Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics, a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

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Chapter 1
Commutation monoids
and
heaps of pieces:

basic definitions
(2)

IMSc, Chennai

9 January 2017

from the previous lecture

commutation relation C antireflexive symmetric

$\equiv_C$ congruence of A^* generated by the commutations

$$ab \equiv ba \quad \text{iff} \quad aCb$$

- $aCb \Leftrightarrow bCa$
- $\cancel{aCa}$

ex : $A = \{a, b, c, d\}$

$C \quad \begin{cases} ad = da \\ bc = cb \\ cd = dc \end{cases}$

equivalence class

w = abcad — abcda

acbad

abdca

acbda

commutation monoid

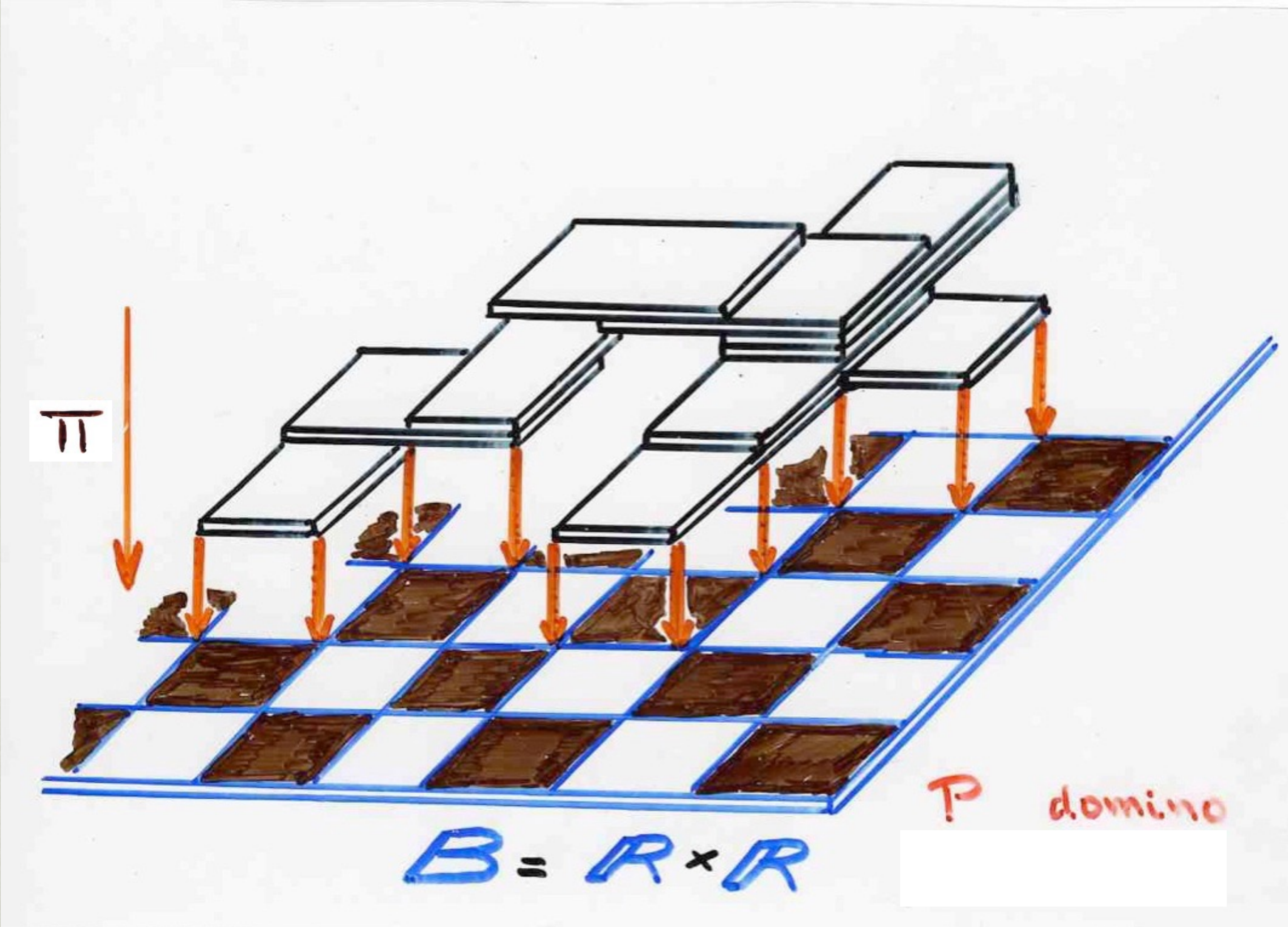
$$A^* /_{\equiv_C}$$

$[w]$ equivalence class of the word $w \in A^*$

$A^* /_{\equiv_C}$ • product in the commutation monoid

$$[u] \cdot [v] = [uv]$$

independant of the choices of representants u and v



π
P
domino
B = ℝ × ℝ

heap definition

- P set (of basic pieces)
- $\mathcal{C}$ binary relation on P $\begin{cases} \text{symmetric} \\ \text{reflexive} \end{cases}$

  (dependency relation)
- heap E, finite set of pairs

  (α, i) $\alpha \in P$, $i \in \mathbb{N}$ (called pieces)

  projection level

(i) $(\alpha, i), (\beta, j) \in E$, $\alpha \,\mathcal{C}\, \beta \Longrightarrow i \neq j$

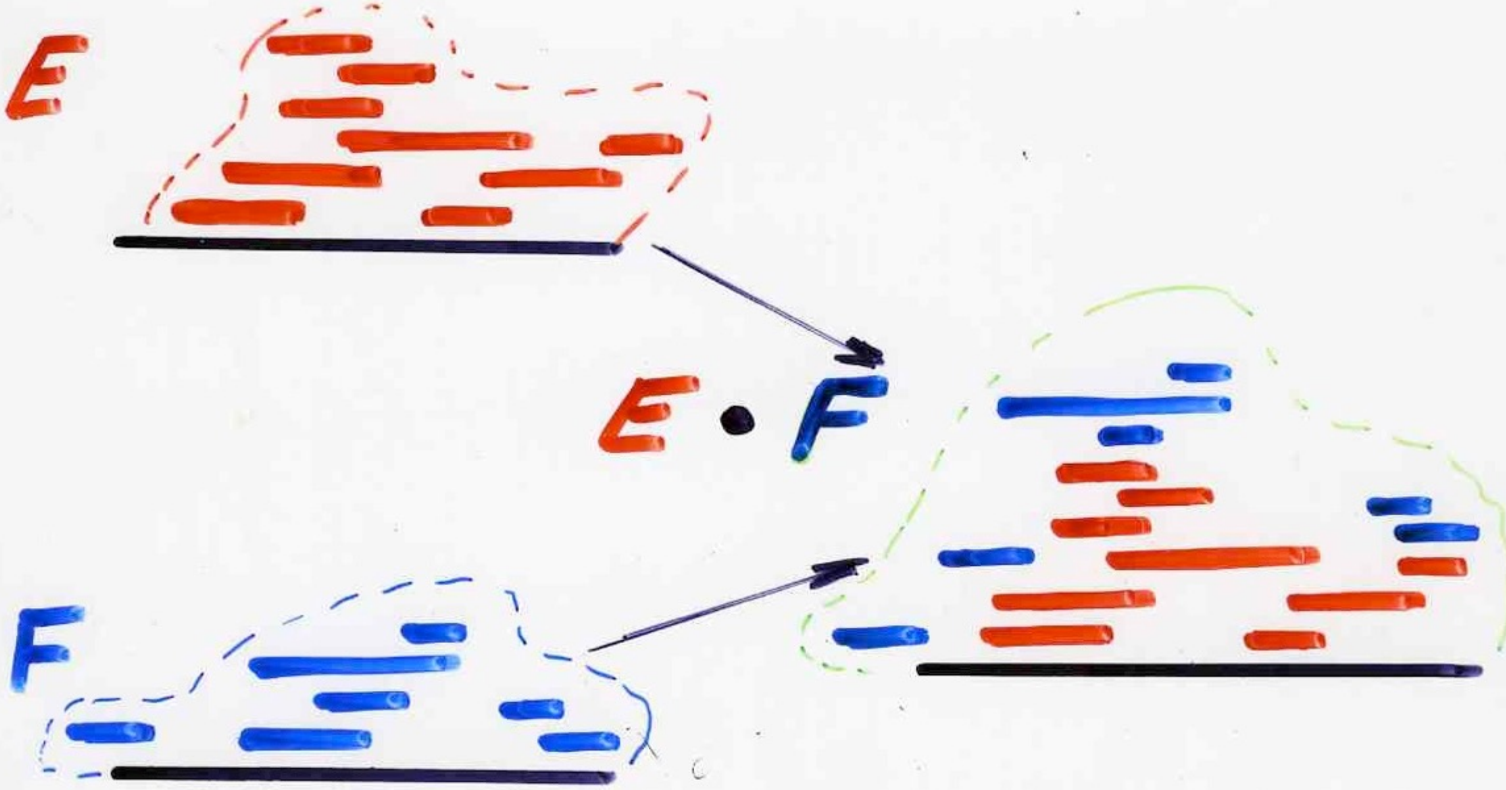
(ii) $(\alpha, i) \in E$, $i > 0 \Longrightarrow \exists \beta \in P$, $\alpha \,\mathcal{C}\, \beta$, $(\beta, i-1) \in E$

Heaps monoid

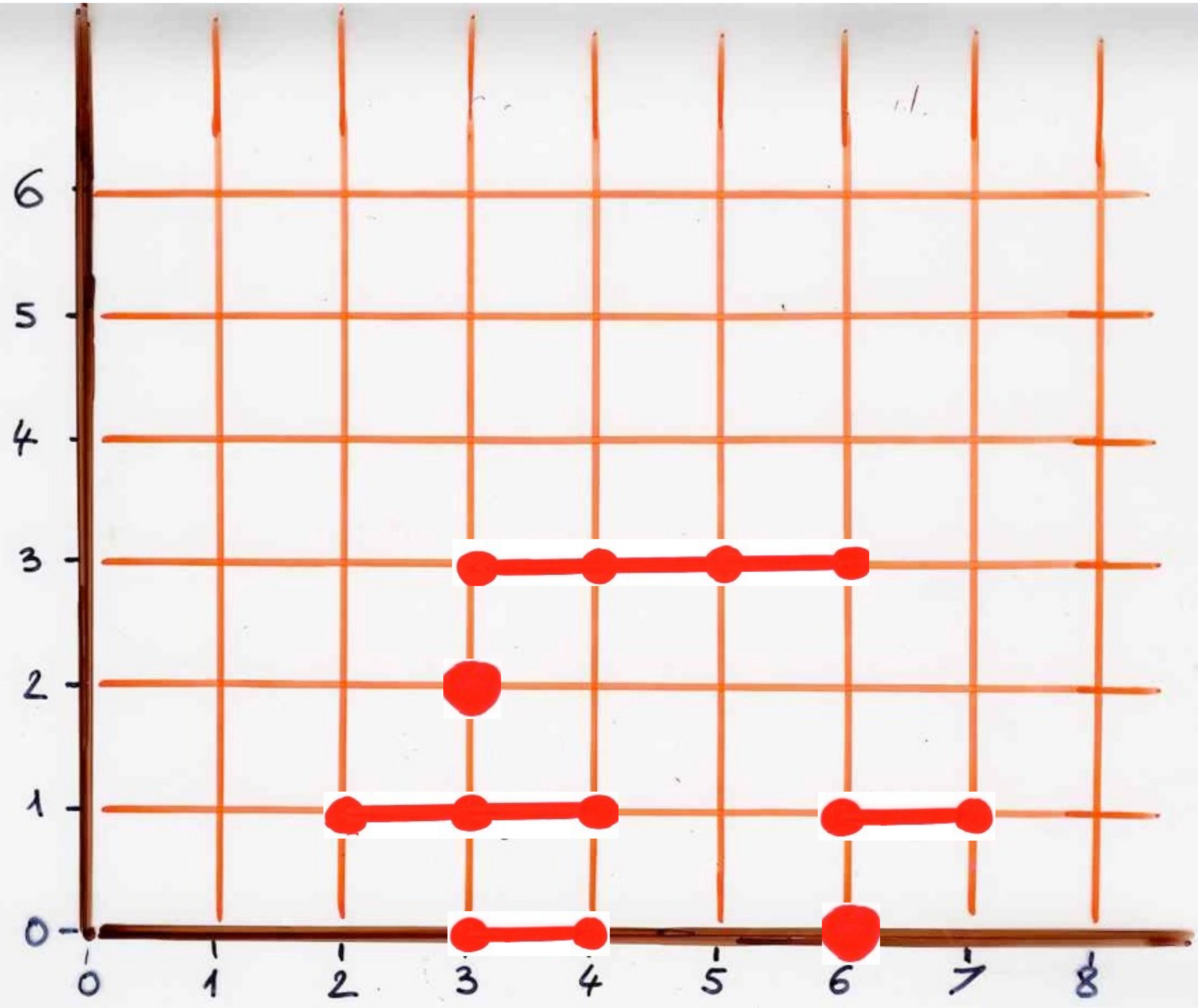
$H(P, \mathcal{C})$

product of two heaps

$E \circ F$



$$E \cdot F = E_{\sim}(E \cup T(F))$$



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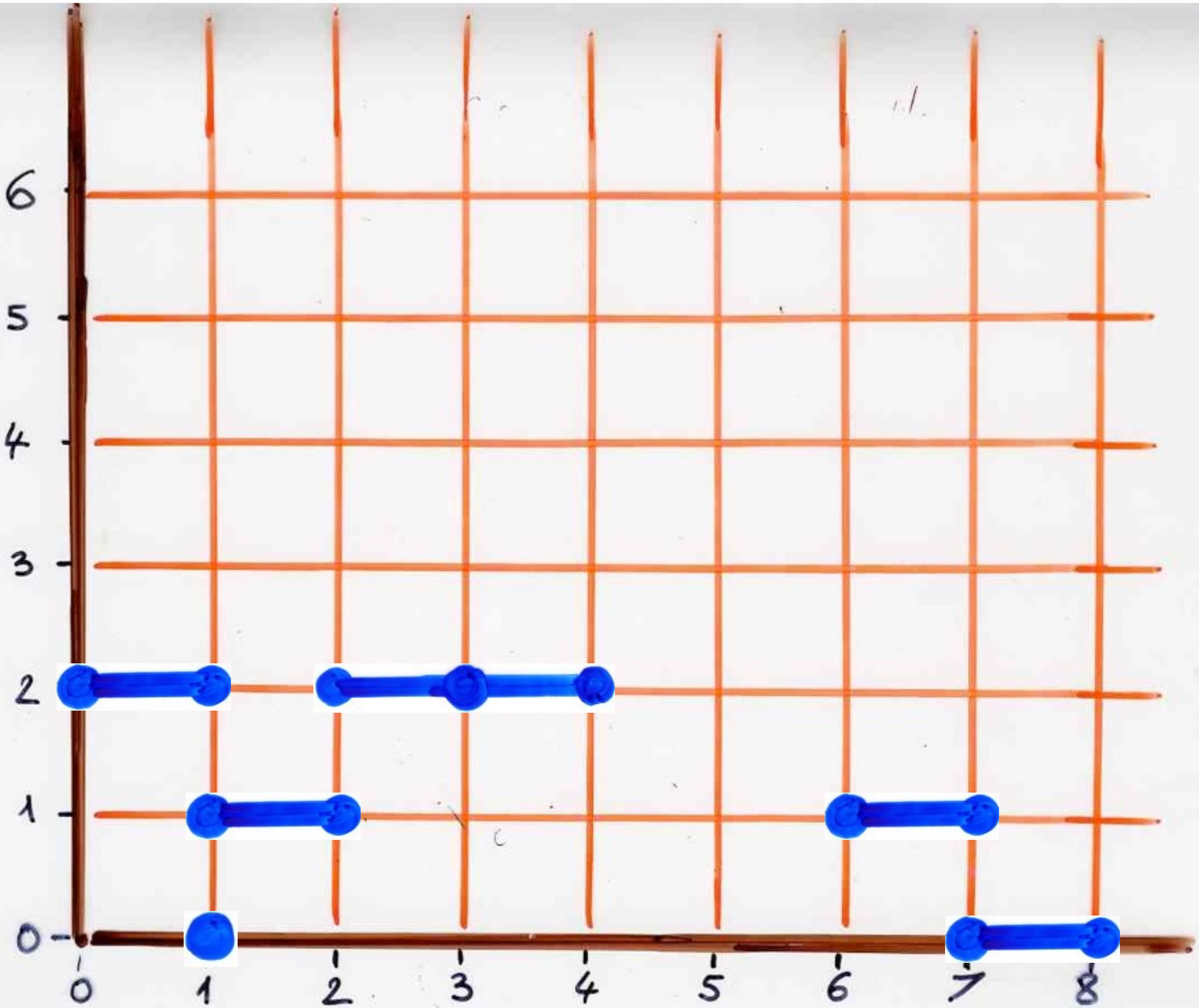
0 1 2 3 4 5 6 7 8



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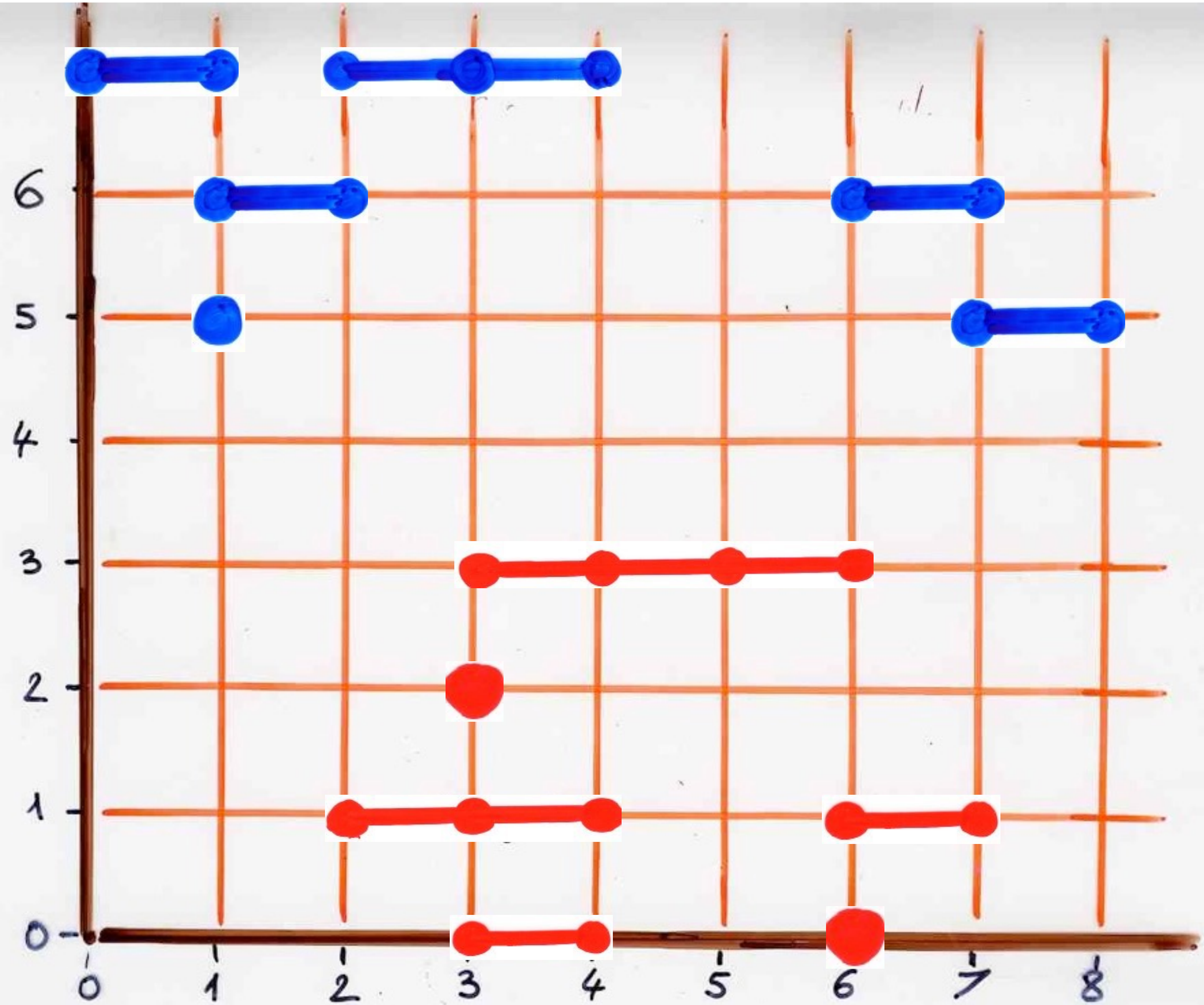

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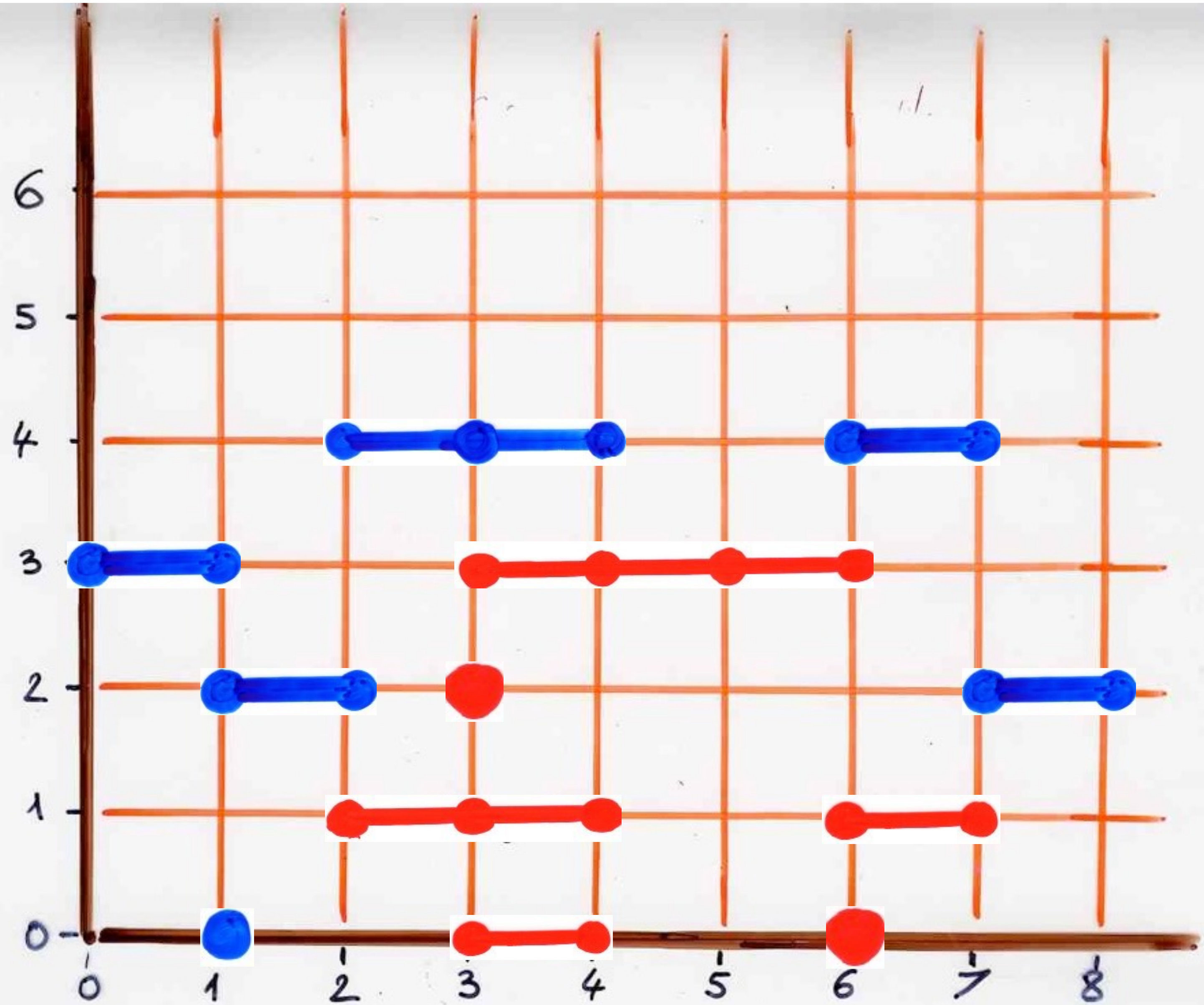

6
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Equivalence commutation monoids and heaps monoids

$$\mathcal{P} \subseteq \mathrm{Heap}(\mathcal{P}, \mathcal{C})$$

$$\alpha \longleftrightarrow \{(\alpha, 0)\}$$

$$\varphi : \mathcal{P}^{*} \longrightarrow \mathrm{Heap}(\mathcal{P}, \mathcal{C})$$

$$w = \alpha_1 \alpha_2 \cdots \alpha_n \longrightarrow \alpha_1 \odot \alpha_2 \odot \cdots \odot \alpha_n$$

word   heap

$$C = \overline{\mathcal{C}}$$

commutation relation

complementary of the dependency relation

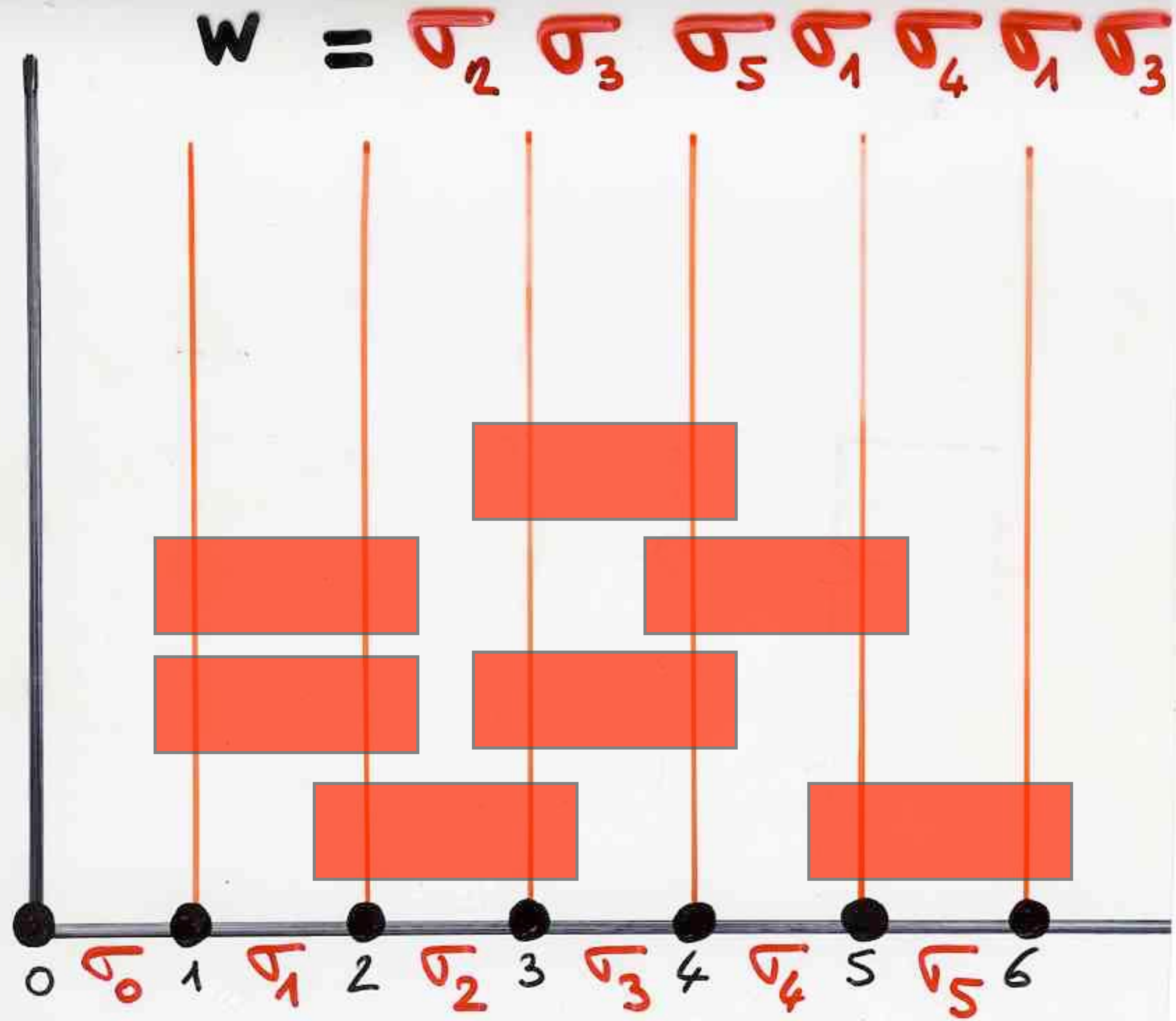
ex : heaps of dimers on $\mathbb{N}$

$$P = \{ [i, i+1] = \sigma_i \ , i \geq 0 \}$$

$\mathcal{C}$

C commutations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{iff } |i - j| \geq 2$$



W = σ2 σ3 σ5 σ1 σ4 σ1 σ3
0 σ0 1 σ1 2 σ2 3 σ3 4 σ4 5 σ5 6

$$P \subseteq \mathrm{Heap}(P, \mathcal{C})$$

$$\alpha \longleftrightarrow \{(\alpha, 0)\}$$

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$$w = \alpha_1 \alpha_2 \cdots \alpha_n \longrightarrow \alpha_1 \odot \alpha_2 \odot \cdots \odot \alpha_n$$

word — heap

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Lemma 1

$$u \underset{C}{\equiv} v \Longrightarrow \varphi(u) = \varphi(v)$$

Lemma 2

$$\varphi(u) = \varphi(v) \Longrightarrow u \underset{C}{\equiv} v$$

Definition $\overline{\varphi}([u]) = \varphi(u)$

$$
\begin{tikzcd}
\mathcal{P}^* \arrow[r, "\varphi"] \arrow[d] & \mathbf{H}(\mathcal{P}, \mathcal{E}) \\
\mathcal{P}^* / \equiv_{\mathcal{C}} \arrow[ur, dashed, "\overline{\varphi}"'] &
\end{tikzcd}
$$

Proposition $\overline{\varphi}$ is an isomorphism of monoids

$$\mathrm{Heap}(P, \mathcal{E}) \simeq P^* / \equiv_C$$

heaps monoid

commutation monoid

$$C = \overline{\mathcal{E}}$$

complementary relation

Proofs of Lemma 1, 2 and Proposition

$$P \subseteq \mathrm{Heap}(P, C)$$

$$\alpha \longleftrightarrow \{(\alpha, 0)\}$$

$$\varphi : P^* \longrightarrow \mathrm{Heap}(P, C)$$

$$w = \alpha_1 \alpha_2 \cdots \alpha_n \longrightarrow \alpha_1 \odot \alpha_2 \odot \cdots \odot \alpha_n$$

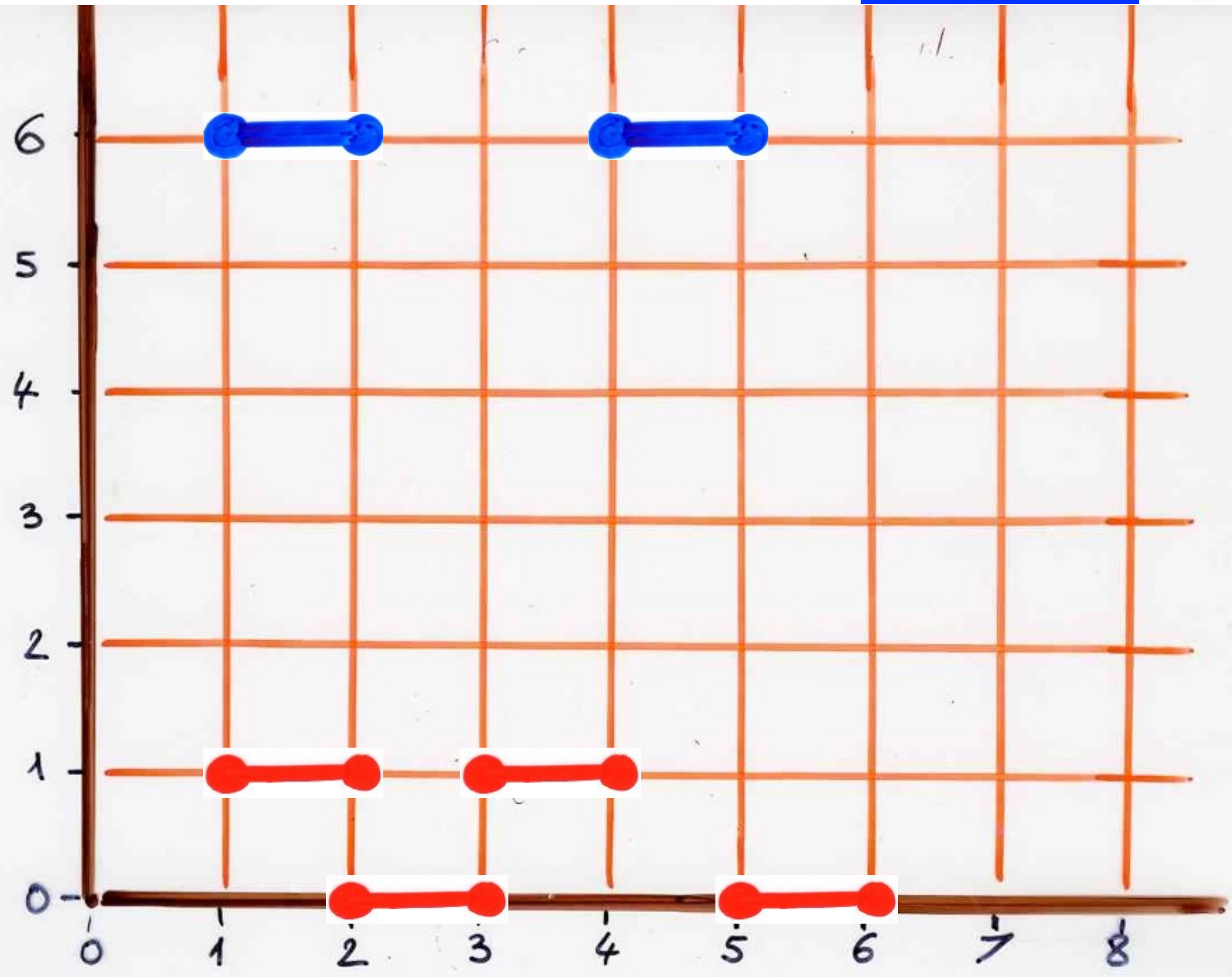
word heap

<u>Lemma 1</u>

$$u \equiv_C v \Rightarrow \varphi(u) = \varphi(v)$$

<u>Proof</u>: obvious

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \boxed{\sigma_4\ \sigma_1}\ \sigma_3$



