An introduction to

enumerative
algebraic combinatorics
bijective

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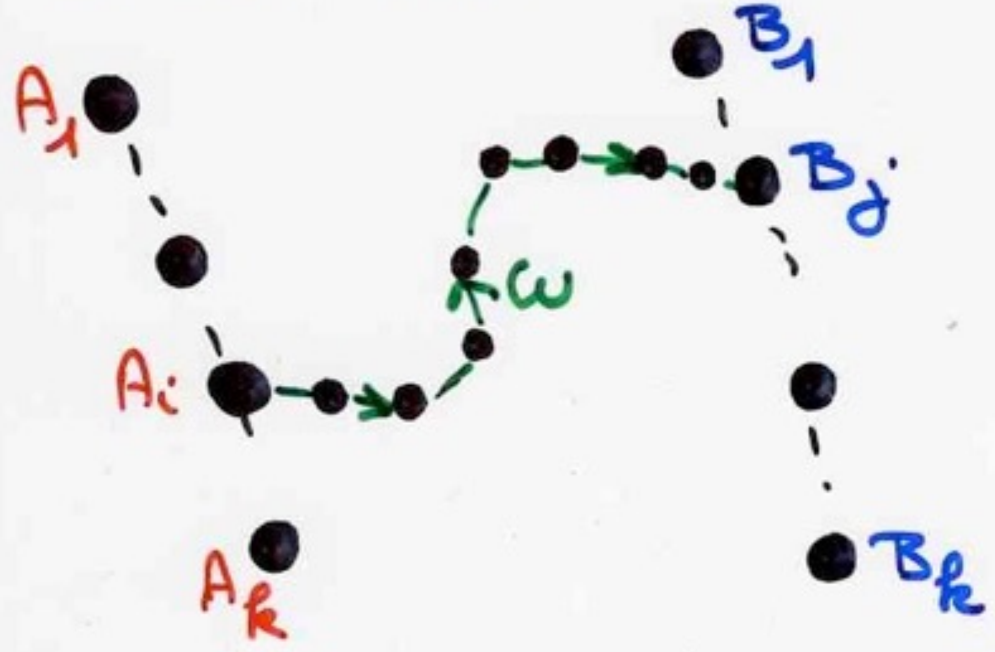
Chapter 5

Tilings, determinants and non-crossing paths (2)

IMSc

3 March 2016

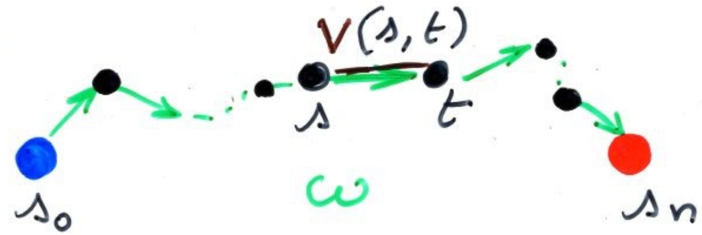
from the previous lecture



$A_1, \dots, A_k$

$B_1, \dots, B_k$

$$a_{ij} = \sum_{A_i \rightsquigarrow B_j} v(\omega)$$



weighted path

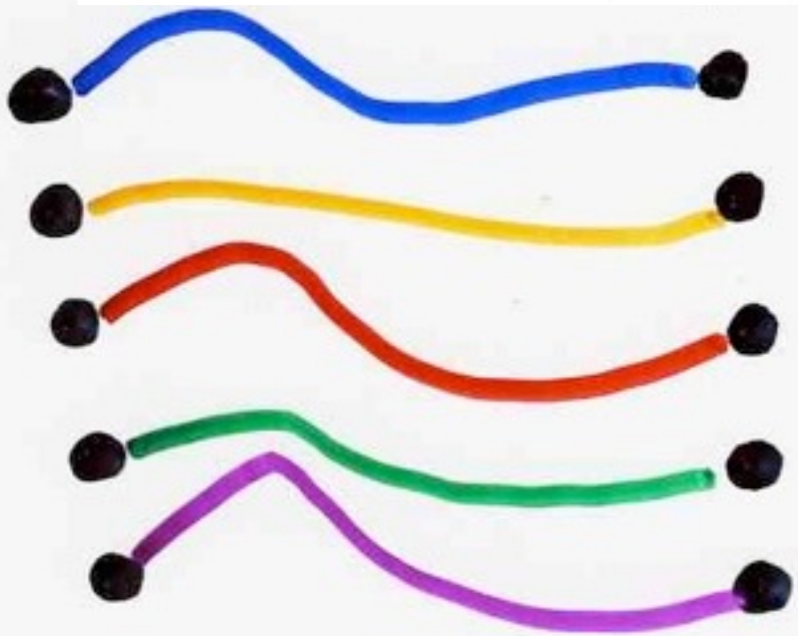
suppose finite sum

Proposition (LGV Lemma) (C) crossing condition

$$\det(a_{ij}) = \sum_{(\omega_1, \dots, \omega_k)} v(\omega_1) \dots (\omega_k)$$

