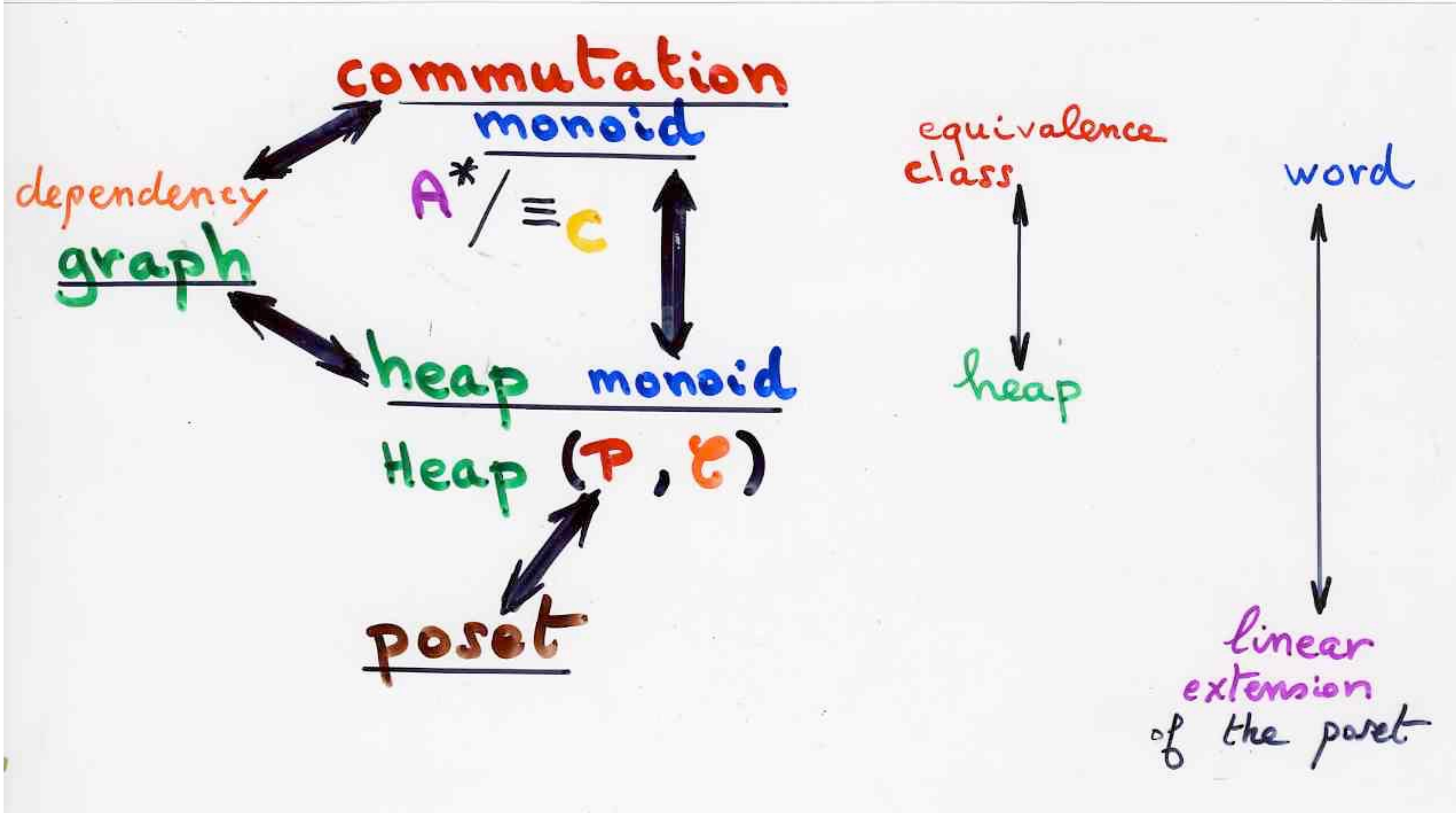


$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$



$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$




commutation monoid
A*/≡C
dependency graph
heap monoid
Heap (P, C)
poset
equivalence class
heap
word
linear extension of the poset

solution exercise

X set

$P \subseteq \mathcal{P}(X)$

set of subsets of X

dependency relation

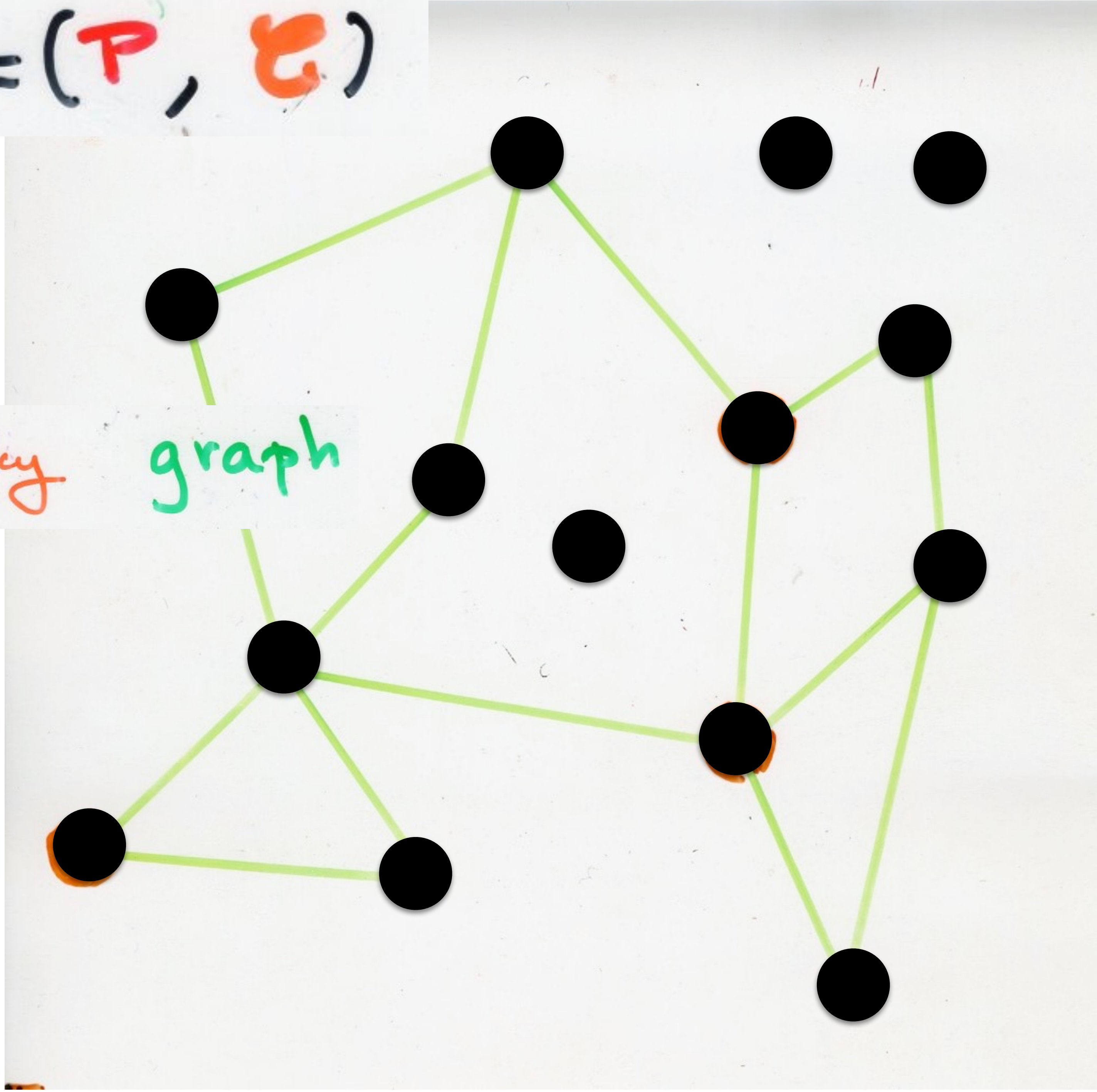
$A, B \in P, \quad A \mathcal{C} B \Leftrightarrow A \cap B \neq \emptyset$

$H(P)$

heaps of subsets

Proposition Every heap monoid is isomorphic to a "heap of subsets of a set X" monoid.

$G = (P, E)$



dependency graph

$G = (P, E)$

dependency graph

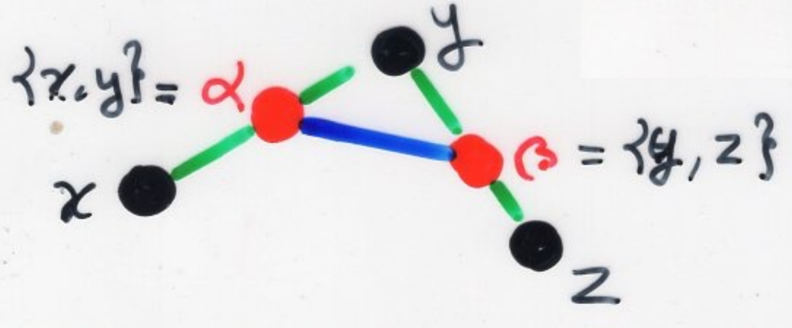
median graph $MG = (E, \alpha)$

vertices of MG = edges E of G

$x, y \in P$

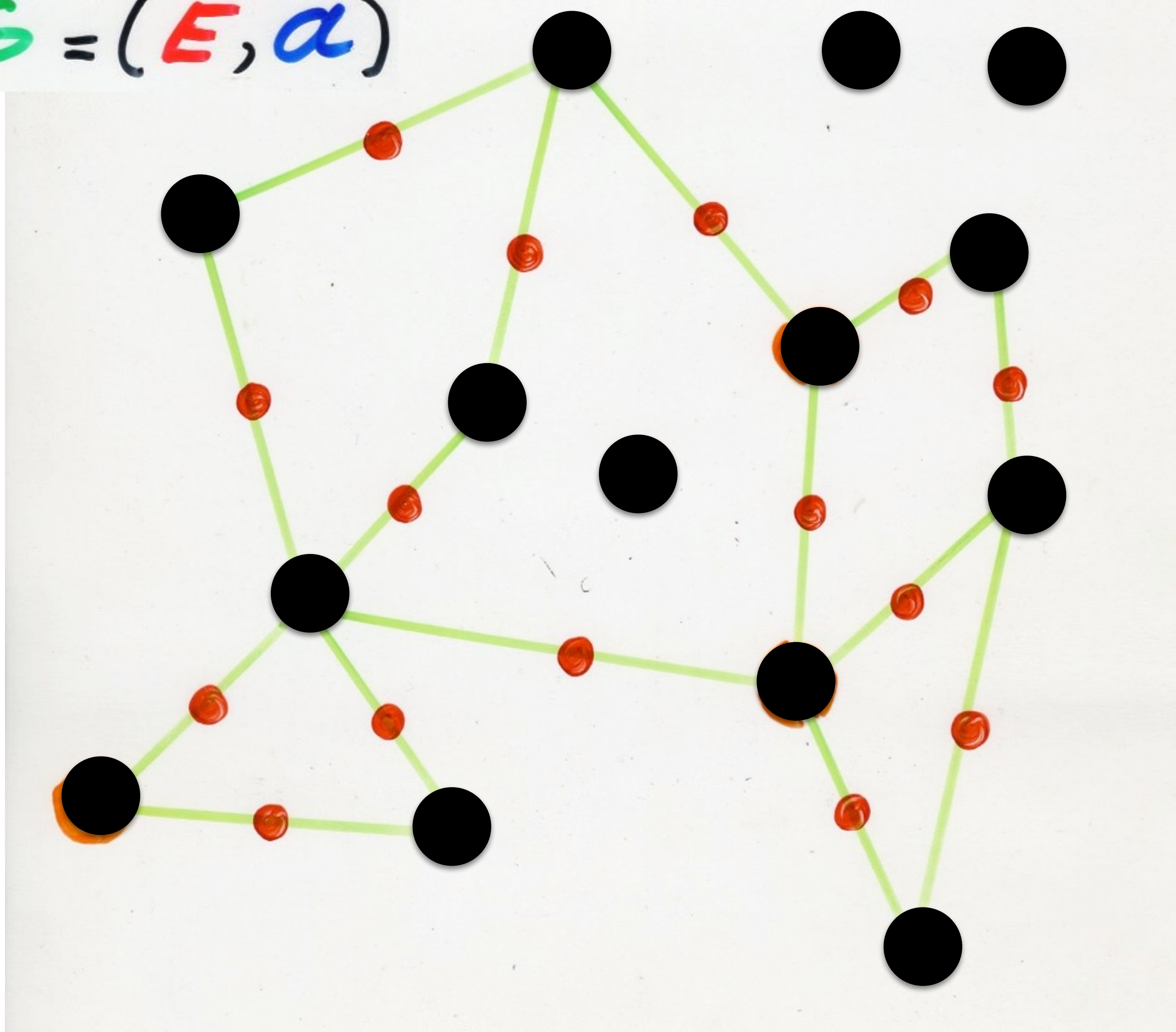
edges of MG $\alpha = \{x, y\}$, $\beta = \{z, t\}$ in relation α