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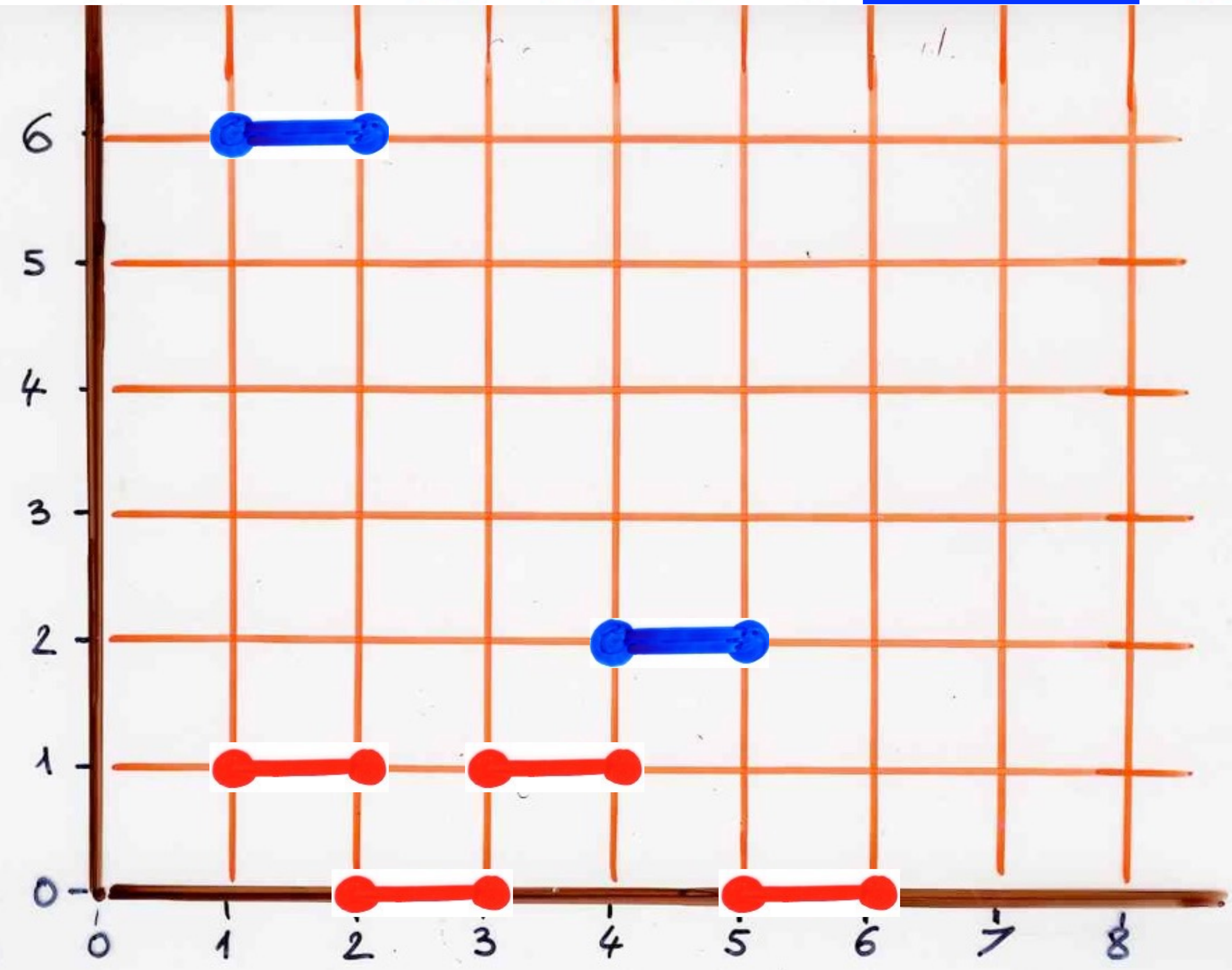

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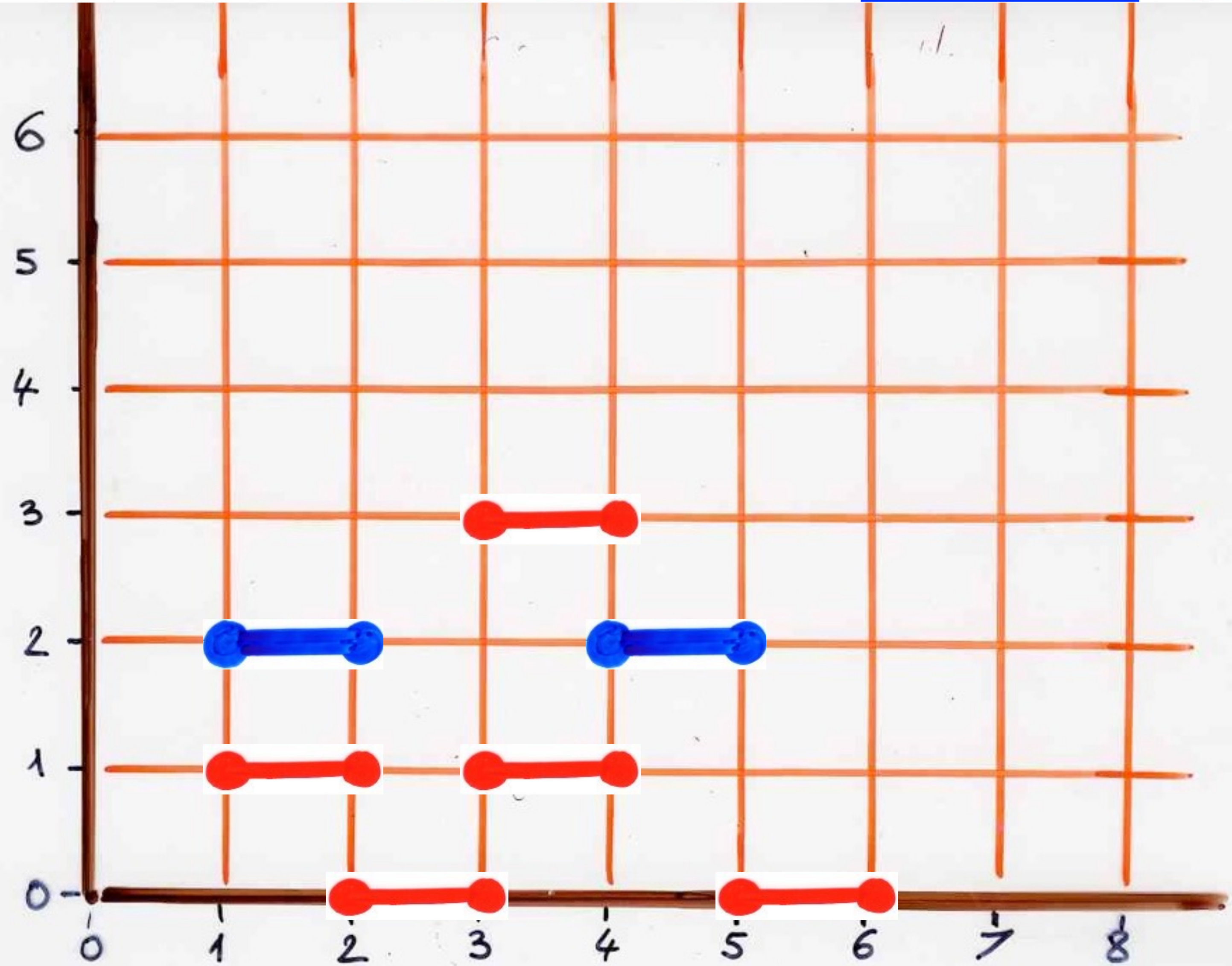

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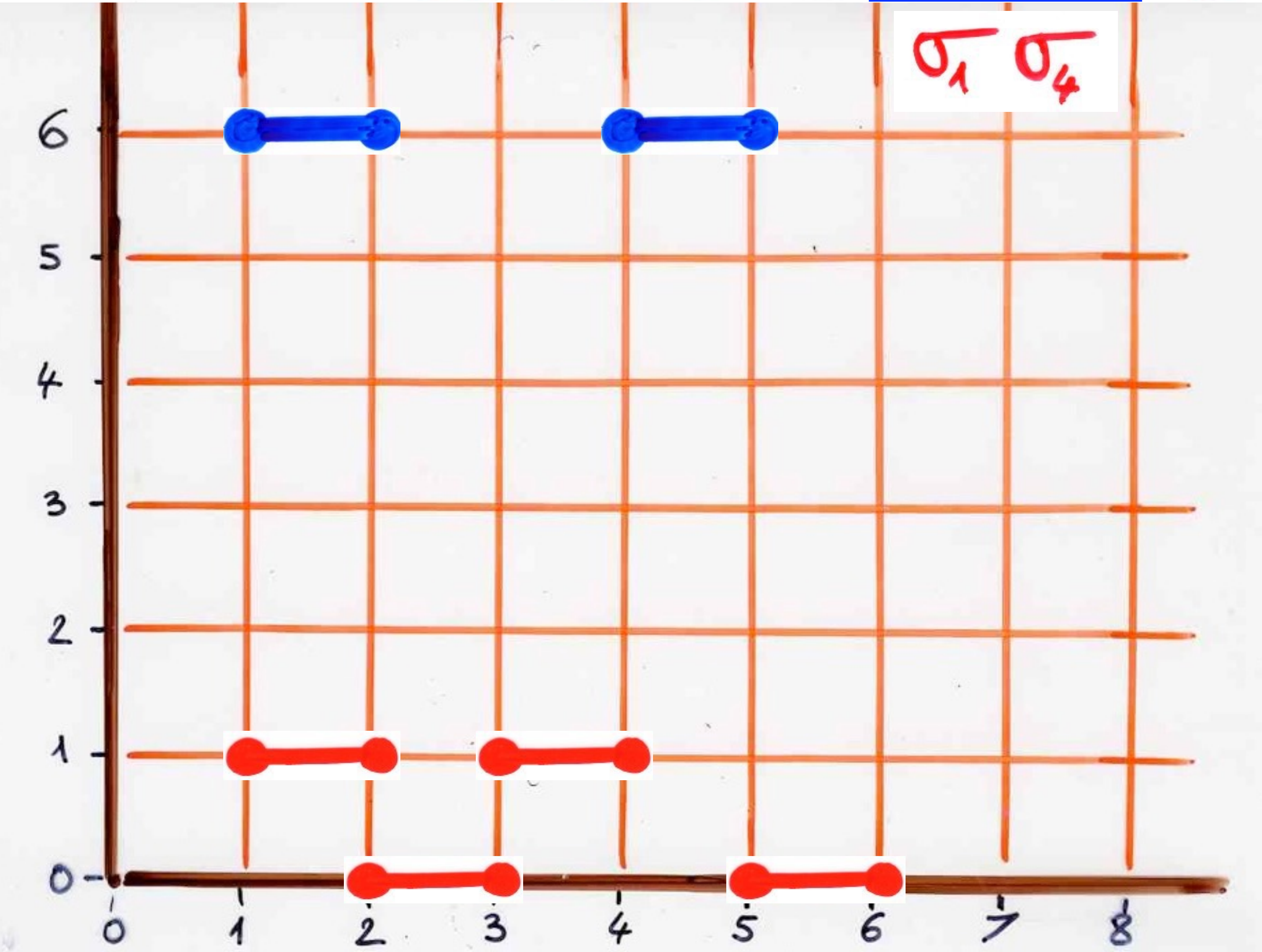
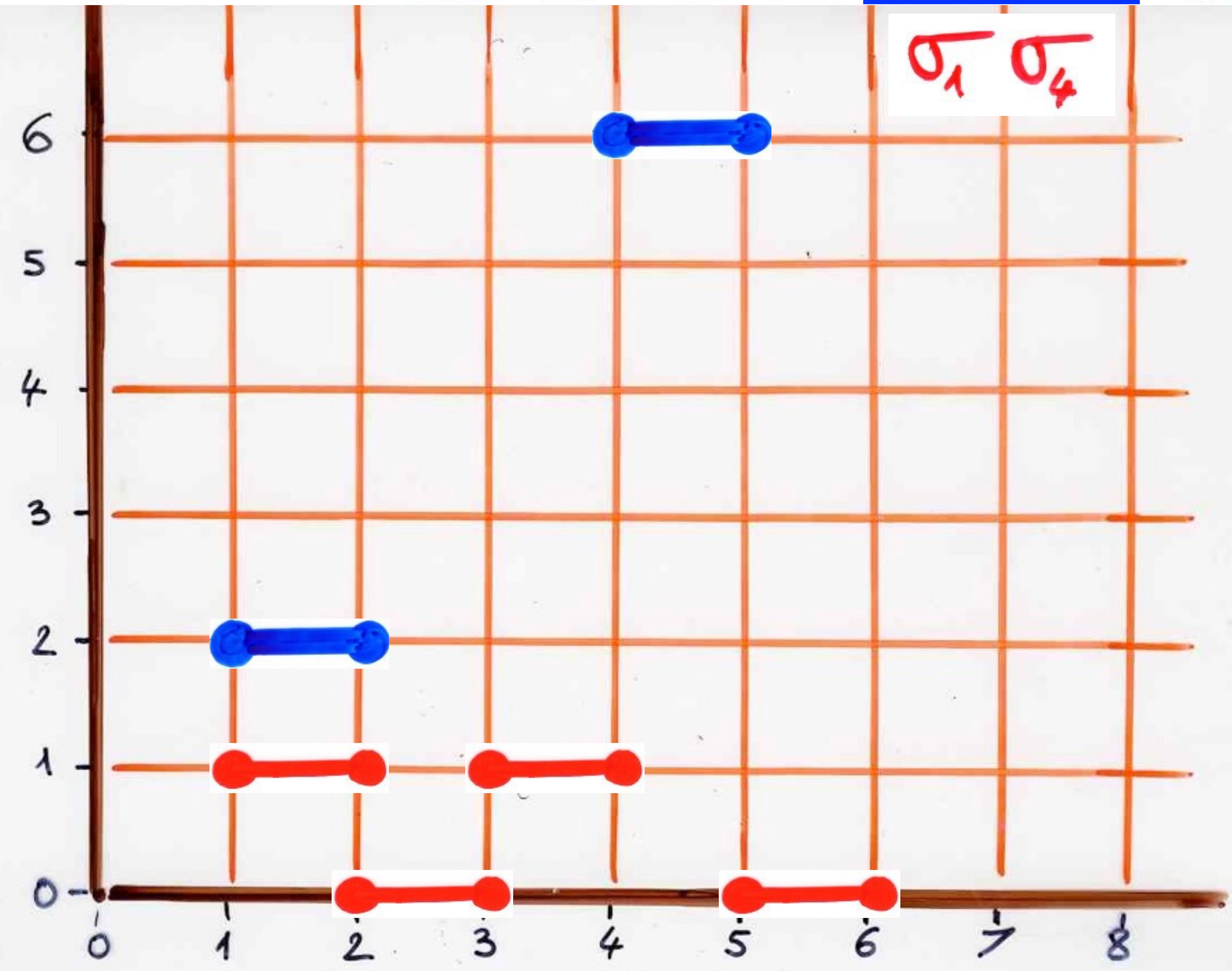
$\sigma_1\ \sigma_4$



example $w = \sigma_2 \, \sigma_3 \, \sigma_5 \, \sigma_1 \, \boxed{\sigma_4 \, \sigma_1} \, \sigma_3$

$\sigma_1 \, \sigma_4$



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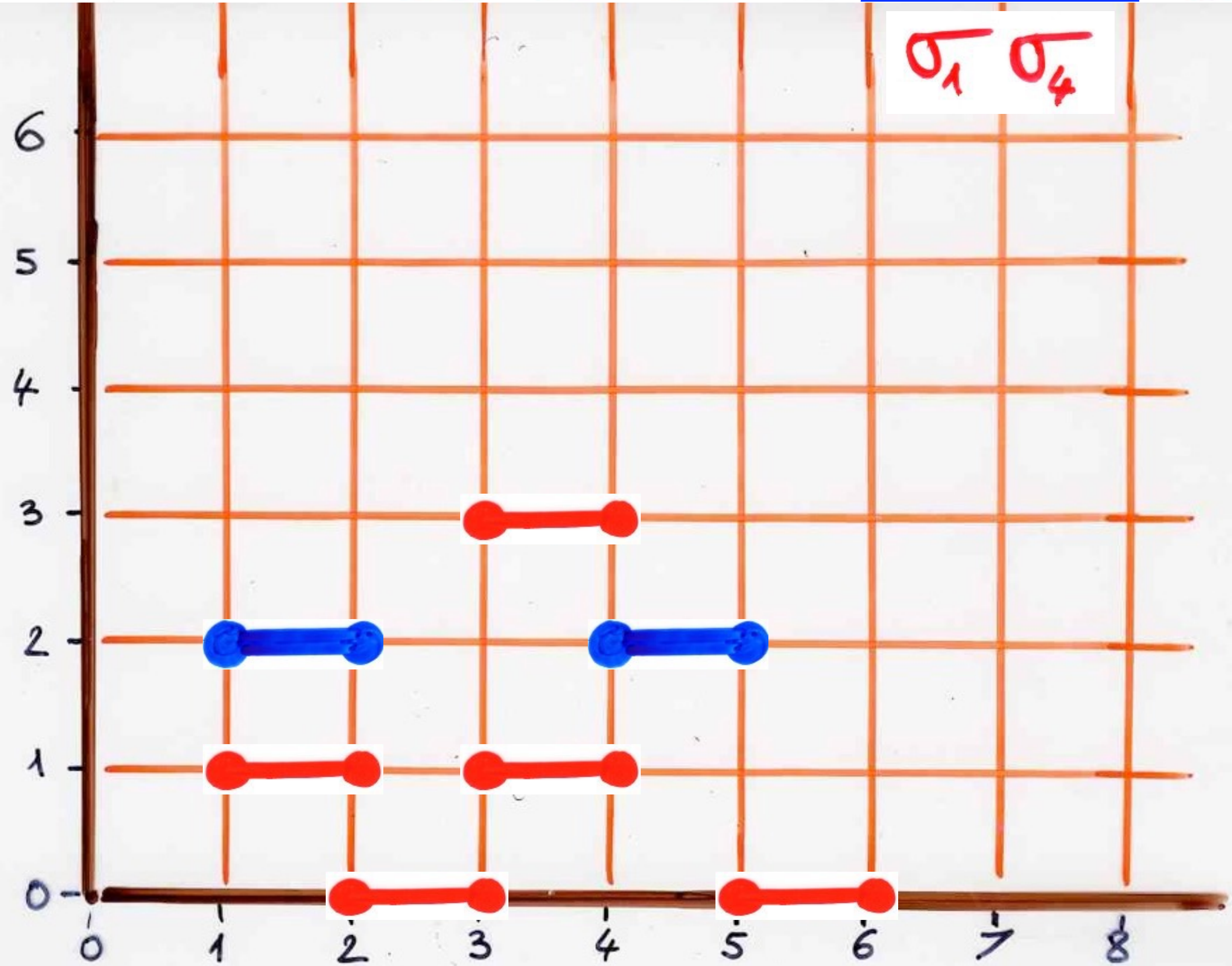

$$P \subseteq \mathrm{Heap}(P, \mathcal{C})$$

$$\alpha \longleftrightarrow \{(\alpha, 0)\}$$

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word heap

$$C = \overline{\mathcal{C}}$$

commutation relation

complementary of the dependency relation

Lemma 2

$$\varphi(u) = \varphi(v) \Rightarrow u \equiv_C v$$

proof of Lemma 2 with

Cartier -Foata normal form

Cartier-Foata normal form

Lemma Every element $[w] \in L(A, C)$ has a unique decomposition into blocks

$$[w] = [w_1]\,[w_2] \cdots [w_r]$$

- where each w_i is a word where the letters "commute" two by two ($y\,C\,z$)
- for every letter z of the $(j+1)^{th}$ block, there exist a letter y of the j^{th} block such that $\cancel{y\,C\,z}$ (i.e. does not "commute")

in particular all the letters of each w_i are distinct (C antireflexive)

Proof Let F_1 the set of letters y of w such that $w \equiv_C y\, v_1$

(i.e. applying commutations y can be put as the first letter)

- the letters of F_1 "commute" 2 by 2

$$w = \cdots y \cdots z \cdots \qquad y, z \in F_1$$

$[w_1]$ is the equivalence class of the product of the letters of F_1

- Let F_2 be the set of letters y of w such that $w \equiv_C w_1\, y\, v_2$

$[w_2]$ is the equivalence class of the product of the letters of F_2, and these letters "commute" 2 by 2

- .... etc... for $F_2, \ldots, F_r$

<u>example</u> $w = \sigma_2\, \sigma_3\, \sigma_5\, \sigma_1\, \sigma_4\, \sigma_1\, \sigma_3$

$$\equiv \sigma_2\, \sigma_5 \,/\, \sigma_3\, \sigma_1\, \sigma_4\, \sigma_1\, \sigma_3$$

$$\equiv \sigma_2\, \sigma_5 \,/\, \sigma_3\, \sigma_1 \,/\, \sigma_4\, \sigma_1\, \sigma_3$$

$$\equiv \sigma_2\, \sigma_5 \,/\, \sigma_1\, \sigma_3 \,/\, \sigma_1\, \sigma_4 \,/\, \sigma_3$$

<u>unicity</u> ... obvious

Lemma

Let $[w] = [w_1] \dots [w_r]$

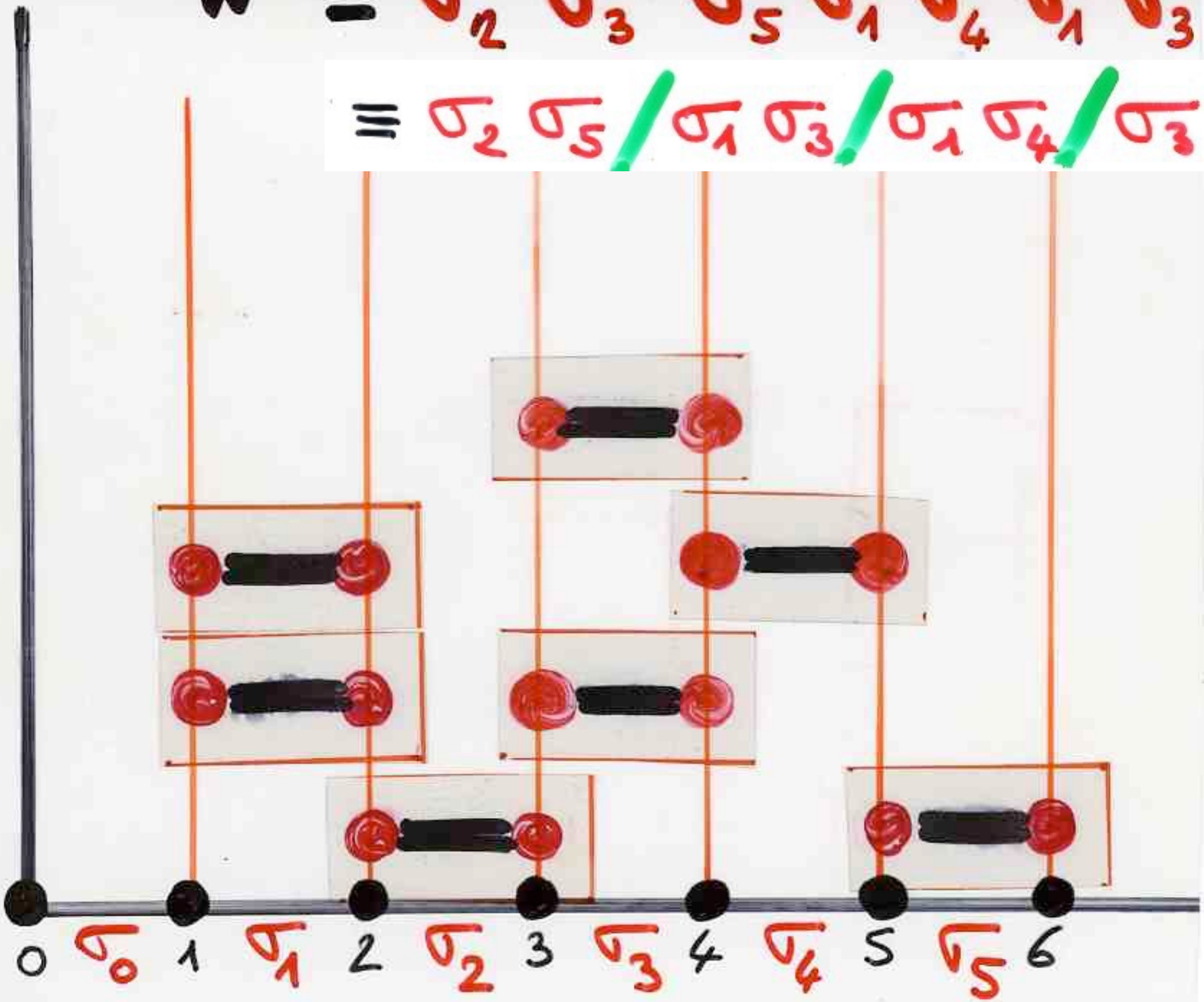
be the Cartier-Foata normal form

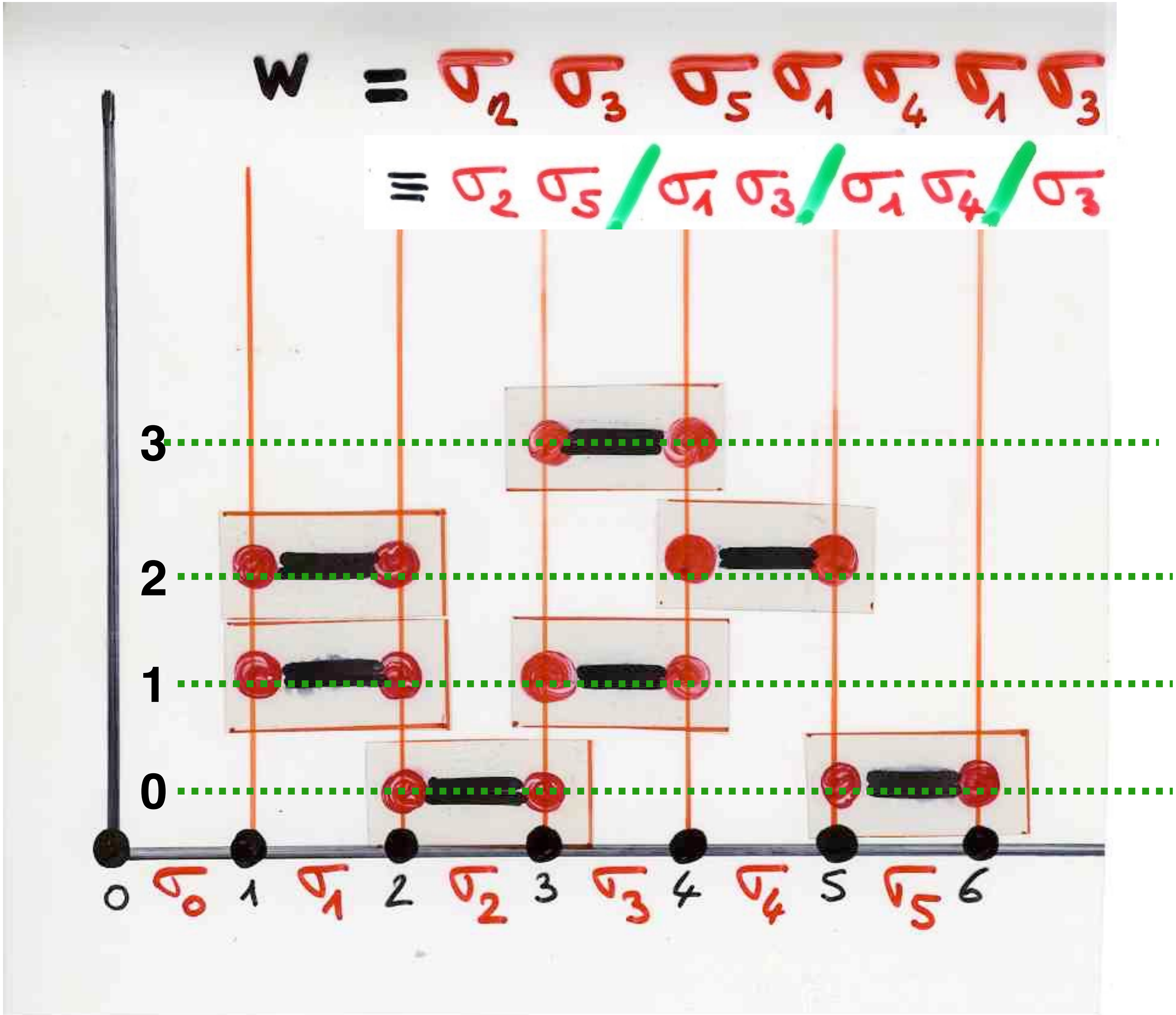
of $[w] \in L(A,C)$.