$\omega_i : A_i \rightsquigarrow B_i$

non-intersecting

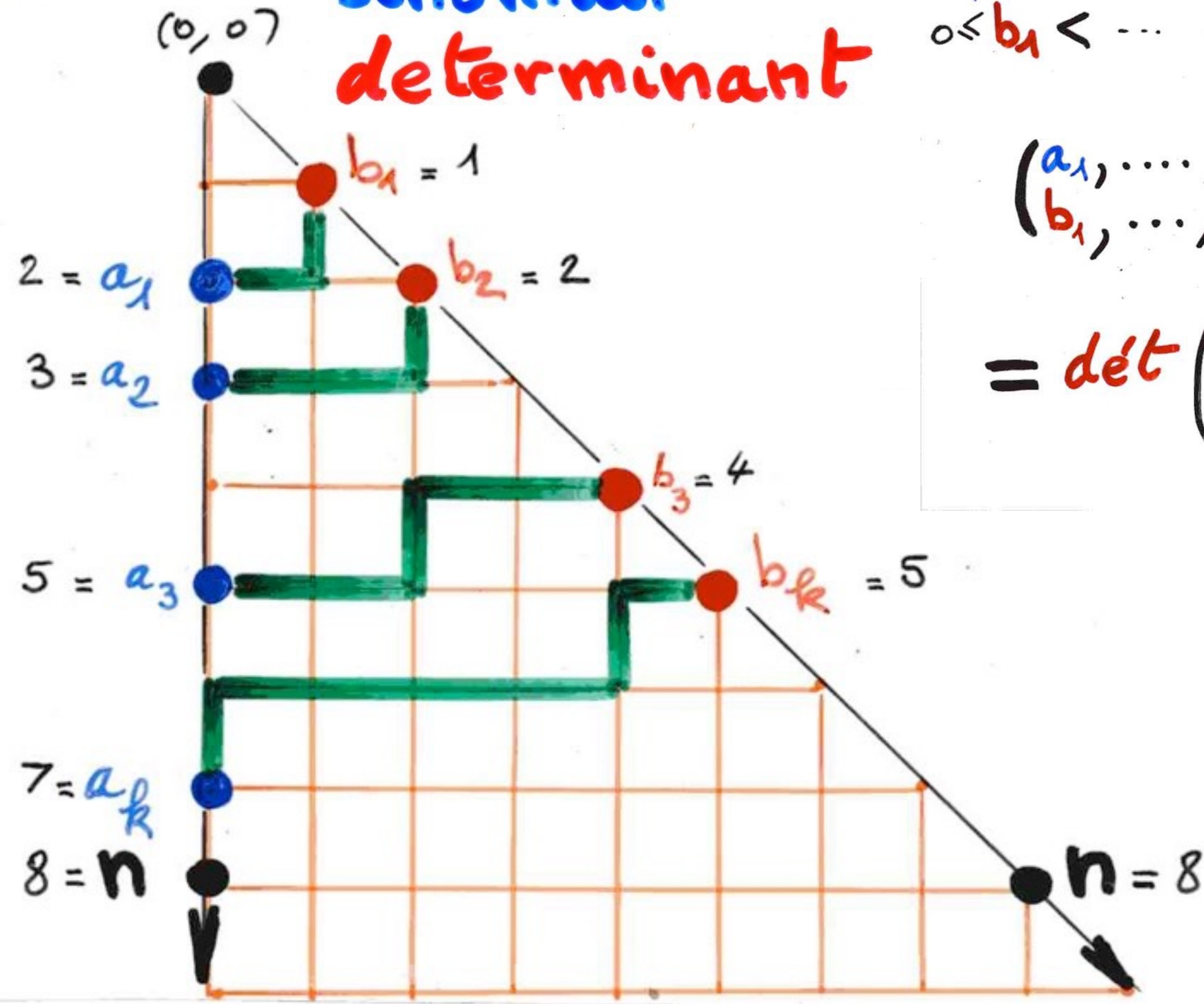


binomial determinant

$0 \le a_1 < \cdots < a_k$

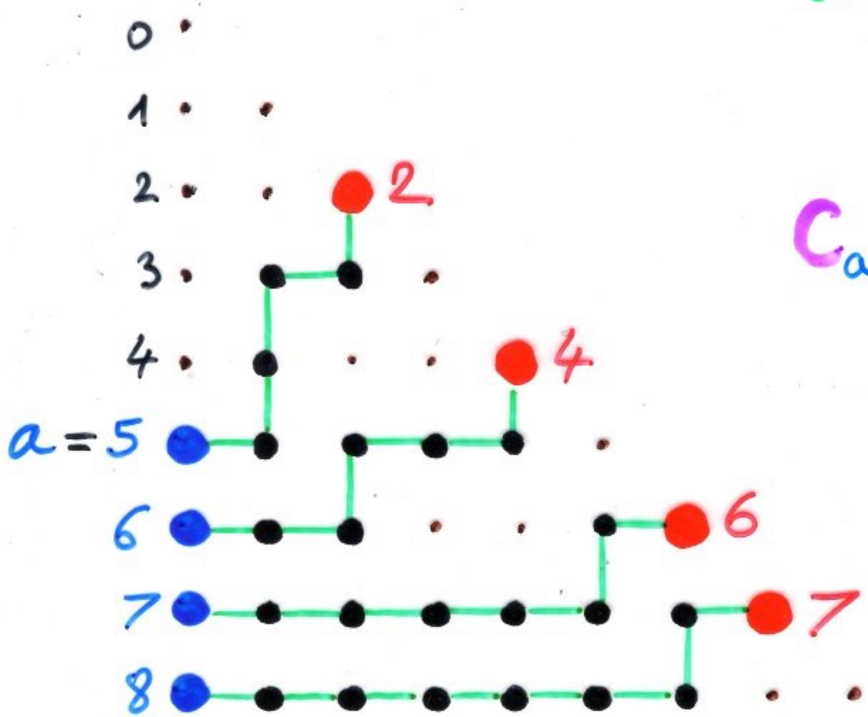
$0 \le b_1 < \cdots < b_k$

$$\binom{a_1, \ldots, a_k}{b_1, \ldots, b_k} = \text{dét}\left(\binom{a_i}{b_j}\right)_{1 \le i,j \le k}$$



Proposition

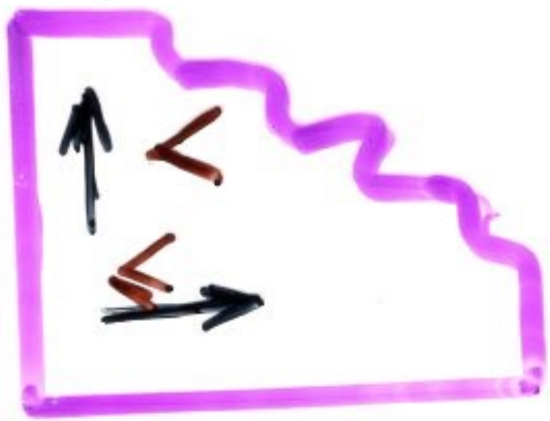
$$\binom{a,\ a+1,\ \dots\ ,\ a+k-1}{b_1,\ b_2,\ \dots\ ,\ b_k} = \frac{C_a(\mu)}{H(\mu)}$$



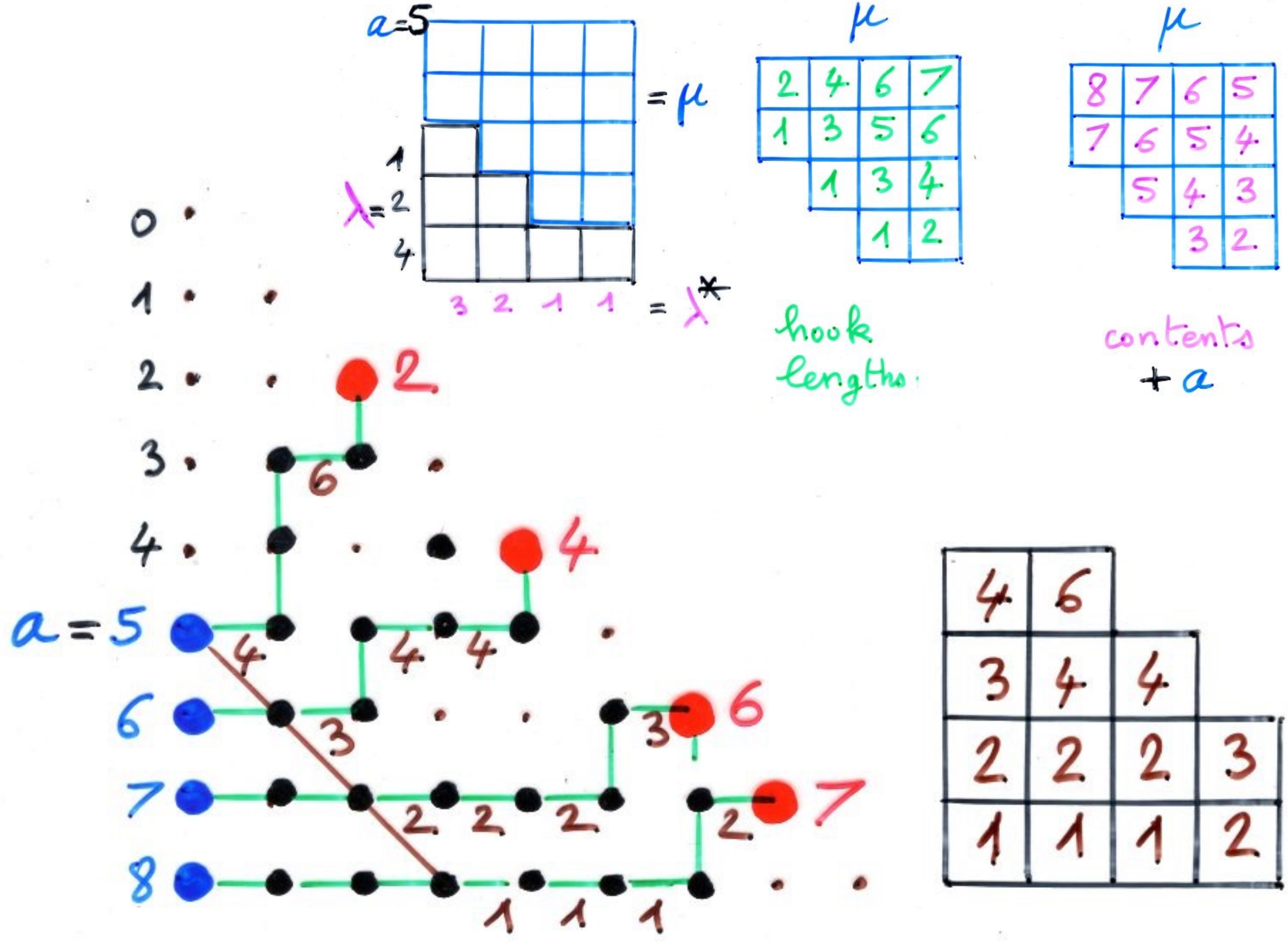
$H(\mu)$ = product of hook-lengths of μ

$C_a(\mu)$ = product of contents of μ augmented by a

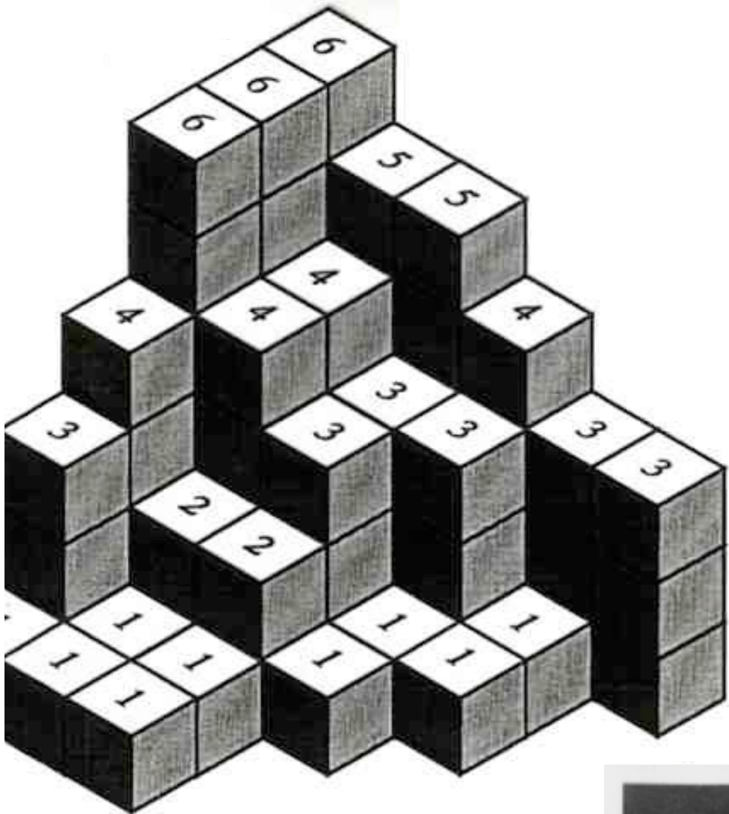
semi-standard
Young tableaux



4	6		
3	4	4	
2	2	2	3
1	1	1	2



a=5
= μ
λ = 1 2 4
3 2 1 1 = λ*
μ
2 4 6 7
1 3 5 6
1 3 4
1 2
hook lengths
μ
8 7 6 5
7 6 5 4
5 4 3
3 2
contents + a
0 1 2 3 4
a=5
6 7 8
2 4 6 7
4 6
3 4 4
2 2 2 3
1 1 1 2