iff α, β adjacents



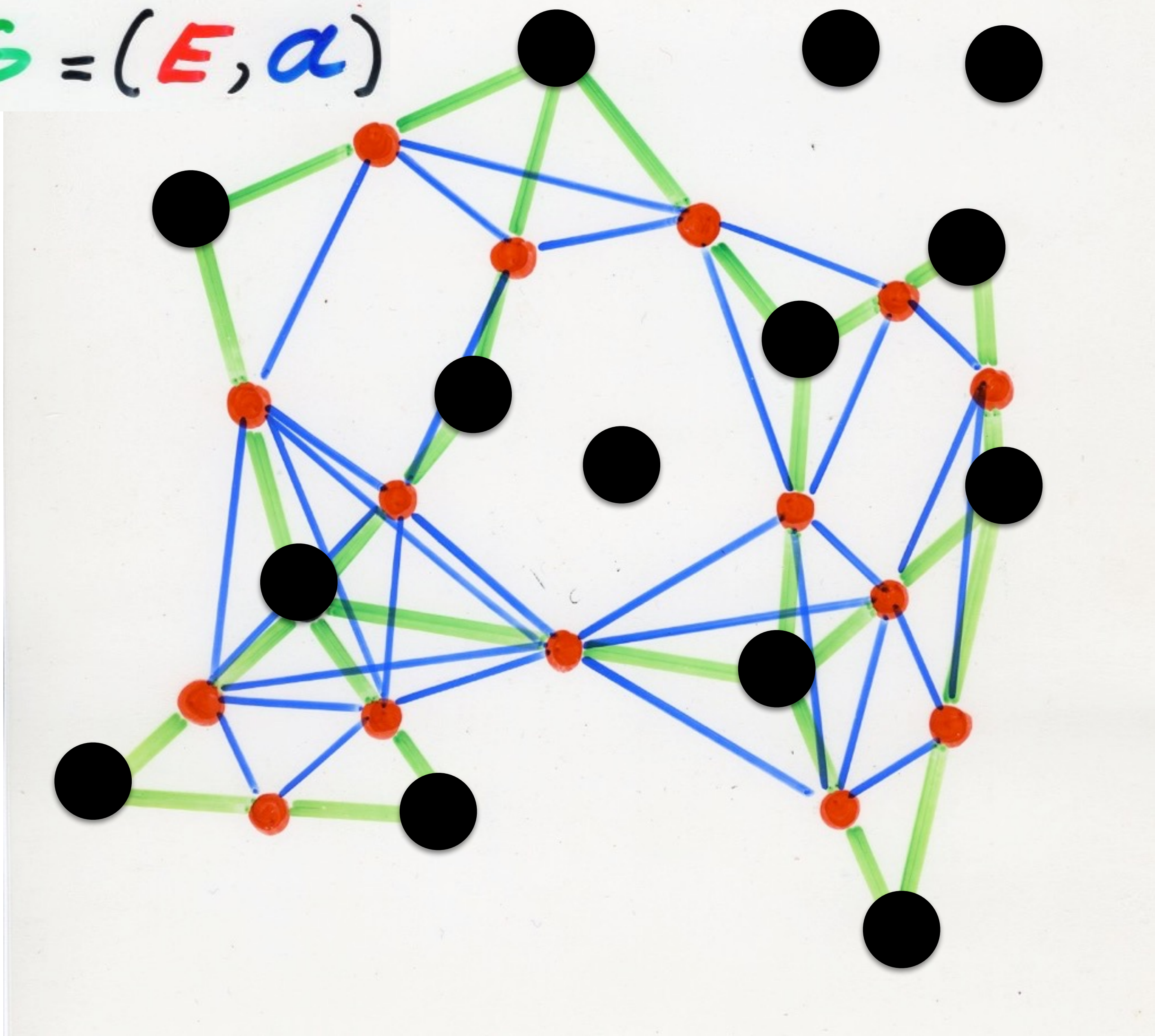
median graph

$MG = (E, \alpha)$

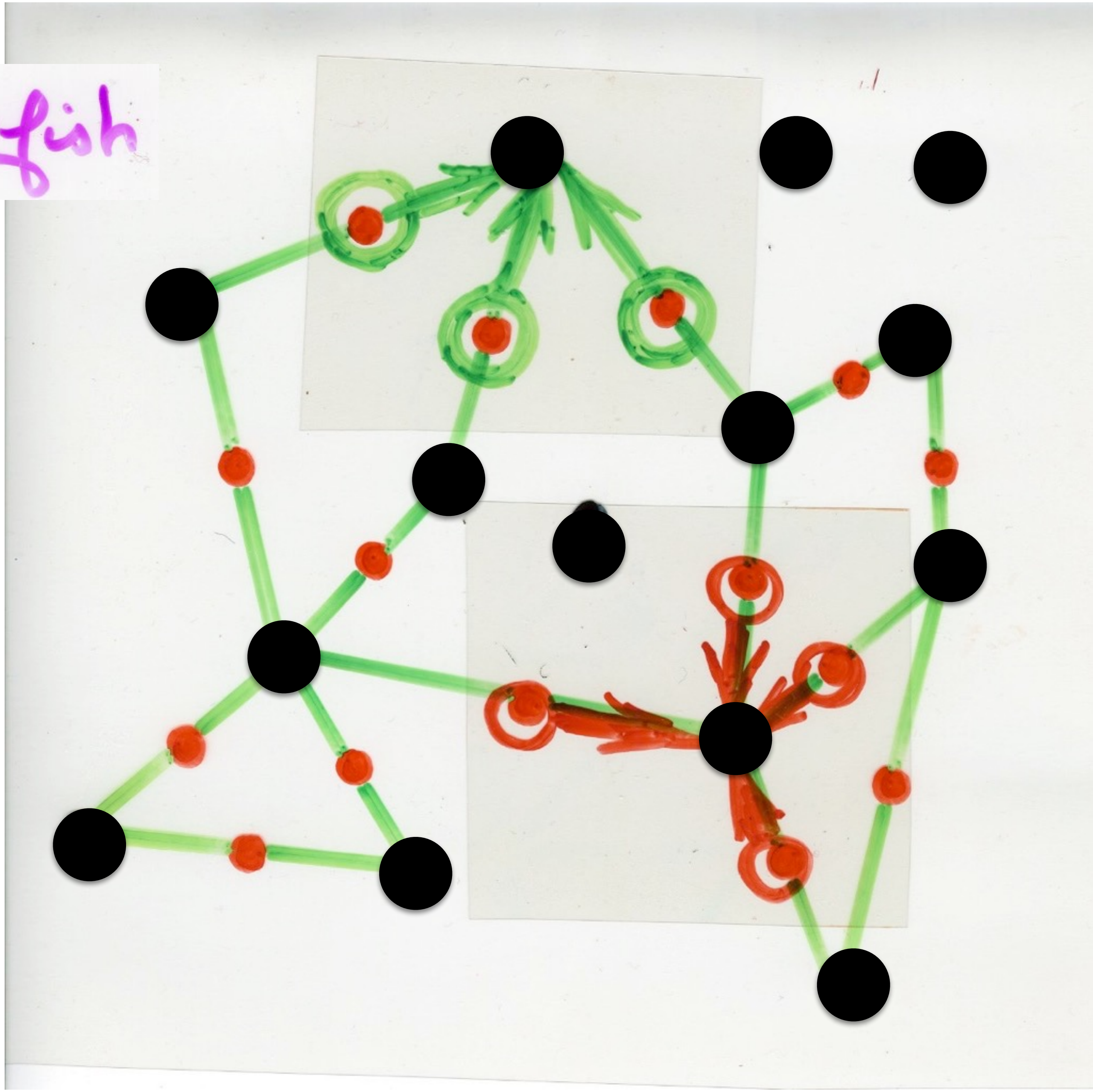


median graph

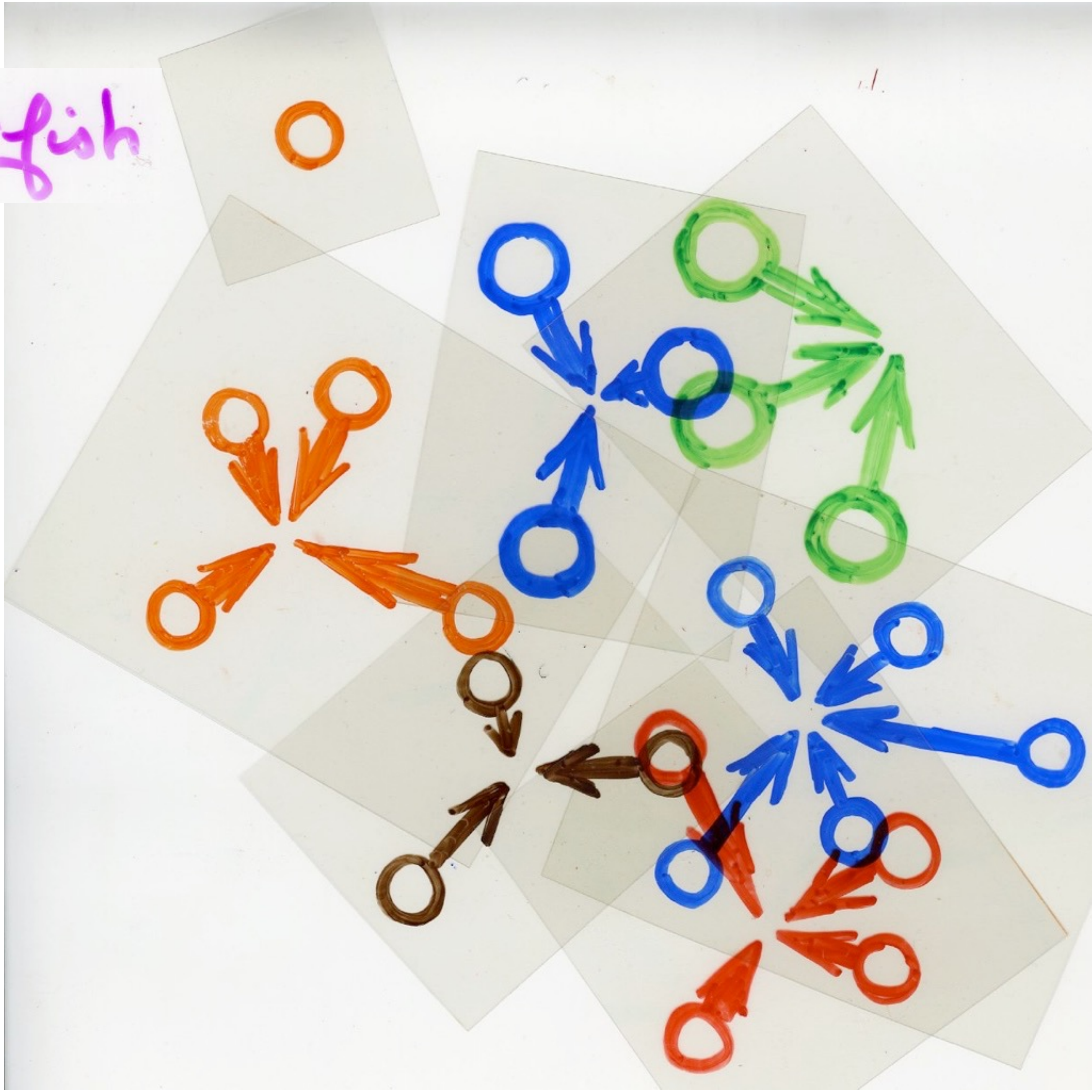
$MG = (E, \alpha)$



starfish

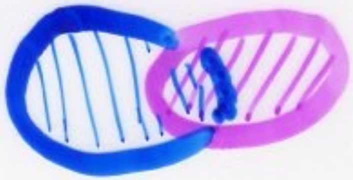