Each block $[w_i]$ corresponds to the elements of the heap $\varphi(w)$ located at level $i-1$

$$W = \sigma_2\,\sigma_3\,\sigma_5\,\sigma_1\,\sigma_4\,\sigma_1\,\sigma_3$$

$$\equiv \sigma_2\,\sigma_5 \,/\, \sigma_1\,\sigma_3 \,/\, \sigma_1\,\sigma_4 \,/\, \sigma_3$$





W = σ2 σ3 σ5 σ1 σ4 σ1 σ3
≡ σ2 σ5 / σ1 σ3 / σ1 σ4 / σ3
3
2
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0
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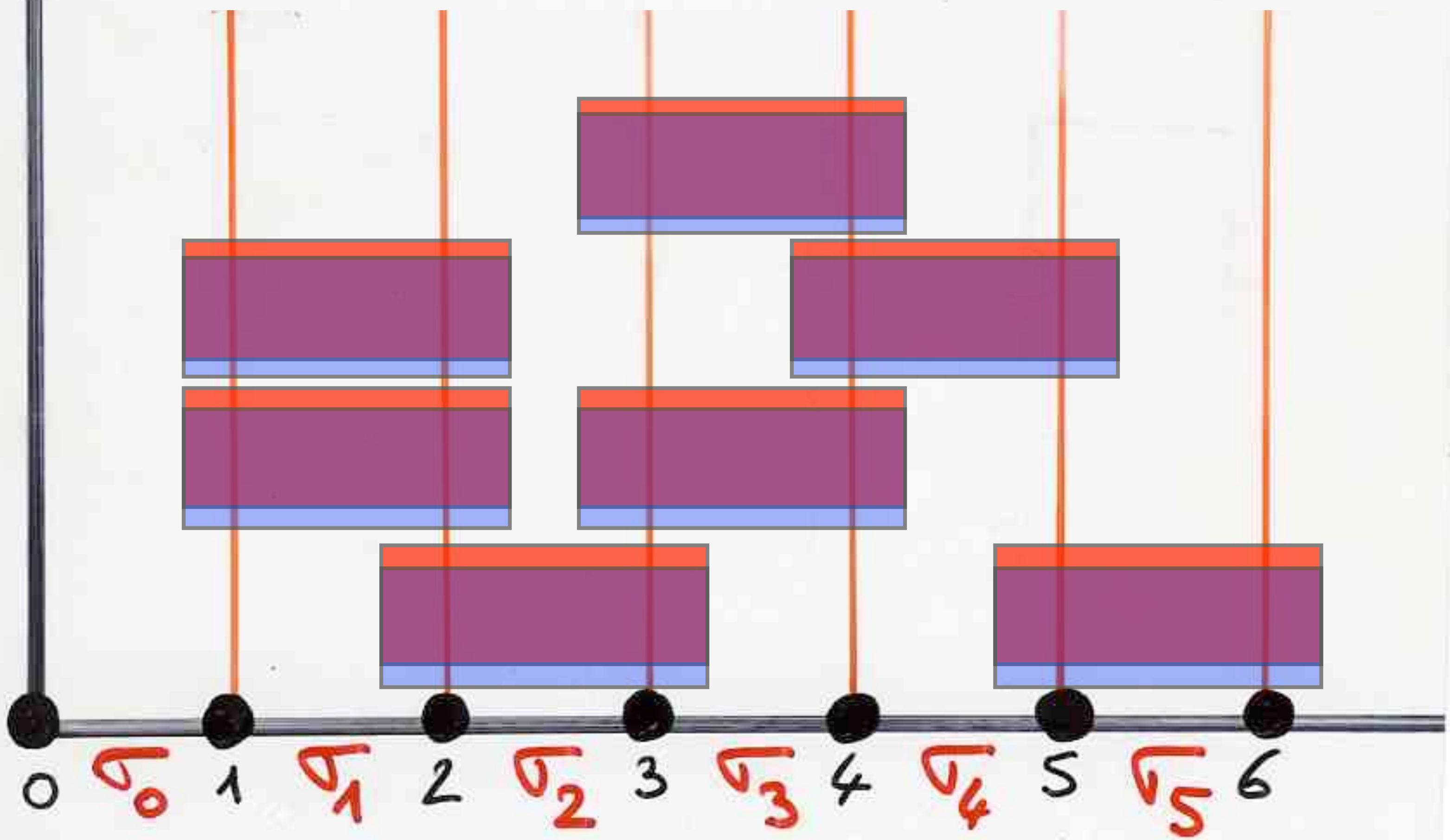
$$\varphi(u) = \varphi(v) \Rightarrow u \equiv_C v$$

Proof of Lemma 2

$$W = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$$

$$\equiv \sigma_2\ \sigma_5 \,/\, \sigma_1\ \sigma_3 \,/\, \sigma_1\ \sigma_4 \,/\, \sigma_3$$

$$W = \sigma_5\ \sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_4\ \sigma_3$$



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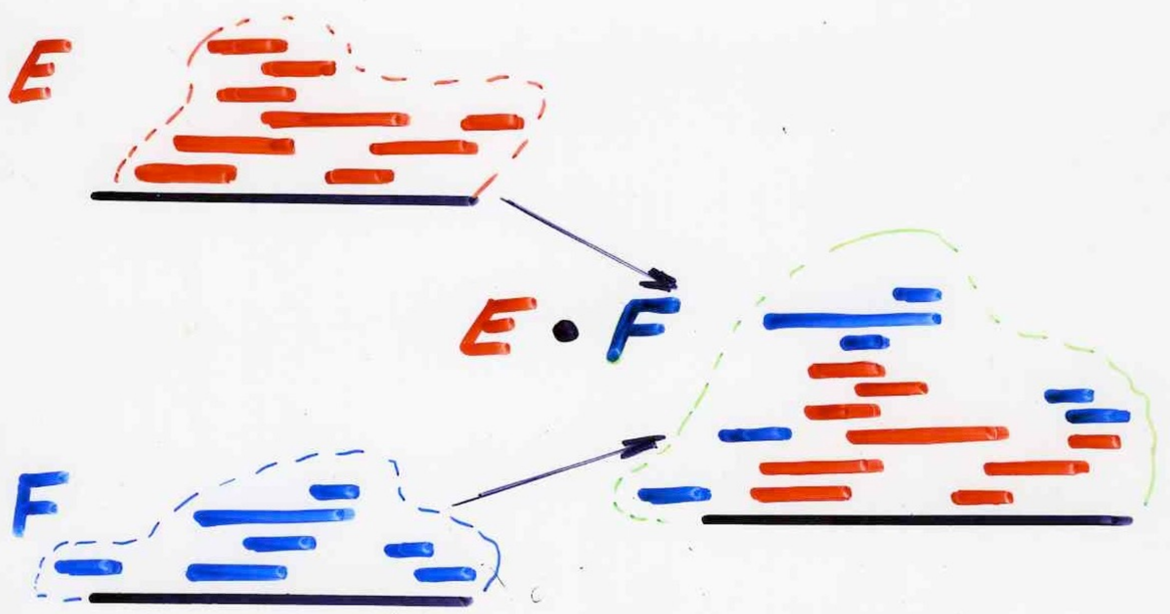
$[w]$ equivalence class of the word $w \in A^*$

$A^*/\equiv_C$ • product in the commutation monoid

$$[u]\cdot[v] = [uv]$$

independant of the choices of representants u and v



$$E \cdot F = E_{\sim}(E \cup T(F))$$

exercise using Cartier-Foata normal form prove that the commutation monoid $L(A, C)$ is simplificable, i.e.

$$uv = uv' \Rightarrow v = v'$$

$$uv = u'v \Rightarrow u = u'$$

lexicographic normal form

(« Knuth »)

lexicographic normal form

(Knuth)

$H(\mathcal{P}, \mathcal{E})$

total order on $\mathcal{P}$

set of basic pieces

minimal letter of a class $[w]$

$$[w] = [y\, y_1]$$

Cartier-Foata normal form

$$[w] = [w_1][w_2] \cdots [w_r]$$

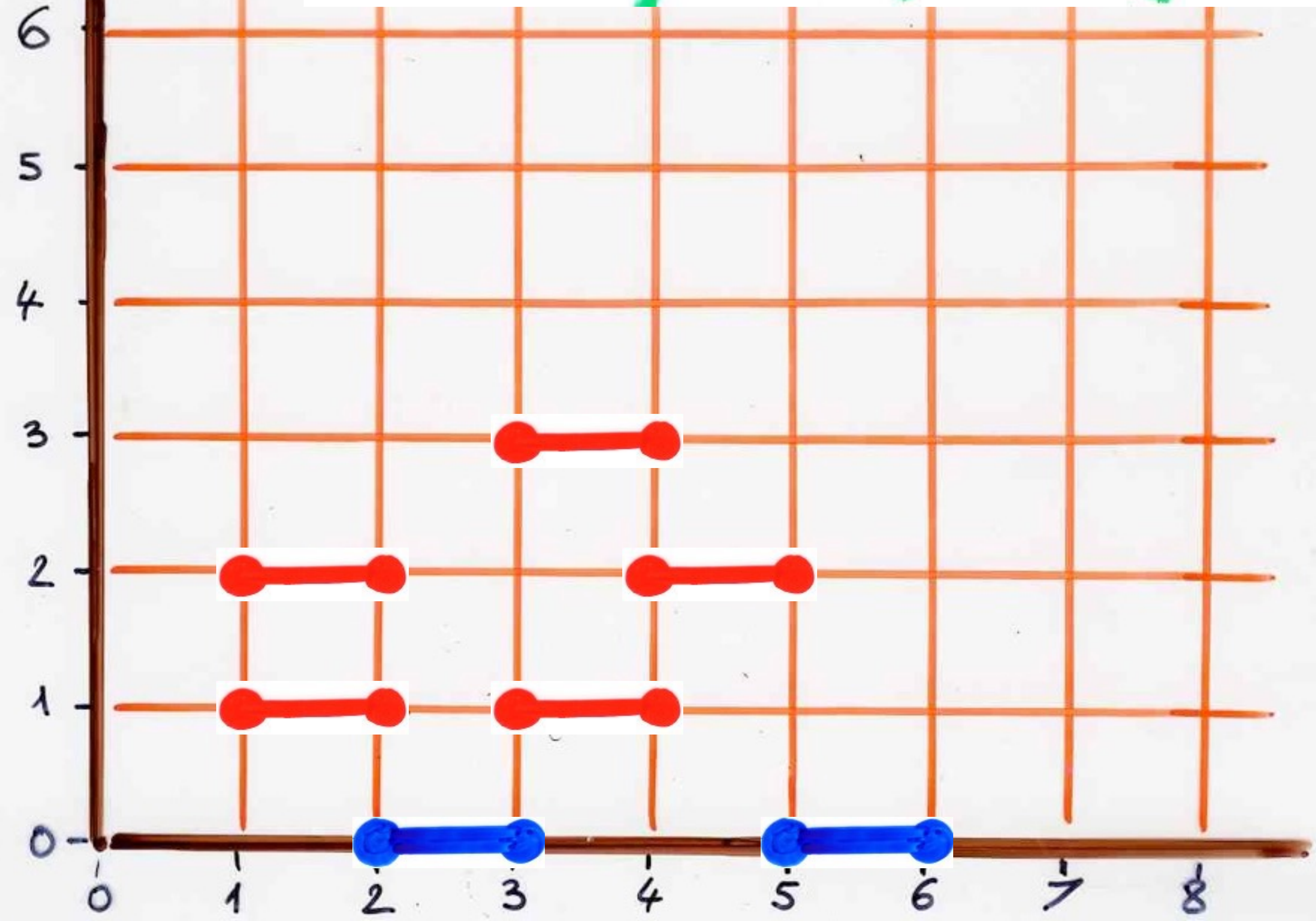
minimal letter : any letter of w_1

$\updownarrow$

piece of the associated heap at level 0

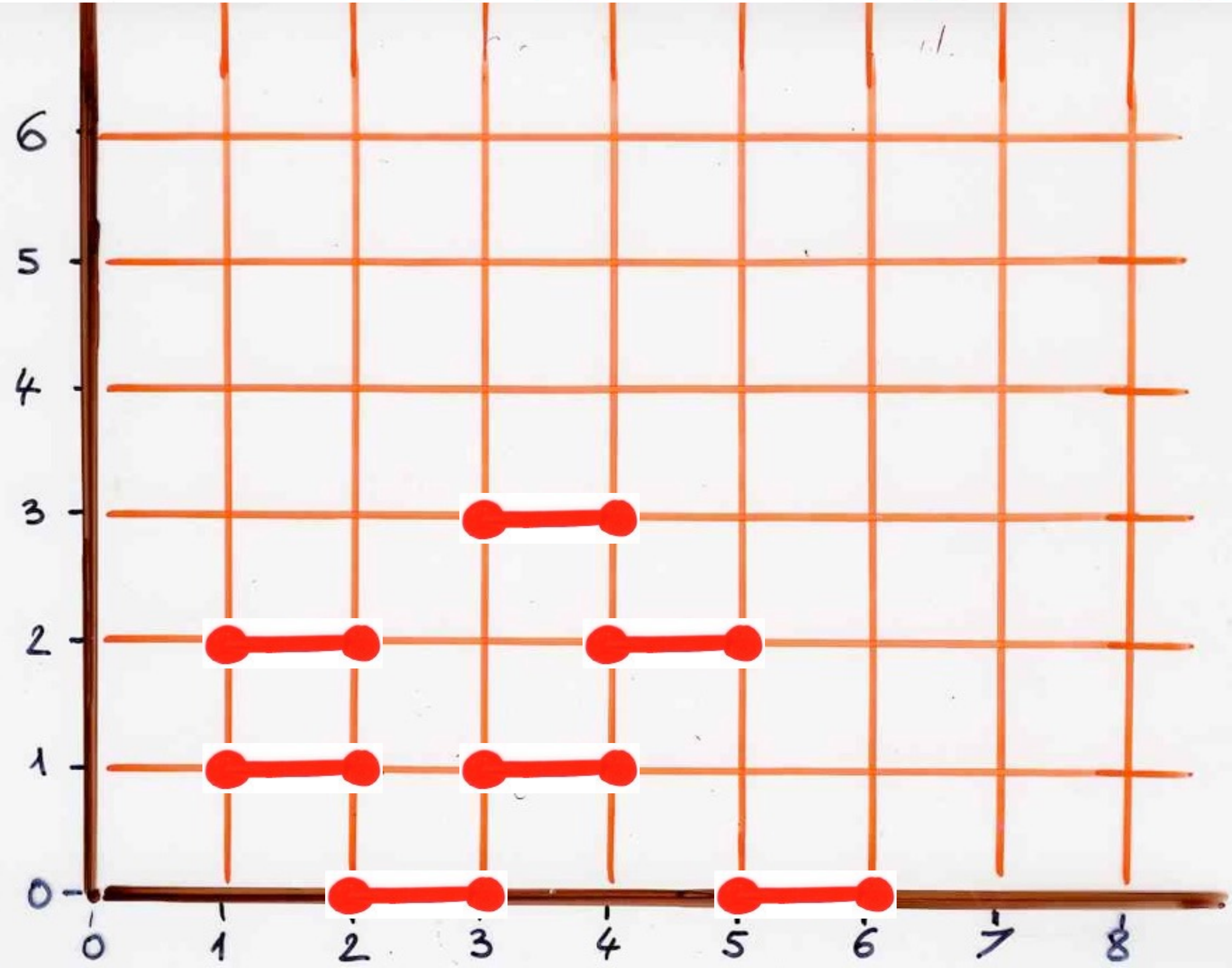
example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$

$\equiv \sigma_2 \sigma_5 / \sigma_1 \sigma_3 / \sigma_1 \sigma_4 / \sigma_3$



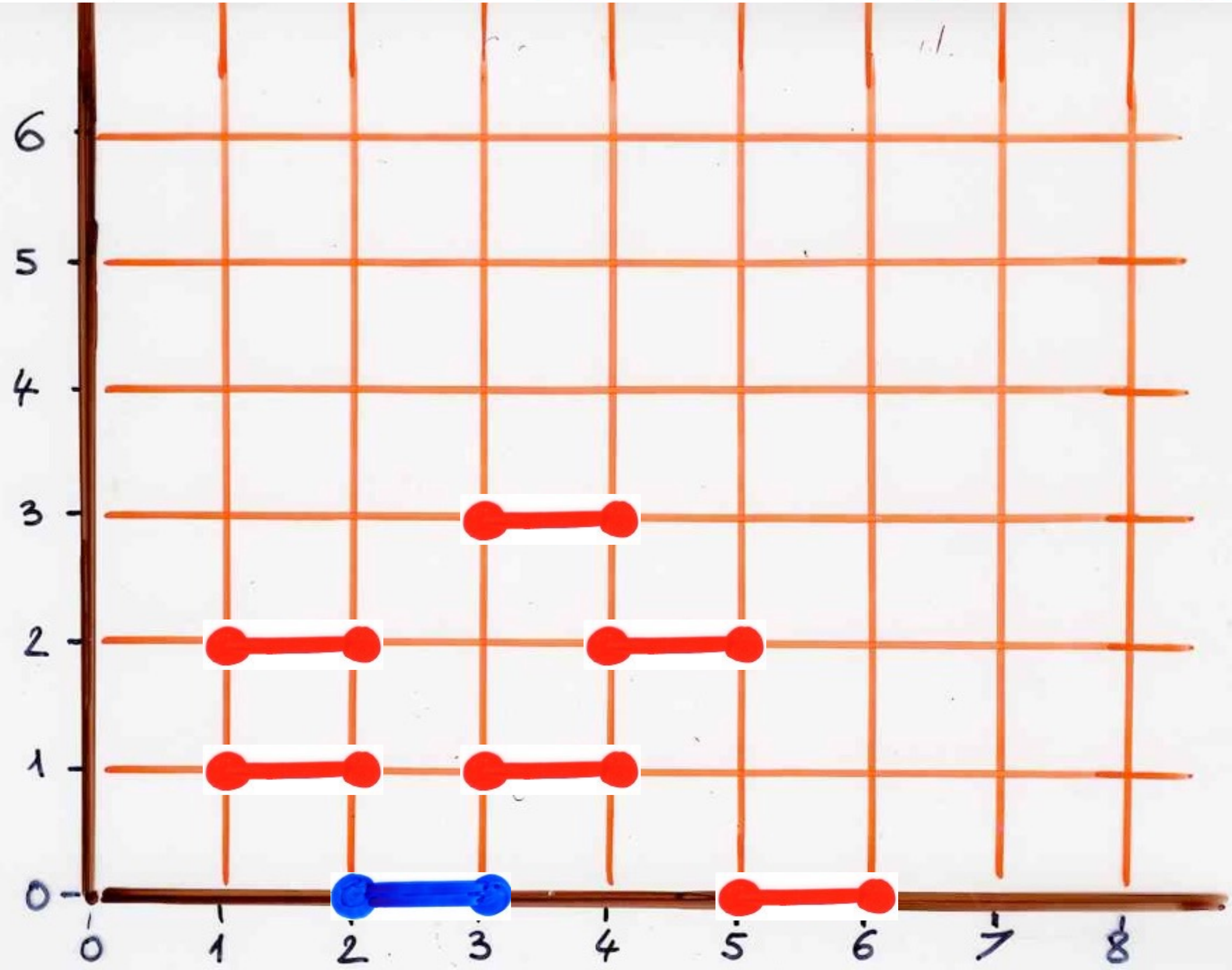
Lemma In the commutation class $[w]$, the smallest word $v = v_1 \cdots v_n$ for the lexicographic order is obtained by taking v_1 the smallest minimal letter of $[w]$, $w = v_1 w_2$, then v_2 the smallest minimal letter of $[w_2]$, then ....

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



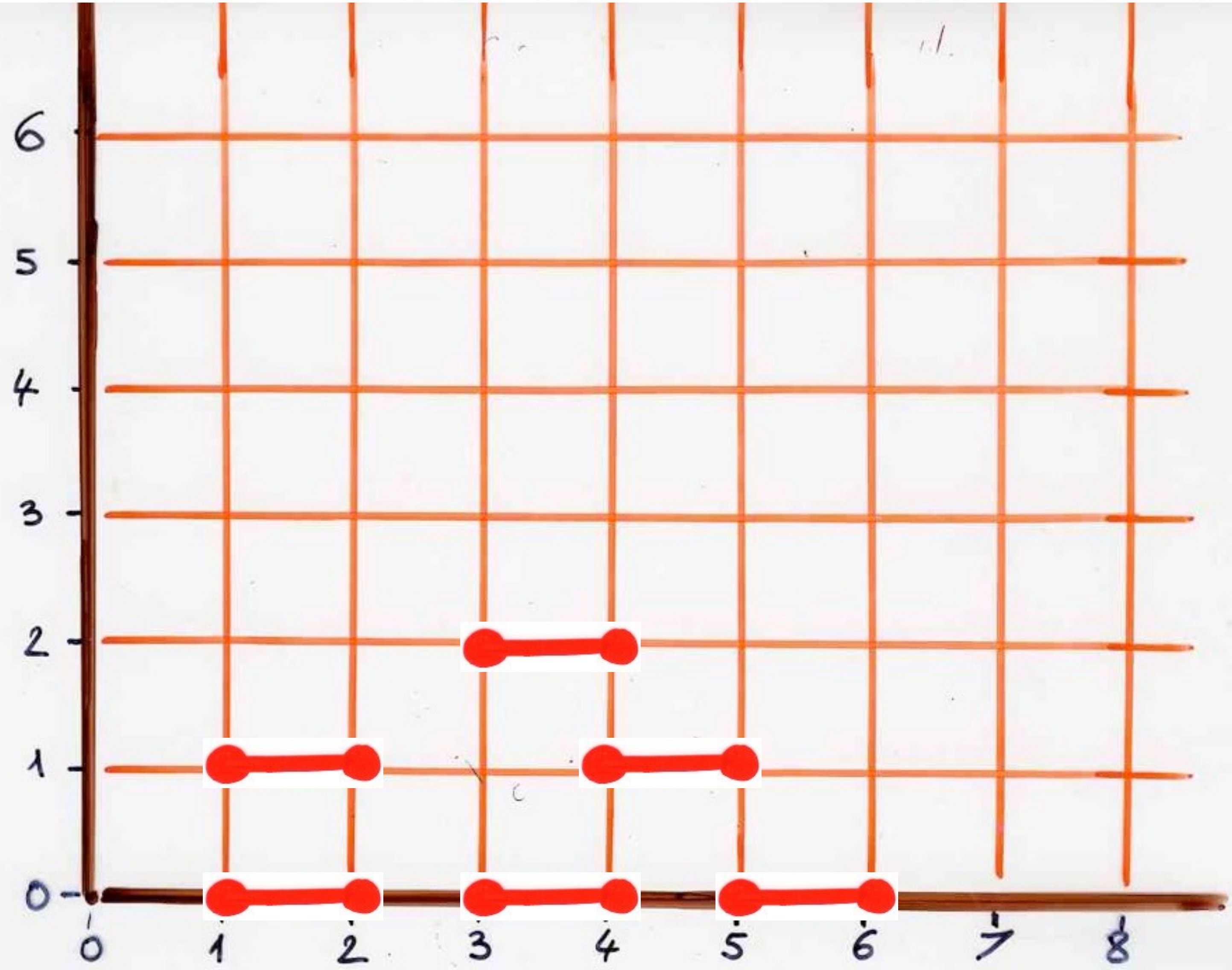
$\sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_5$

example $w = \sigma_2\, \sigma_3\, \sigma_5\, \sigma_1\, \sigma_4\, \sigma_1\, \sigma_3$



σ_2

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



σ_2

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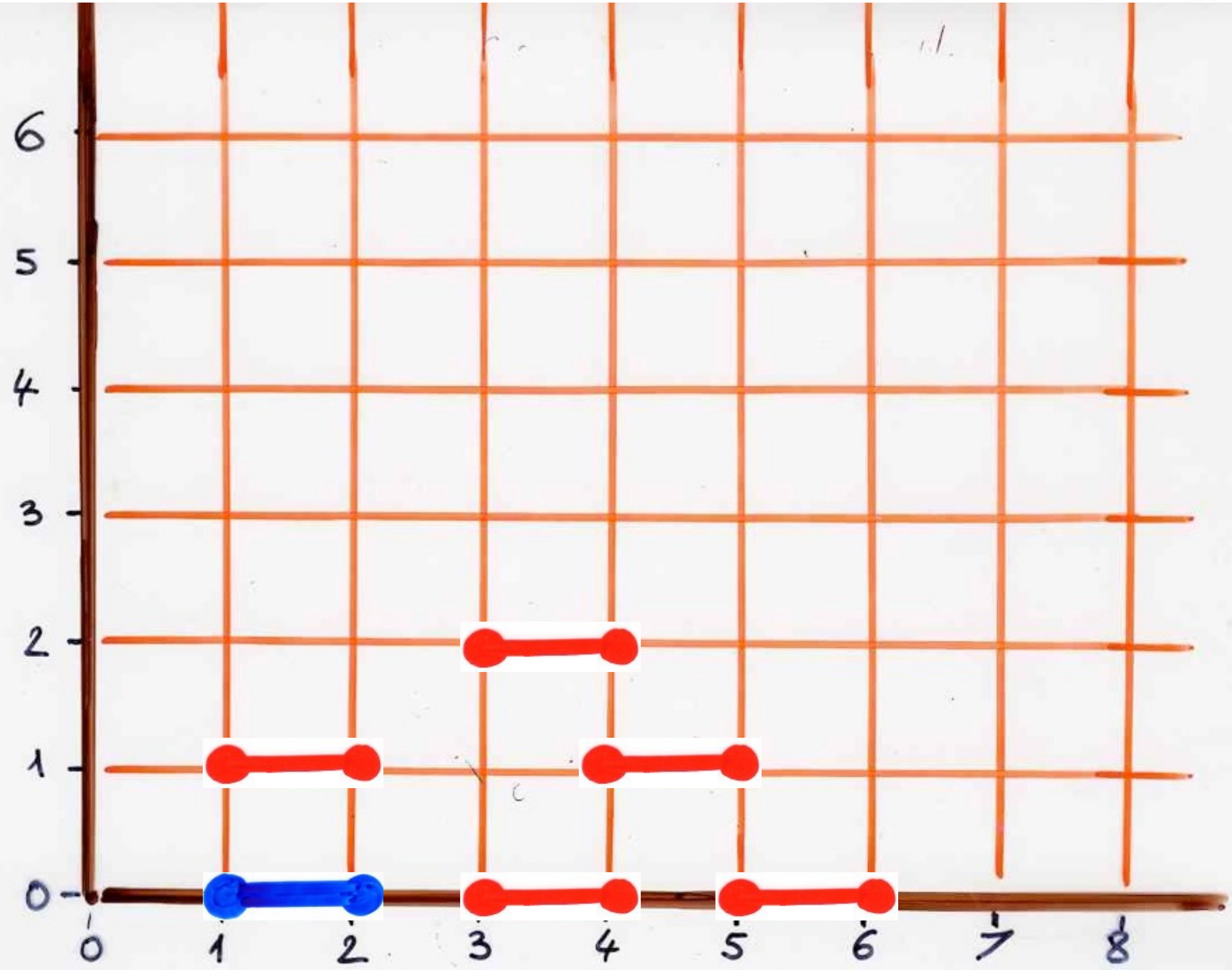
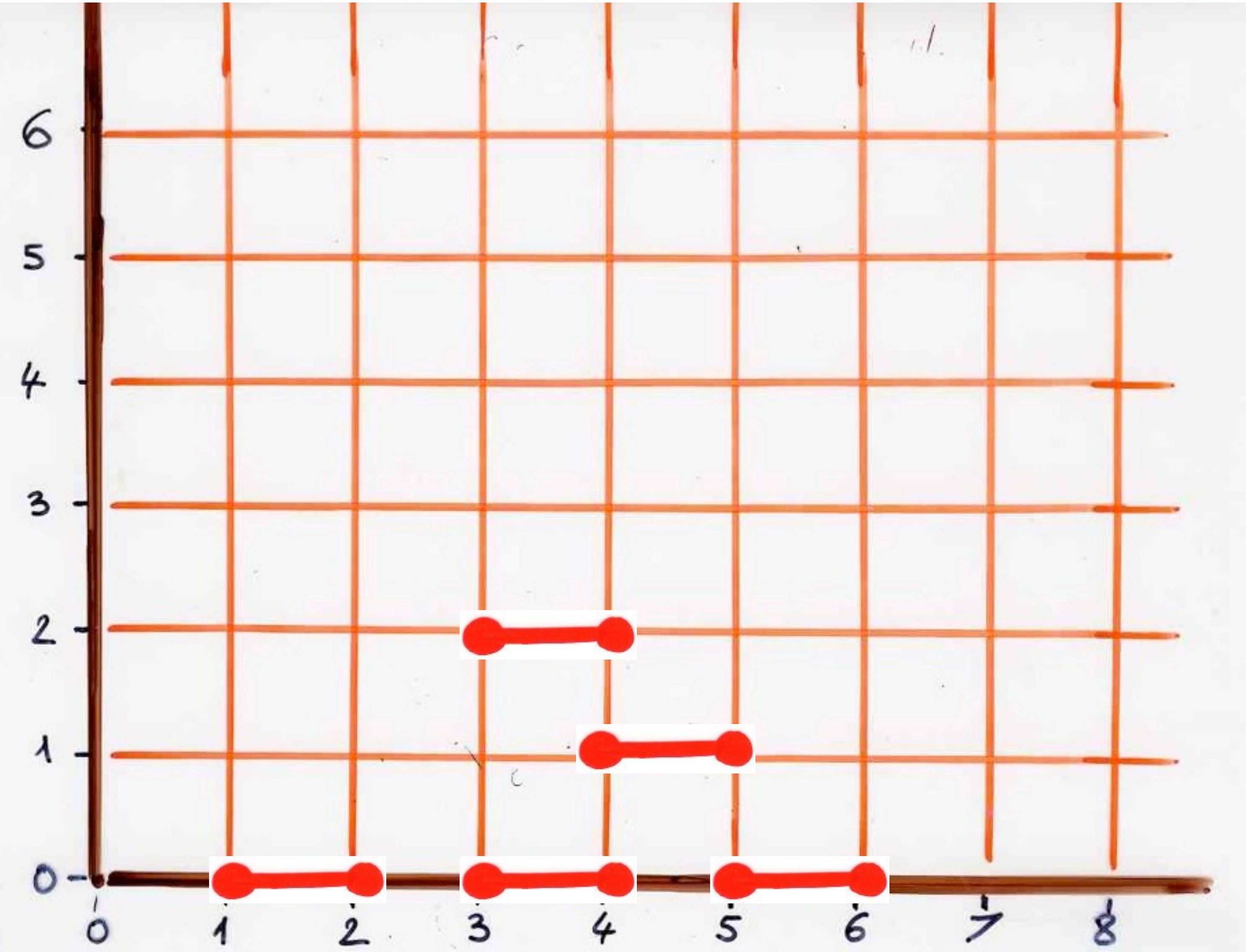


$\sigma_2 \sigma_1$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



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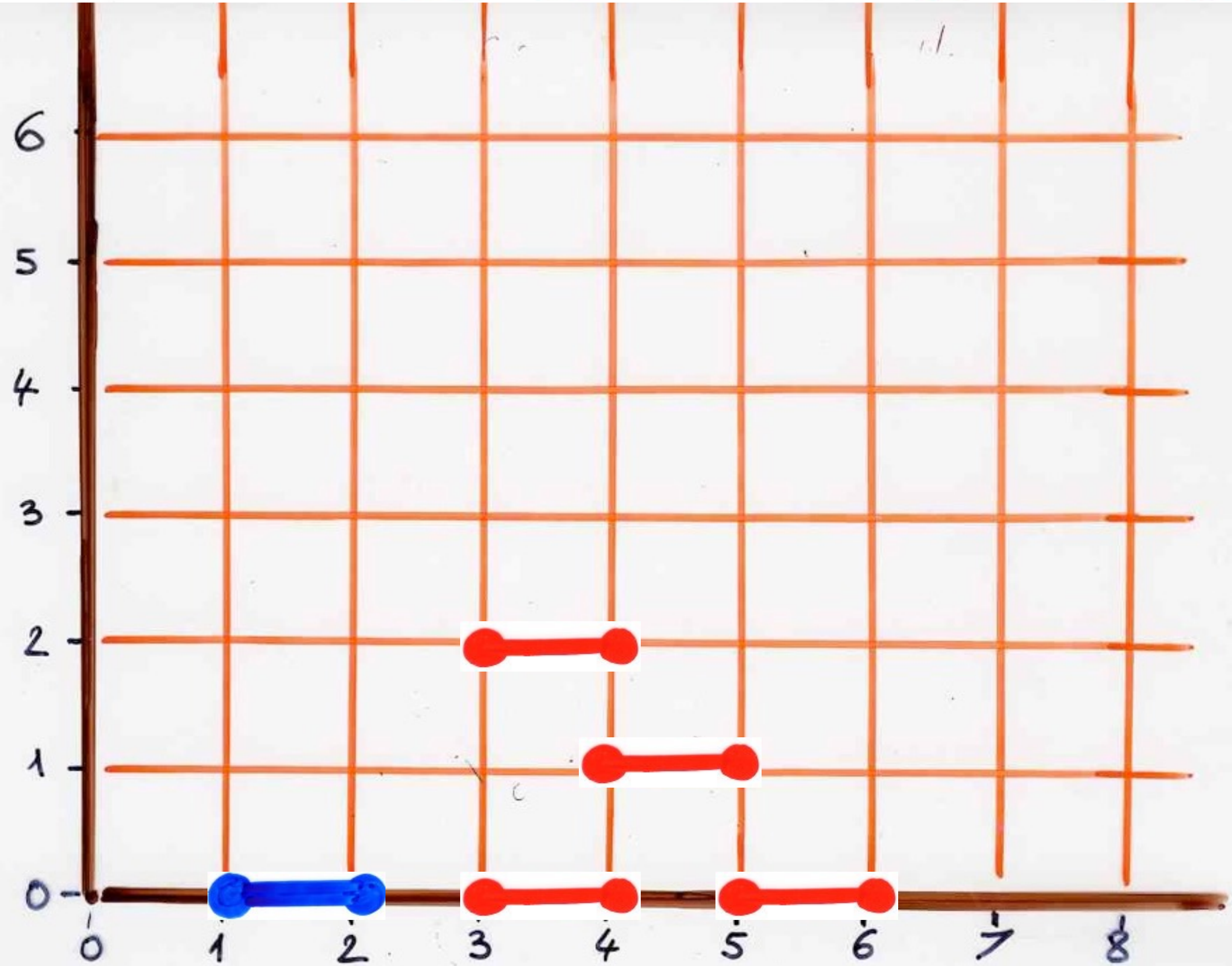
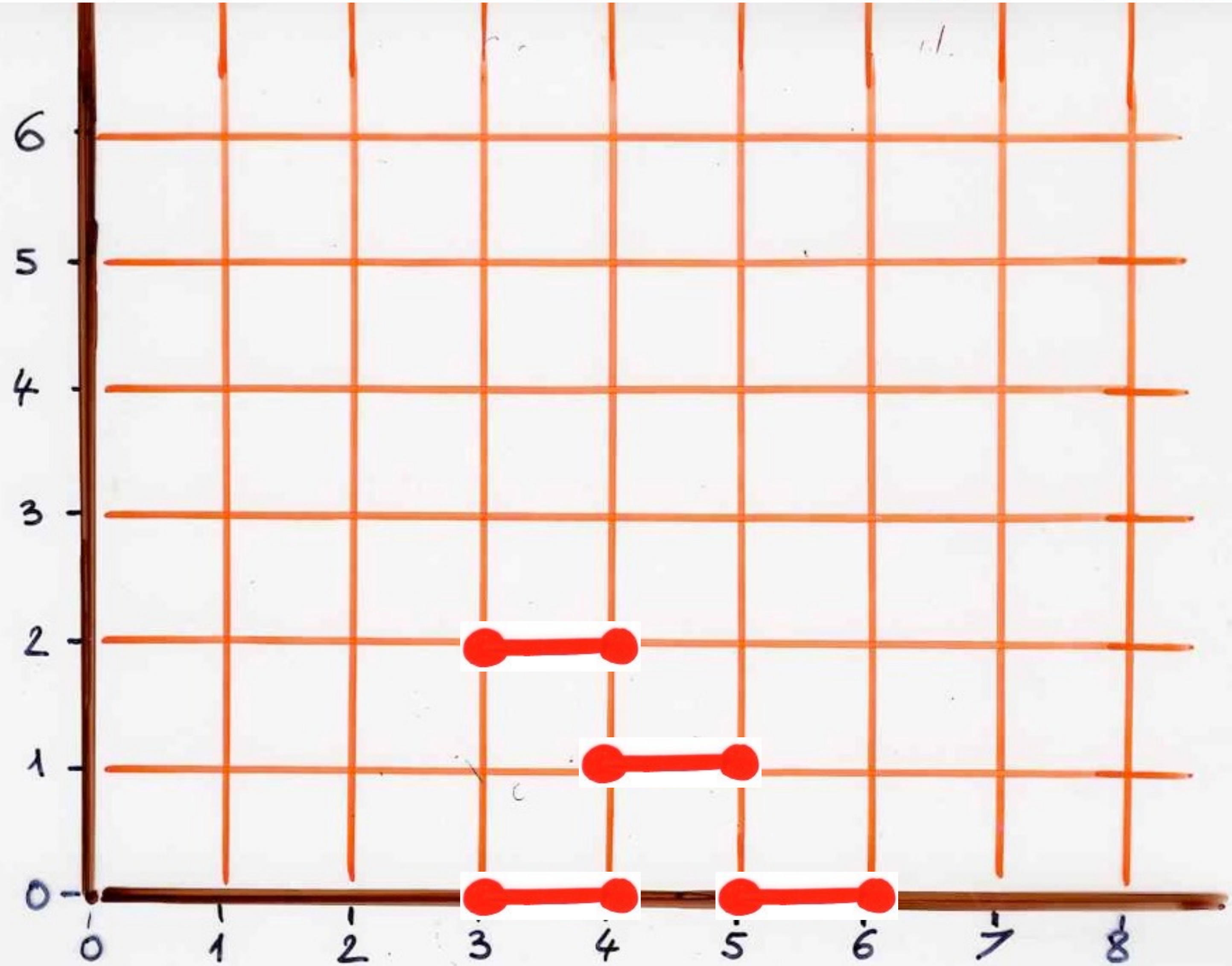