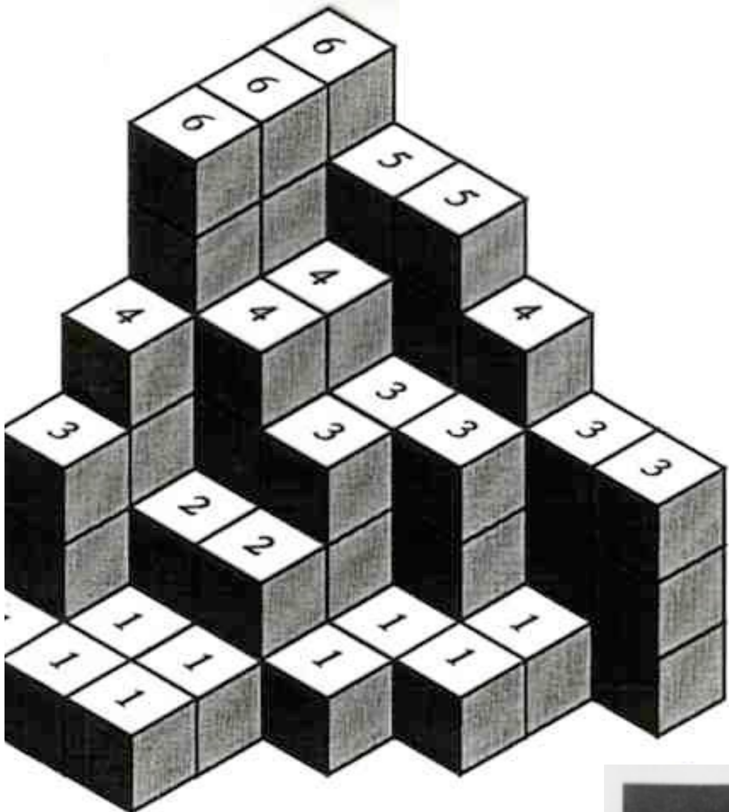


6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

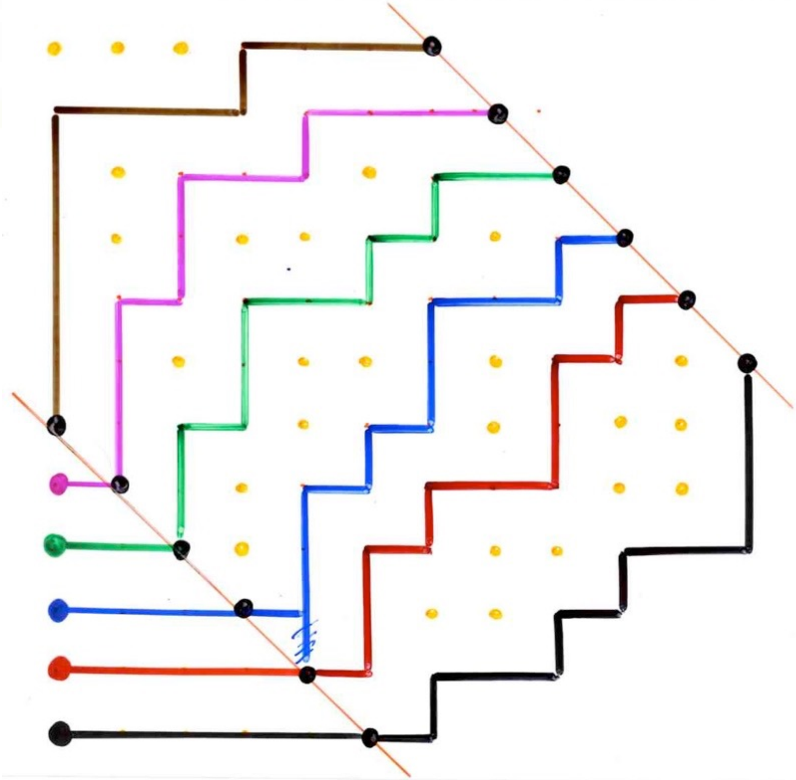
plane partitions

3D Ferrers diagrams

in a box $\mathcal{B}(a,b,c)$



6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			



$$\prod_{\substack{1\le i\le a\\ 1\le j\le b\\ 1\le k\le c}} \frac{i+j+k-1}{i+j+k-2}$$

Jacobi identities
for
Schur functions

symmetric polynomials $\mathbb{K}[x_1, \ldots, x_n]$

$$P(x_1, \ldots, x_n)$$

$\sigma \in \mathfrak{S}_n$

$$P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = P(x_1, \ldots, x_n)$$

$\rightarrow$ complements Ch4c
plactic monoid, product of Schur functions

Def. Homogeneous (or complete) symmetric functions

$$h_p(x_1,\dots,x_m) = \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_m) \\ \text{compositions of } p \\ (\alpha_i \geq 0\ ,\ \alpha_1+\dots+\alpha_m=p)}} x_1^{\alpha_1}\cdots x_m^{\alpha_m}$$

$\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ partition

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}$$

basis of the space of symmetric polynomials in $x_1,\dots,x_n$

Def: symmetric elementary function

$$e_p = \sum_{1 \le i_1 < i_2 < \cdots < i_p \le m} x_{i_1} \cdots x_{i_p}$$

$\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ partition

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$$

basis of the space of symmetric polynomials in $x_1, \dots, x_n$

Schur Functions

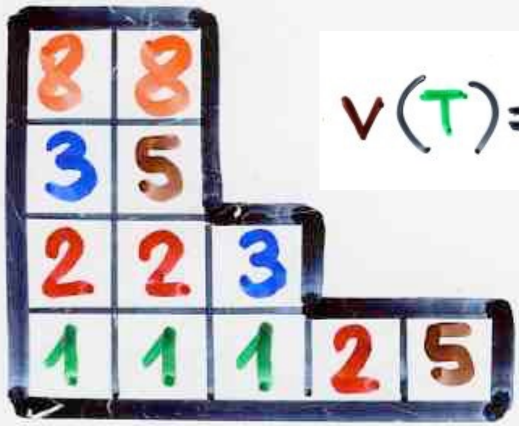
$$s_\lambda(x_1, x_2, \dots, x_m) = \sum_{T} v(T)$$

T Young tableau
shape λ
entries $1, 2, \dots, m$

Jacobi (1841)
Schur (1901)

Littlewood-Richardson (1934)

$$v(T) = \prod_{\substack{\text{cells}\\ \text{of } \lambda}} x_{i(\text{cell})}$$



$$v(T) = x_1^3\, x_2^3\, x_3^2\, x_5^2\, x_8^2$$

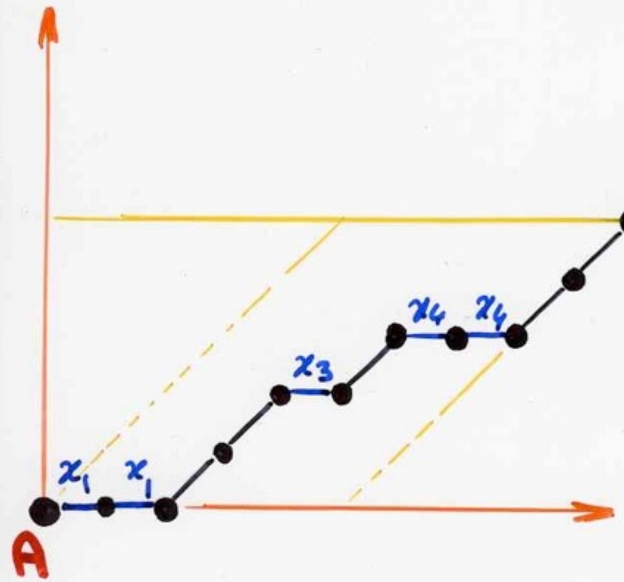
$$\underset{\text{cell}}{\boxed{i}} \longrightarrow x_i$$

basis of the space of symmetric polynomials in $x_1, \dots, x_n$

Def. Homogeneous (or complete) symmetric functions

$$h_p(x_1,\dots,x_m) = \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_m) \\ \text{compositions of } p \\ (\alpha_i \geq 0,\ \alpha_1+\dots+\alpha_m=p)}} x_1^{\alpha_1}\cdots x_m^{\alpha_m}$$

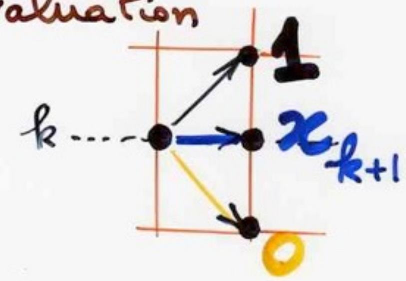
Lemma $$h_p(x_1,\dots,x_m) = \sum_{\substack{\omega \\ \text{Motzkin path} \\ \omega : A \rightsquigarrow B}} v(\omega)$$