starfish



$$H(P, \tau) \simeq H(SF)$$

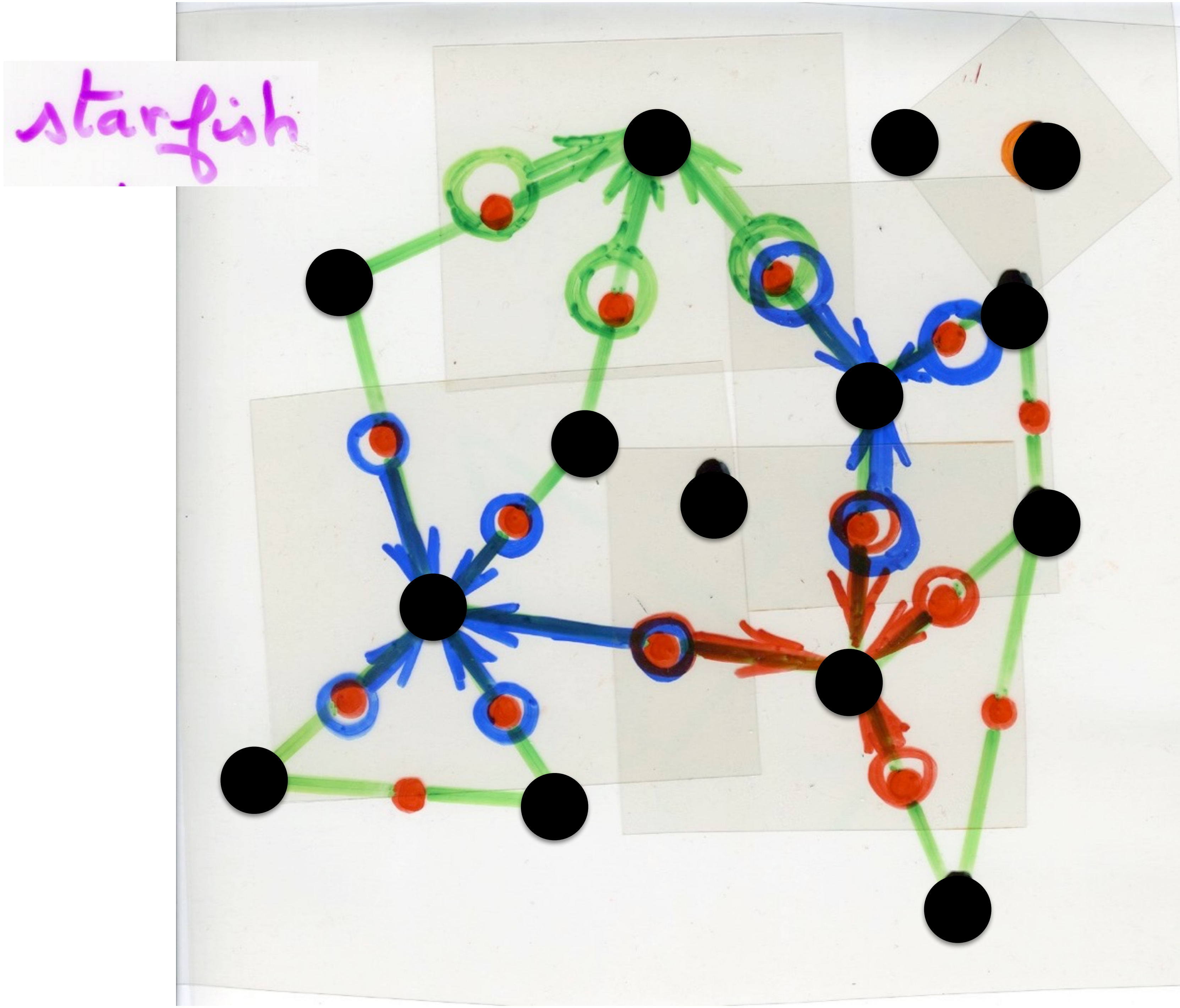
heaps of starfishes



$x \in P \longrightarrow$
- $\{x\}$ if x isolated vertex of G

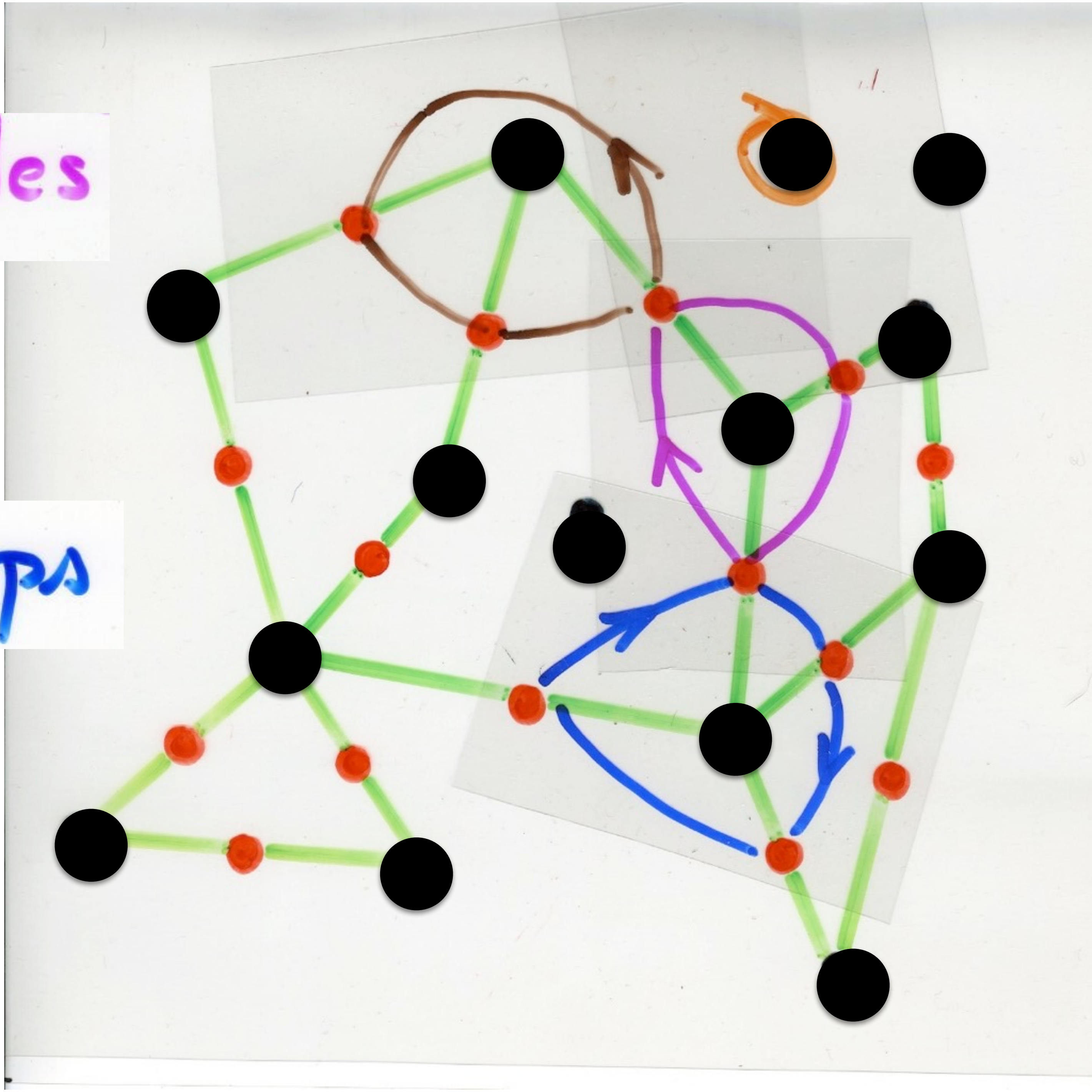
  else
- $\{\alpha = \{x, y\}$ vertices of $MG\}$