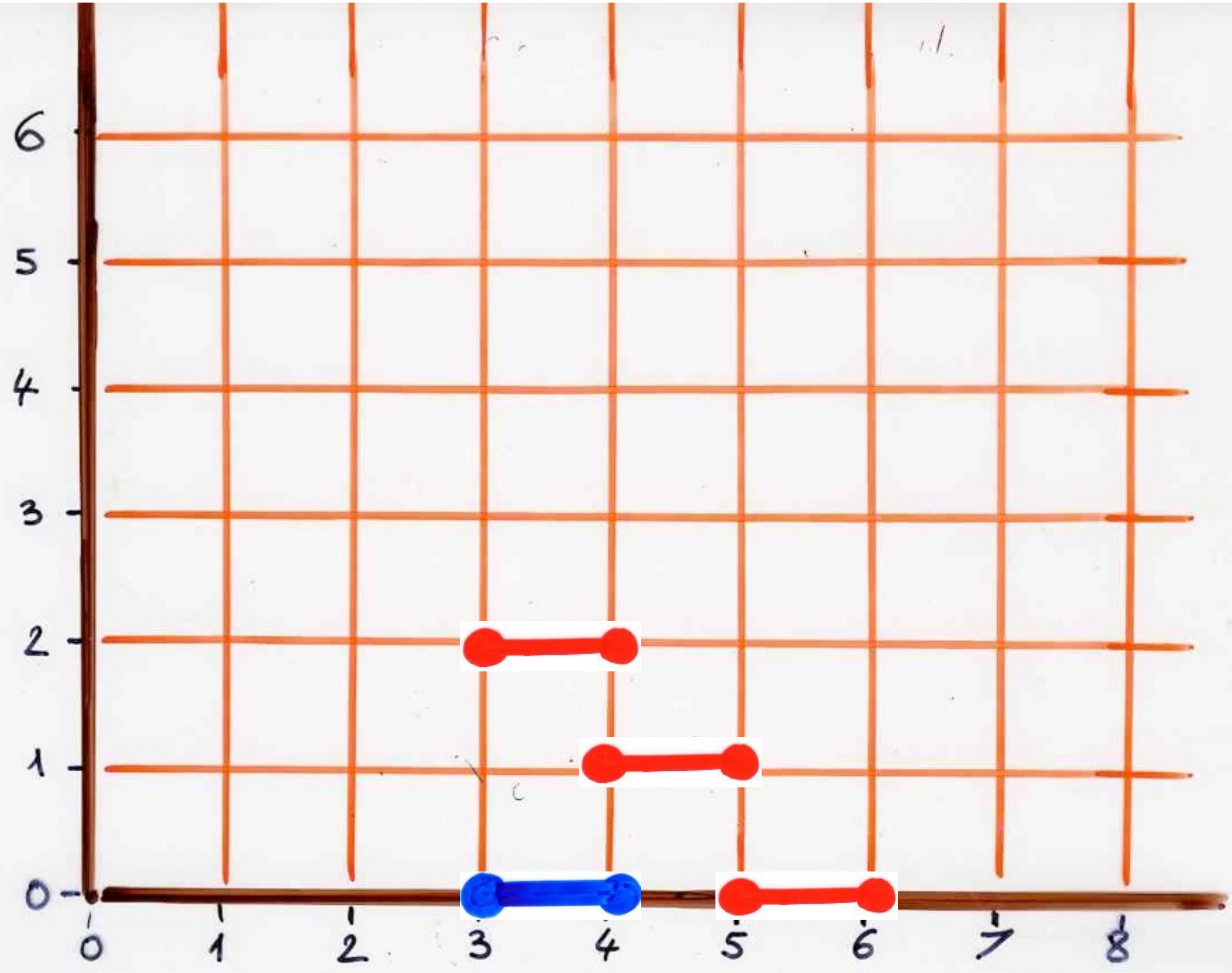
$\sigma_2\ \sigma_1\ \sigma_1$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



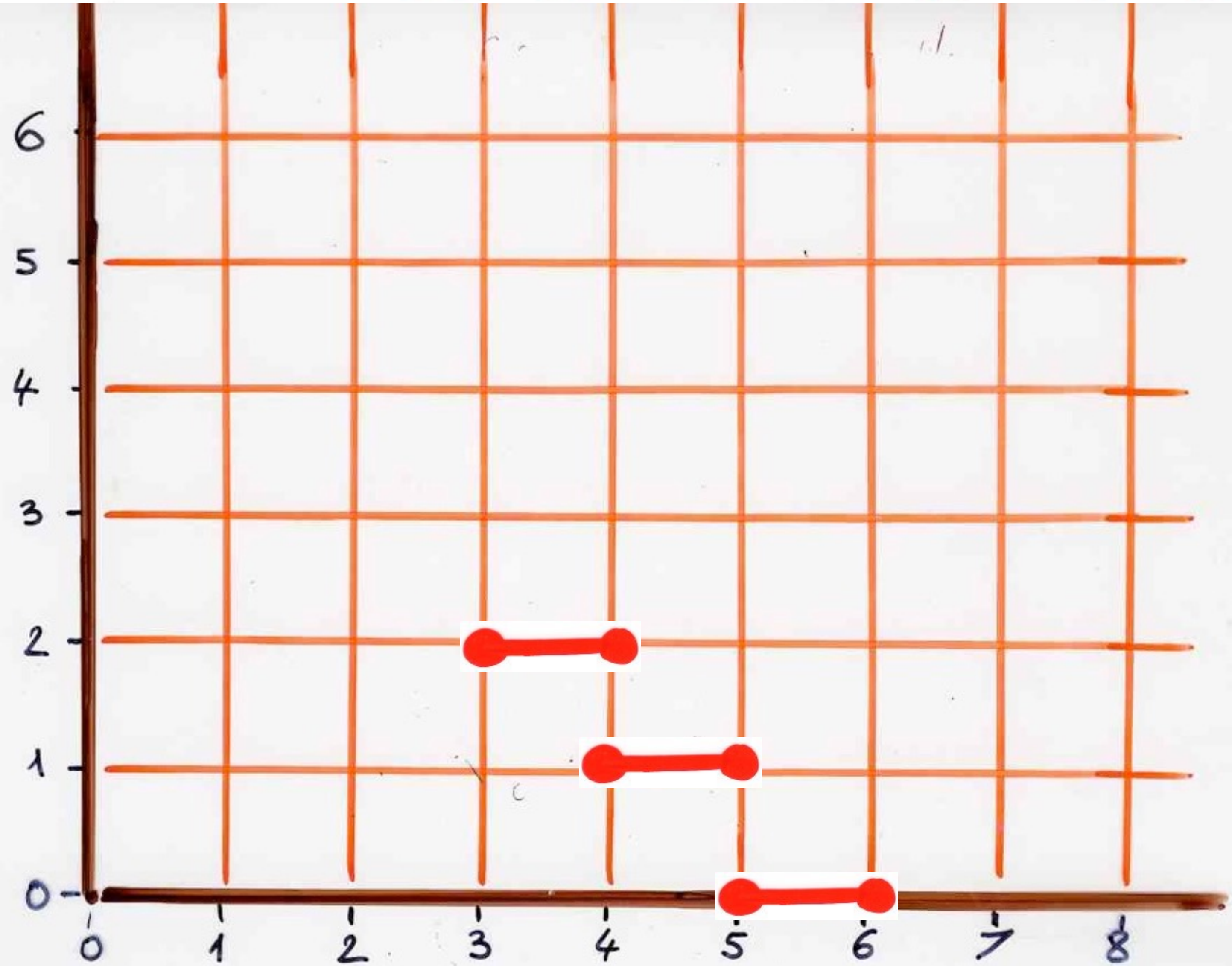
$\sigma_2 \sigma_1 \sigma_1$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



$\sigma_2 \sigma_1 \sigma_1 \sigma_3$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3$

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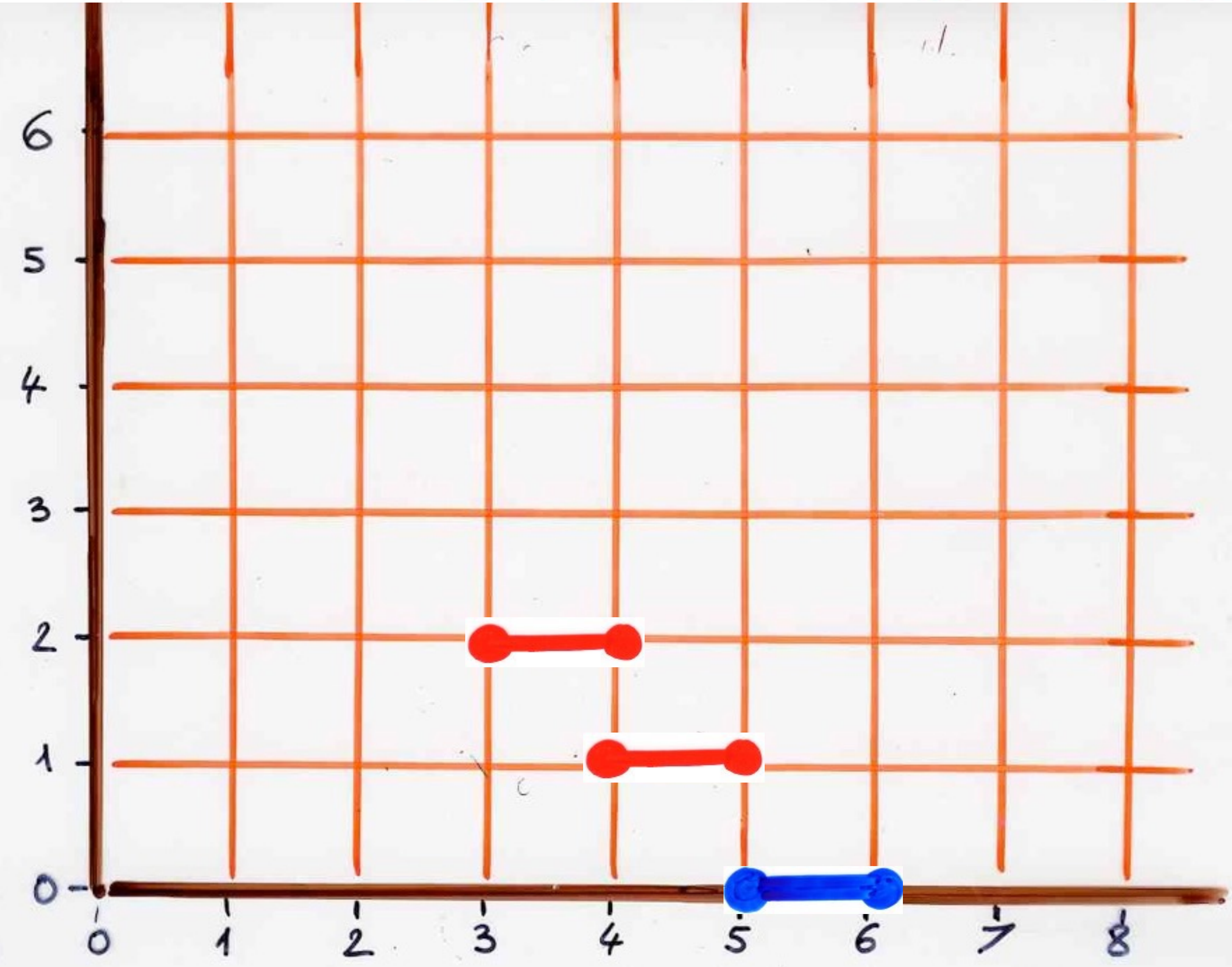
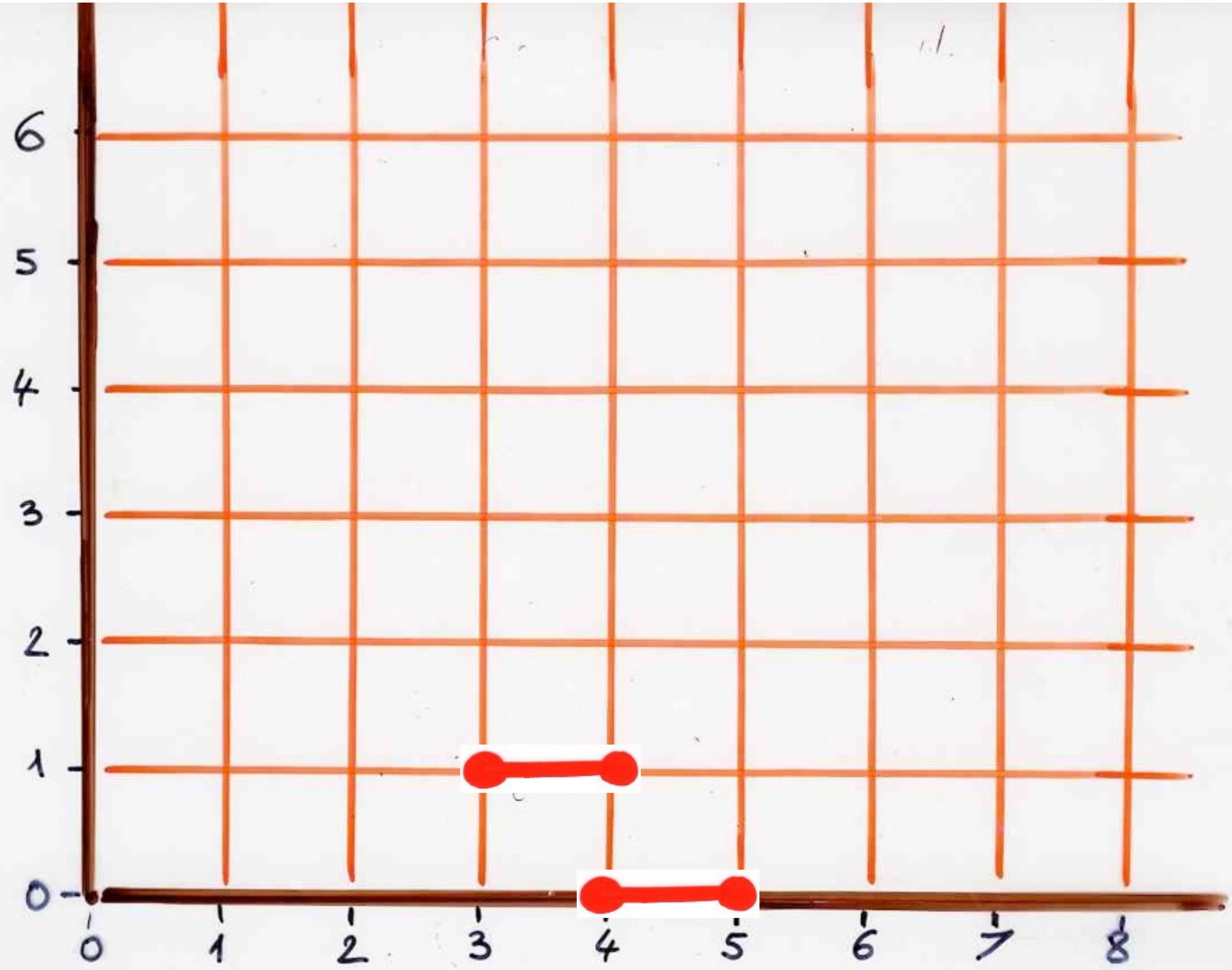


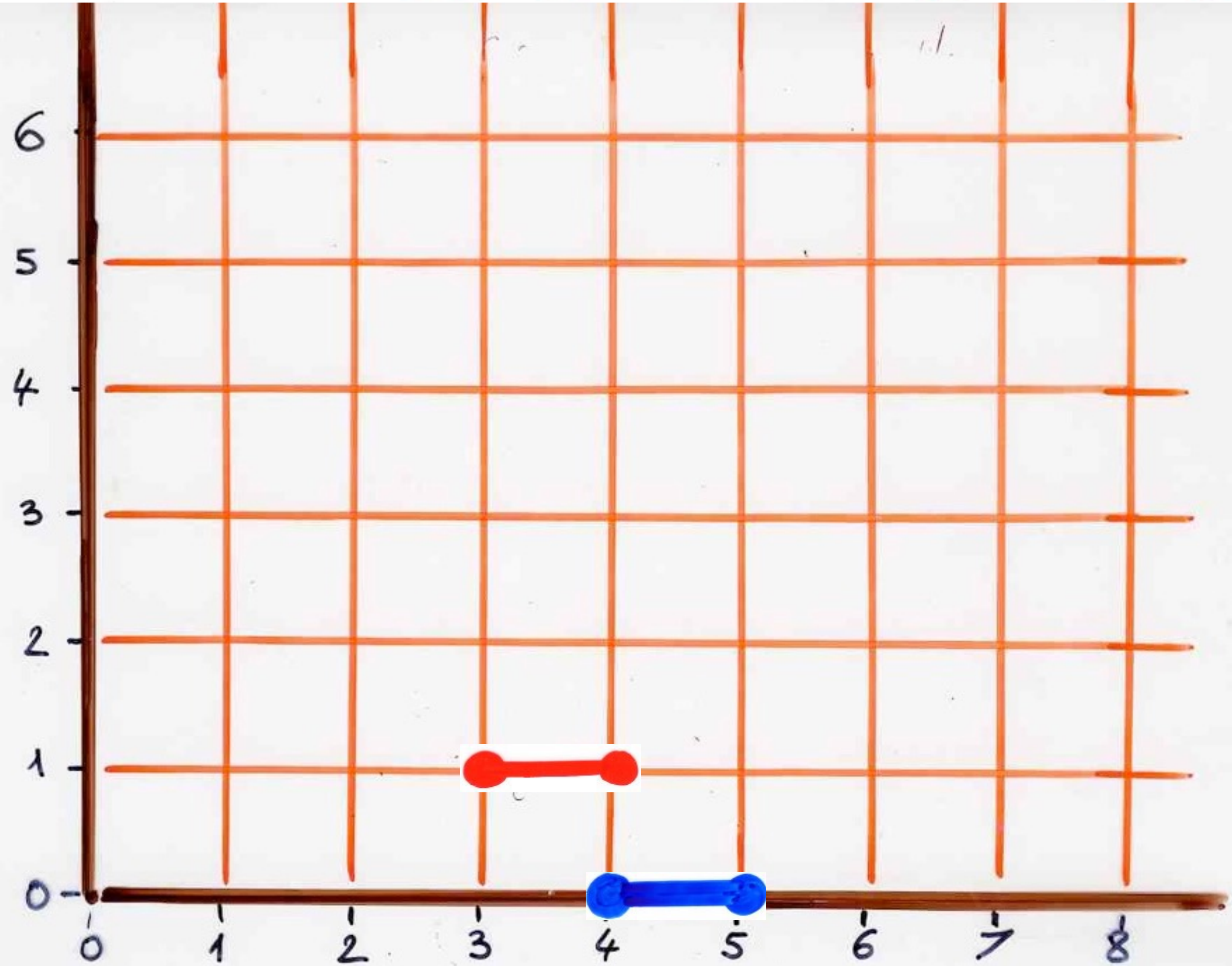
$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



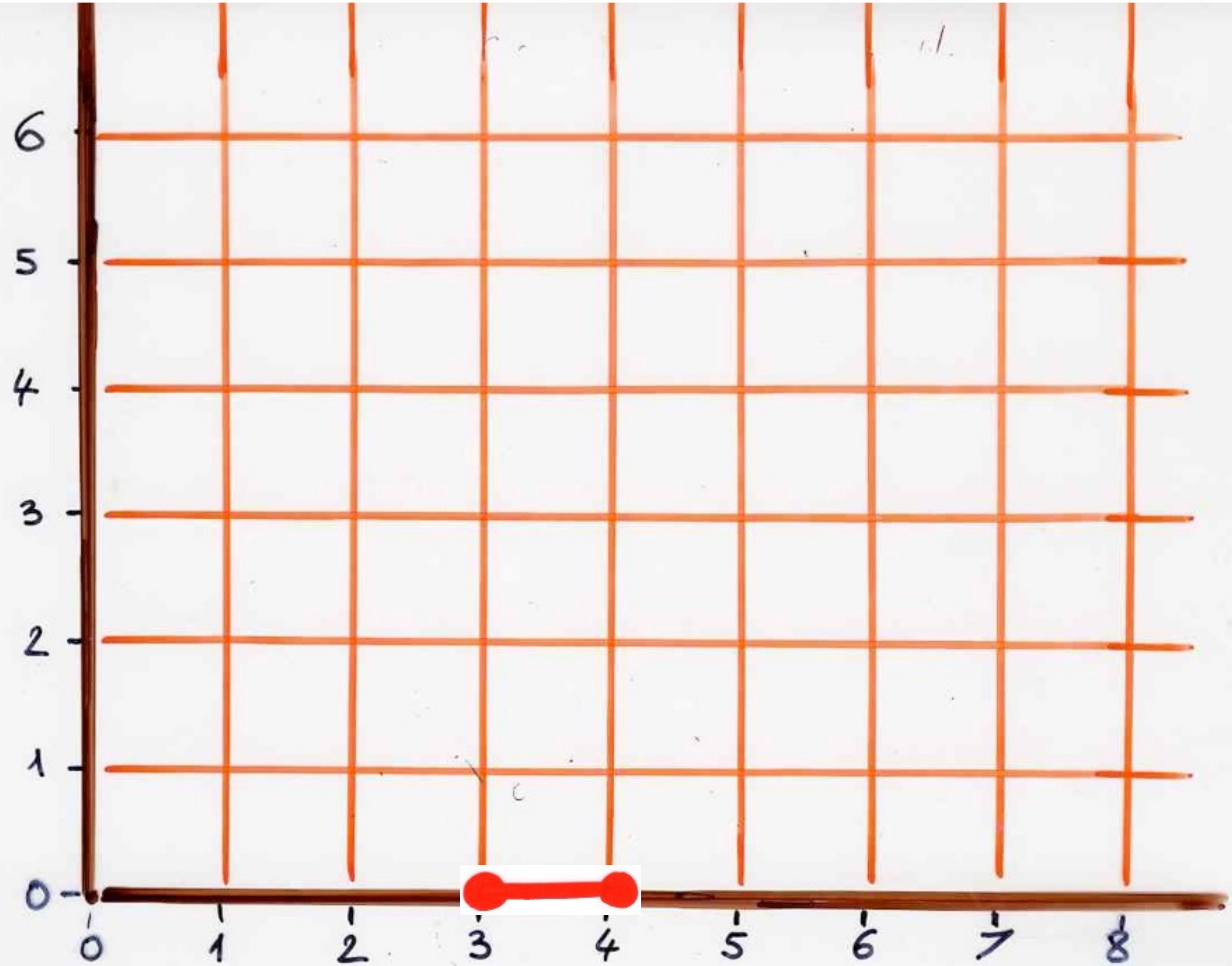
$\sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_5$

example $\quad w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



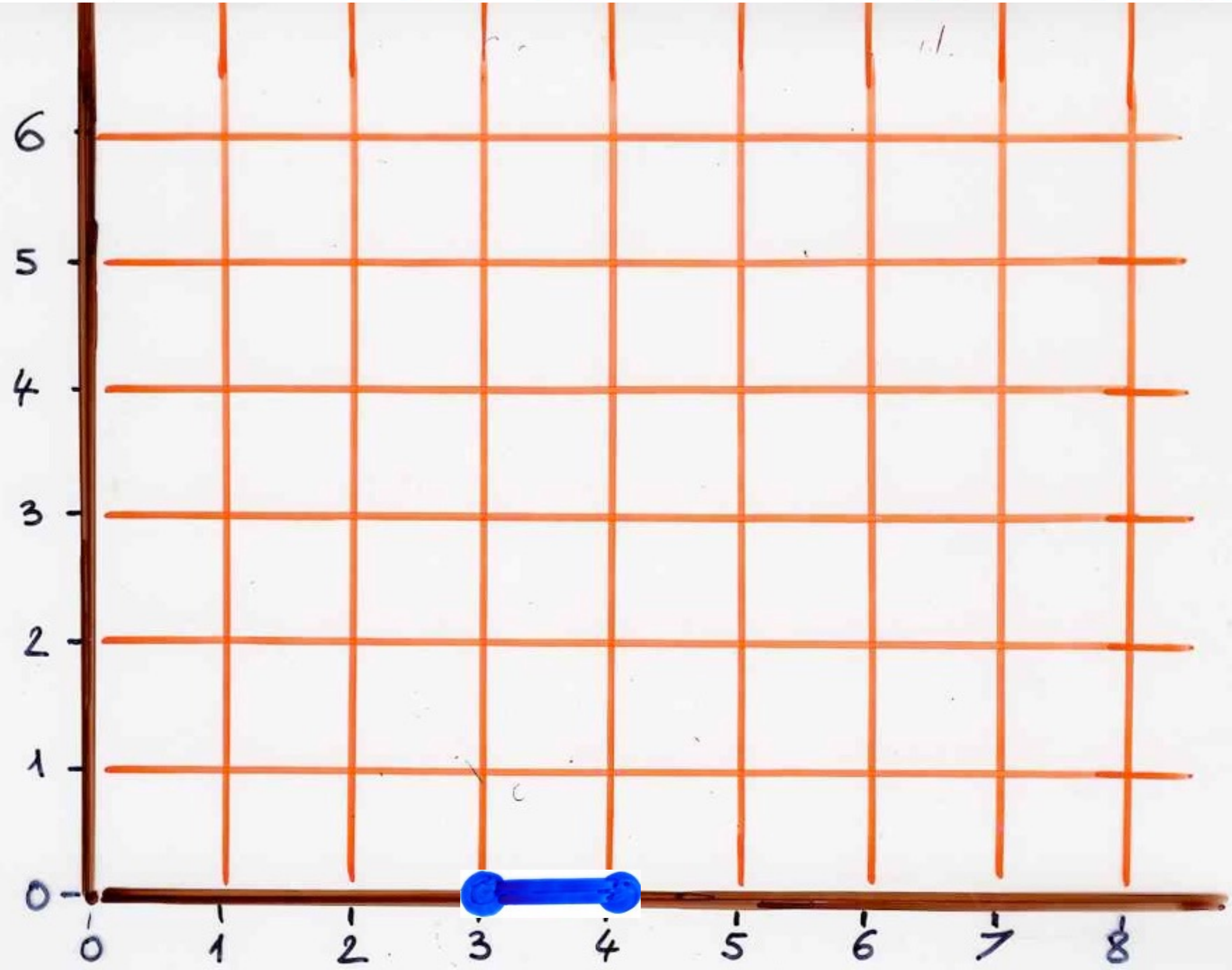
$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5\ \sigma_4$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



$\sigma_2 \sigma_1 \sigma_1 \sigma_3 \sigma_5 \sigma_4$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5\ \sigma_4\ \sigma_3$