$A=(0,0)$

$B=(p+m-1,\ m-1)$

valuation



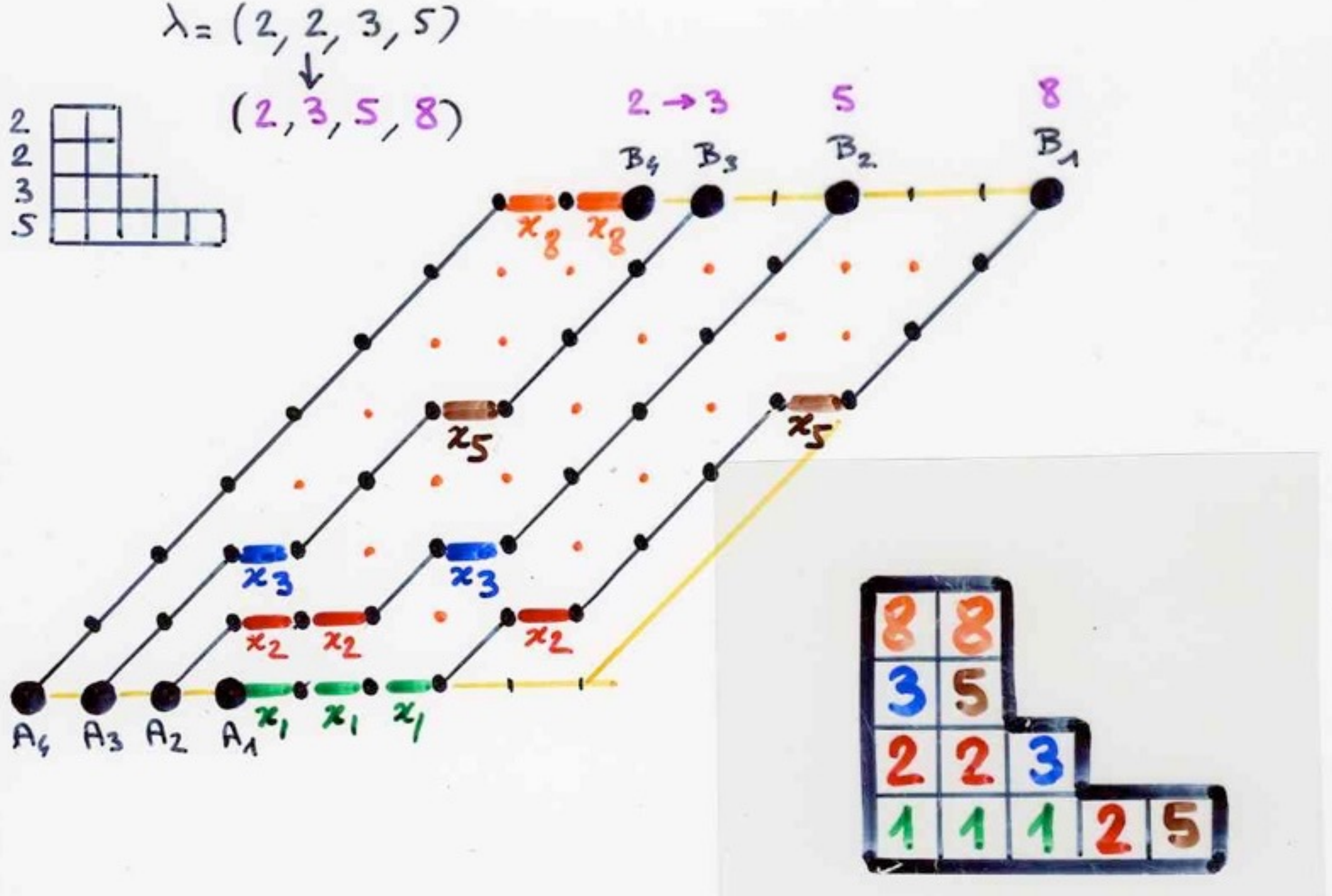
Jacobi - Trudi

$$\det\left(h_{\lambda_i - i + j}\right)_{1 \le i, j \le r} = S_\lambda(x_1, \dots, x_m)$$

Schur

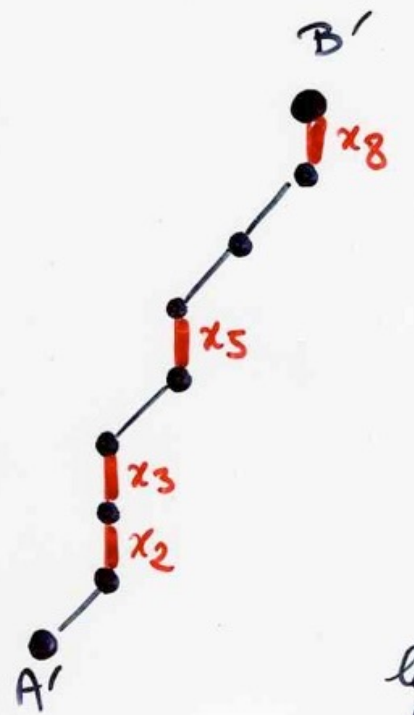
$$\begin{vmatrix} h_5 & h_6 & h_7 & h_8 \\ h_2 & h_3 & h_4 & h_5 \\ h_0 & h_1 & h_2 & h_3 \\ h_{-1} & h_0 & h_1 & h_2 \end{vmatrix}$$

transpose


λ= (2, 2, 3, 5)
(2, 3, 5, 8)
2
2
3
5
2 → 3
5
8
B4
B3
B2
B1
x8
x8
x5
x5
x3
x3
x2
x2
x2
x1
x1
x1
A4
A3
A2
A1
8 8
3 5
2 2 3
1 1 1 2 5

Def: symmetric elementary function

$$e_p = \sum_{1 \le i_1 < i_2 < \cdots < i_p \le m} x_{i_1} \cdots x_{i_p}$$



$$e_p = \sum_{\omega} v(\omega)$$

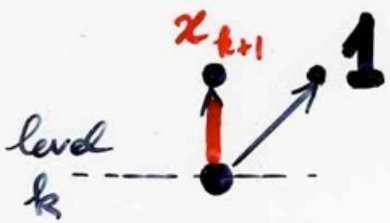
"Favard" path

$\omega : A' \rightsquigarrow B'$

$A' = (0,0)$

$B' = (m, m-p)$

valuation :