  starfish centered on x

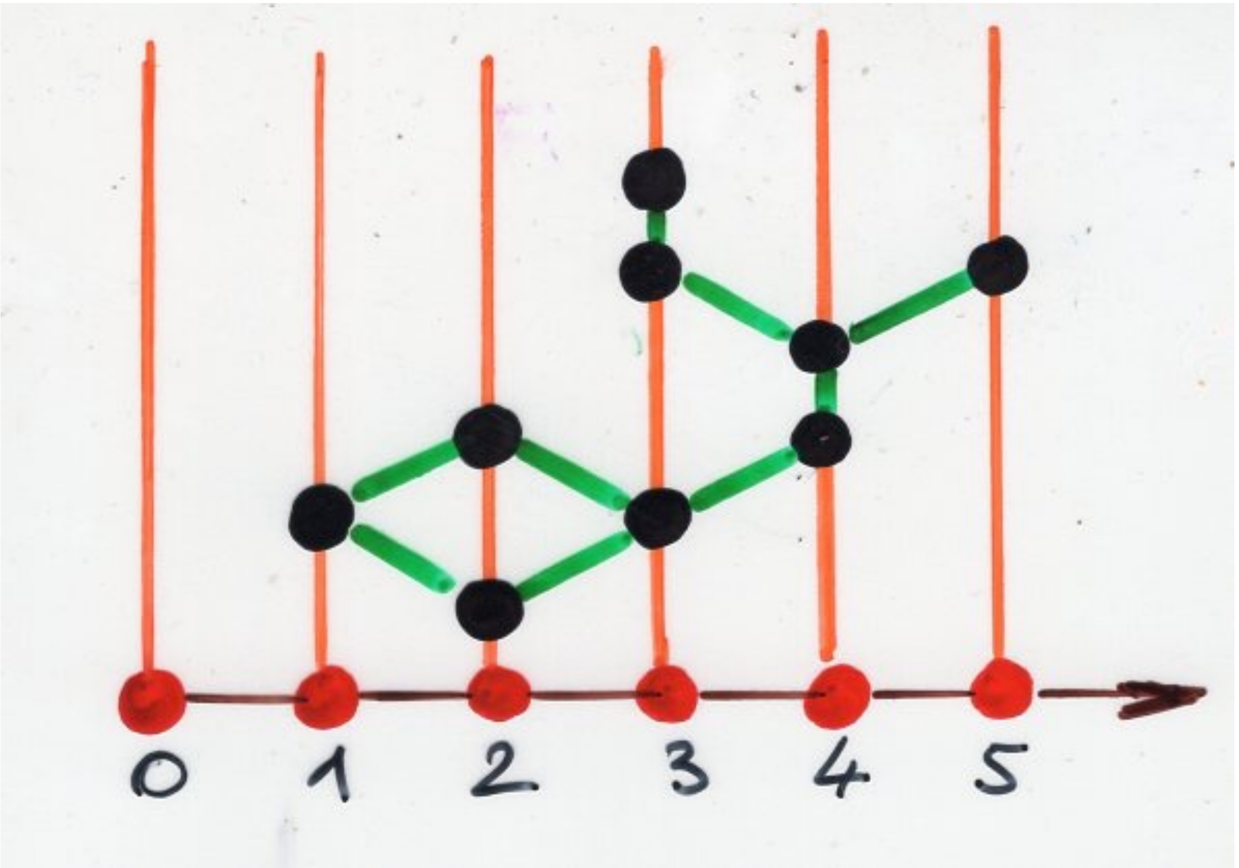

starfish

cycles

loops



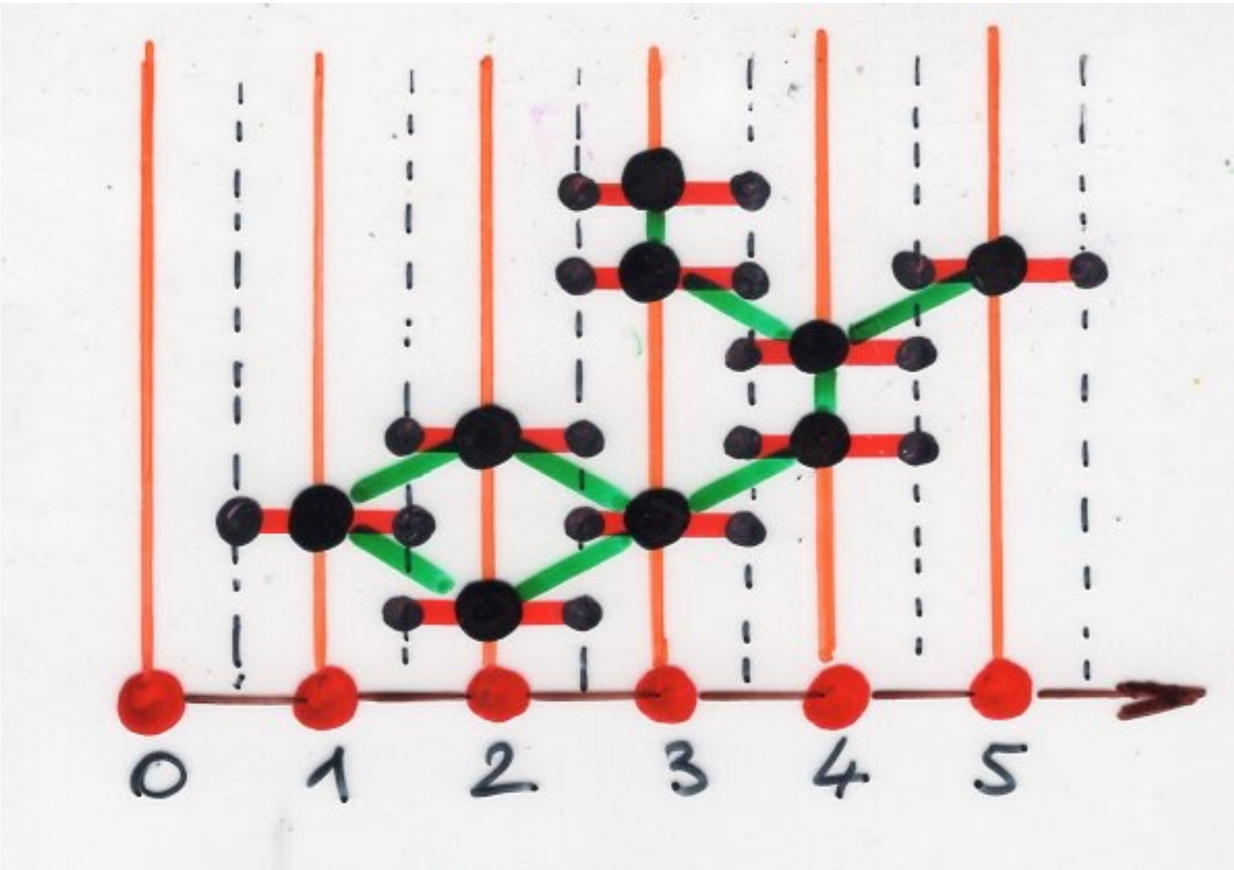
example 1



$\mathcal{P} = \mathbb{N}$

$i \,\mathcal{C}\, j \iff |i-j| \leqslant 1$

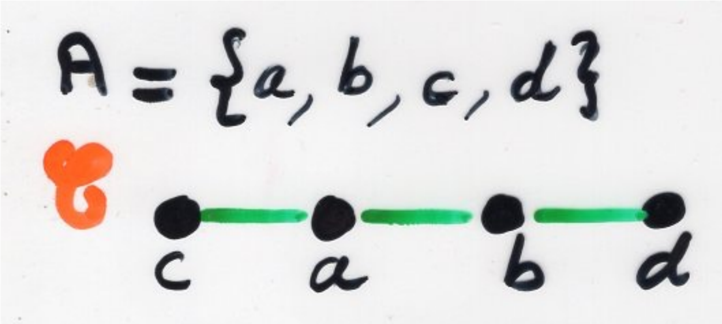
example 1

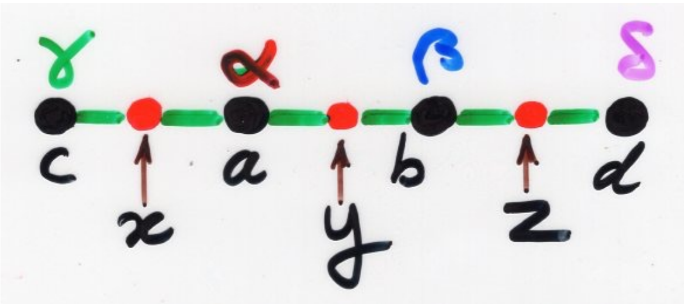


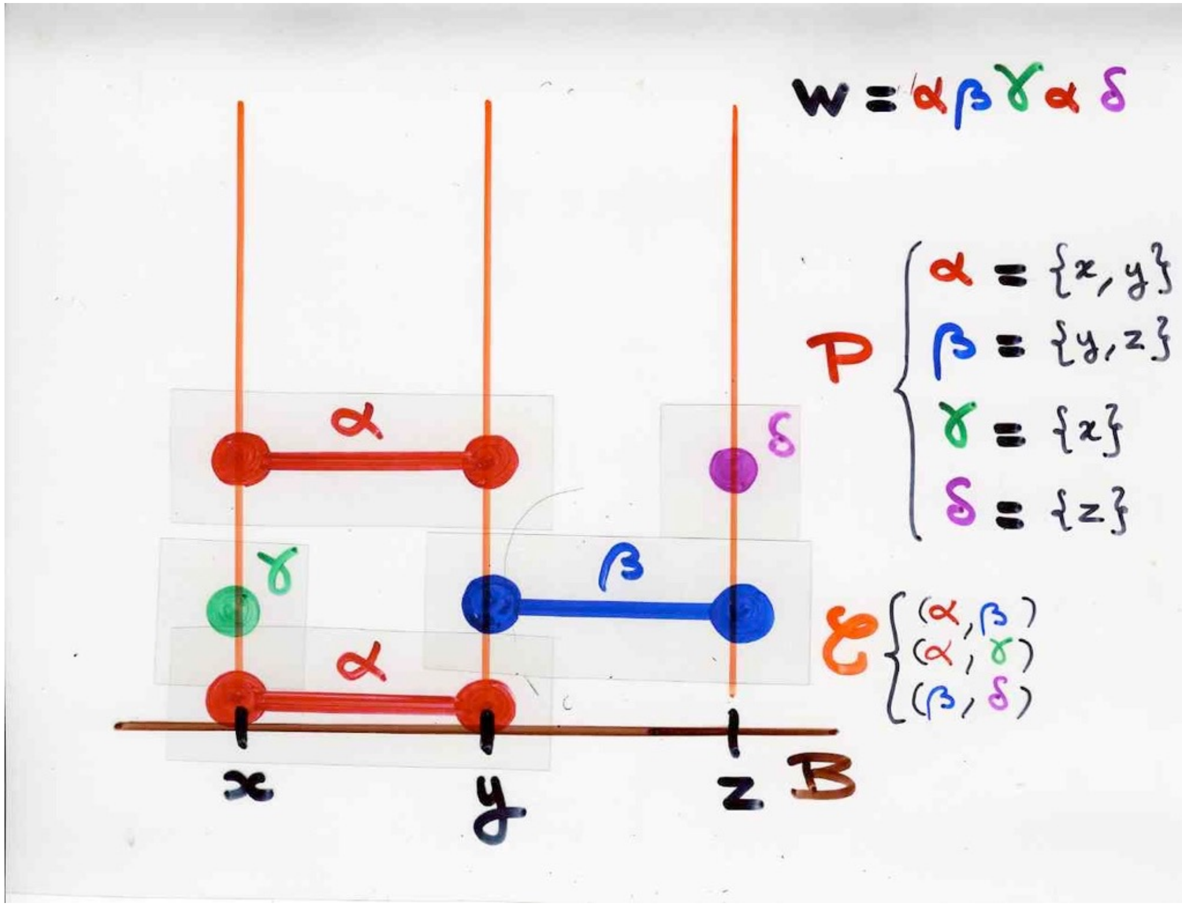
$P = N$

$i \sim j \Leftrightarrow |i-j| \leq 1$

example 2


A = {a, b, c, d}
c a b d


γ α β δ
c a b d
x y z


w = αβγαδ
P
α = {x, y}
β = {y, z}
γ = {x}
δ = {z}
(α, β)
(α, γ)
(β, δ)
α
δ
γ
β
α
x y z B

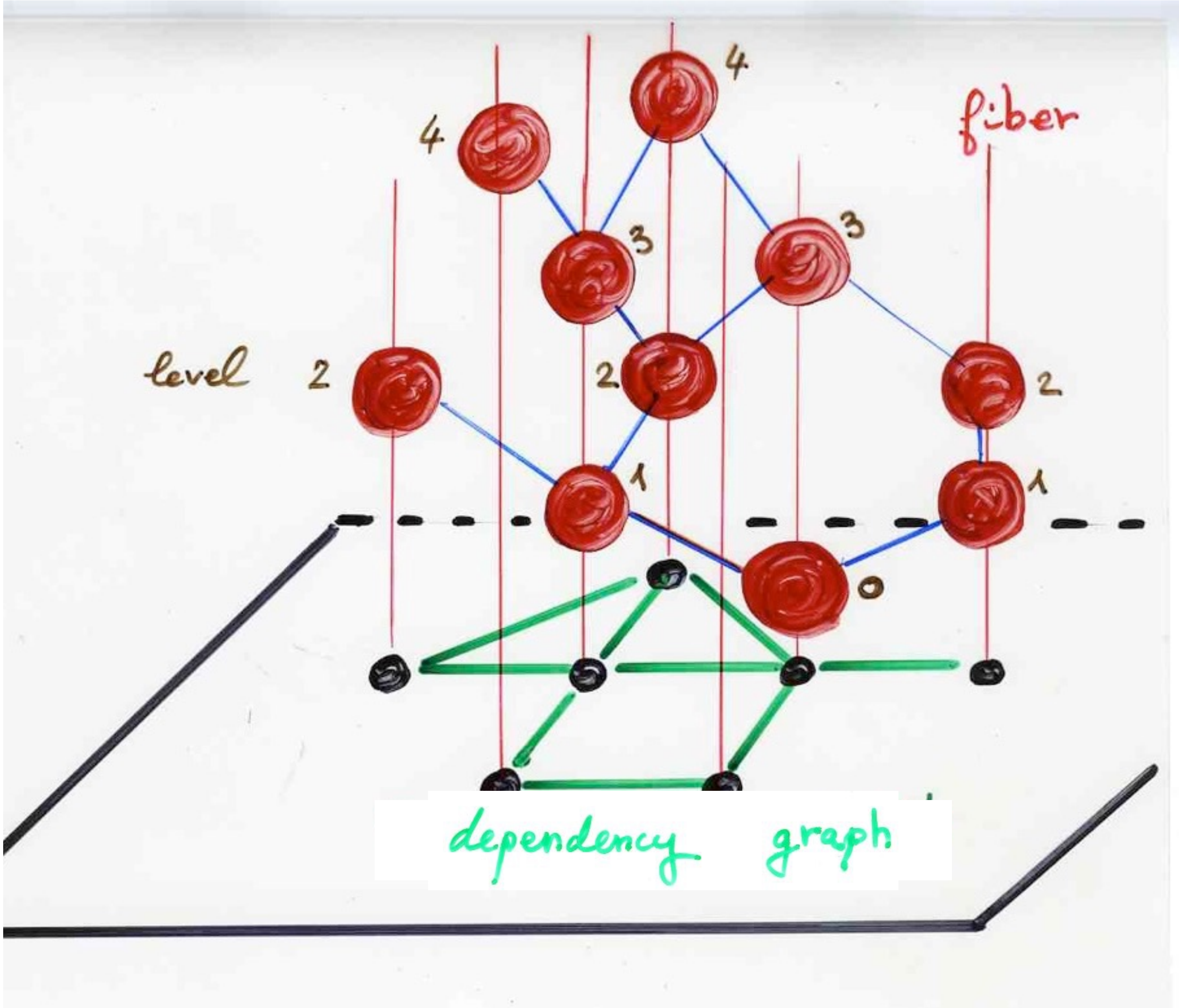
other definitions for the notion heaps of pieces