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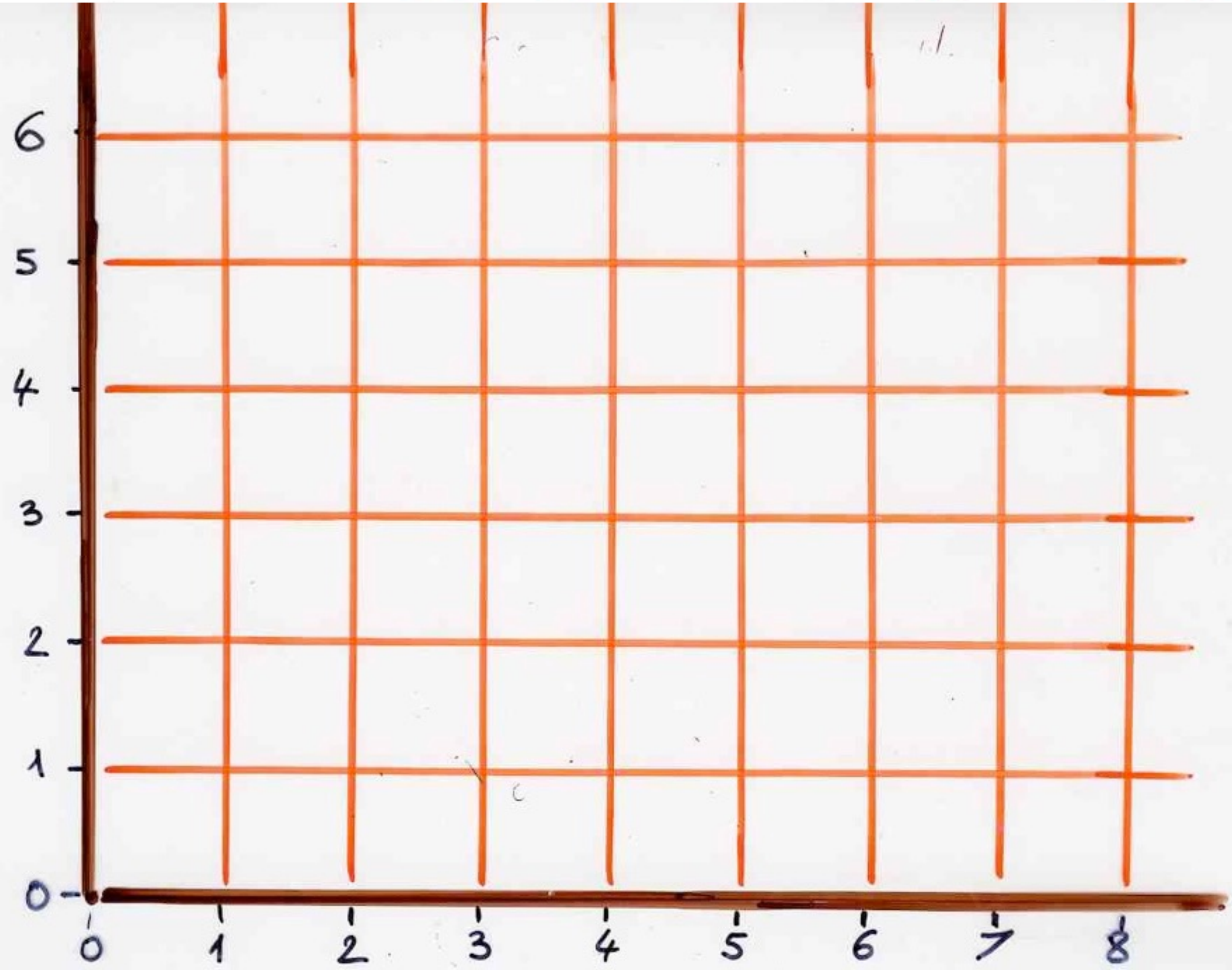
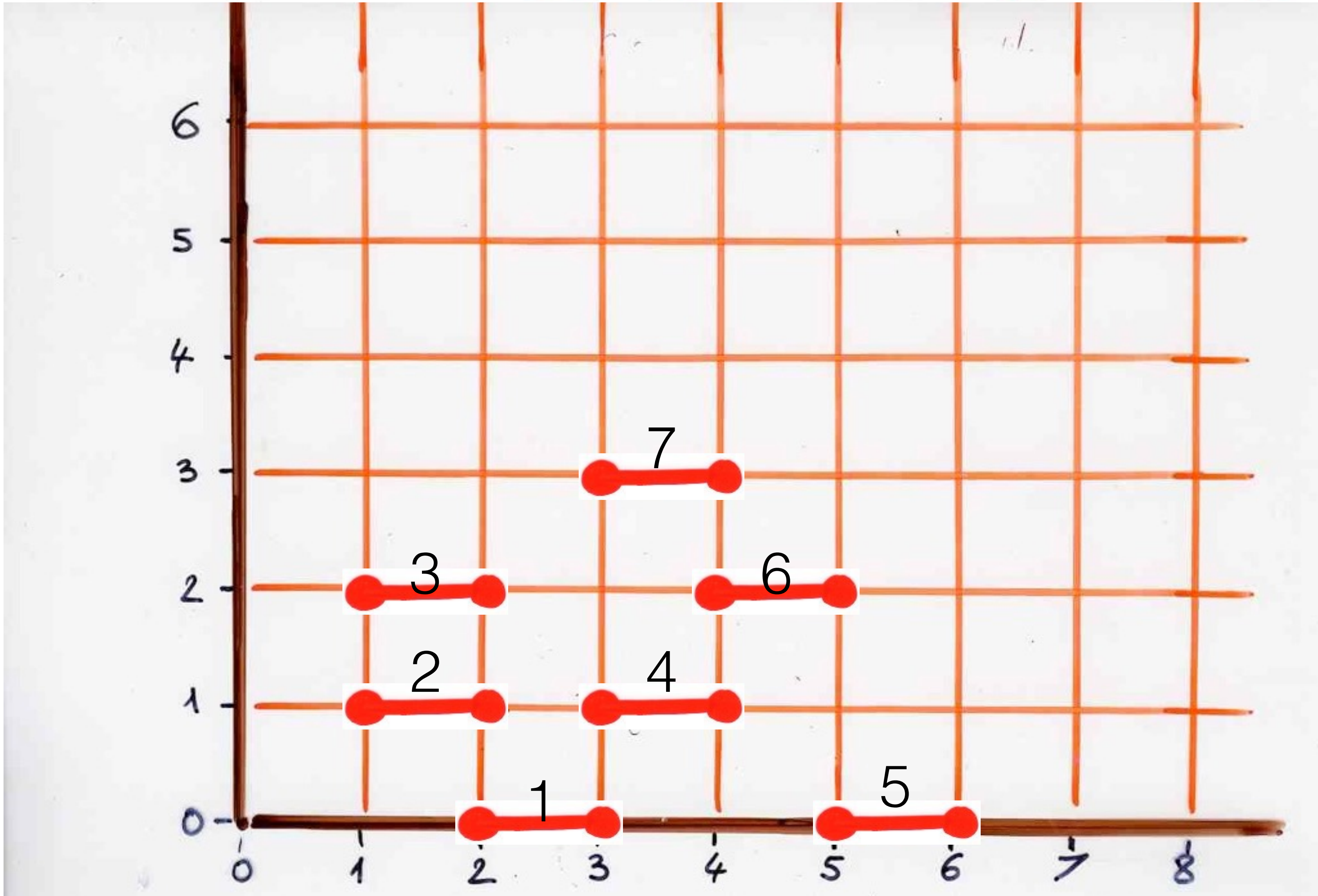


$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5\ \sigma_4\ \sigma_3$

example $w = \sigma_2\ \sigma_3\ \sigma_5\ \sigma_1\ \sigma_4\ \sigma_1\ \sigma_3$



$\sigma_2\ \sigma_1\ \sigma_1\ \sigma_3\ \sigma_5\ \sigma_4\ \sigma_3$

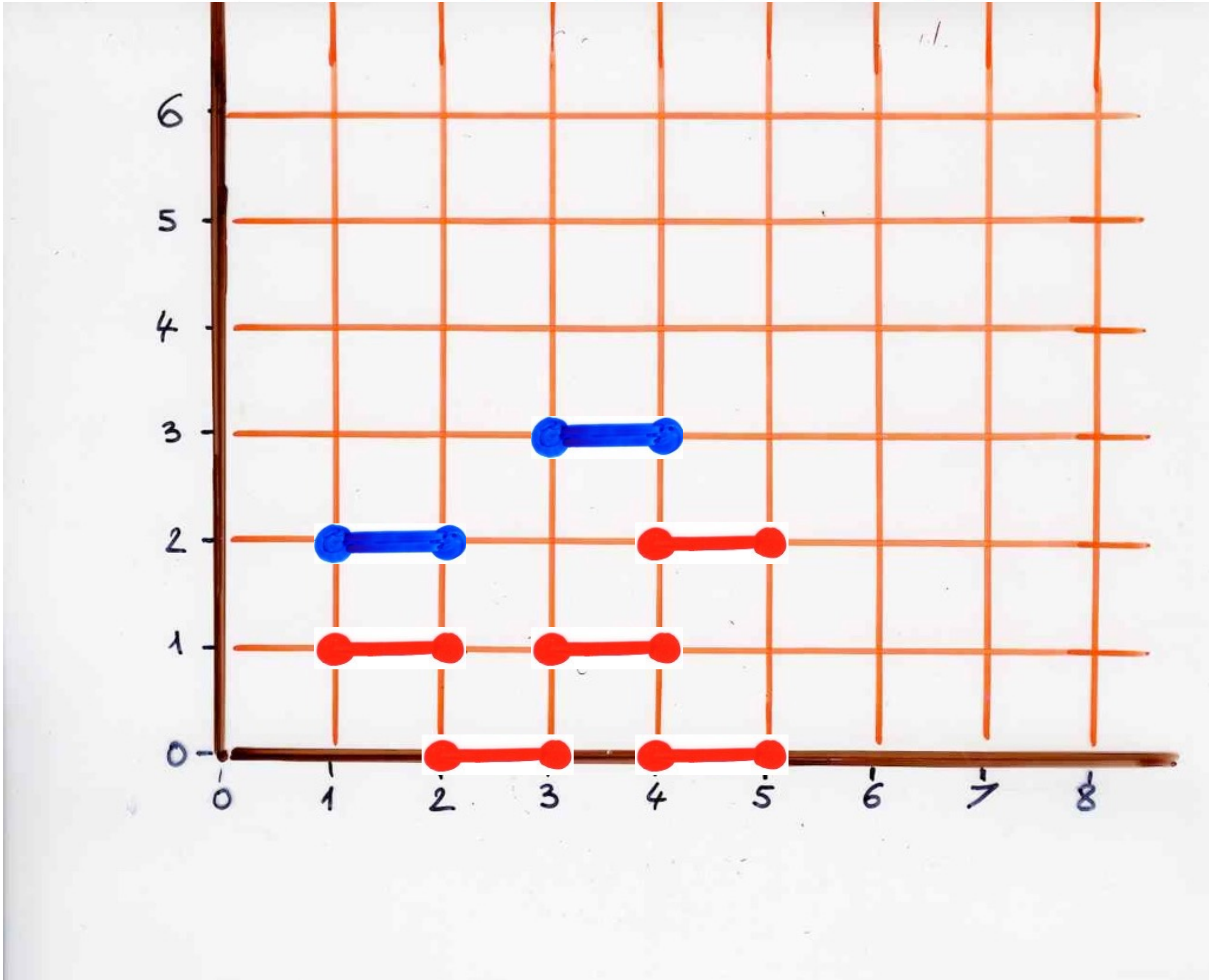
minimal letter of a class $[w]$

$$[w] = [y\, y_1]$$

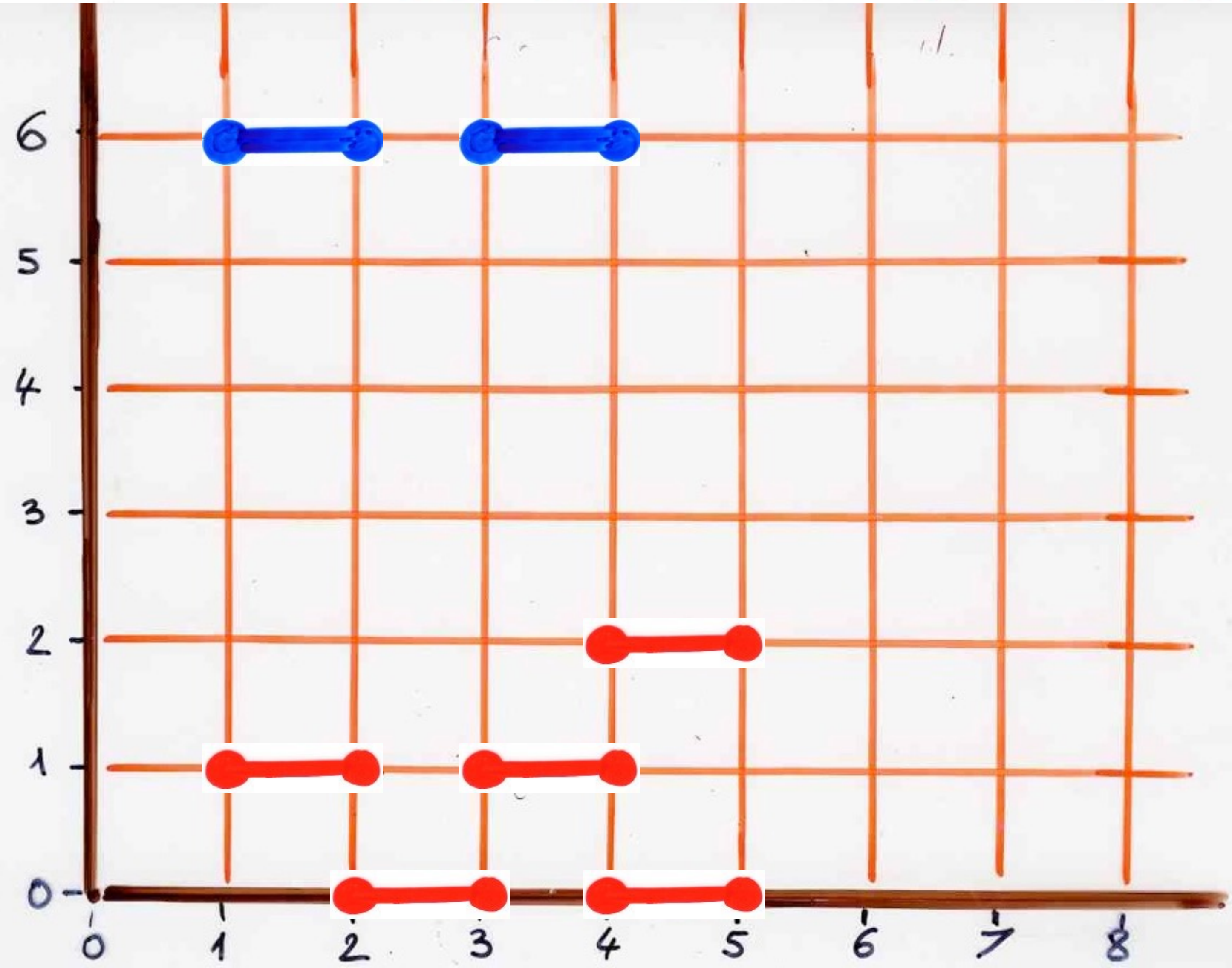
maximal letter of a class $[w]$

$$[w] = [u_1\, z]$$

example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



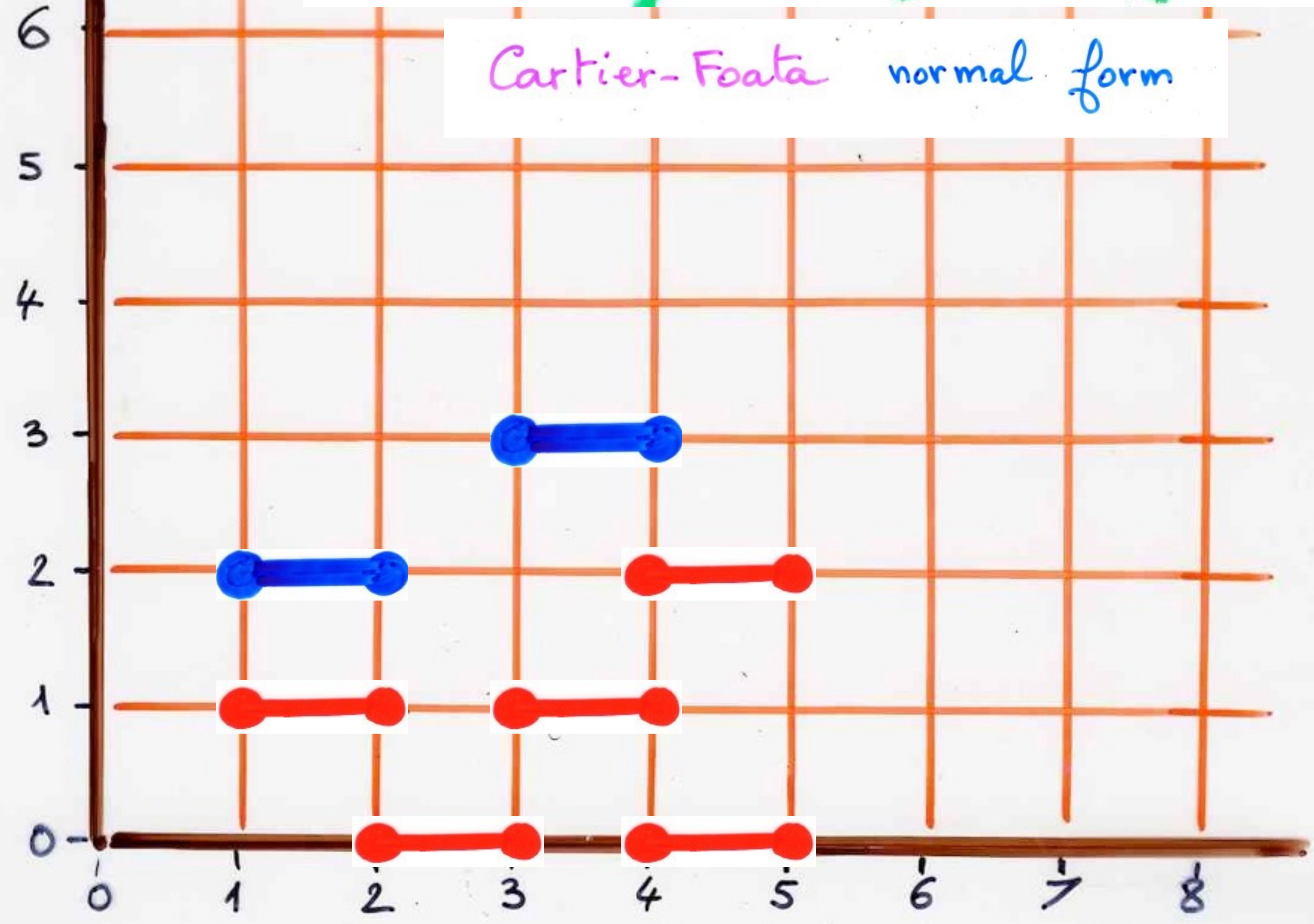
example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$



example $w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$

$\equiv \sigma_2 \sigma_5 / \sigma_1 \sigma_3 / \sigma_1 \sigma_4 / \sigma_3$

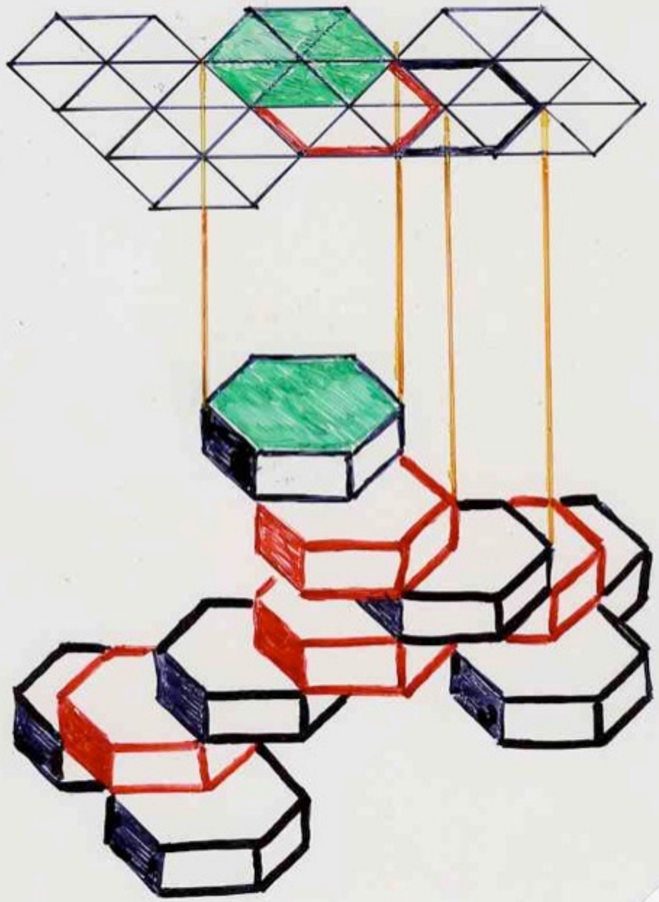
Cartier-Foata normal form



Pyramid

Def- Heap having only one maximal piece

$$-p(-t) = y$$



KC.

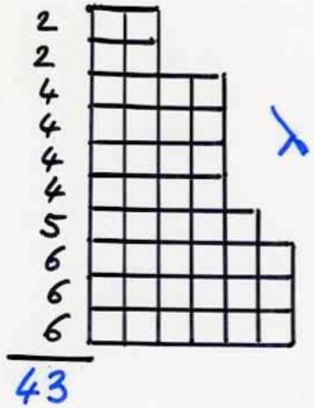
exercise

quasi-partition of integers

partition of an integer n

$$\lambda = (6, 6, 6, 5, 4, 4, 4, 4, 2, 2)$$

$$n = 43 = 6+6+6+5+4+4+4+4+2+2$$



Ferrers diagram

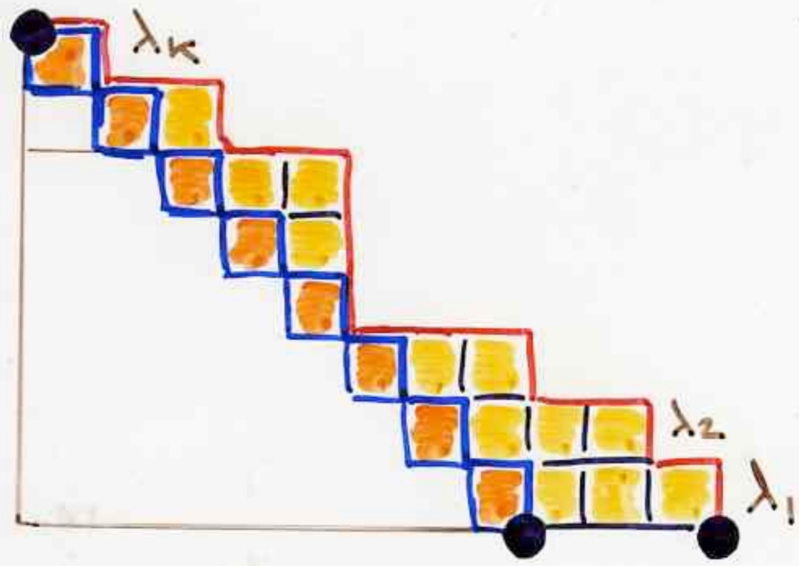
quasi-partition de n

$$n = \lambda_1 + \lambda_2 + \dots + \lambda_k$$

$$\lambda_i \geq \lambda_{i+1} - 1$$

$i = 1, \dots, k-1$

$\lambda = (4, 4, 3, 1, 2, 3, 2, 1)$



quasi-partitions

Auluck 1951

Andrews 1981 reciprocal of Rogers-Ramanujan identities

<u>exercise 1</u> using lexicographic normal form find a bijection between heaps of dimers on $\mathbb{N}_+ = \{1, 2, \dots\}$ and quasi-partitions $(\lambda_1, \dots, \lambda_k)$

- the number k of parts will be the number of dimers of the heap

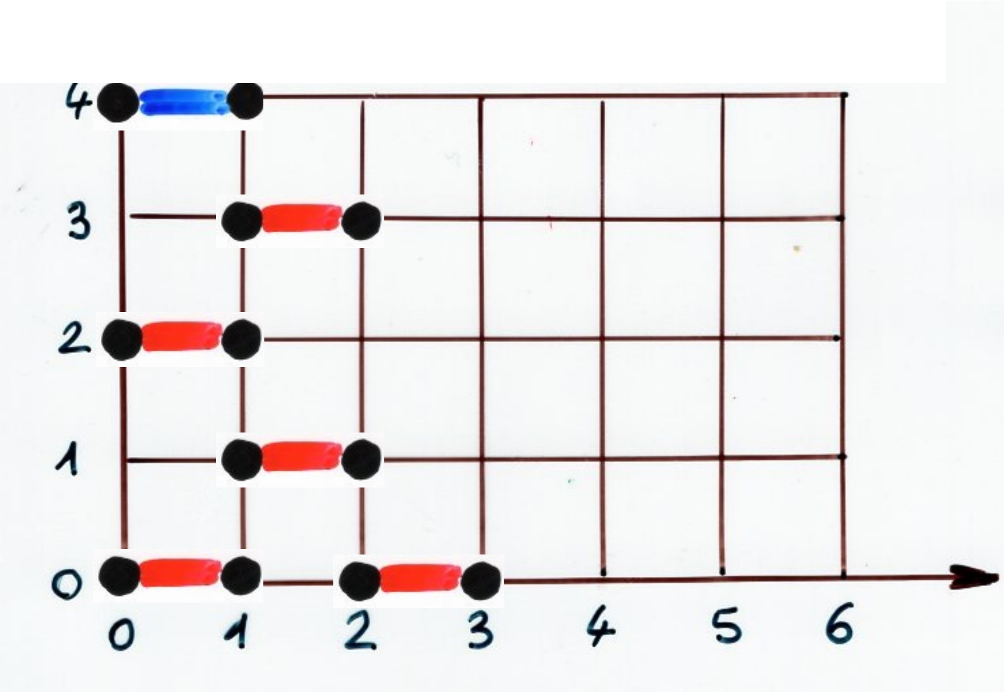
- find an interpretation with the corresponding heap of dimers of $n = \lambda_1 + \dots + \lambda_k$

exercises

pyramids and semi-pyramids of dimers

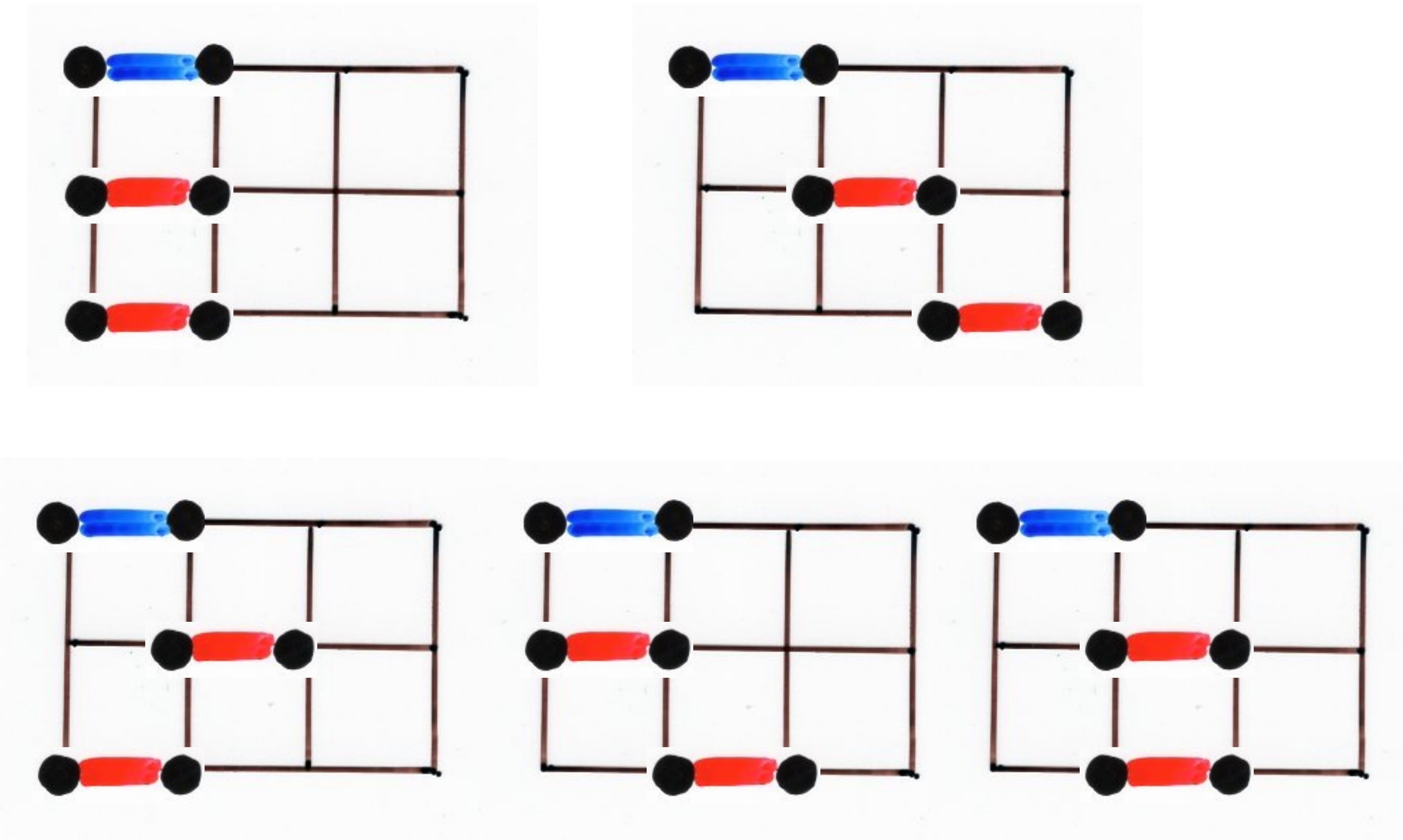
exercise semi-pyramid of dimers on $\mathbb{N}$

the unique maximal piece has projection $[0,1]$

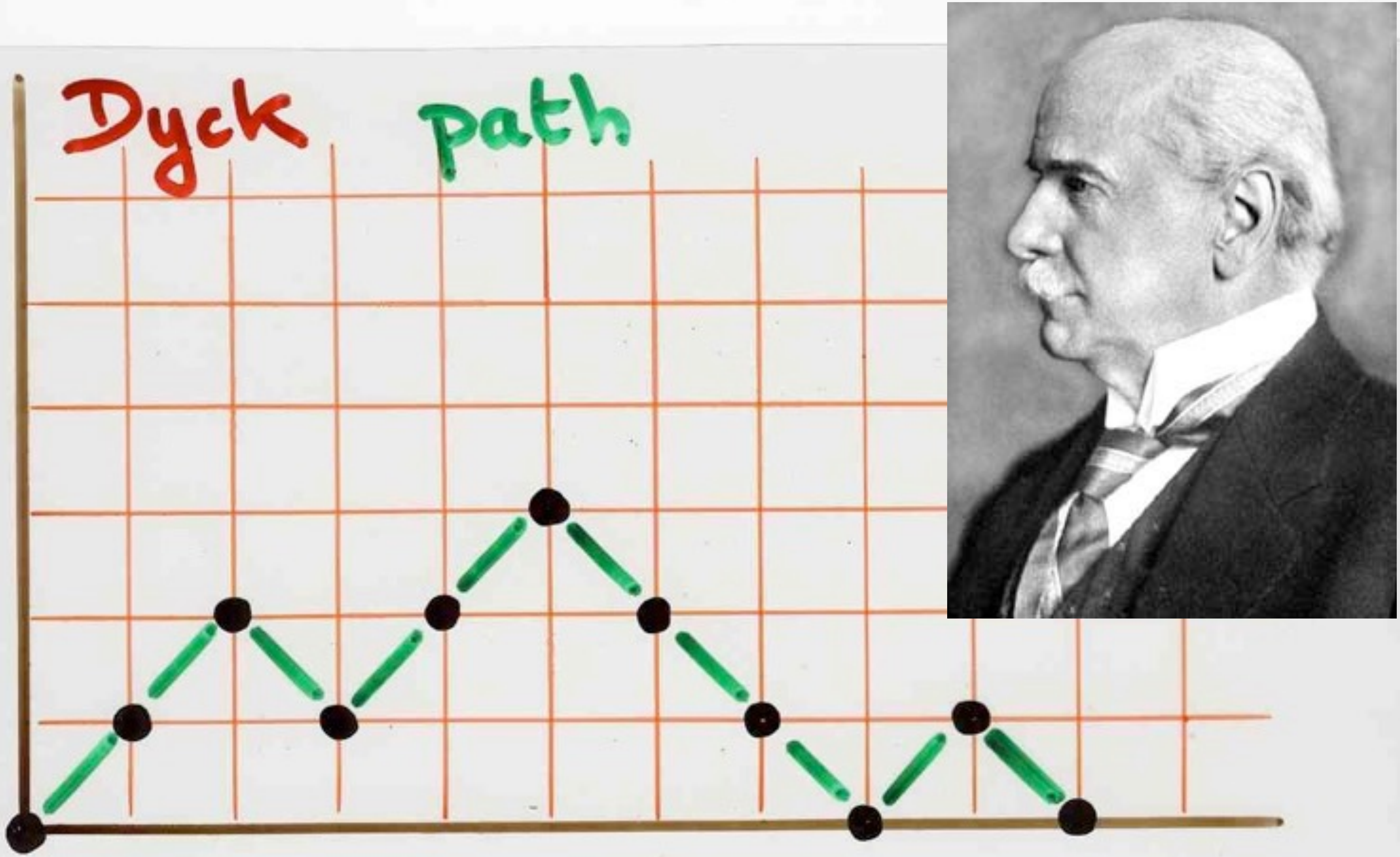


Catalan number

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$



$C_3 = 5$



Catalan number

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

exercise 2 Using exercise 1 (about quasi-partitions and lexicographic normal form) find a bijection between semi-pyramids of dimers on $\mathbb{N}$ having n dimers and Dyck paths of length $2n$.

exercise 3 — **pyramid of dimers on $\mathbb{Z}$ up to translation**

(more difficult)

bilateral Dyck path

bilateral Dyck word

number of bilateral Dyck paths ω of length $2n$ $= \binom{2n}{n}$

exercise 3 (more difficult)

pyramid of dimers on $\mathbb{Z}$ up to translation

- using exercise 2 find a bijection between pyramids of dimers on $\mathbb{Z}$ such that the projection of the maximal piece is $[0,1]$ and bilateral Dyck paths starting with step

- thus the number of pyramid of dimers on $\mathbb{Z}$ up to translation is

$$\frac{1}{2}\binom{2n}{n}$$

Posets

Poset (partially ordered set)

$(E, \preccurlyeq)$

$\preccurlyeq$ order relation

$\preccurlyeq$ order relation on E

- reflexive $x \preccurlyeq x$ all $x \in E$
- antisymmetric $x \preccurlyeq y$ and $y \preccurlyeq x \Rightarrow x = y$
- transitive $x \preccurlyeq y$ and $y \preccurlyeq z \Rightarrow x \preccurlyeq z$

for all $x, y, z \in E$

Poset (partially ordered set)