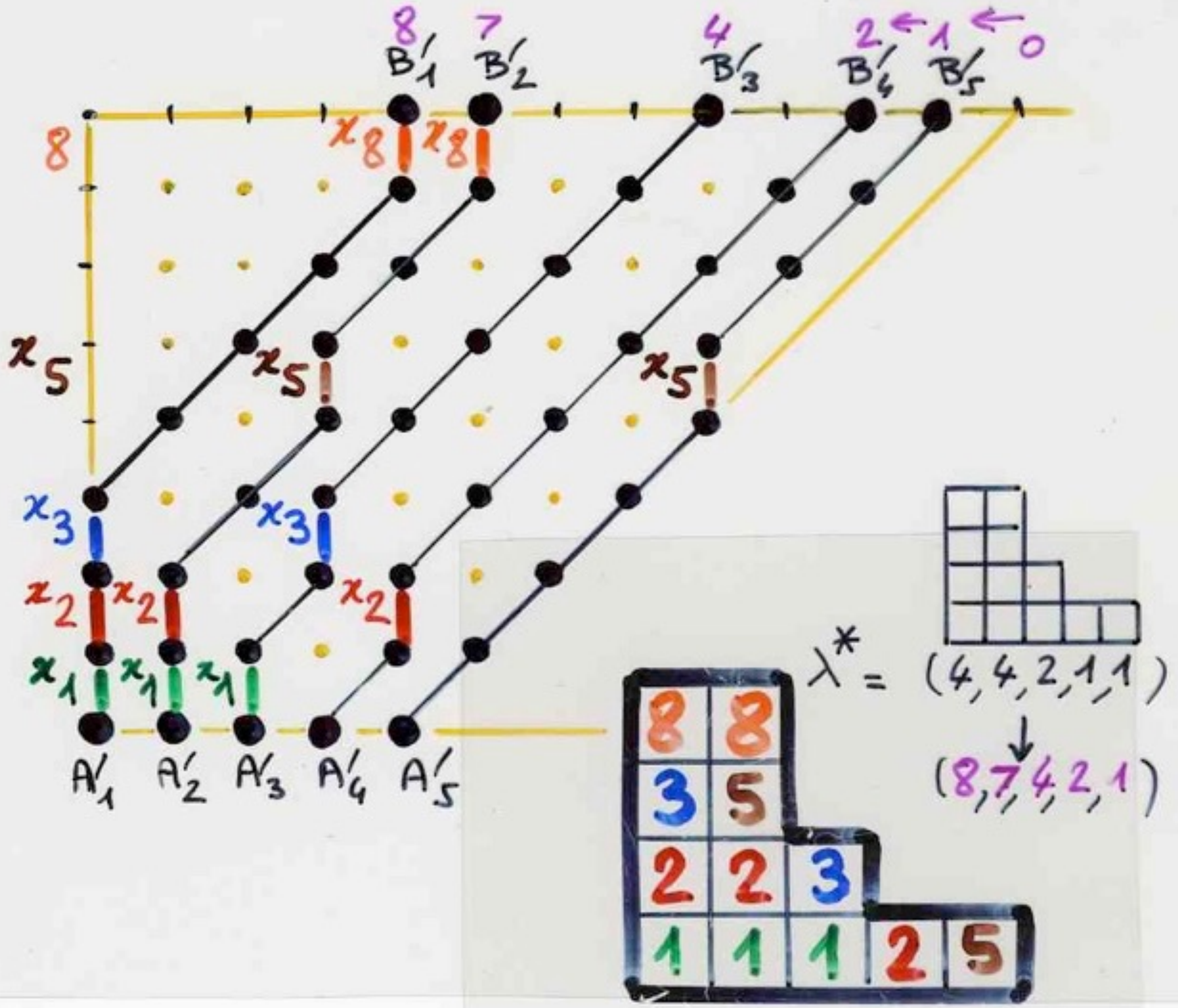


$$\det\left(e_{\lambda'_i - i + j}\right) = S_\lambda(x_1, \dots, x_m)$$

Schur

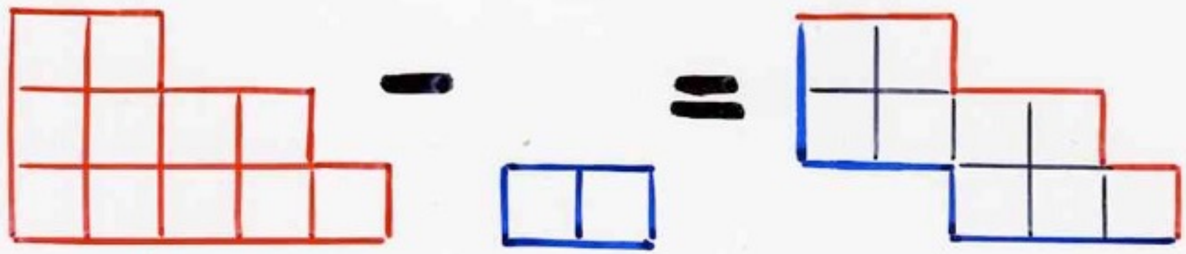
$$\begin{vmatrix} e_4 & e_5 & e_6 & e_7 & e_8 \\ e_3 & e_4 & e_5 & e_6 & e_7 \\ e_0 & e_1 & e_2 & e_3 & e_4 \\ e_{-2} & e_{-1} & e_0 & e_1 & e_2 \\ e_{-3} & e_{-2} & e_{-1} & e_0 & e_1 \end{vmatrix}$$

transpose


8 7 4 2 1 0
B'1 B'2 B'3 B'4 B'5
8
x8 x8
x5 x5 x5
x3 x3
x2 x2 x2
x1 x1 x1
A'1 A'2 A'3 A'4 A'5
λ* = (4, 4, 2, 1, 1)
(8, 7, 4, 2, 1)
8 8
3 5
2 2 3
1 1 1 2 5

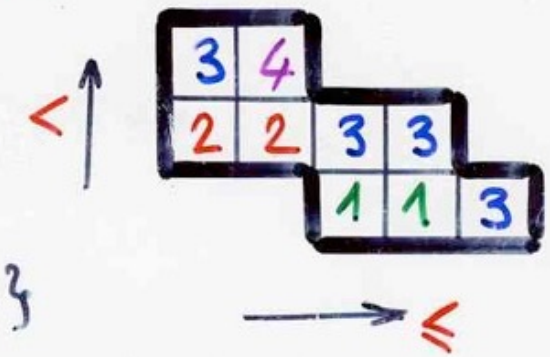
Jacobi identities
for
skew Schur functions

Def- skew-Ferrers diagram λ/μ



λ μ $\mu \leq \lambda$

Def- **Tableau** T



entries
$\{1, 2, 3, \ldots, m\}$

weight
$v(T) = x_1^2 x_2^2 x_3^4 x_4$

Def. skew - **Schur** function

$$S_{\lambda/\mu}(x_1, \dots, x_m) = \sum_{T} v(T)$$

tableau
shape λ/μ

- **Schur** function $S_{\lambda}(x_1, \dots, x_m)$
  $\mu = \emptyset$

Prop. $S_{\lambda/\mu}(x_1,\dots,x_m) = \det\left(h_{\lambda_i-\mu_j-i+j}\right)_{1\le i,j\le r}$

($r \geq$ nb of parts of λ)

$\mu=\emptyset$ **Jacobi-Trudi**

$\lambda=(5,4,2)$

$\mu=(2,0,0)$

$$\det\left(h_{\lambda_i-\mu_j-i+j}\right)_{1\le i,j\le r} = \begin{vmatrix} h_3 & h_6 & h_7 \\ h_1 & h_4 & h_5 \\ h_{-2} & h_1 & h_2 \end{vmatrix}$$

$\tilde{H}$ transpose

ex – $\mu = (2,0,0)$
$\lambda = (5,4,2)$

$$\begin{array}{|c|c|c|c|c|}
\hline
3 & 4 & & & \\
\hline
2 & 2 & 3 & 3 & \\
\hline
 & & 1 & 1 & 3 \\
\hline
\end{array}$$

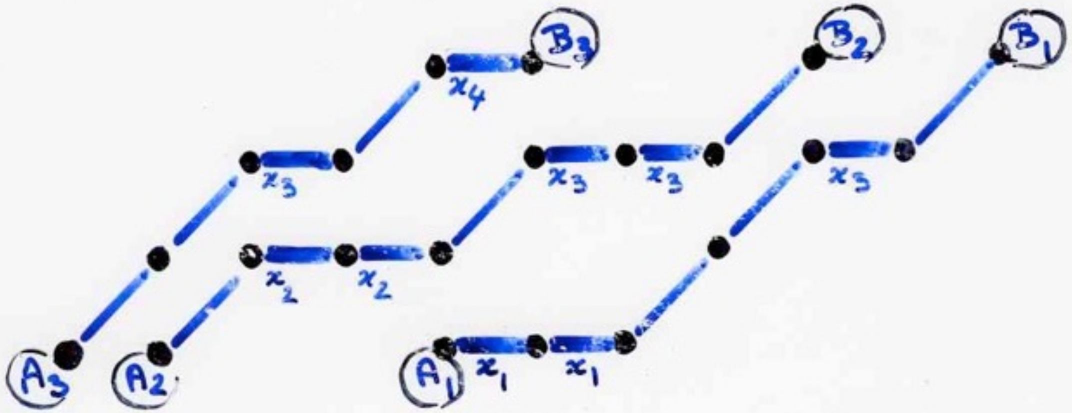
$m = 4$
$r = 3$

$$\mu^{\#} = (2+2,\ 0+1,\ 0+0) = (4,1,0)$$

$$\lambda^{\#} = (5+2,\ 4+1,\ 2+0) = (7,5,2)$$

$A_1 = (4,0)$ $\quad A_2 = (1,0)$ $\quad A_3 = (0,0)$

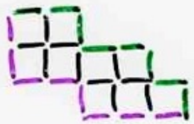
$B_1 = (7+3,3)$ $\quad B_2 = (5+3,3)$ $\quad B_3 = (2+3,3)$



Prop $$s_{\lambda/\mu}(x_1, \dots, x_m) = \det\left(e_{\lambda'_i - \mu'_j - i + j}\right)_{1 \le i,j \le s}$$

$s \geq$ nb of columns λ

$\left.\begin{matrix}\lambda' \\ \mu'\end{matrix}\right)$ conjugate partitions

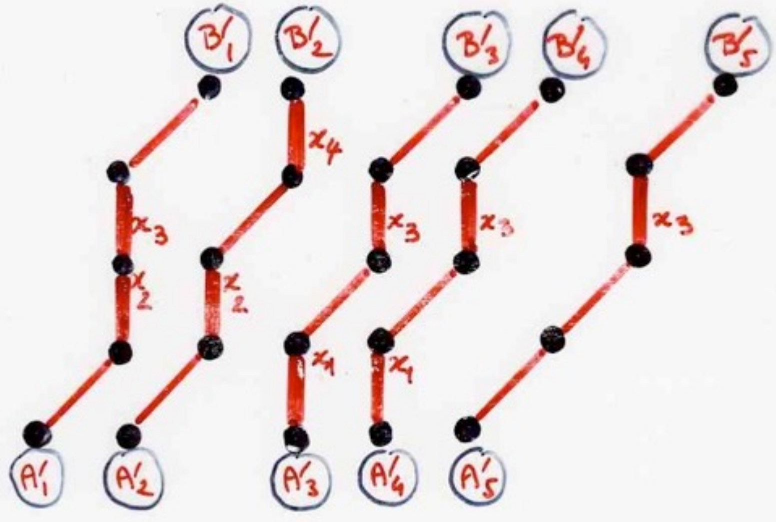


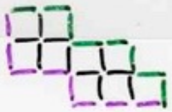
$\lambda' = (3, 3, 2, 2, 1)$

$\mu' = (1, 1, 0, 0, 0)$

$$\det\left(e_{\lambda'_i - \mu'_j - i + j}\right)_{1 \le i,j \le s} = \begin{vmatrix} e_2 & e_3 & e_5 & e_6 & e_7 \\ e_1 & e_2 & e_4 & e_5 & e_6 \\ e_{-1} & e_0 & e_2 & e_3 & e_4 \\ e_{-2} & e_{-1} & e_1 & e_2 & e_3 \\ e_{-4} & e_{-3} & e_{-1} & e_0 & e_1 \end{vmatrix}$$