the orginal definition

(paper X.V. 1986)

X.V. [41] *Heaps of pieces, I: Basic definitions and combinatorial lemma*, in « *Combinatoire énumérative* », eds. G. Labelle et P. Leroux, , Lecture Notes in Maths. n° 1234, Springer-Verlag, Berlin, 1986, p. 321-325.

(downloadable on my main site www.xavierviennot.org)



4
4
fiber
3
3
level
2
2
2
1
1
0
dependency graph

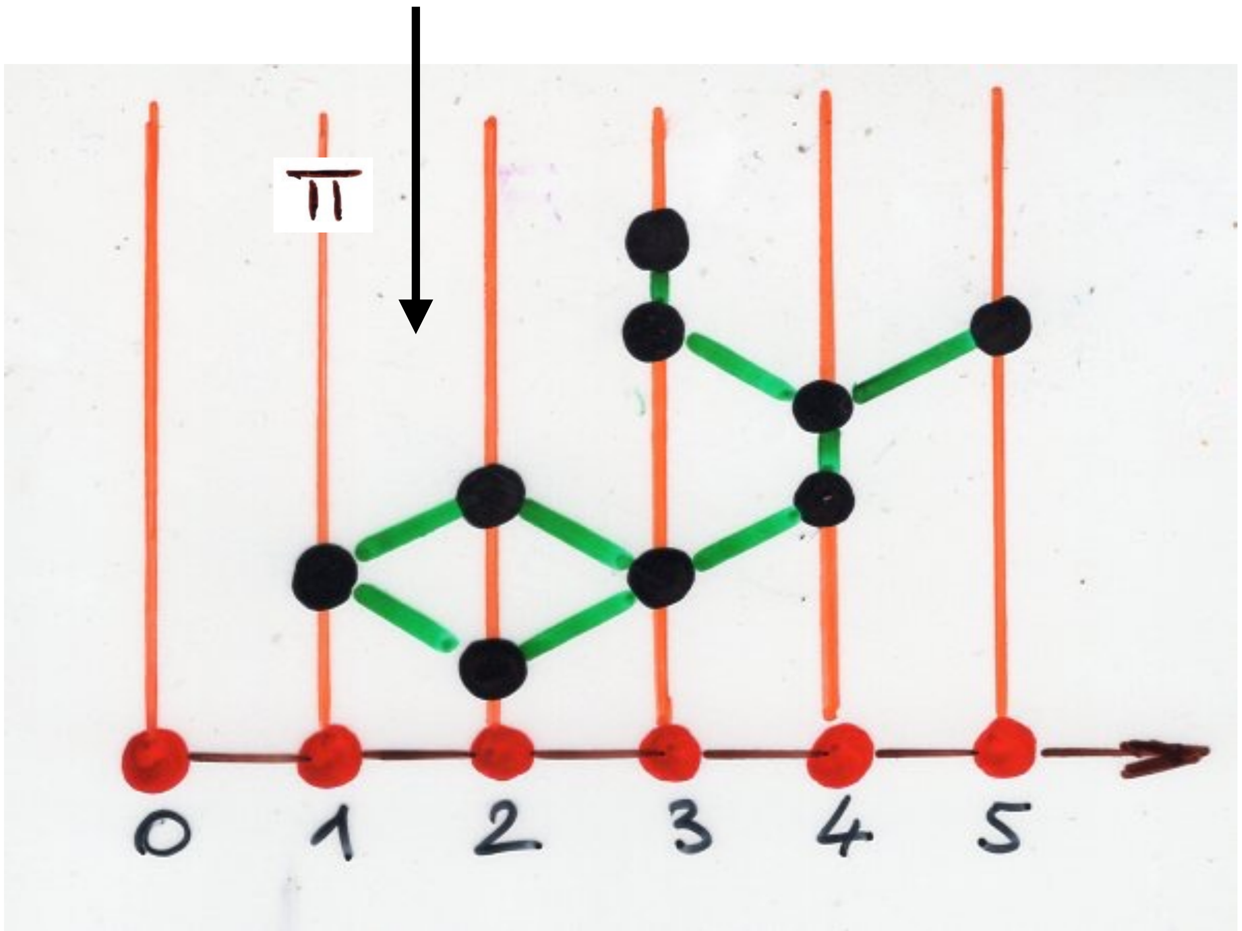
(second definition of heap of pieces)
(in french "empilement de pièces")

E heap of pieces in P

- P set (of basic pieces)
- $\mathcal{C}$ dependency relation on P
  symmetric and reflexive
  - is a poset with order relation $\preceq$
  - $E \xrightarrow{\pi} P$ π projection ("to be above")

(i) $\alpha, \beta \in E$, $\pi(\alpha) \,\mathcal{C}\, \pi(\beta) \Rightarrow$ or $\alpha \preceq \beta$, $\beta \preceq \alpha$

(ii) $\alpha, \beta \in E$, $\alpha \preceq \beta$, β covers α
$\Rightarrow \pi(\alpha) \,\mathcal{C}\, \pi(\beta)$



$$P = N$$

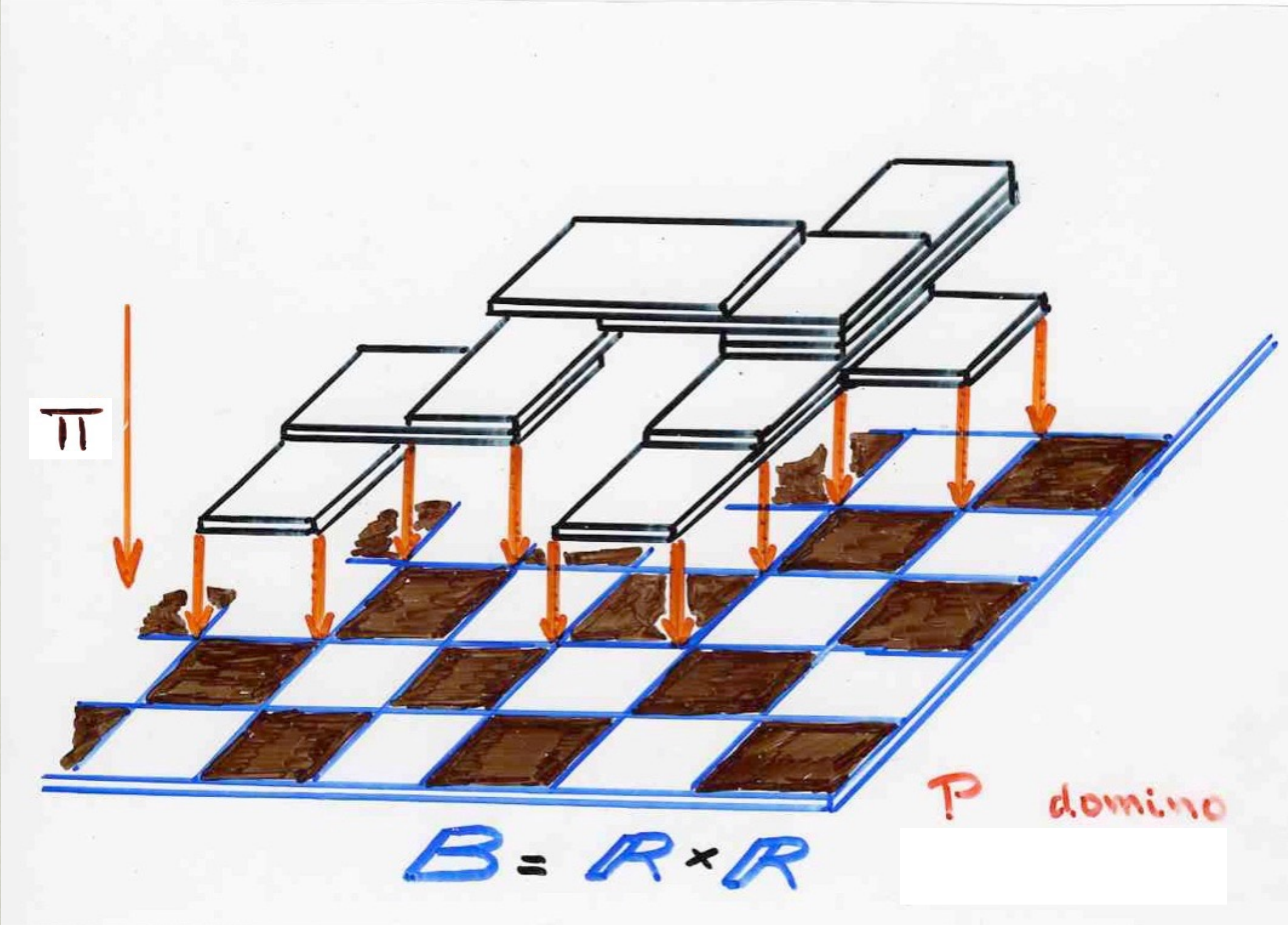
$$i \, \mathcal{C} \, j \iff |i-j| \leq 1$$

equivalent definition

(i) $\alpha, \beta \in E$, $\Pi(\alpha) \,\mathcal{C}\, \Pi(\beta) \Rightarrow \begin{cases} \alpha \preccurlyeq \beta \\ \text{or} \\ \beta \preccurlyeq \alpha \end{cases}$

(ii') $\preccurlyeq$ is the transitive closure of the relation in (i)
$\alpha \preccurlyeq \beta$ and $\Pi(\alpha) \,\mathcal{C}\, \Pi(\beta)$

i.e. $\alpha \preccurlyeq \beta \Leftrightarrow \exists\, \alpha_1 = \alpha \preccurlyeq \alpha_2 \preccurlyeq \cdots \preccurlyeq \alpha_k = \beta$,
with $\Pi(\alpha_i) \,\mathcal{C}\, \Pi(\alpha_{i+1})$ for $i = 1, \dots, k-1$.