$(E, \preccurlyeq)$

$\preccurlyeq$ order relation

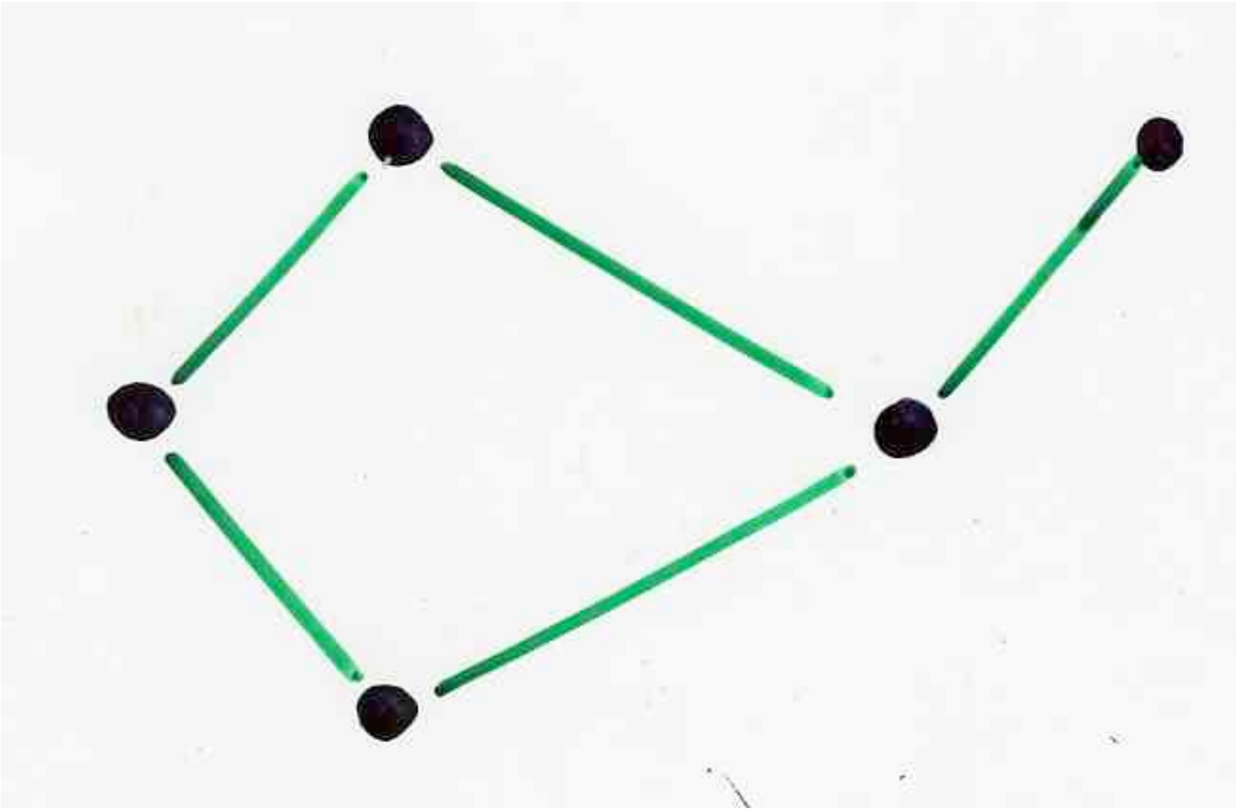
covering relation

$x, y \in E$, y covers x

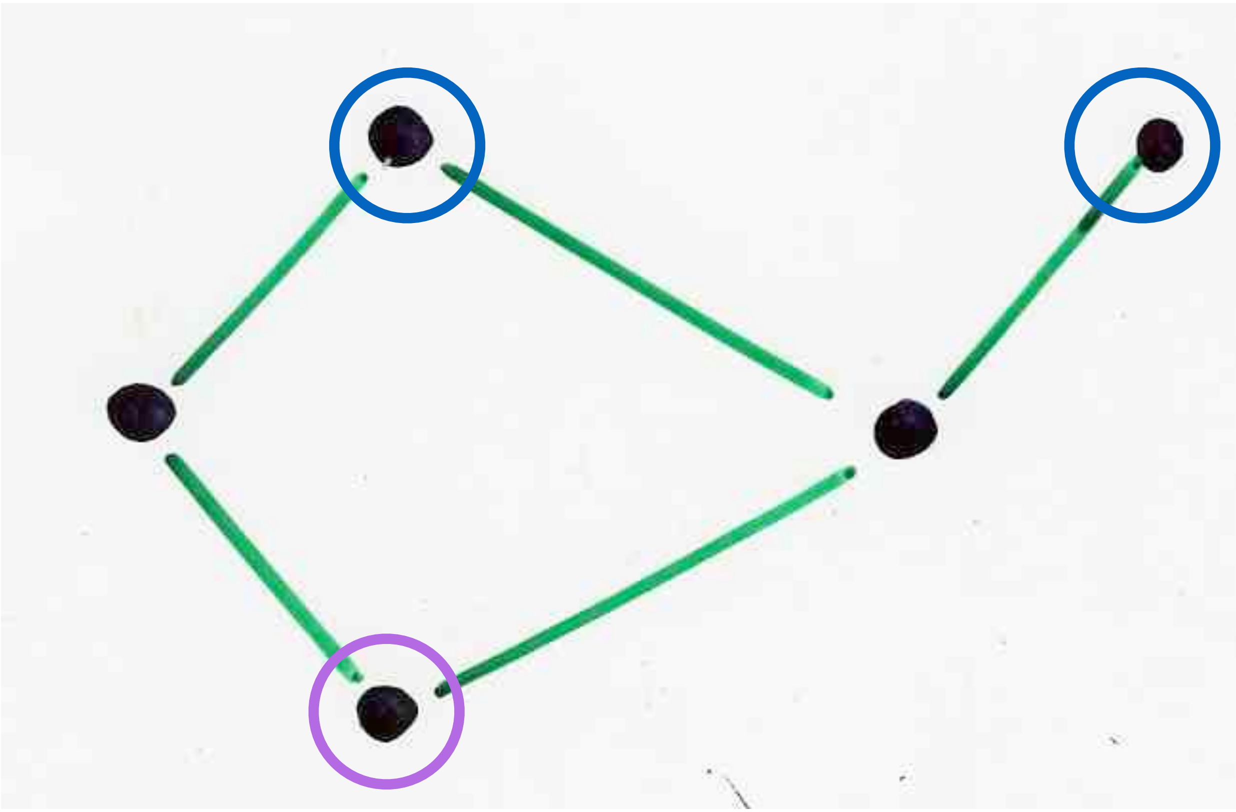
iff $x \prec y$ (strict) and $x \preccurlyeq z \preccurlyeq y \Rightarrow \left\{ \begin{array}{l} z = x \\ \text{or} \\ z = y \end{array} \right.$

the interval $[x, y]$ is reduced to $\{x, y\}$

Hasse diagram of a poset



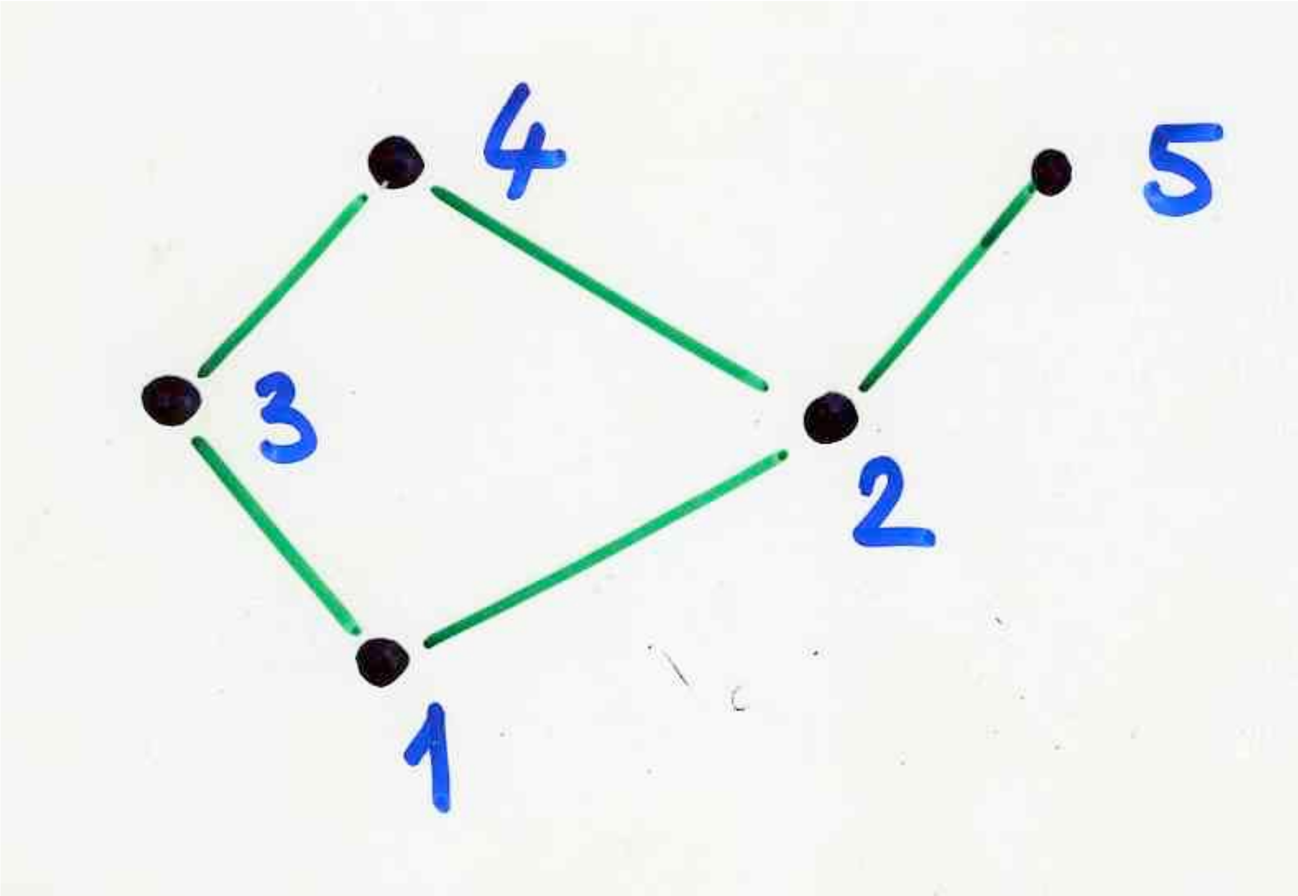
minimal
maximal element of a poset



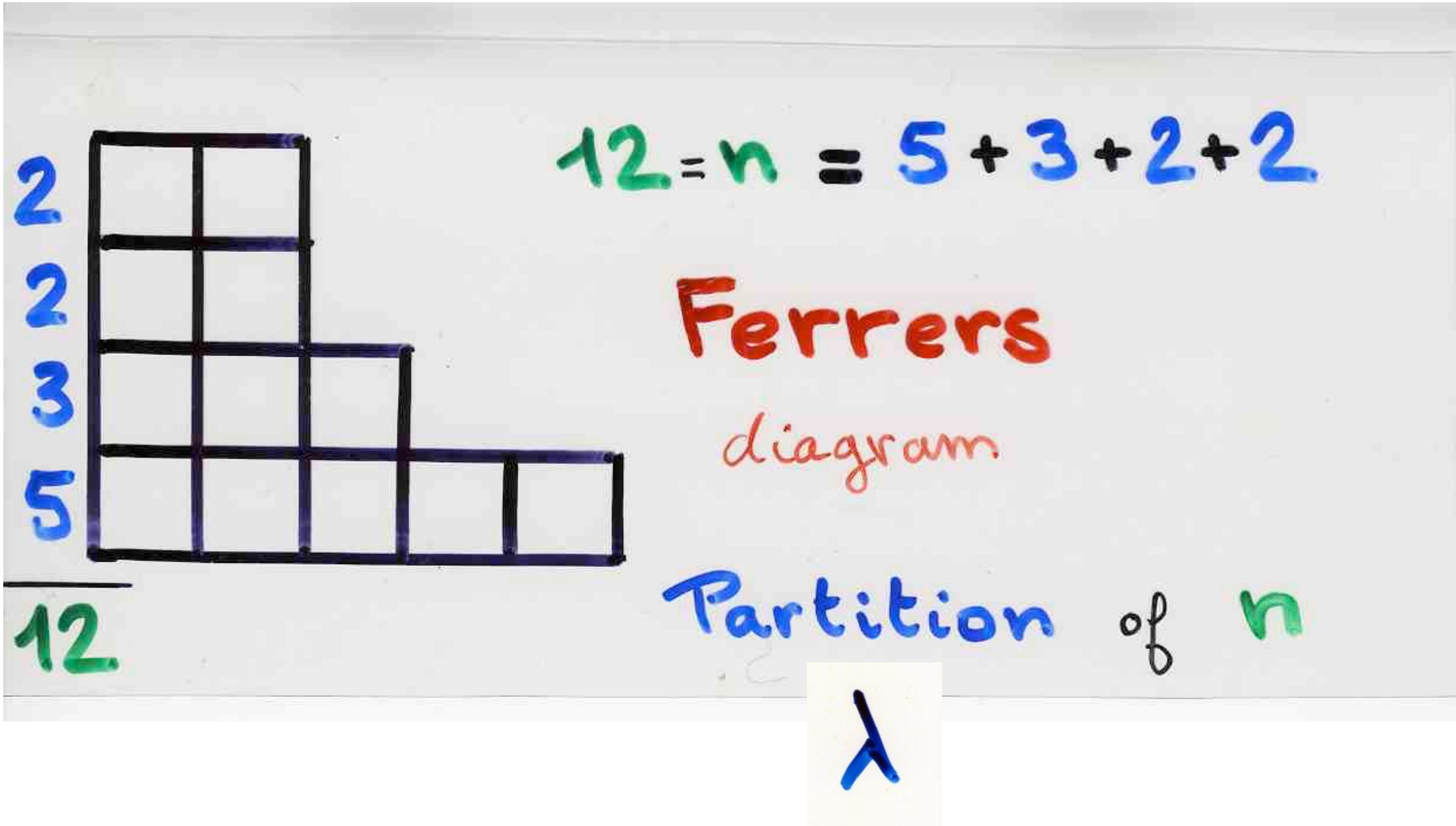
linear extension of a poset $(E, \preccurlyeq)$

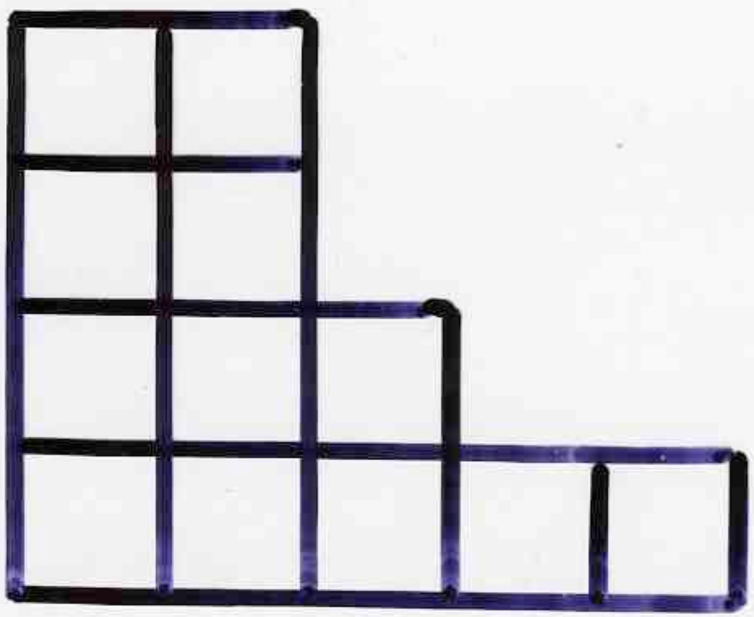
Def. $f : E \longrightarrow [1, n]$ bijection

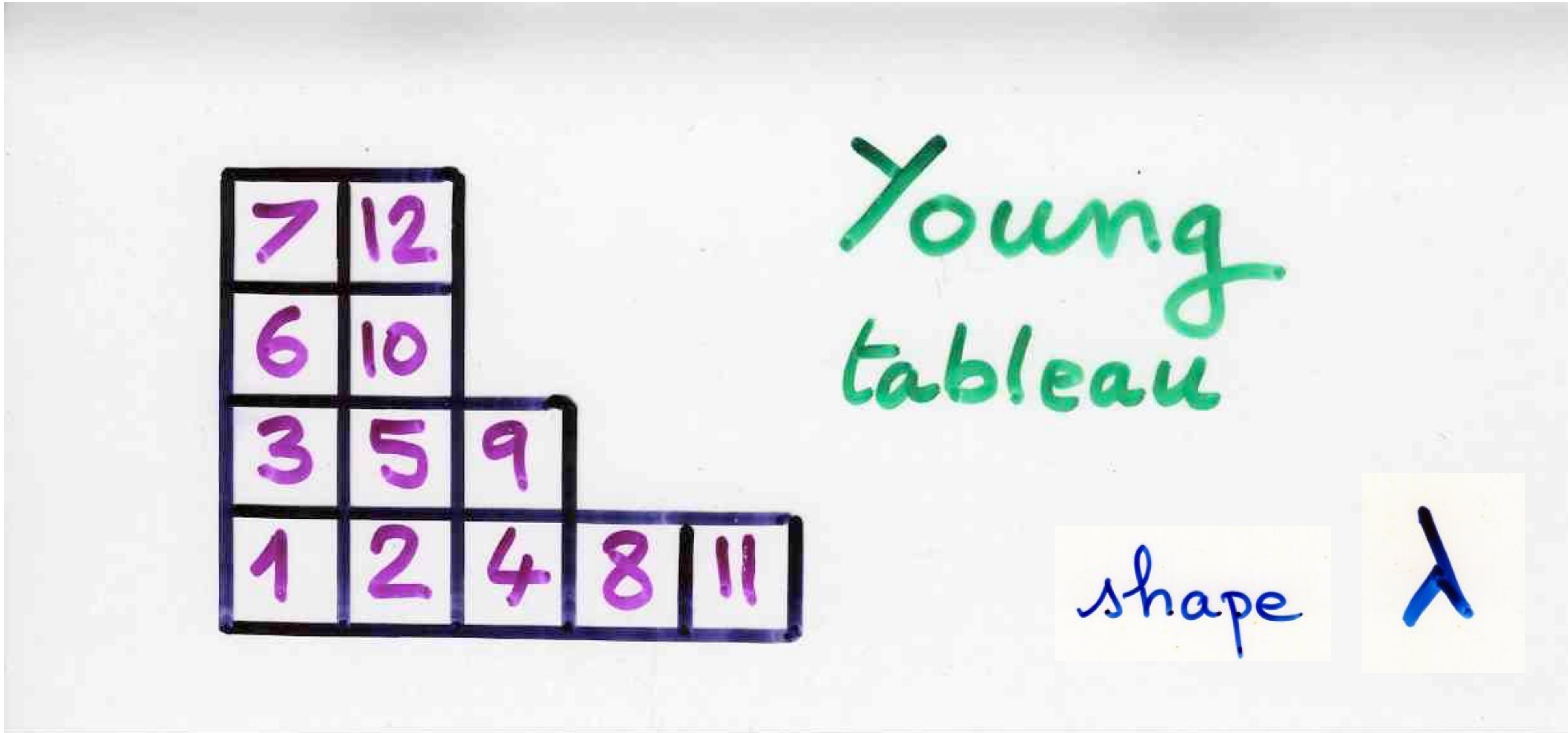
$x \preccurlyeq y \Rightarrow f(x) \leq f(y)$