$\tilde{E}$ transpose



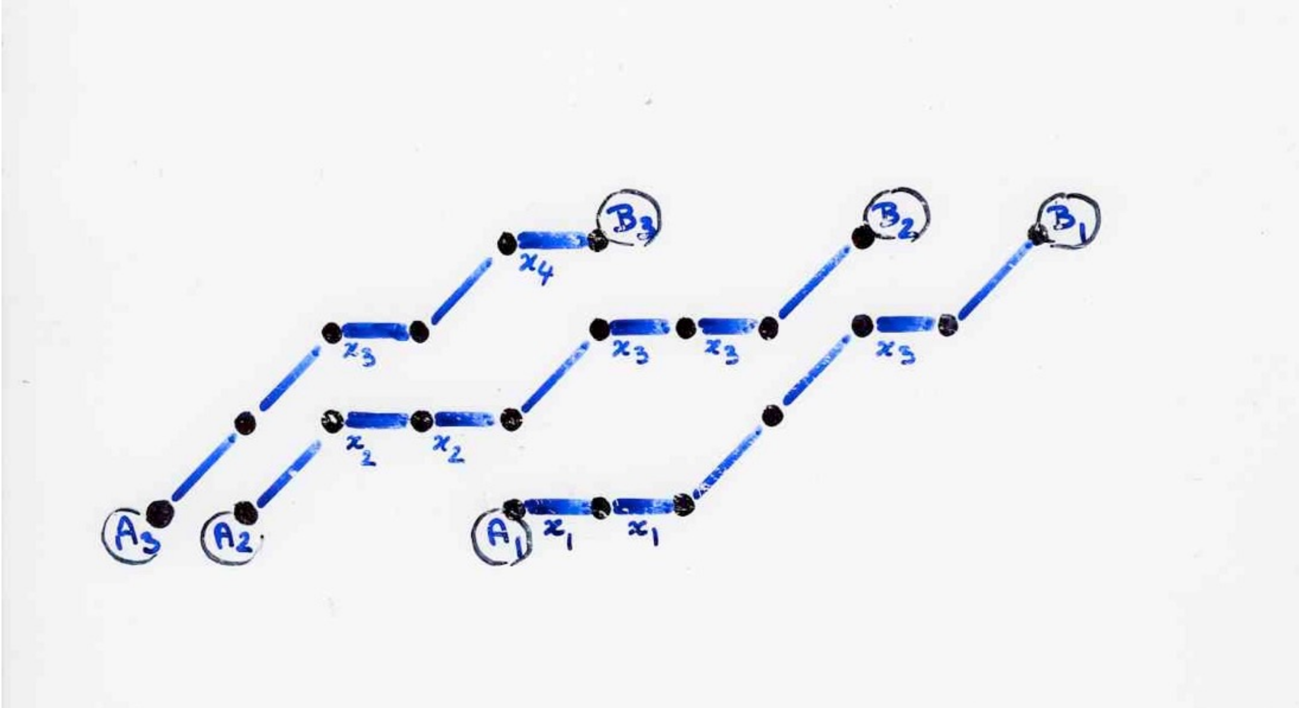


$\lambda' = (3, 3, 2, 2, 1)$

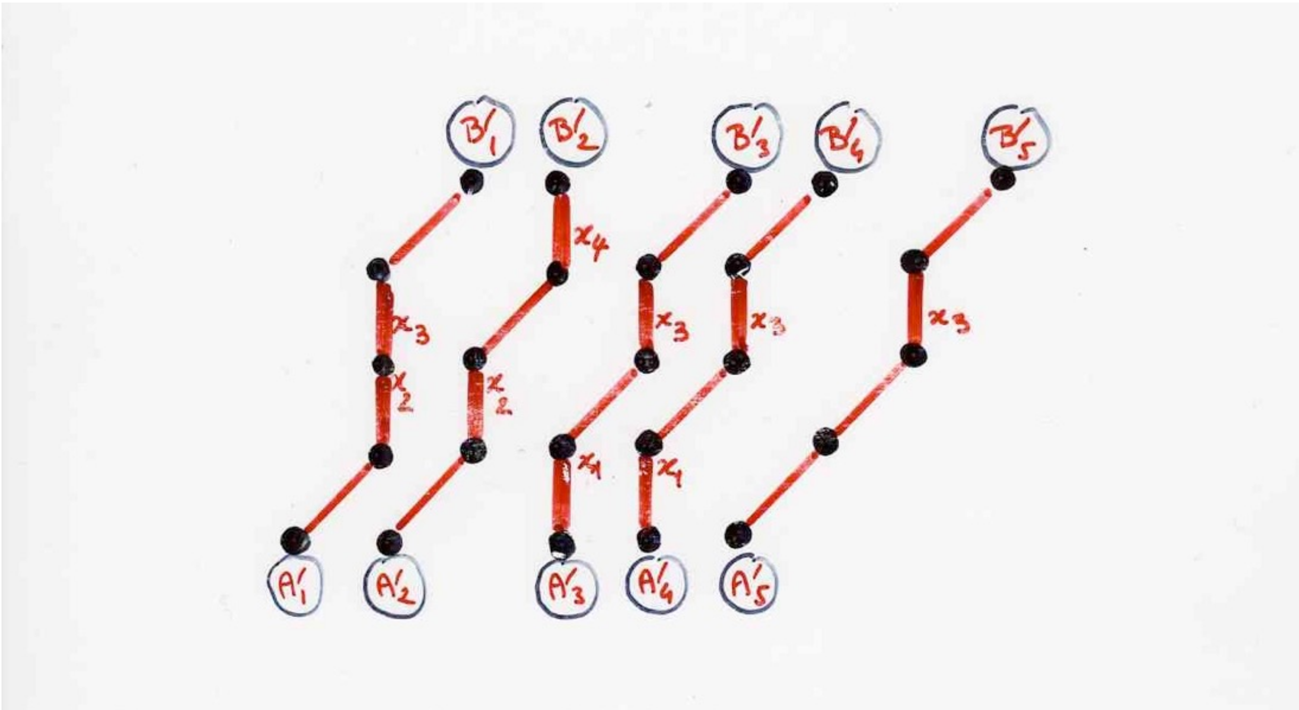
$\mu' = (1, 1, 0, 0, 0)$

$$\det\left(e_{\lambda'_i - \mu'_j - i + j}\right)_{1\le i,j\le 5} = \begin{vmatrix} e_2 & e_3 & e_5 & e_6 & e_7 \\ e_1 & e_2 & e_4 & e_5 & e_6 \\ e_{-1} & e_0 & e_2 & e_3 & e_4 \\ e_{-2} & e_{-1} & e_1 & e_2 & e_3 \\ e_{-4} & e_{-3} & e_{-1} & e_0 & e_1 \end{vmatrix}$$

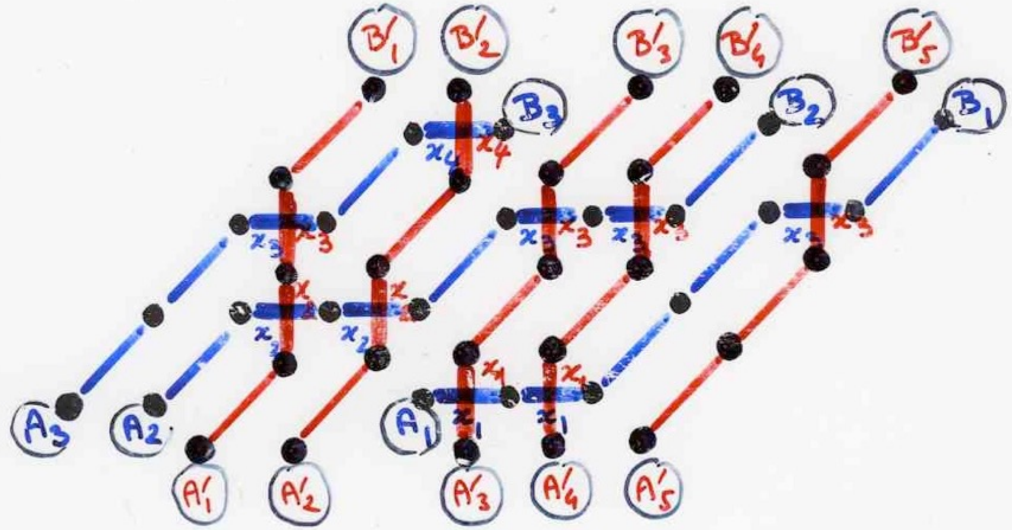
$\tilde{E}$ transpose



B_3
B_2
B_1
x_4
x_3
x_3
x_3
x_3
x_2
x_2
A_3
A_2
A_1
x_1
x_1



B'_1
B'_2
B'_3
B'_4
B'_5
x_4
x_3
x_3
x_3
x_3
x_2
x_2
x_1
x_1
A'_1
A'_2
A'_3
A'_4
A'_5