π
P
domino
B = ℝ × ℝ

E heap (second definition)

<u>Definition</u> level of a $\alpha \in E$ (or height) $h(\alpha)$

- if α minimal element of $(E, \preceq)$
  $h(\alpha) = 0$
- in general $h(\alpha)$ is the length of the longest chain going from a minimal element to α

$$\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_k = \alpha$$

minimal

$$k = h(\alpha)$$

satisfies axioms (i) and (ii) of the first definition

(i) $(\alpha, i), (\beta, j) \in E,\ \alpha \mathcal{C} \beta \Rightarrow i \neq j$

(ii) $(\alpha, i) \in E,\ i > 0 \Rightarrow \exists \beta \in P,\ \alpha \mathcal{C} \beta,\ (\beta, i-1) \in E$

the two definitions are equivalent

by taking the projection π to be $\pi(\alpha, i) = \alpha$ (from the first definition)

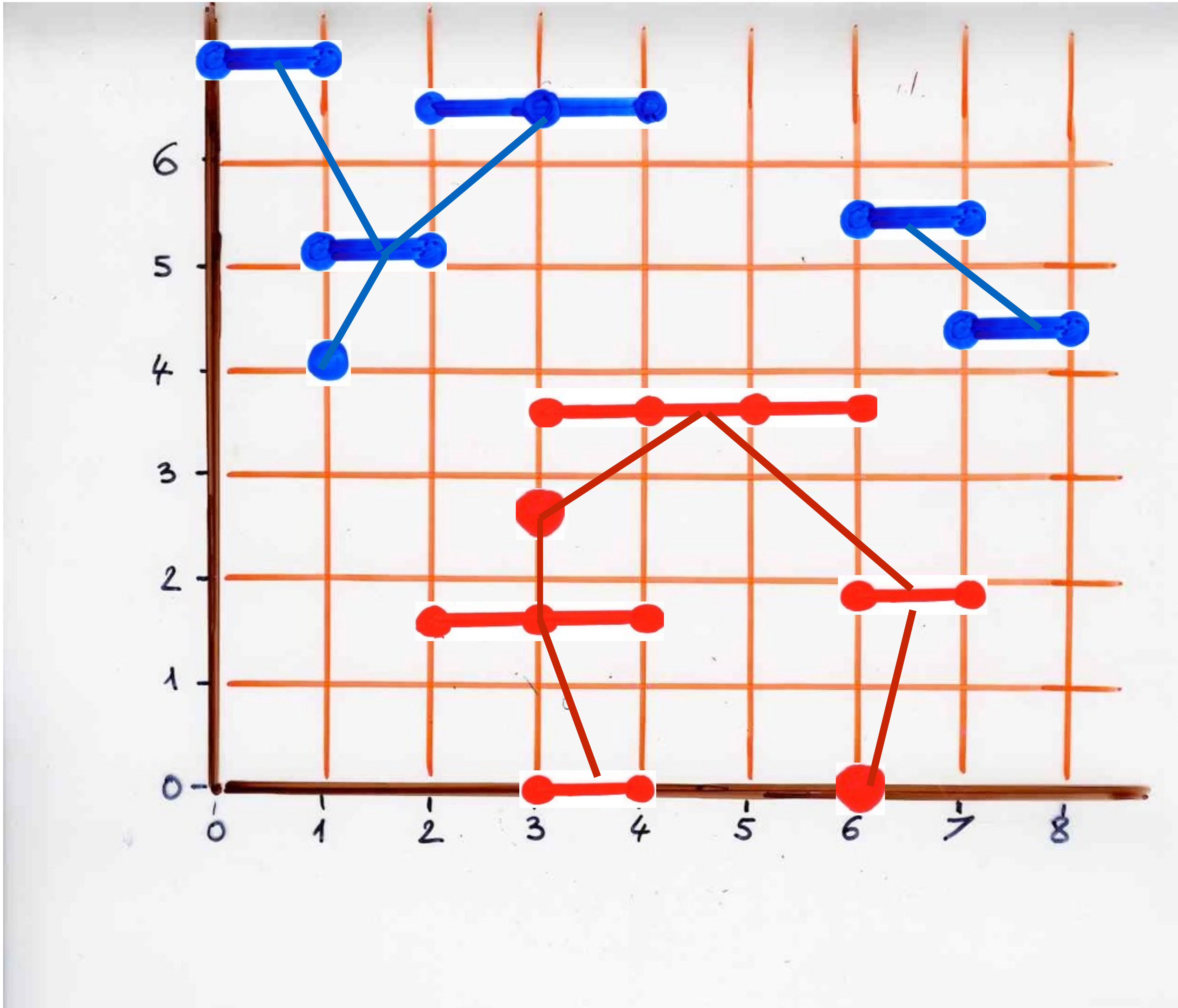
Definition Product $\odot$ of two heaps (with the second definition)

$E_1 = (E_1, \preccurlyeq_1, \pi_1)$, $E_2 = (E_2, \preccurlyeq_2, \pi_2)$

$E_1 \odot E_2 = (E_3, \preccurlyeq_3, \pi_3)$ with

- $E_3 = E_1 + E_2$ (disjoint union)
- $\pi_3 : E_3 \to P$ such that the restrictions $\pi_3/_{E_1} = \pi_1$ $\pi_3/_{E_2} = \pi_2$
- $\preccurlyeq_3$ is the transitive closure of the relation R

for $\alpha, \beta \in E_3$
$\alpha R \beta \iff \begin{cases} \bullet\ \alpha, \beta \in E_1 \text{ and } \alpha \preccurlyeq_1 \beta \\ \bullet\ \alpha, \beta \in E_2 \text{ and } \alpha \preccurlyeq_2 \beta \\ \bullet\ \alpha \in E_1,\ \beta \in E_2 \text{ and } \pi_1(\alpha)\ \mathcal{C}\ \pi_2(\beta) \end{cases}$



6
5
4
3
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0
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4
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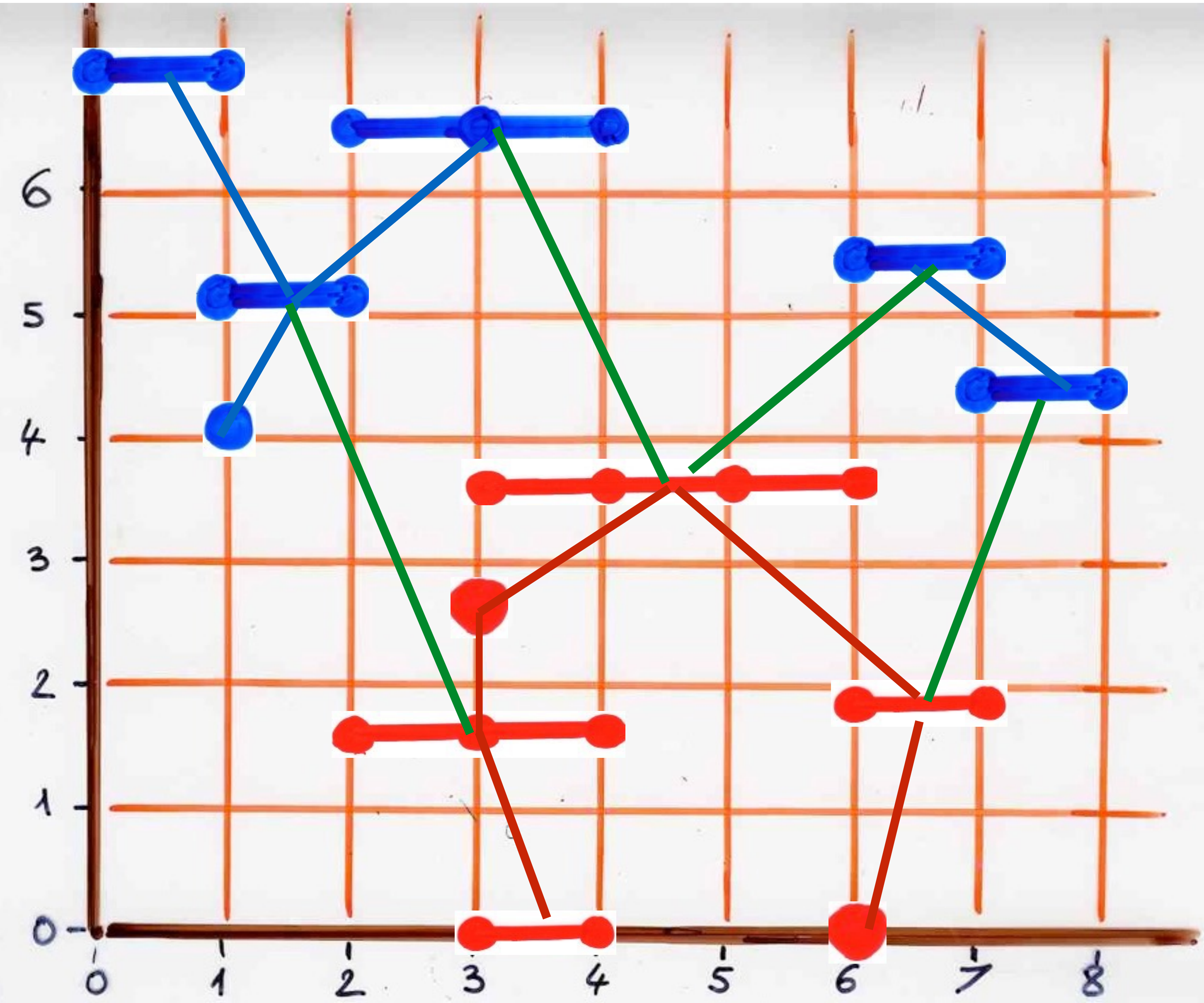
7
8


0
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6
0
1
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4
5
6
7
8

heaps morphism

$$\begin{array}{ccc} (E, \preccurlyeq_E) & \xrightarrow{\pi_E} & P_{\mathcal{C}} \\ f \downarrow & & \downarrow g \\ (F, \preccurlyeq_F) & \xrightarrow{\pi_F} & Q_{\mathcal{D}} \end{array}$$

(i) • commutative diagram

(ii) • $f : E \longrightarrow F$ increasing map

(iii) • $g : P \longrightarrow Q$ such that

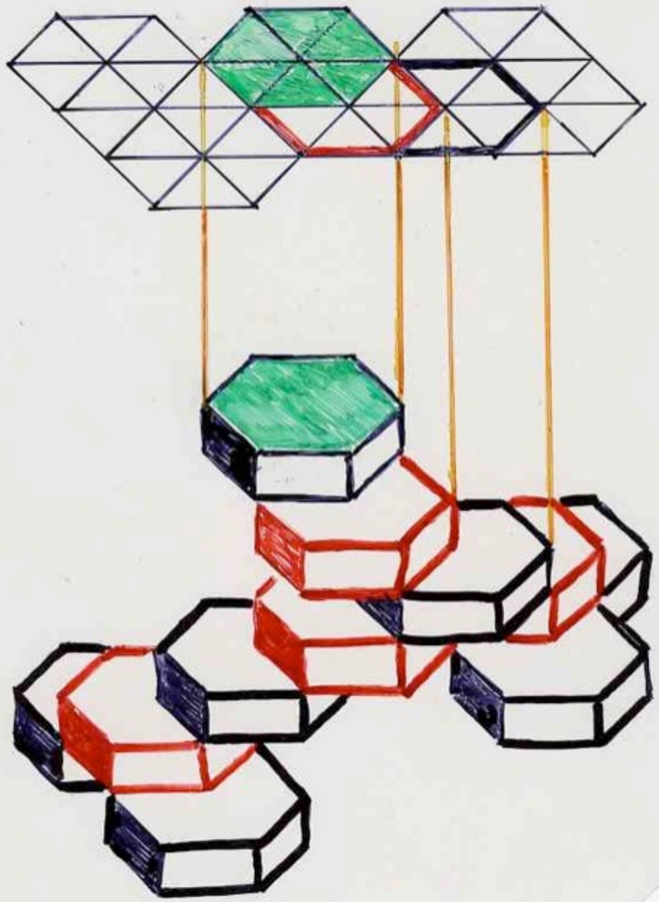
$$\alpha \, \mathcal{C} \, \beta \implies g(\alpha) \, \mathcal{D} \, g(\beta)$$