4
5
3
2
1

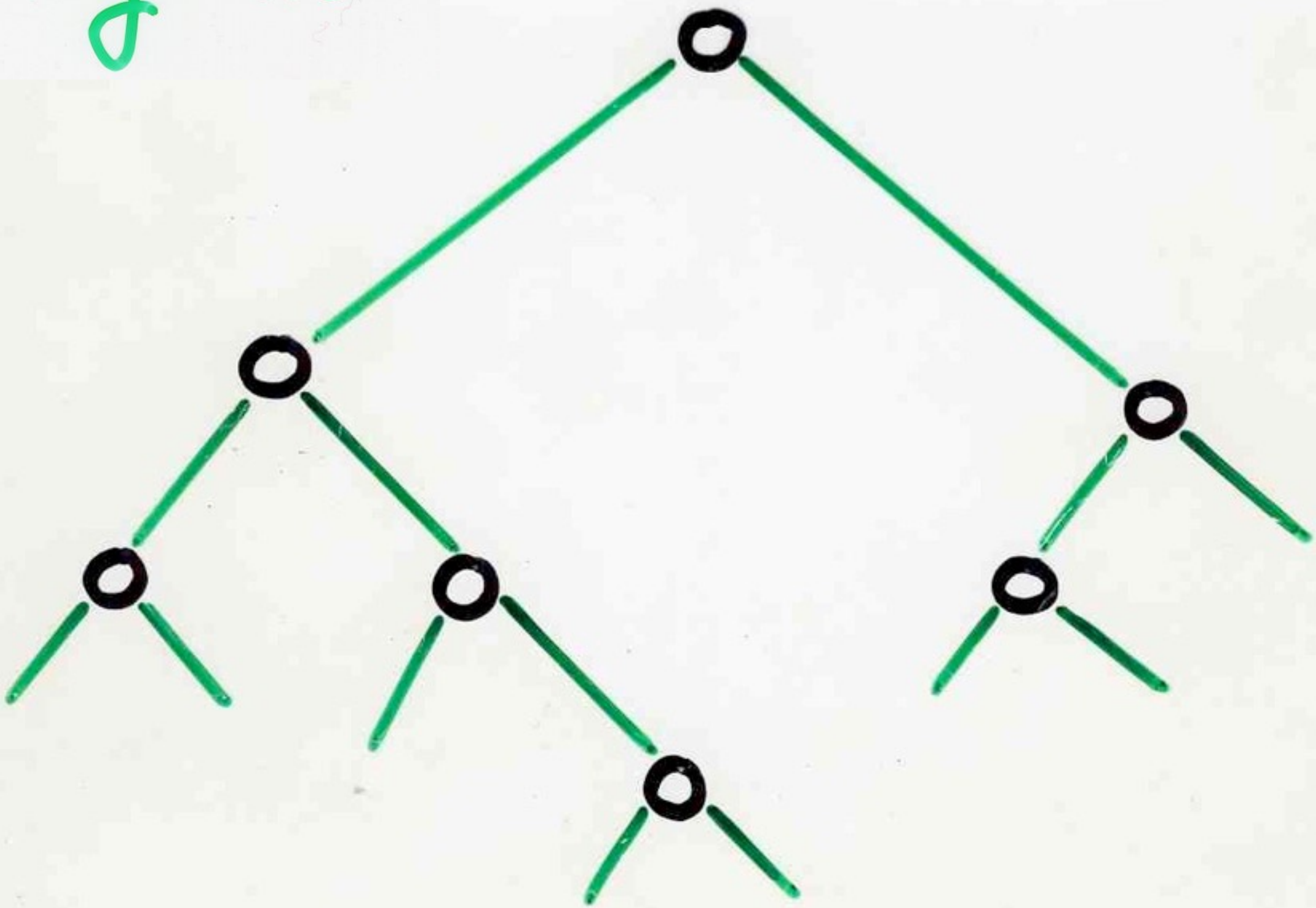
some examples

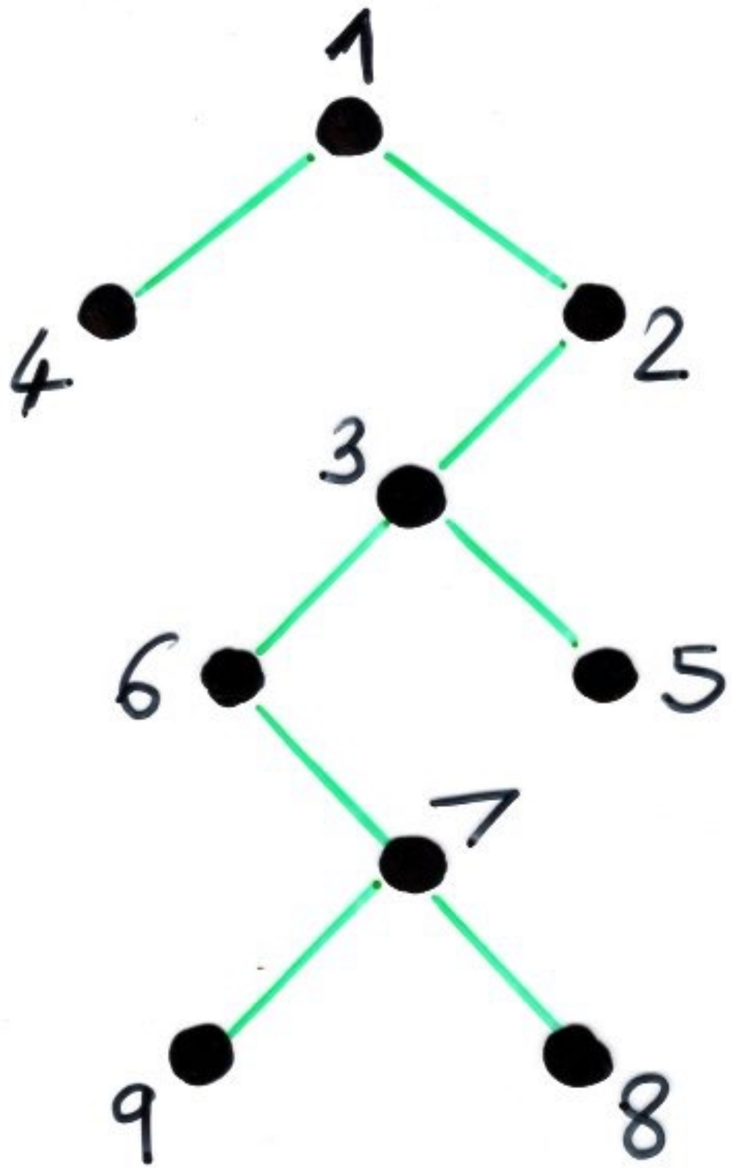





7 12
6 10
3 5 9
1 2 4 8 11
Young
tableau
shape λ

binary tree

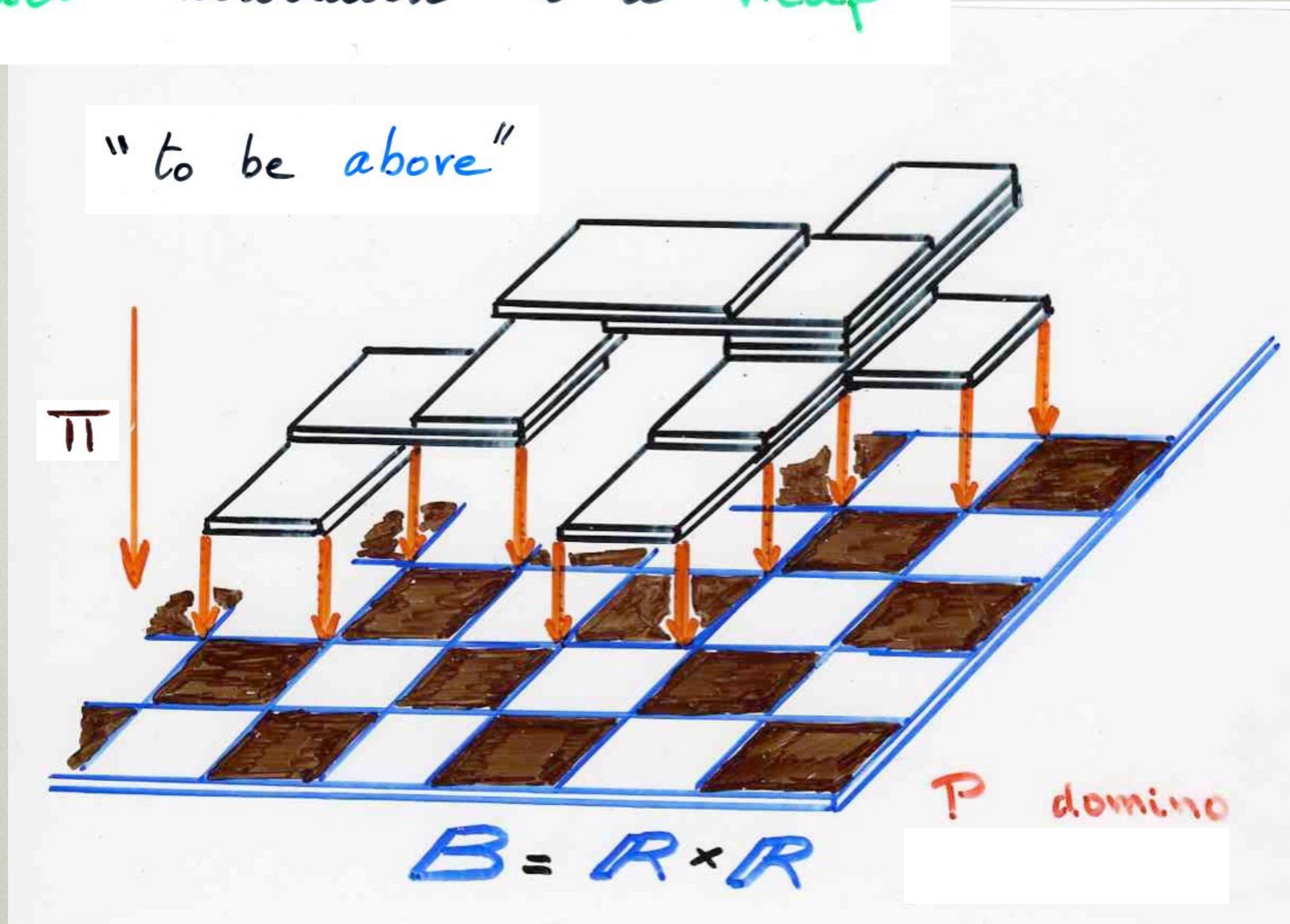




1
4
2
3
6
5
7
9
8

Heaps and posets

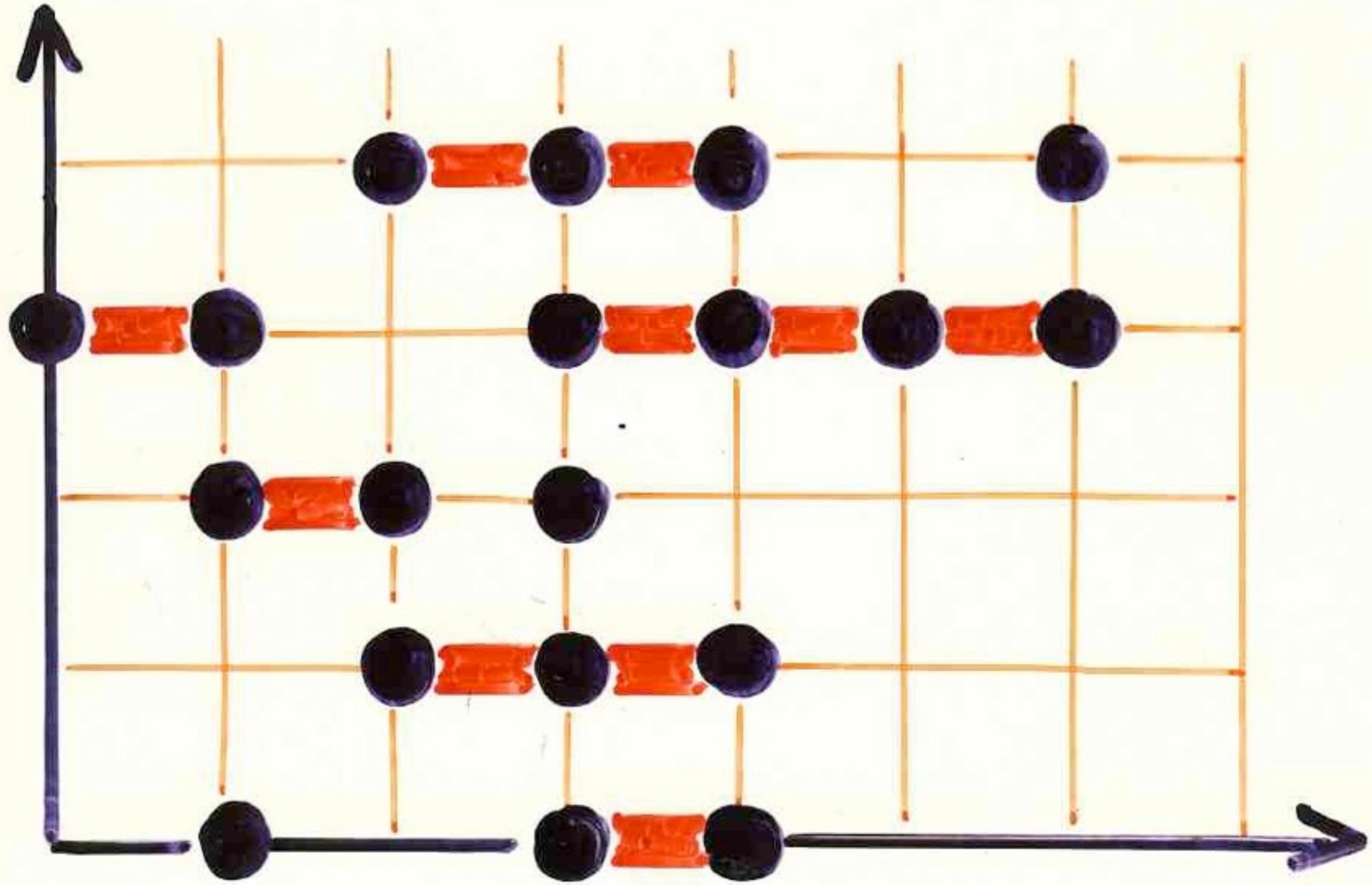
poset associated to a heap

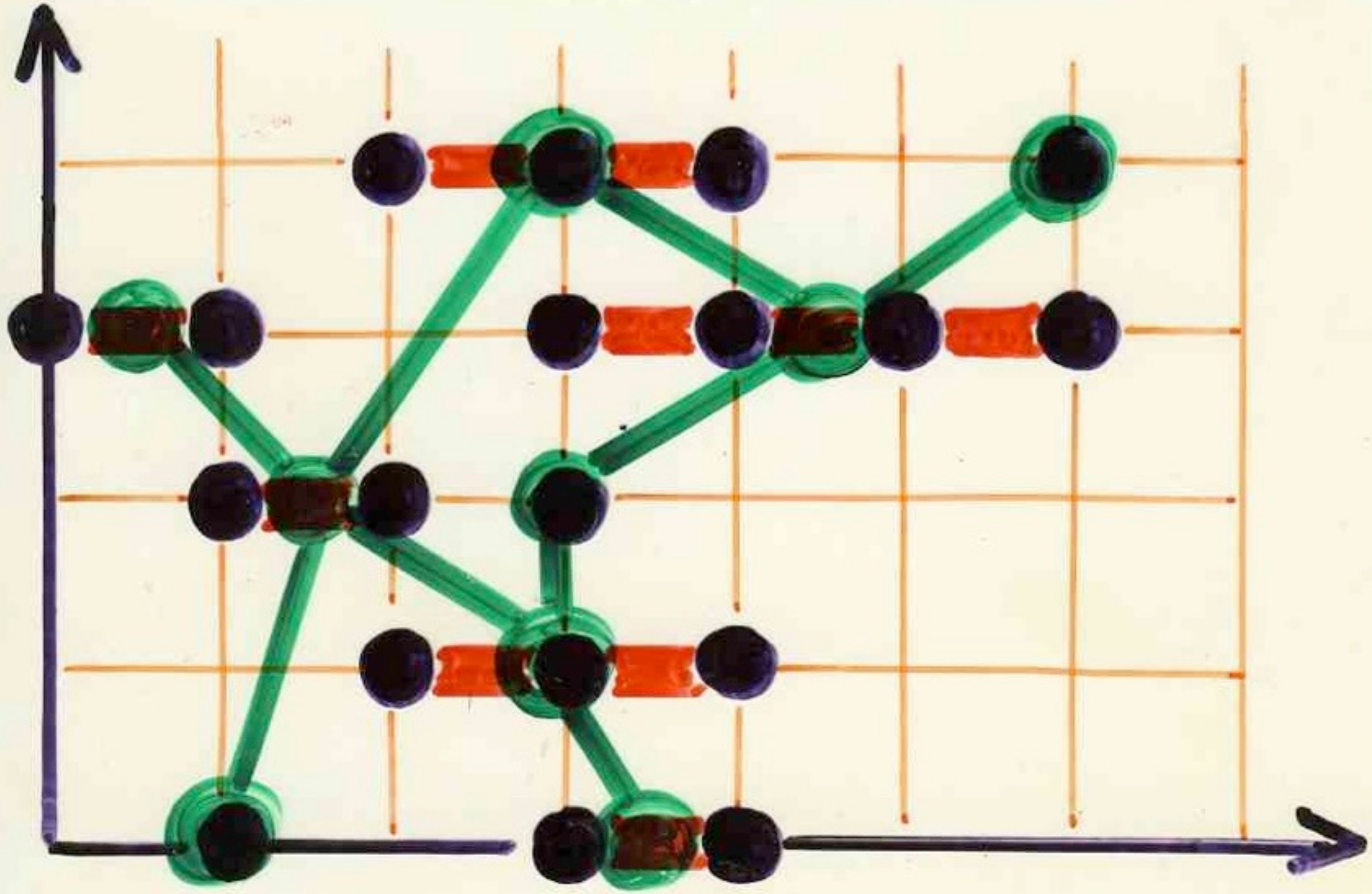


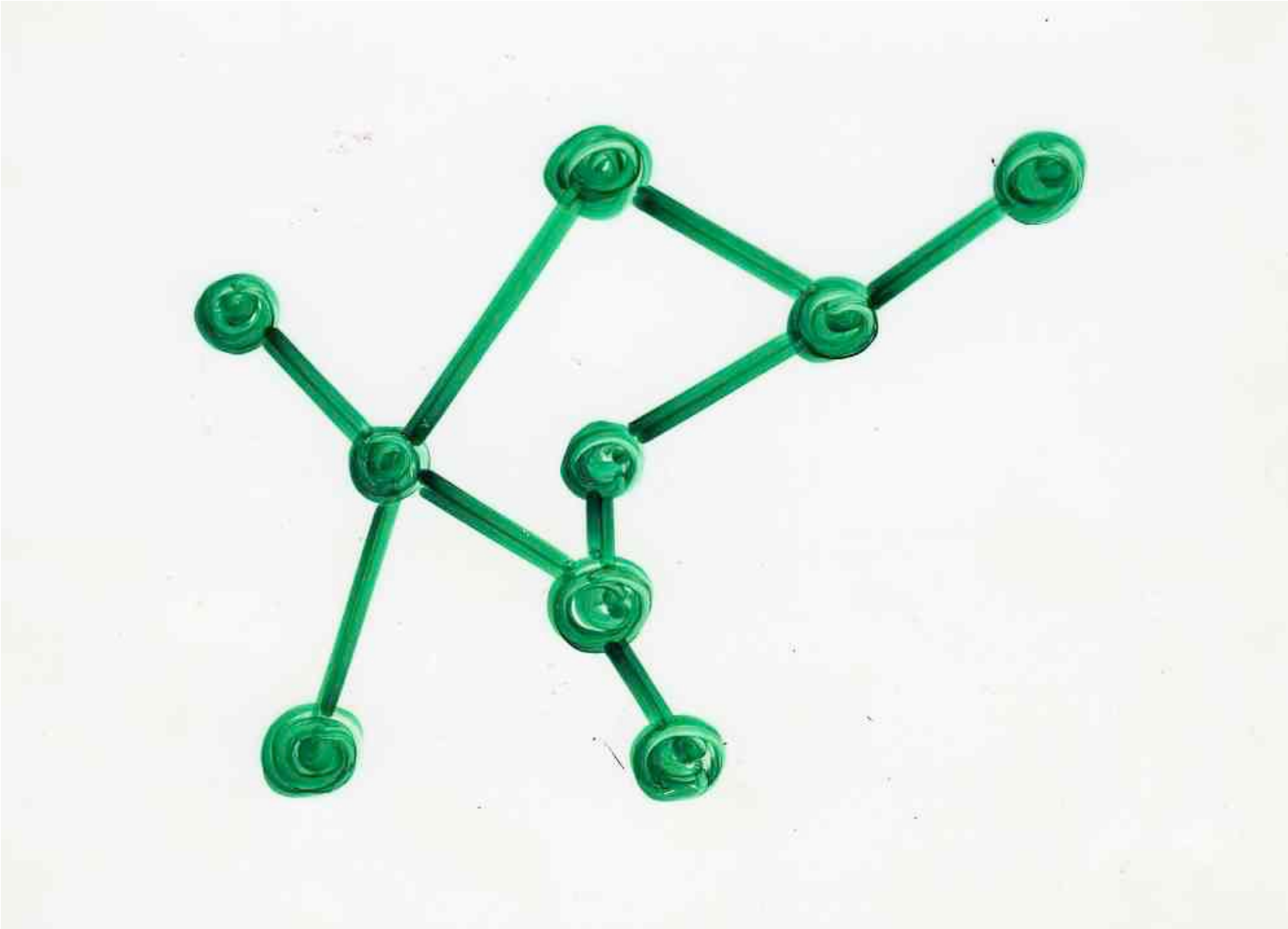
Def. **Poset** $(E, \preceq)$ associated to a heap E

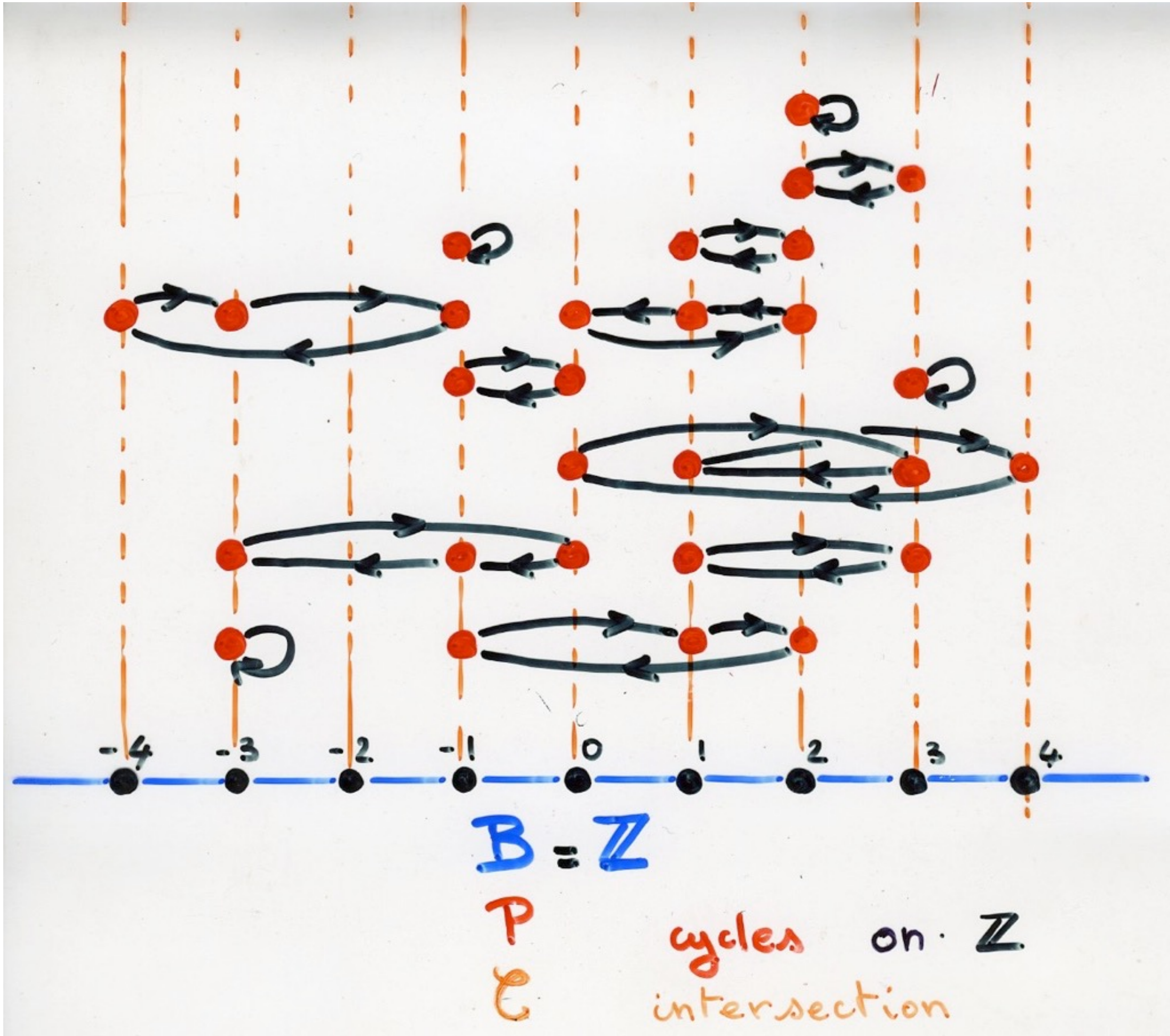
$\preceq$ transitive closure of the relation $\preceq_{\mathcal{C}}$

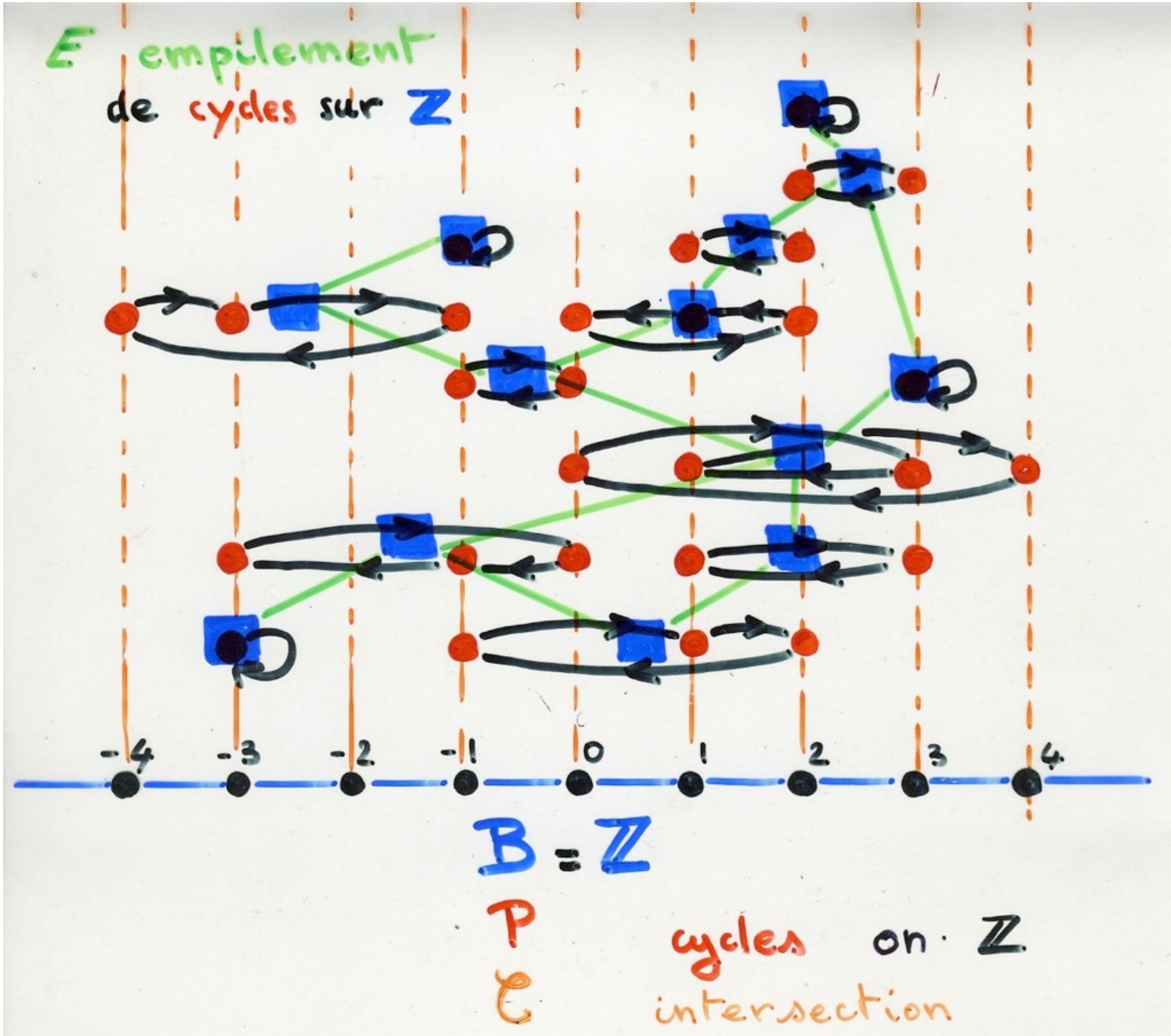
$$(\alpha, i) \preceq_{\mathcal{C}} (\beta, j) \iff \alpha \,\mathcal{C}\, \beta, \; i < j$$







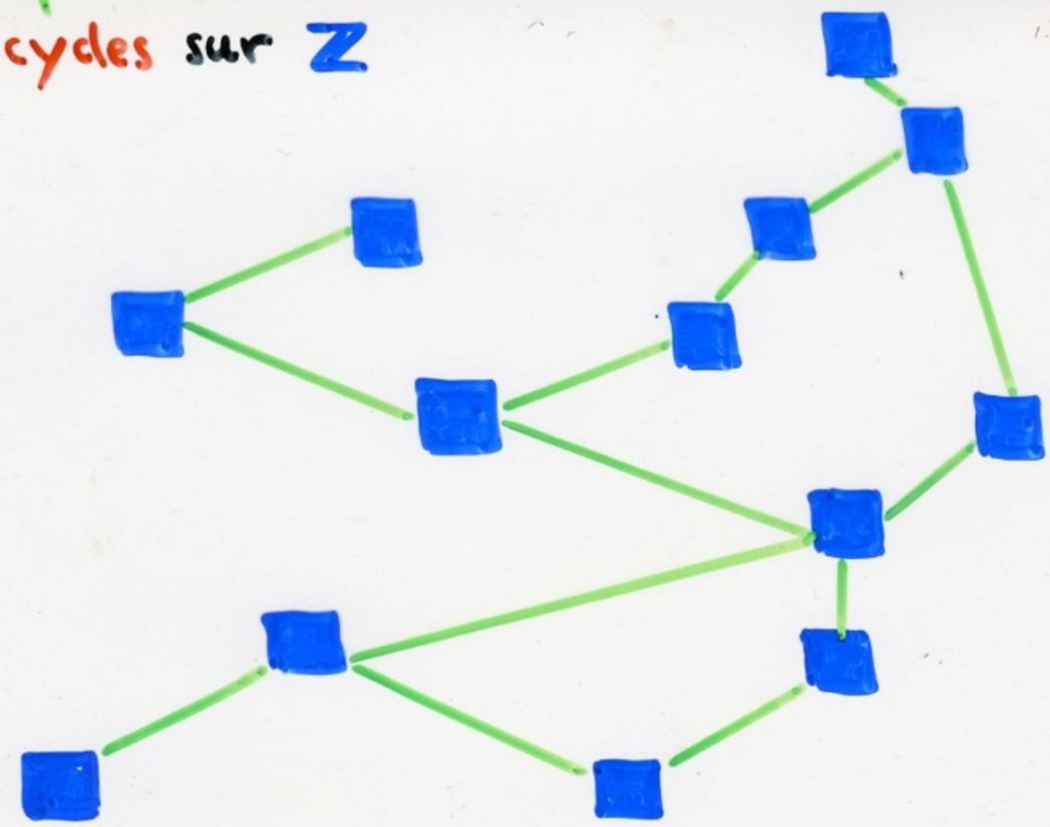

-4
-3
-2
-1
0
1
2
3
4
B = ℤ
P cycles on ℤ
ℭ intersection

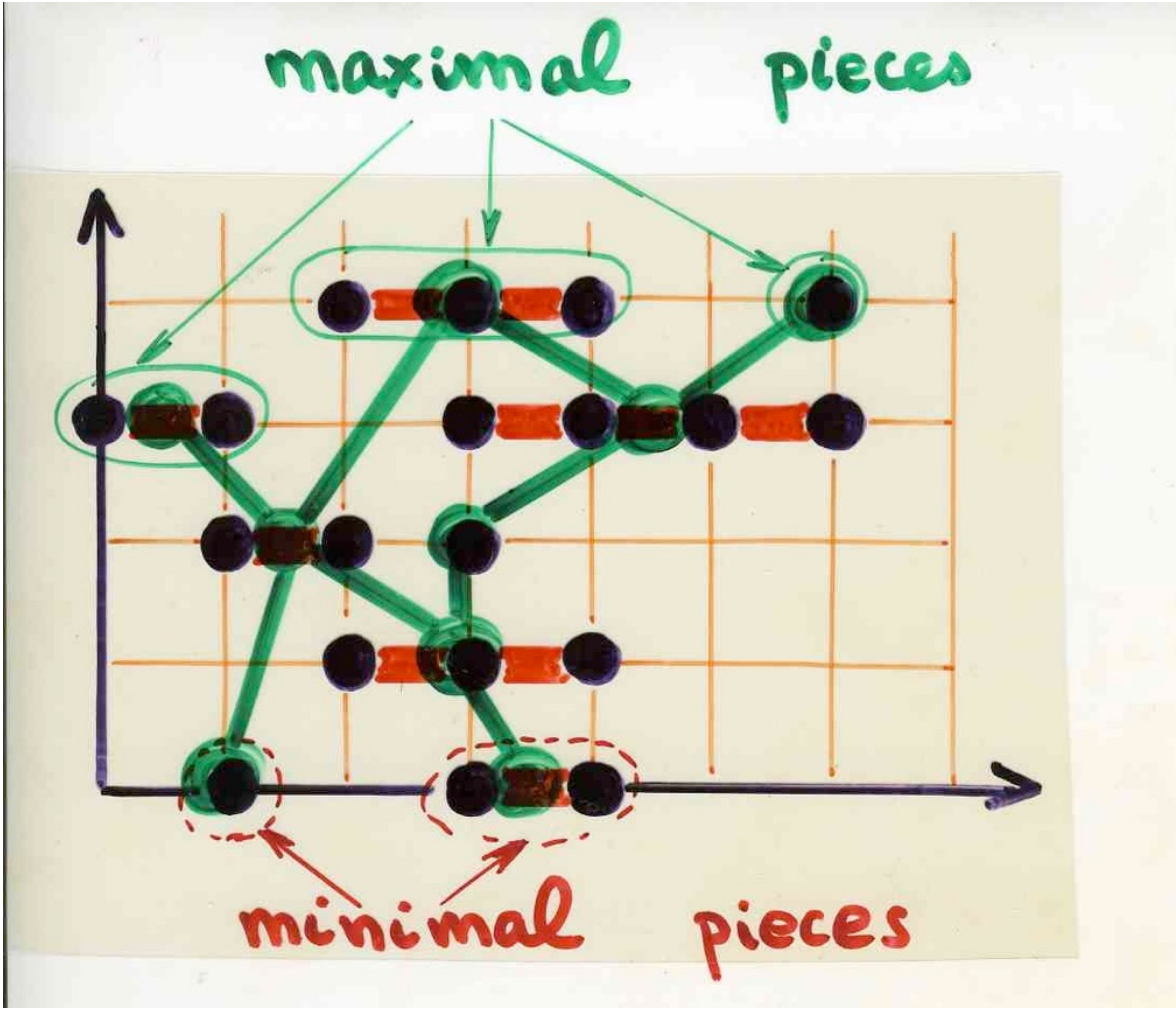


E empilement
de cycles sur Z
-4
-3
-2
-1
0
1
2
3
4
B = ℤ
P cycles on ℤ
𝒞 intersection

E empilement

de cycles sur Z




maximal pieces
minimal pieces

minimal letter of a class $[w]$

$$[w] = [y\, y_1]$$

maximal letter of a class $[w]$

$$[w] = [u_1\, z]$$

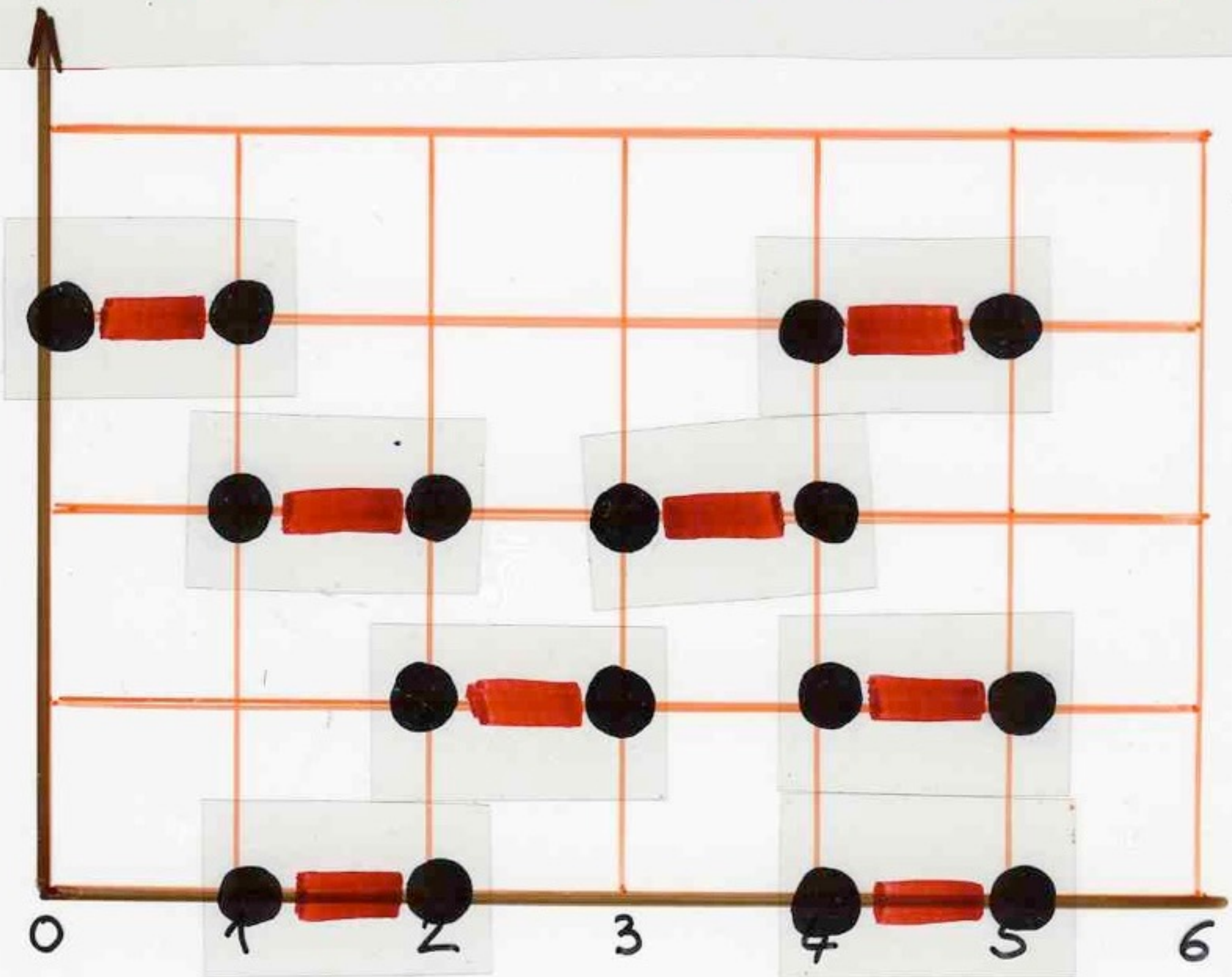
heaps and linear extensions

Proposition Let $w \in A^*$ and $[w]$ the equivalence class $[w] \in L(A, C)$.

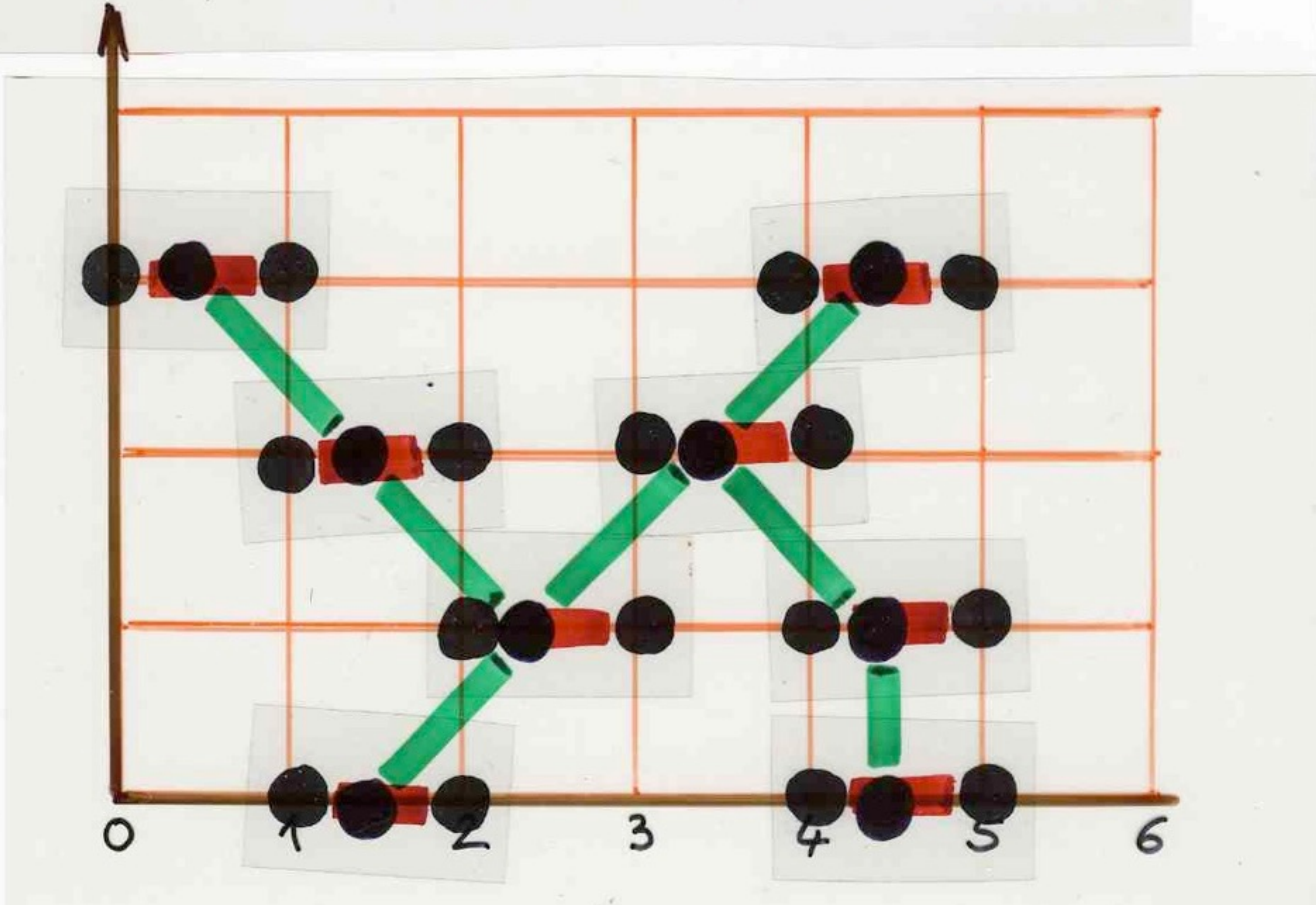
For $P = A$ and $\mathcal{C} = \overline{C}$, let $E \in H(P, \mathcal{C})$ the associated heap $E = \overline{\varphi}(w)$ and poset $(E, \preceq)$.