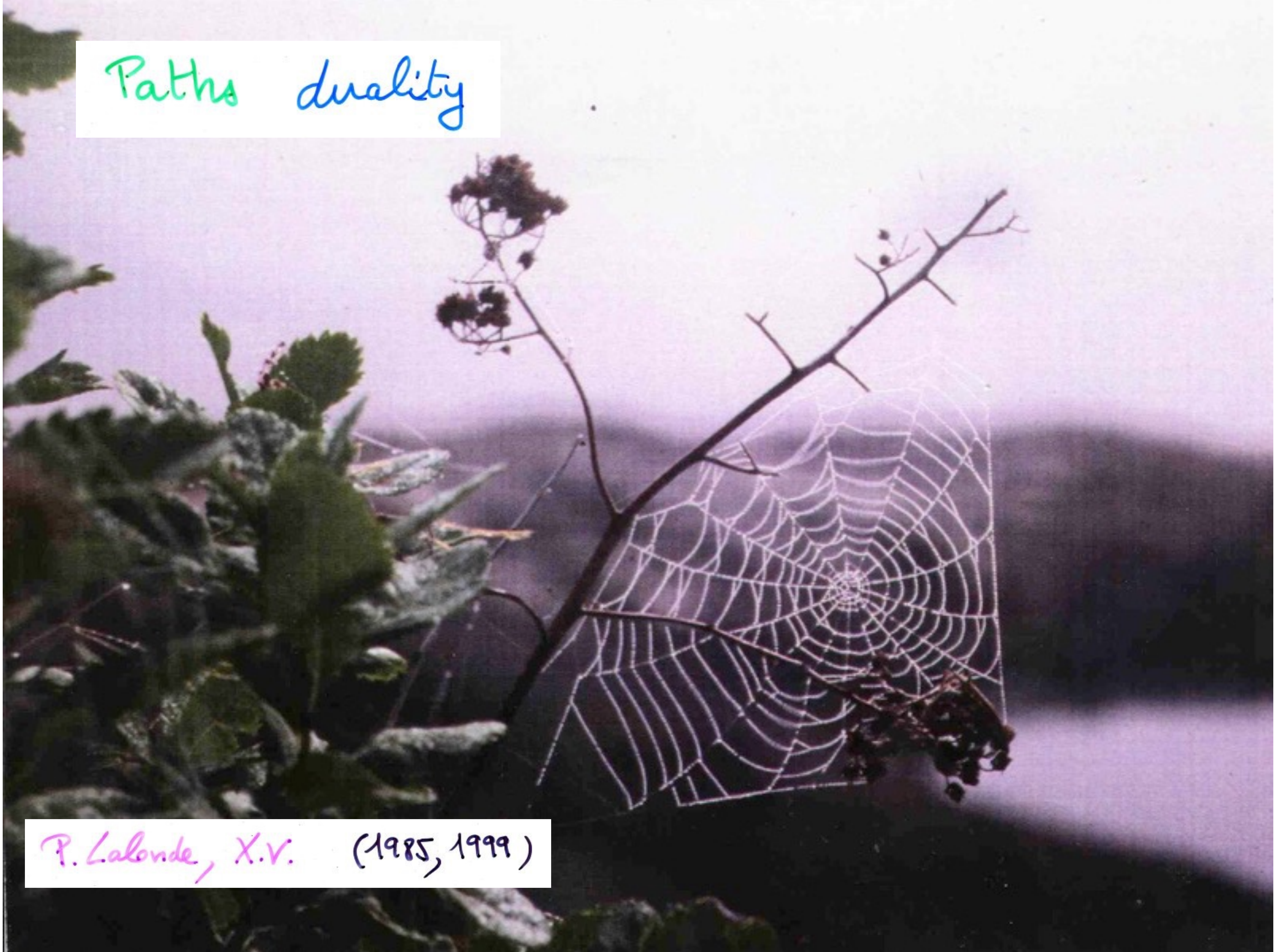


B'1
B'2
B'3
B'4
B'5
B3
B2
B1
A3
A2
A1
A'1
A'2
A'3
A'4
A'5

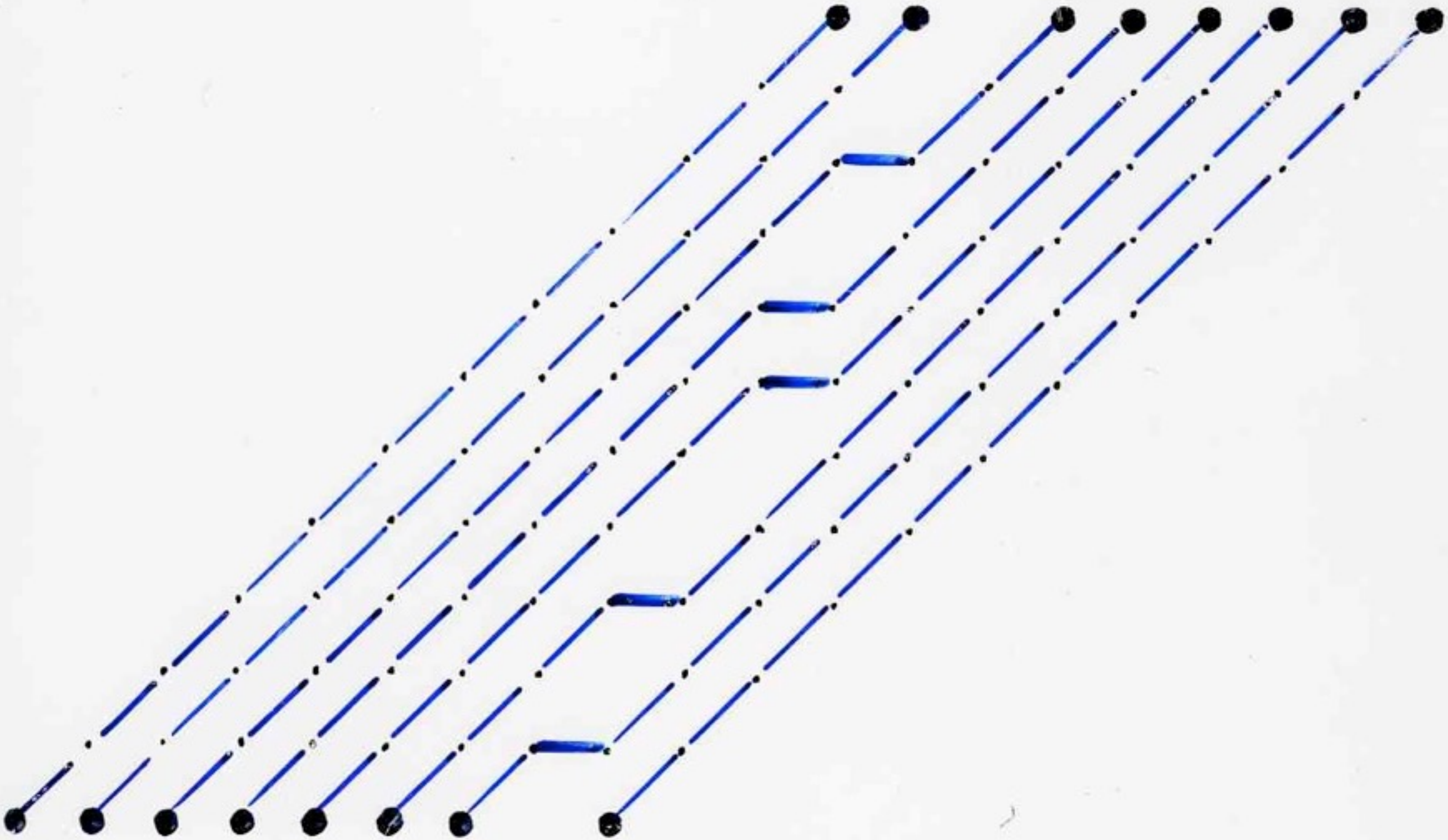
Duality

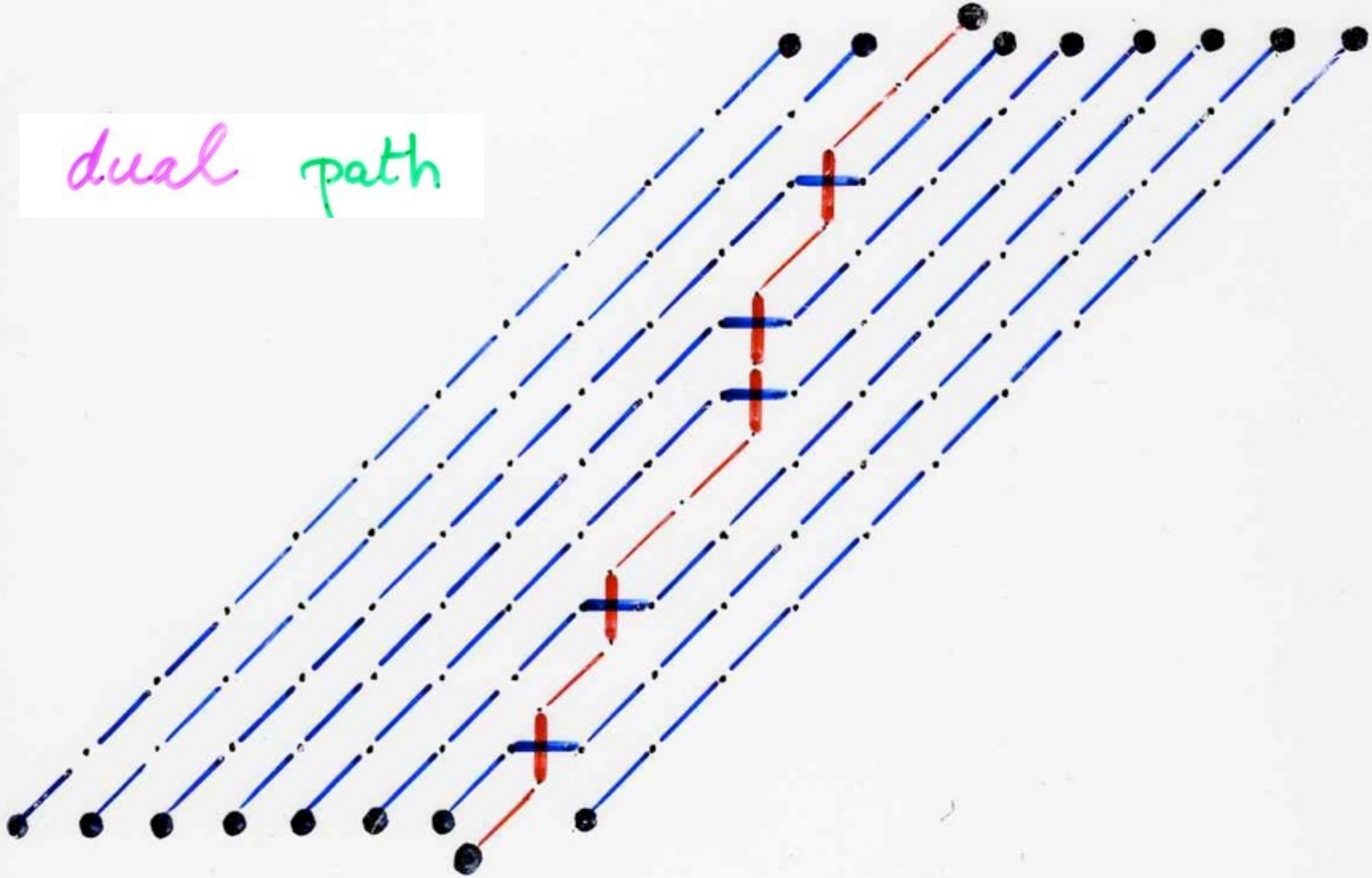
(the idea of duality in paths)