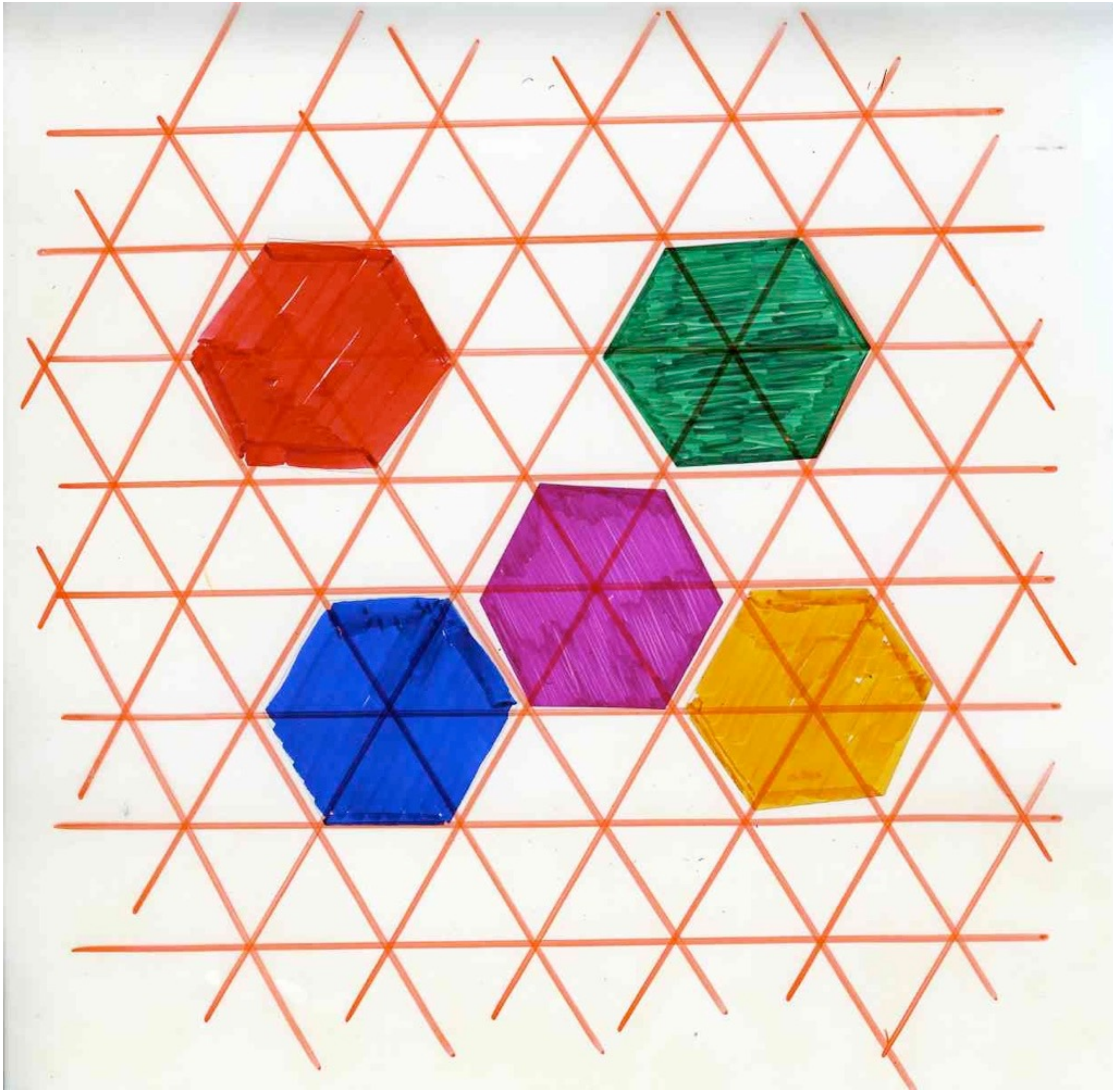
isomorphism

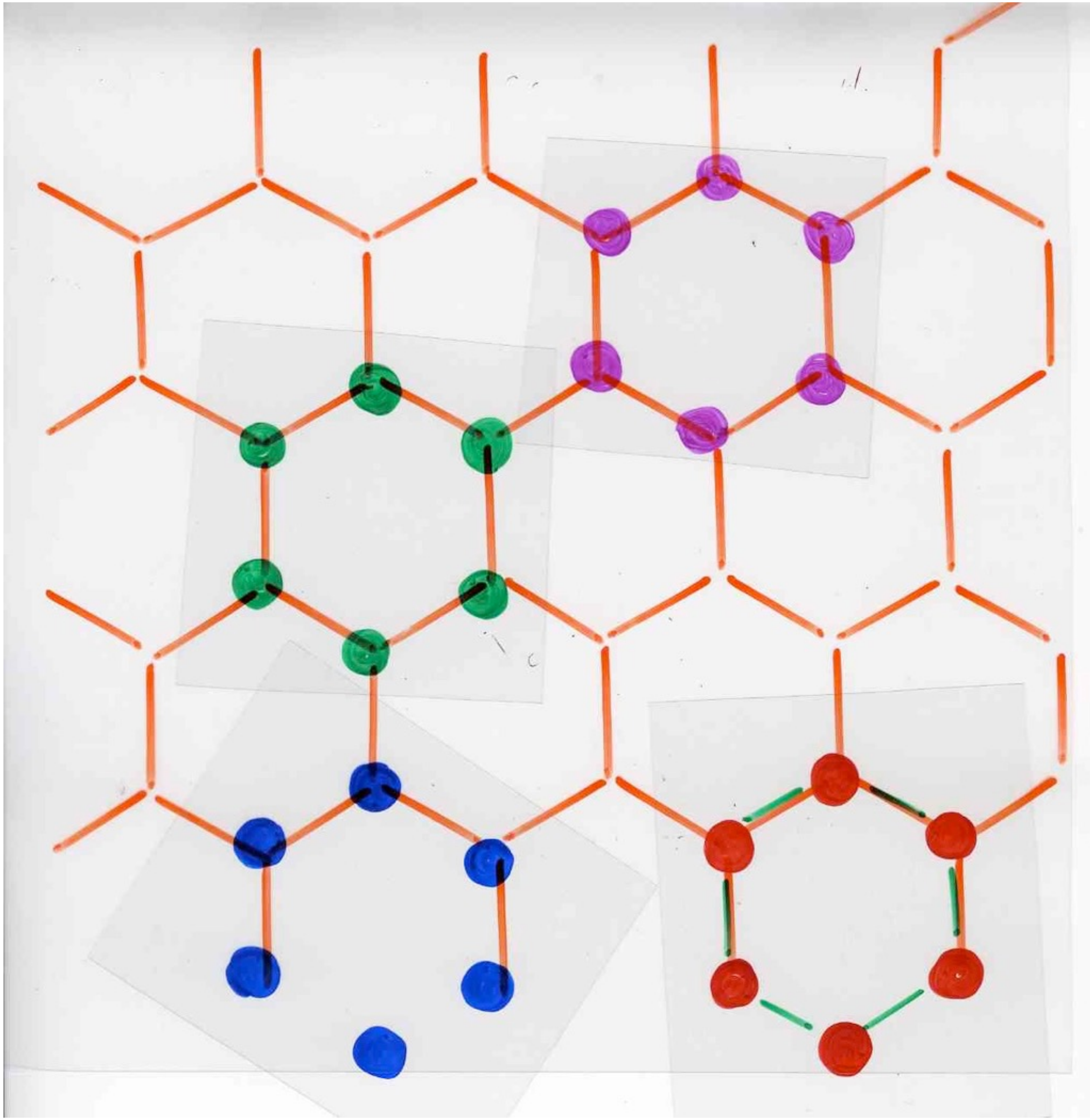
f and g are bijections

f^{-1}, g^{-1} satisfies (ii) and (iii)

$$-p(-t) = y$$







Heaps of dimers

symmetric group

more in chapter 9

heaps of dimers $(i, i+1)$

on $\{0, 1, \ldots, n-1\}$

generators $\{\sigma_0, \sigma_1, \ldots, \sigma_{n-1}\}$

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

iff $|i-j| \geq 2$

Symmetric group $\mathfrak{S}_n$

S_n

$n!$ permutations

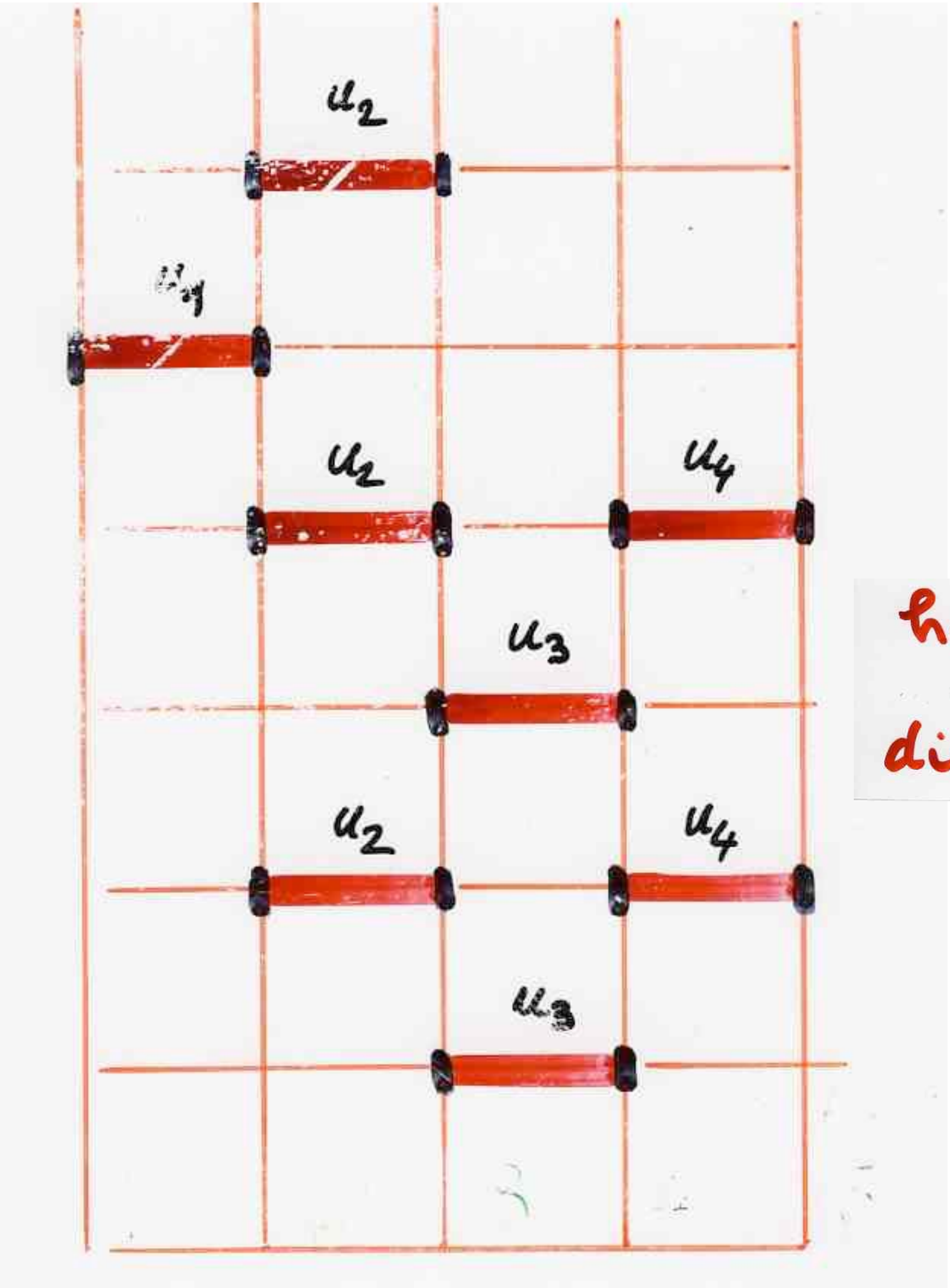
$$\sigma_i = (i, i+1) \qquad i = 1, 2, \ldots, n-1$$

transposition of two consecutive elements

$$\begin{cases} \text{(i)} & \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| \geq 2 \\ \text{(ii)} & \sigma_i^2 = 1, \\ \text{(iii)} & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \end{cases}$$