The words of $[w]$ are in bijection with the linear extensions of $(E, \preceq)$

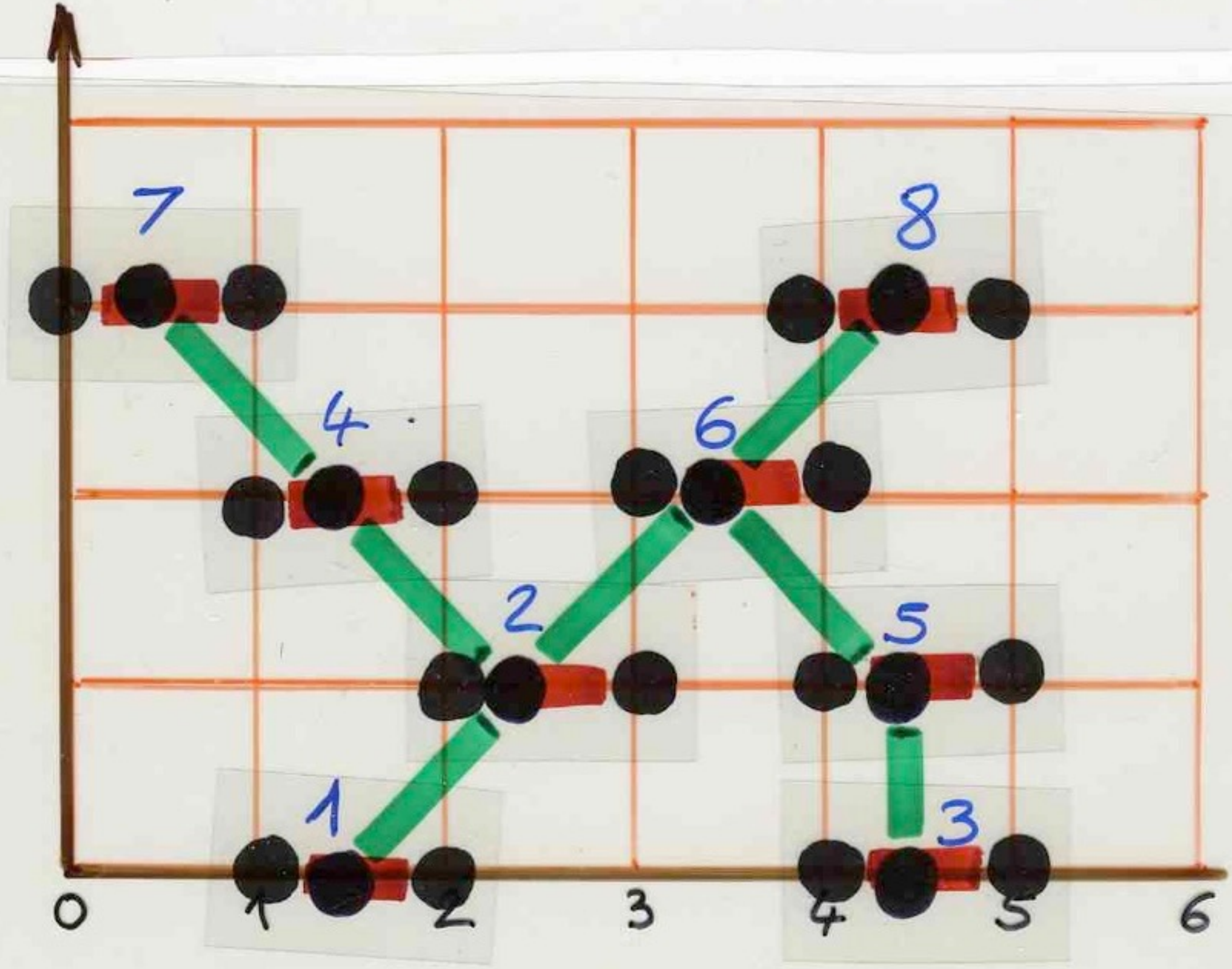
$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$

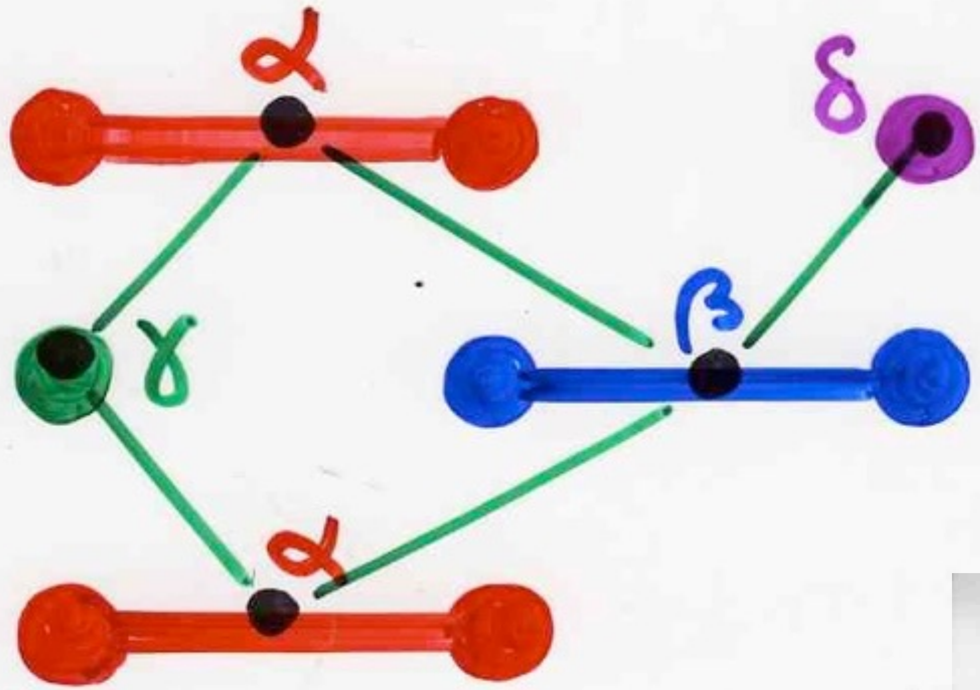


$$w = \sigma_1 \sigma_2 \sigma_4 \sigma_1 \sigma_4 \sigma_3 \sigma_0 \sigma_4$$



$$W = \sigma_1\, \sigma_2\, \sigma_4\, \sigma_1\, \sigma_4\, \sigma_3\, \sigma_0\, \sigma_4$$

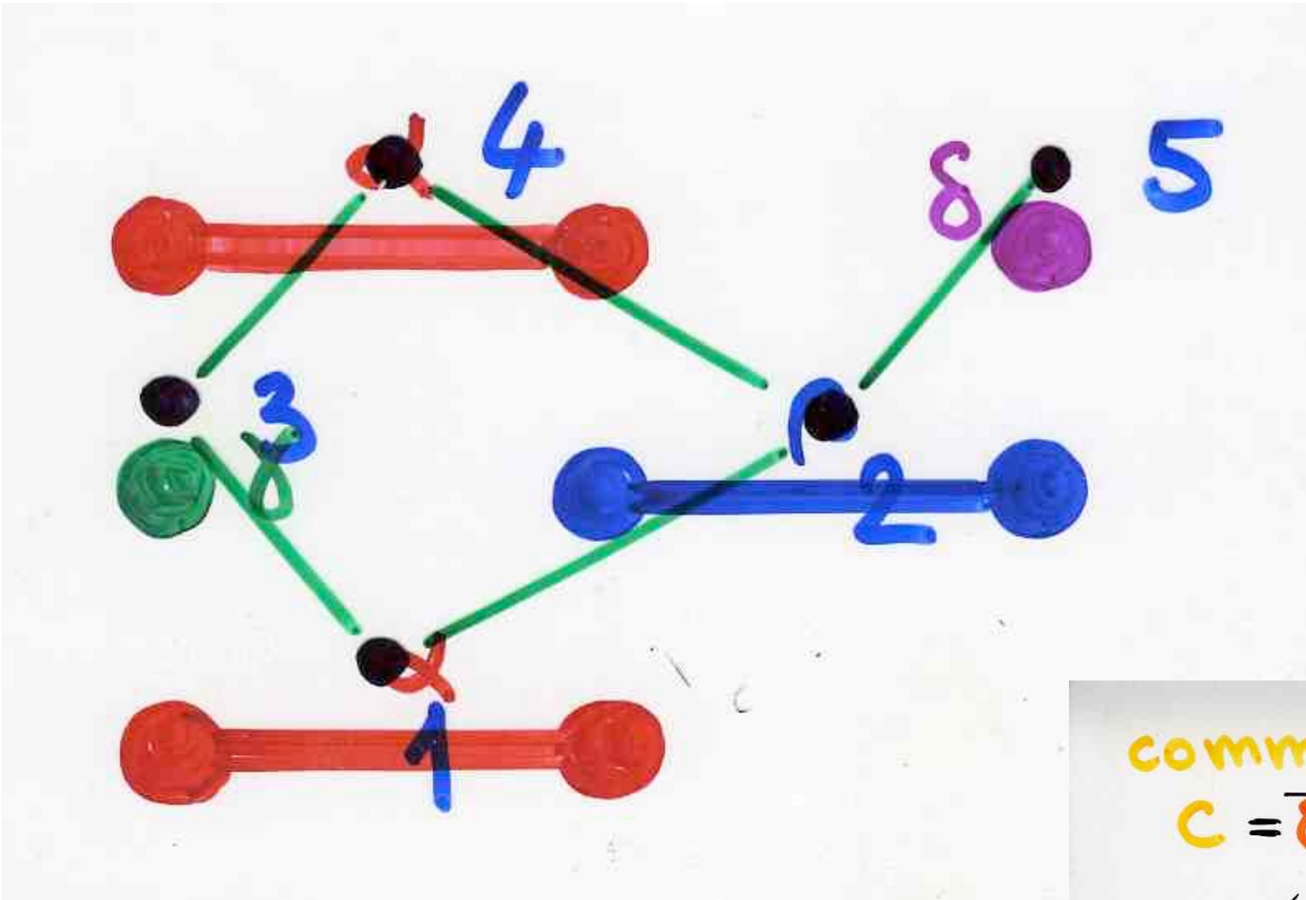




commutations

$C = \overline{C}$

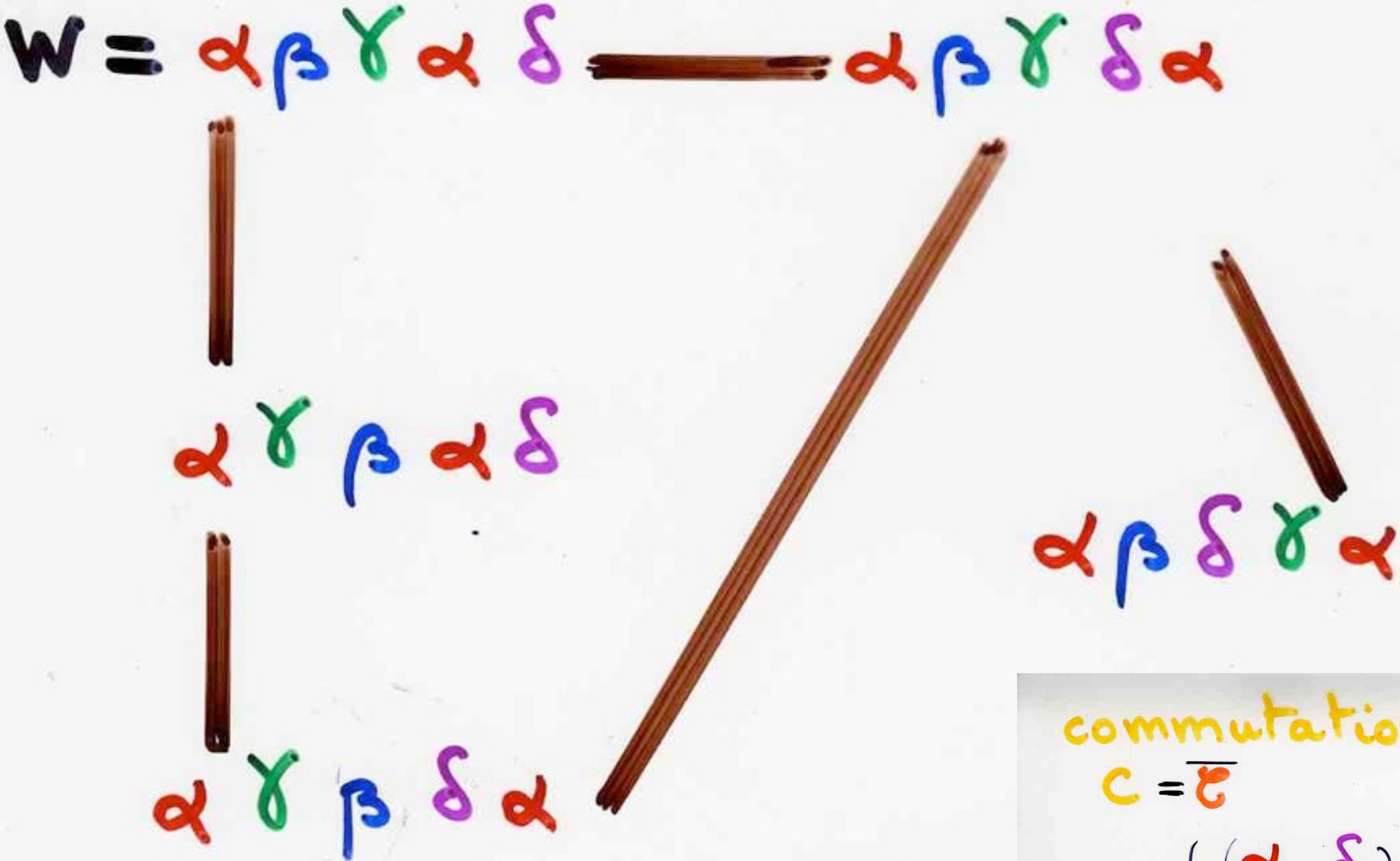
$$C \left\{ \begin{array}{l} (\alpha, \delta) \\ (\beta, \gamma) \\ (\gamma, \delta) \end{array} \right.$$



commutations

$c = \overline{\tau}$

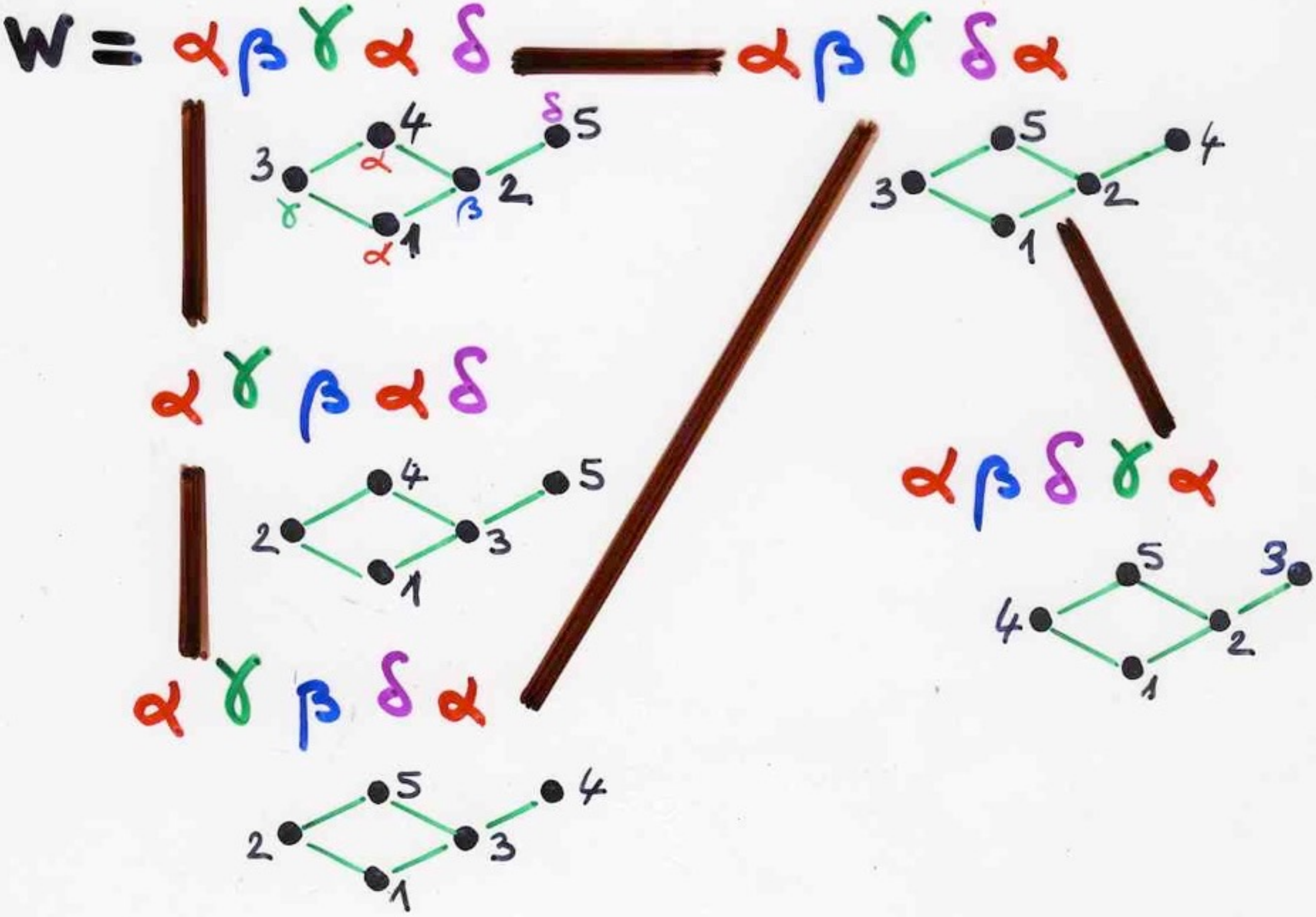
$$C \begin{cases} (\alpha, \delta) \\ (\beta, \gamma) \\ (\gamma, \delta) \end{cases}$$



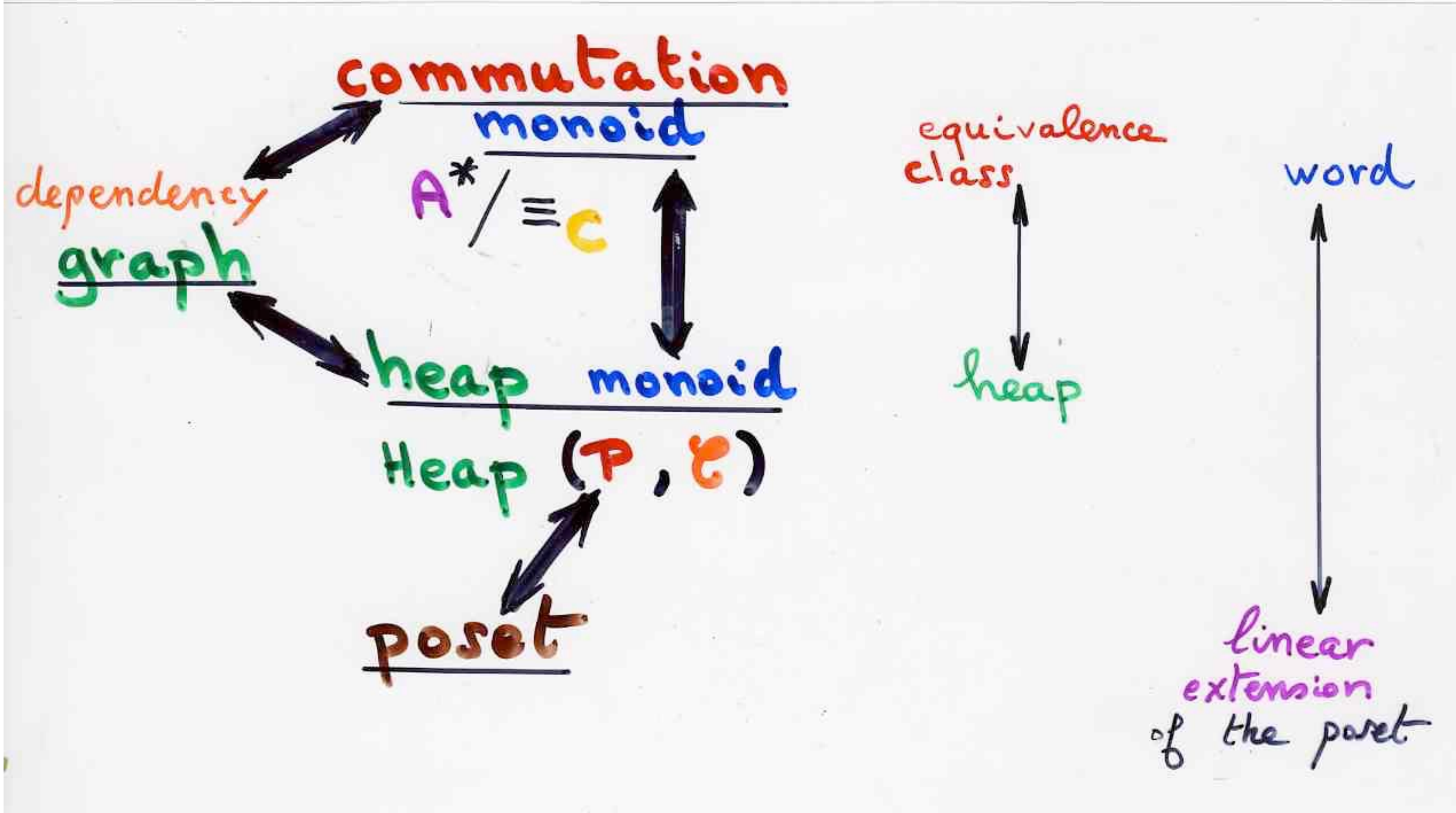
commutations

$C = \overline{\mathcal{C}}$

$$C \begin{cases} (\alpha, \delta) \\ (\beta, \gamma) \\ (\gamma, \delta) \end{cases}$$



W = αβγαδ
αβγδα
αγβαδ
αβδγα
αγβδα
1
2
3
4
5
α
β
γ
δ


commutation monoid
A*/≡C
dependency graph
heap monoid
Heap (P, C)
poset
equivalence class
heap
word
linear extension of the poset

Complements

Heaps, poset and graphs

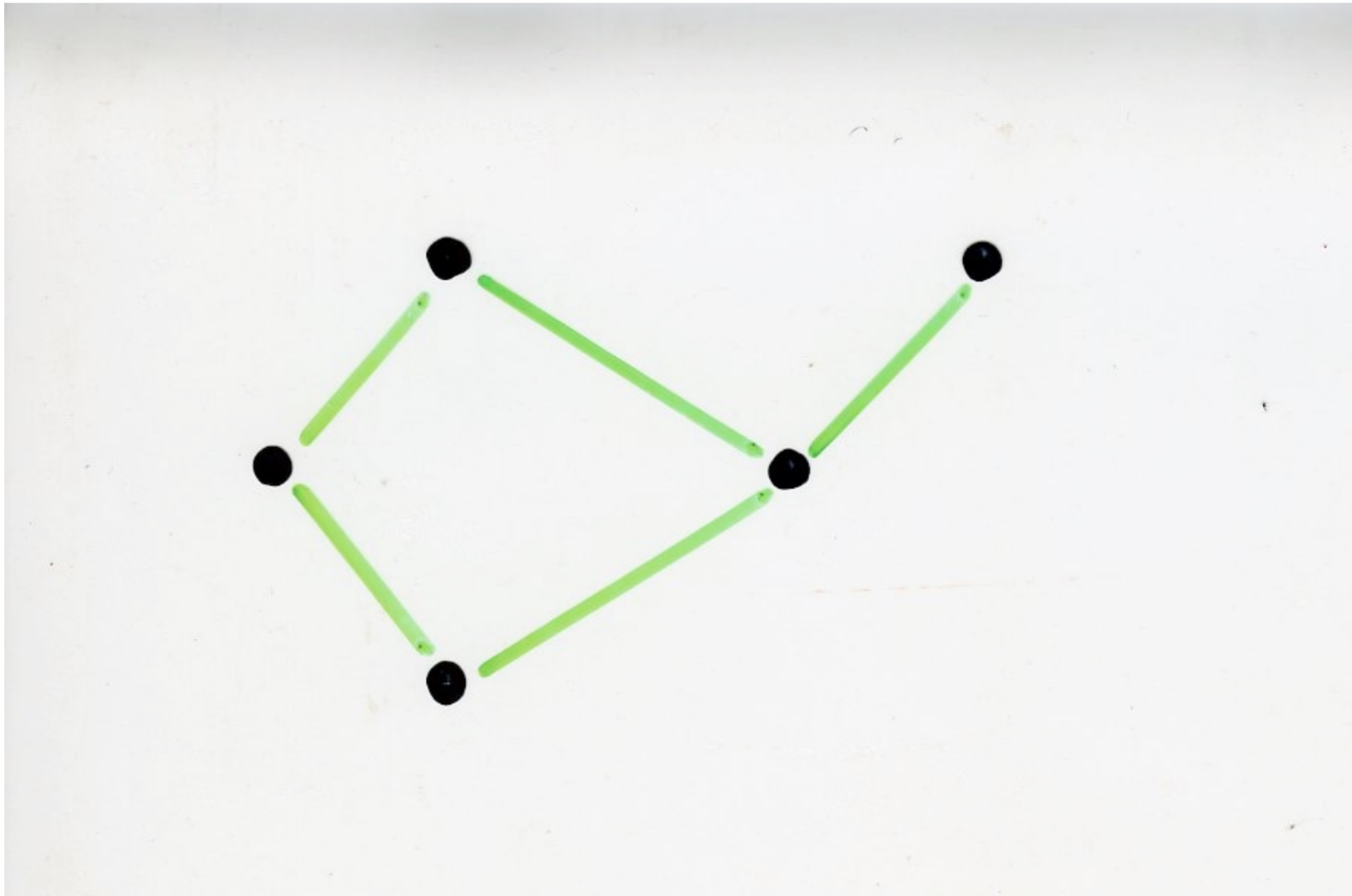
Prop. every poset can be realized as a heap of pieces

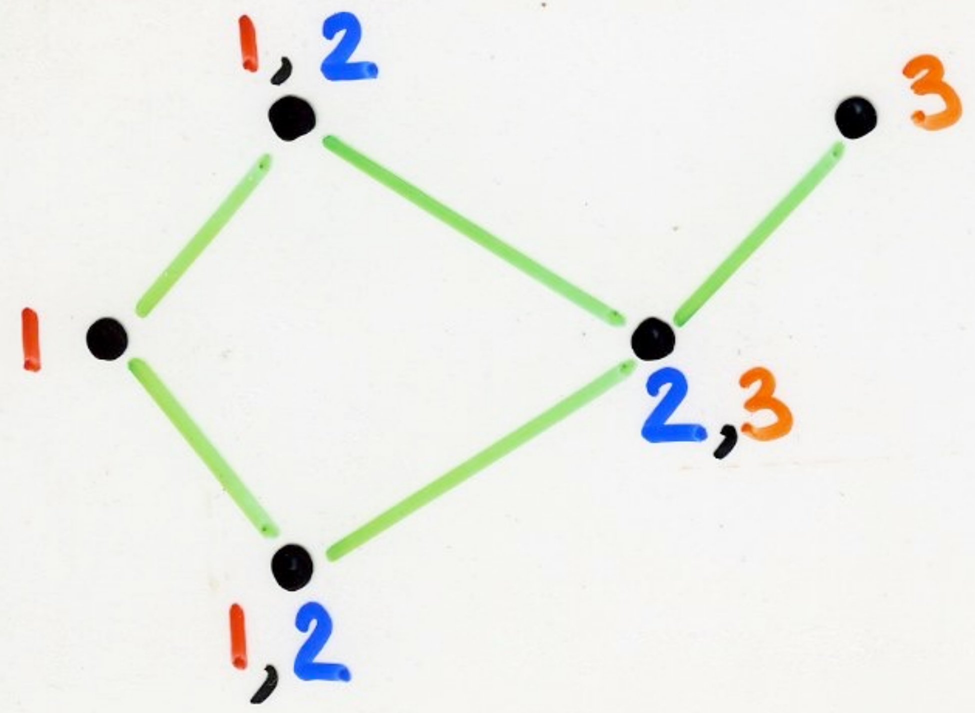
Def E poset

Γ set of chains of E

Γ strongly covers E iff :

$\forall s,t \in E$, $s \prec t$ and t covers s, $\exists \gamma \in \Gamma$ such that $s,t \in \gamma$

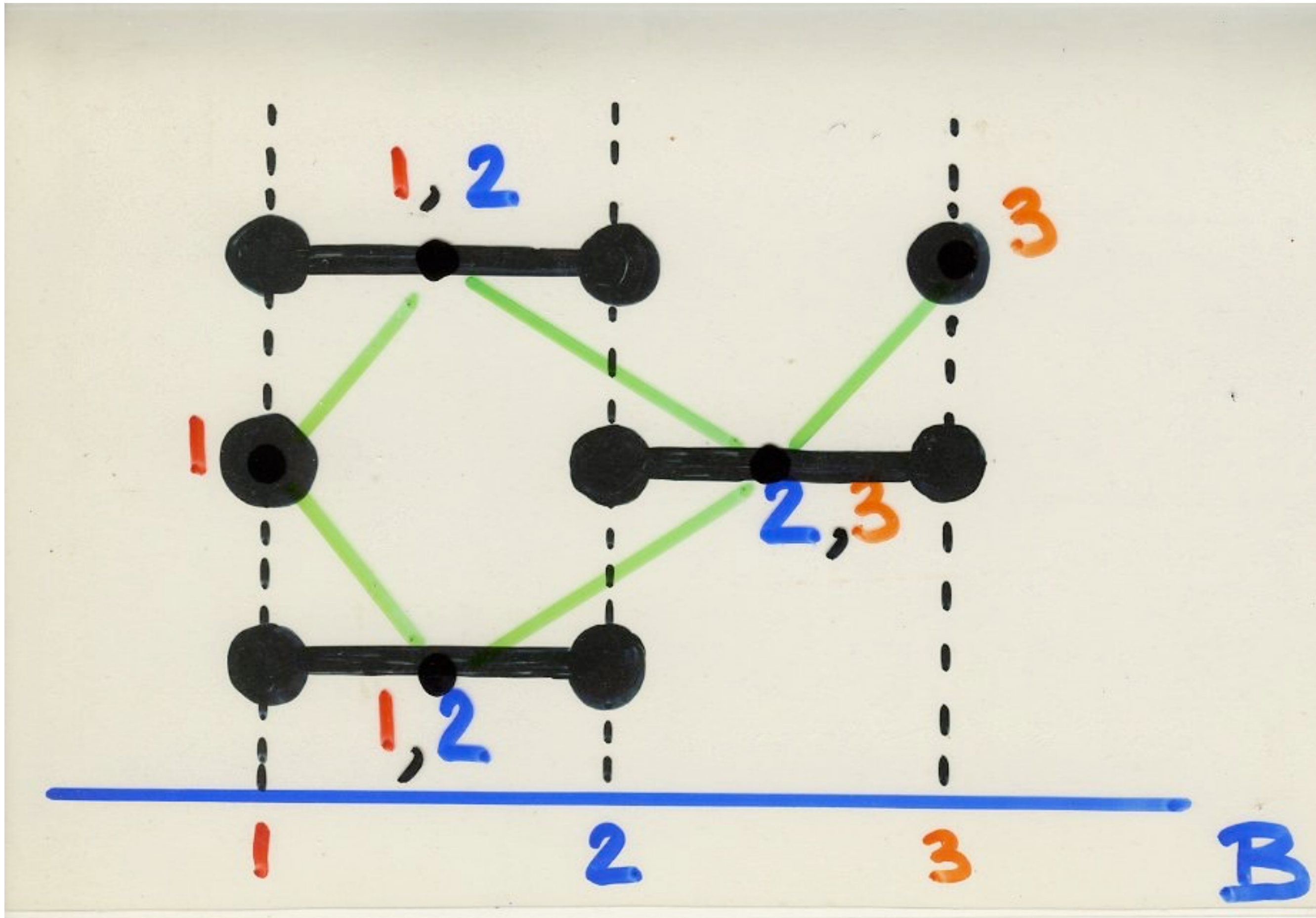



1, 2
3
1
2, 3
1, 2

E base $B = \Gamma$

pièces $P = \mathcal{P}(B)$ $\pi = \mathrm{Id}$

$s \in E \longrightarrow P_s = \{ \gamma \in \Gamma,\ s \in \gamma \}$



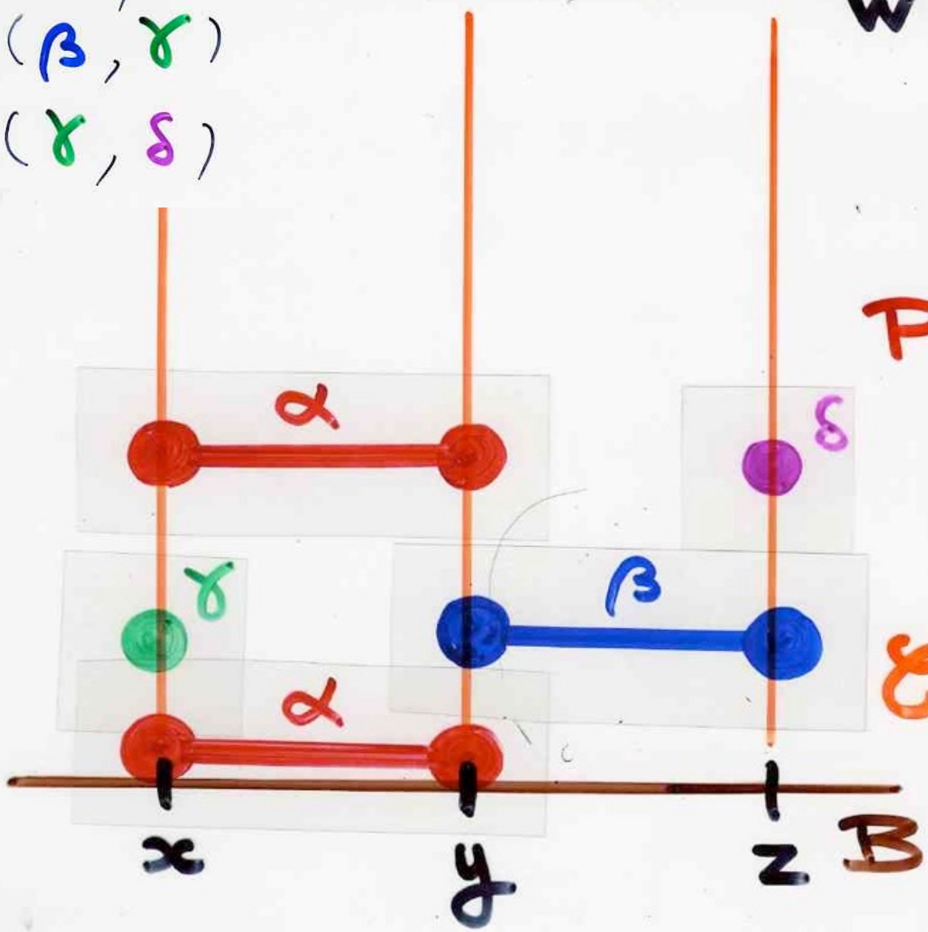
commutations

$C = \overline{\tau}$

$$C \begin{cases} (\alpha, \delta) \\ (\beta, \gamma) \\ (\gamma, \delta) \end{cases}$$

$$w = \alpha\beta\gamma\alpha\delta$$

$$P \begin{cases} \alpha = \{x, y\} \\ \beta = \{y, z\} \\ \gamma = \{x\} \\ \delta = \{z\} \end{cases}$$



$$\tau \begin{cases} (\alpha, \beta) \\ (\alpha, \gamma) \\ (\beta, \delta) \end{cases}$$

Corollary For any poset E, counting the number of linear extensions of E is the same problem as counting the number of words in a commutation class of a commutation monoid.

→ number of Young tableaux
hook-length formula ....

Proposition Every heap monoid is isomorphic to a "heap of subsets of a set X" monoid.

ex: subsets of a set X

- $\mathcal{P}$ set of subsets of X

  basic pieces

  $$\mathcal{P} \subseteq \mathscr{P}(X)$$

- $\mathcal{C}$ dependency relation

  $$A, B \in \mathcal{P}, \quad A \,\mathcal{C}\, B \Leftrightarrow A \cap B \neq \emptyset$$