Moore-Coxeter

Yang-Baxter


u2
u2
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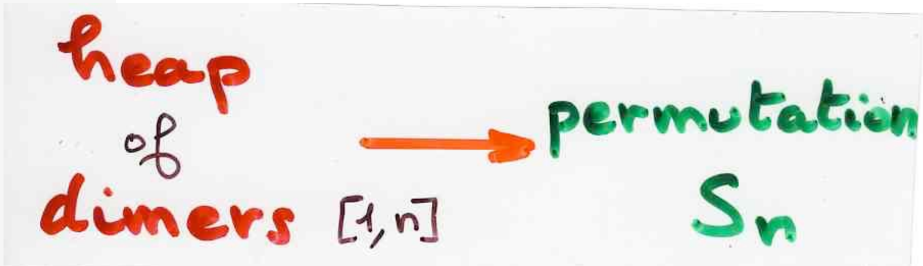
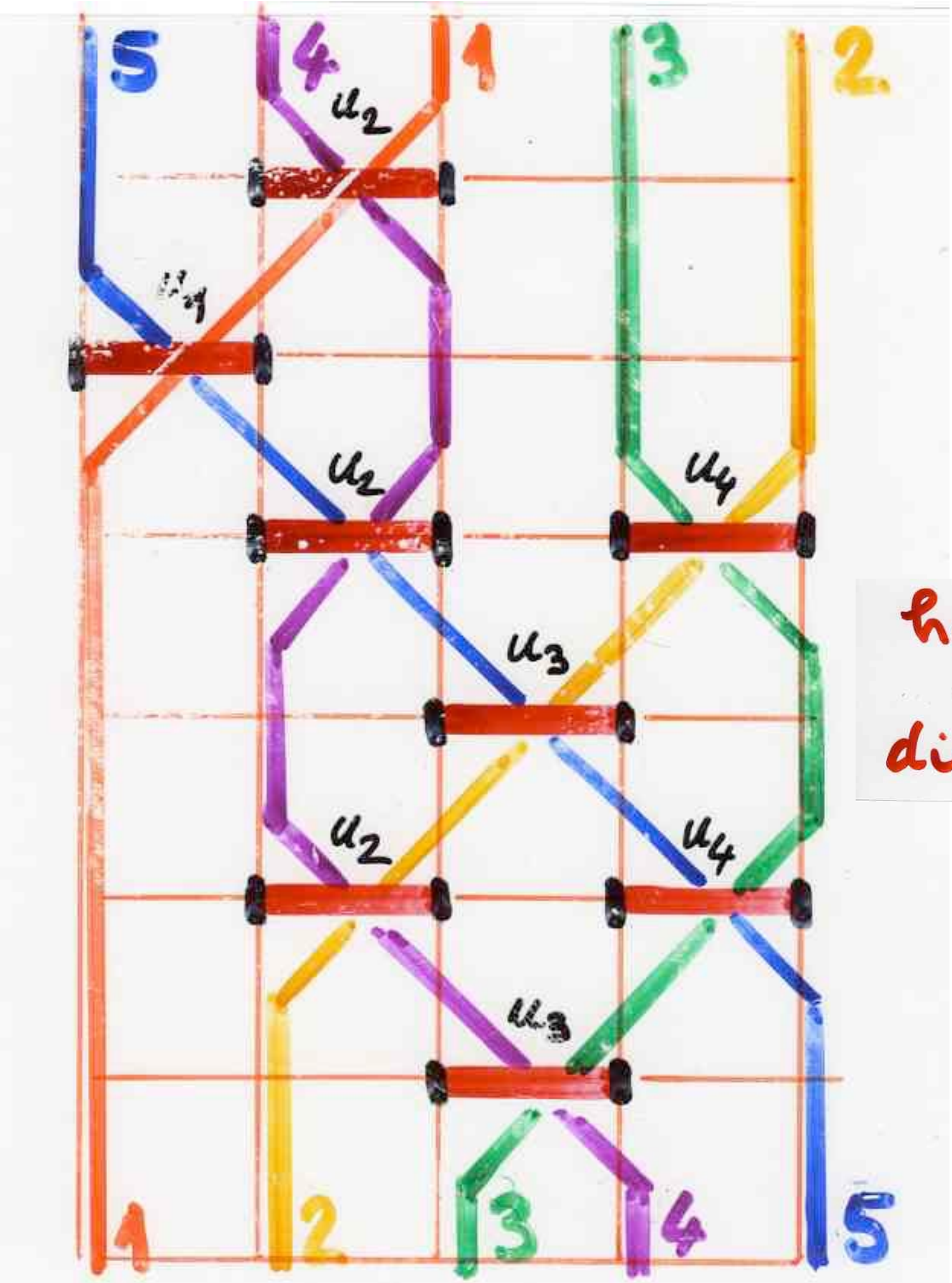
u3
u2
u4
u3


heap of dimers [1, n]
permutation Sn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u3
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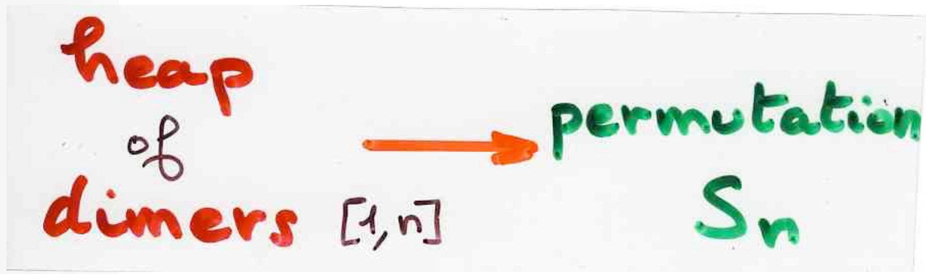

heap
of
dimers [1, n]
permutation
Sn

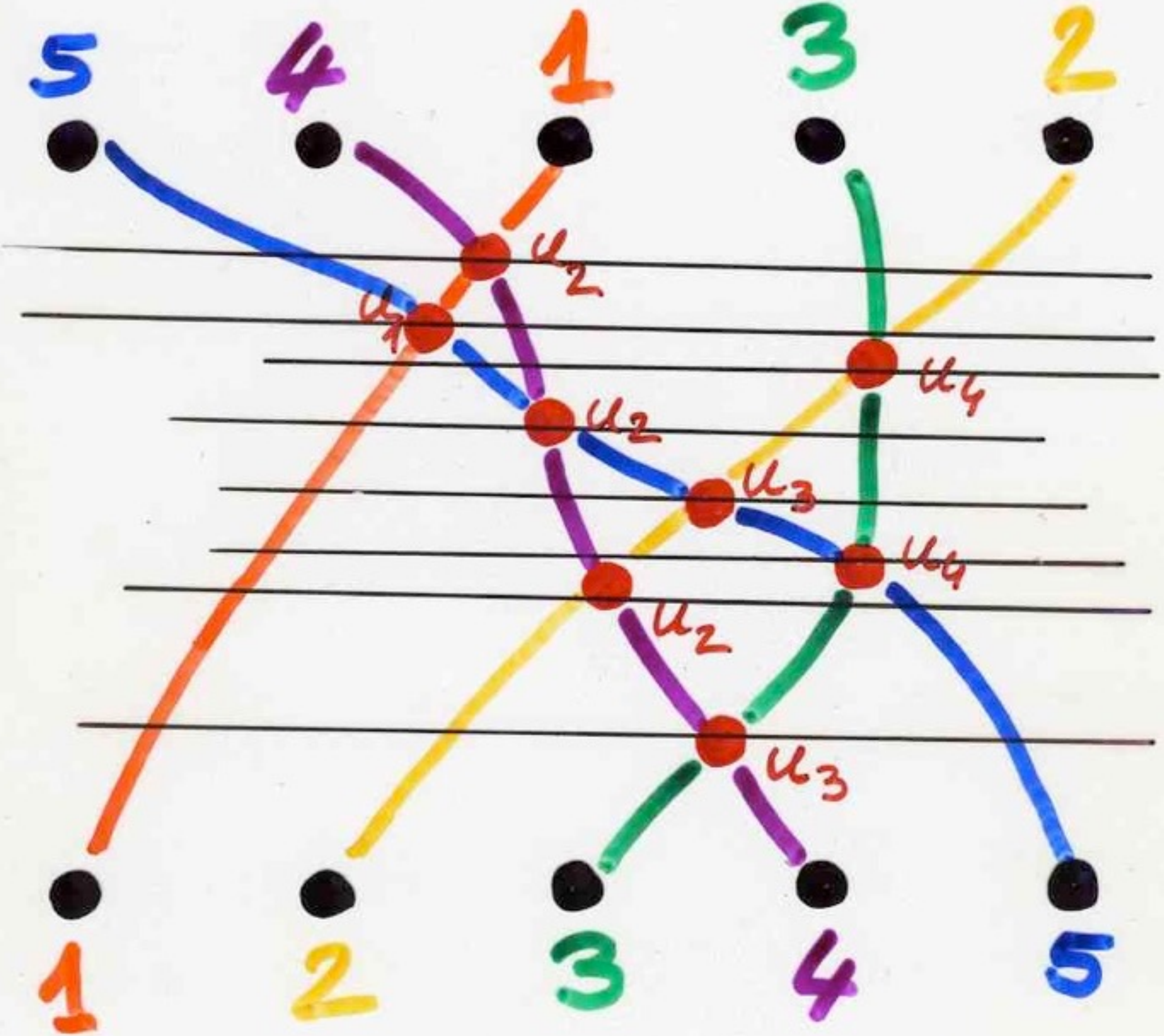
$$u_i(a_1 \cdots a_i\, a_{i+1} \cdots a_n)$$
$$= (a_1 \cdots a_{i+1}\, a_i \cdots a_n)$$

reduced decomposition
of a permutation

$$\sigma = u_{i_1} \cdots u_{i_k}$$

k minimum

(nb of inversion)


5 4 1 3 2
u2
u1
u4
u2
u3
u4
u2
u3
1 2 3 4 5

$$u_i(a_1 \cdots a_i\, a_{i+1} \cdots a_n)$$
$$= (a_1 \cdots a_{i+1}\, a_i \cdots a_n)$$

equivalently :

if $\quad \sigma = u_{i_1} \cdots u_{i_k}(1\,2 \cdots n)$

$$\sigma^{-1} = s_{i_1} \cdots s_{i_k}$$

$$s_i = (i, i+1)$$

transposition

braid group B_n

$$\begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{cases}$$

symmetric group S_n

$$\begin{cases} \sigma_i^2 = 1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{cases}$$

Temperley-Lieb algebra $A_n(\tau)$

$$\begin{cases} e_i^2 = e_i \\ e_i e_j = e_j \sigma_i & |i-j| \geq 2 \\ e_i e_{i\pm1} e_i = \tau e_i \end{cases}$$

Hecke algebra $H_n(q)$

$$\begin{cases} g_i^2 = (q-1) g_i + q \\ g_i g_j = g_j g_i & |i-j| \geq 2 \\ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \end{cases}$$

more in chapter 9