Paths duality

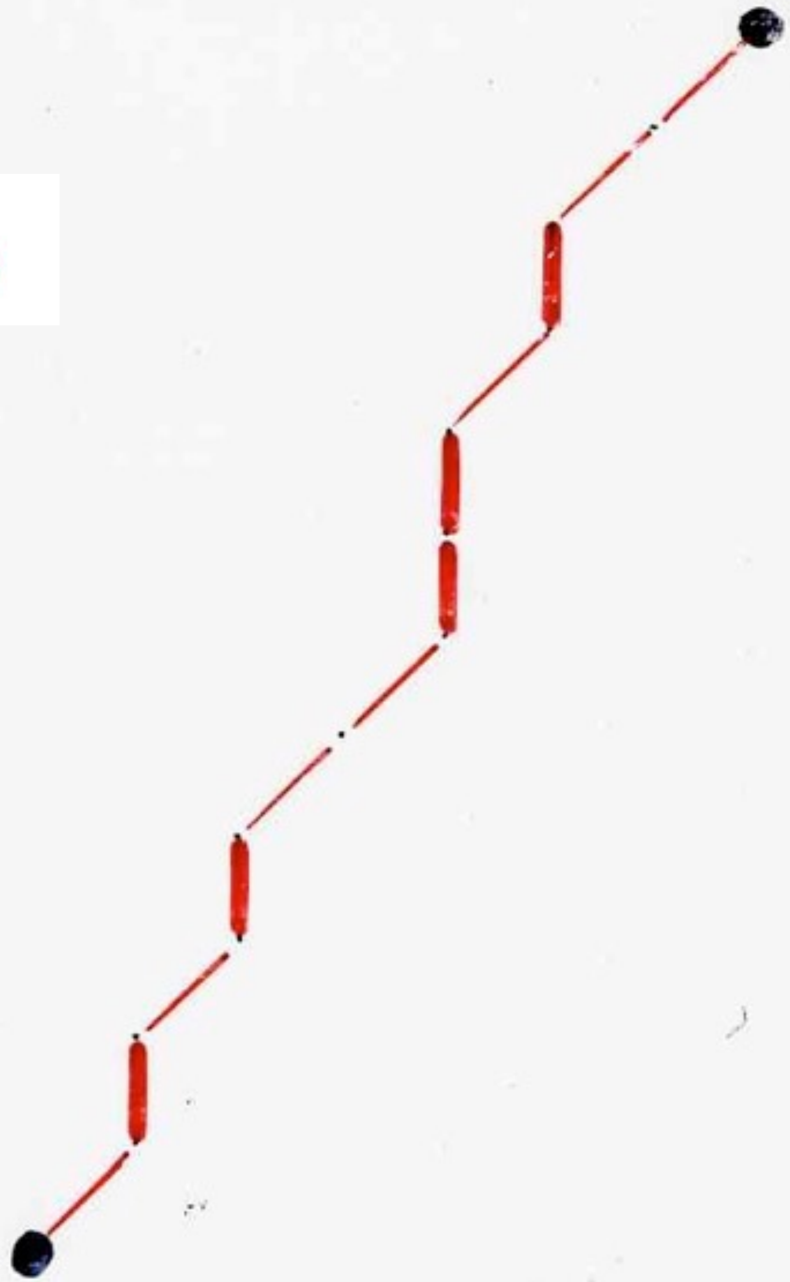


P. Lalonde, X.V. (1985, 1999)

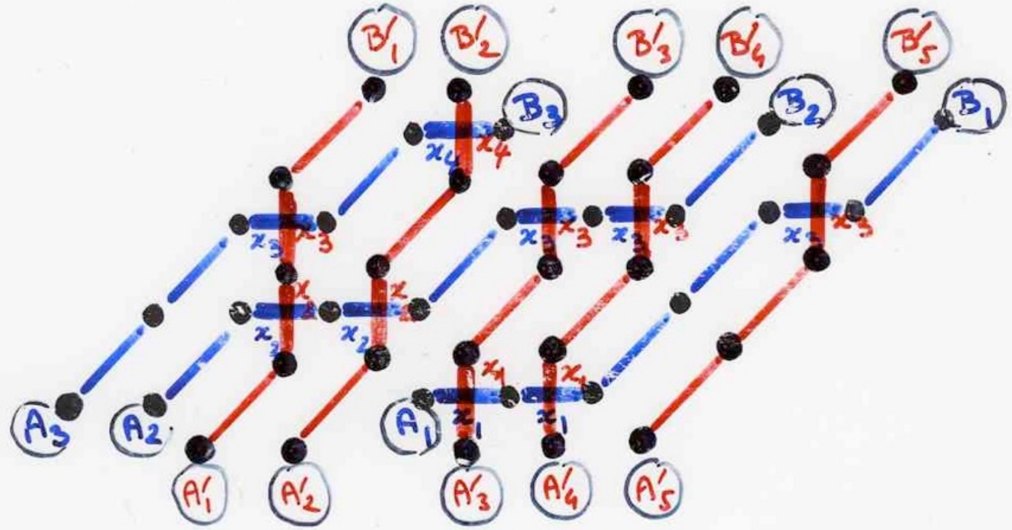


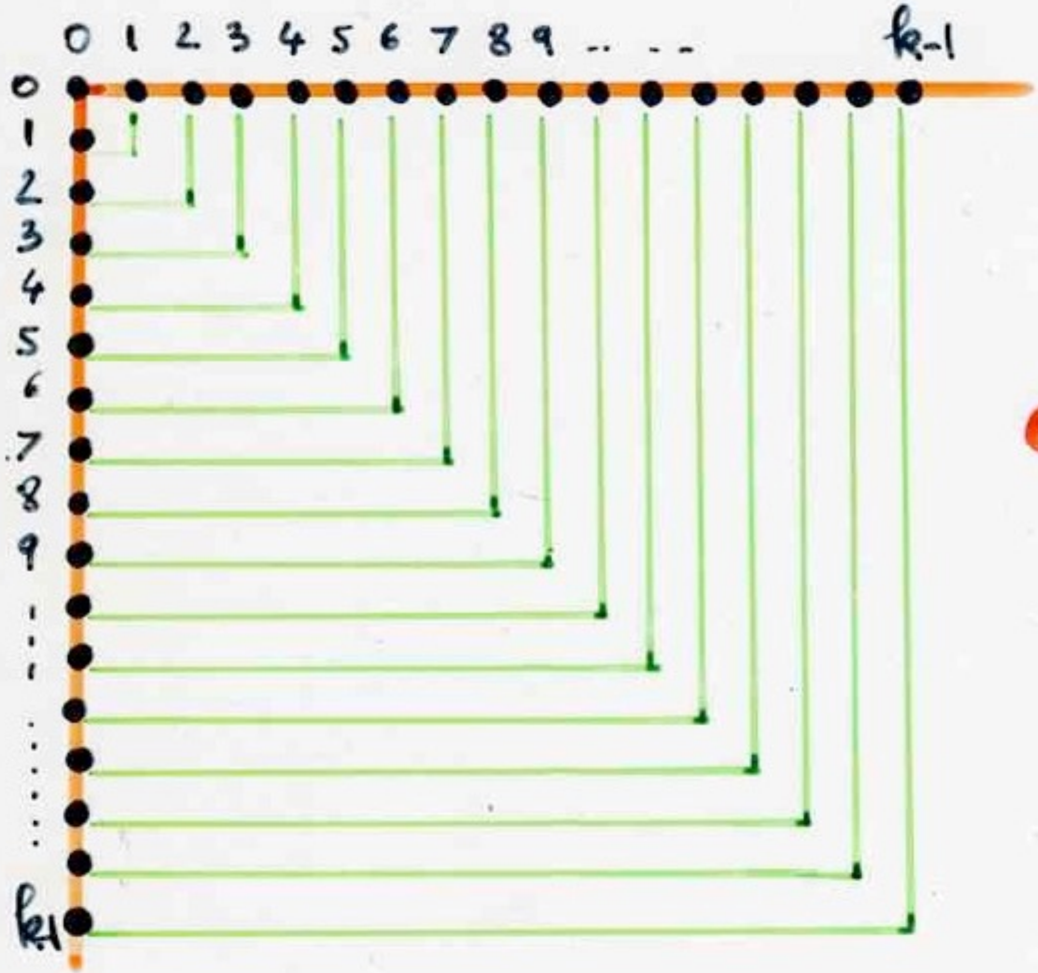

dual path

dual path



dual configurations of non-intersecting paths





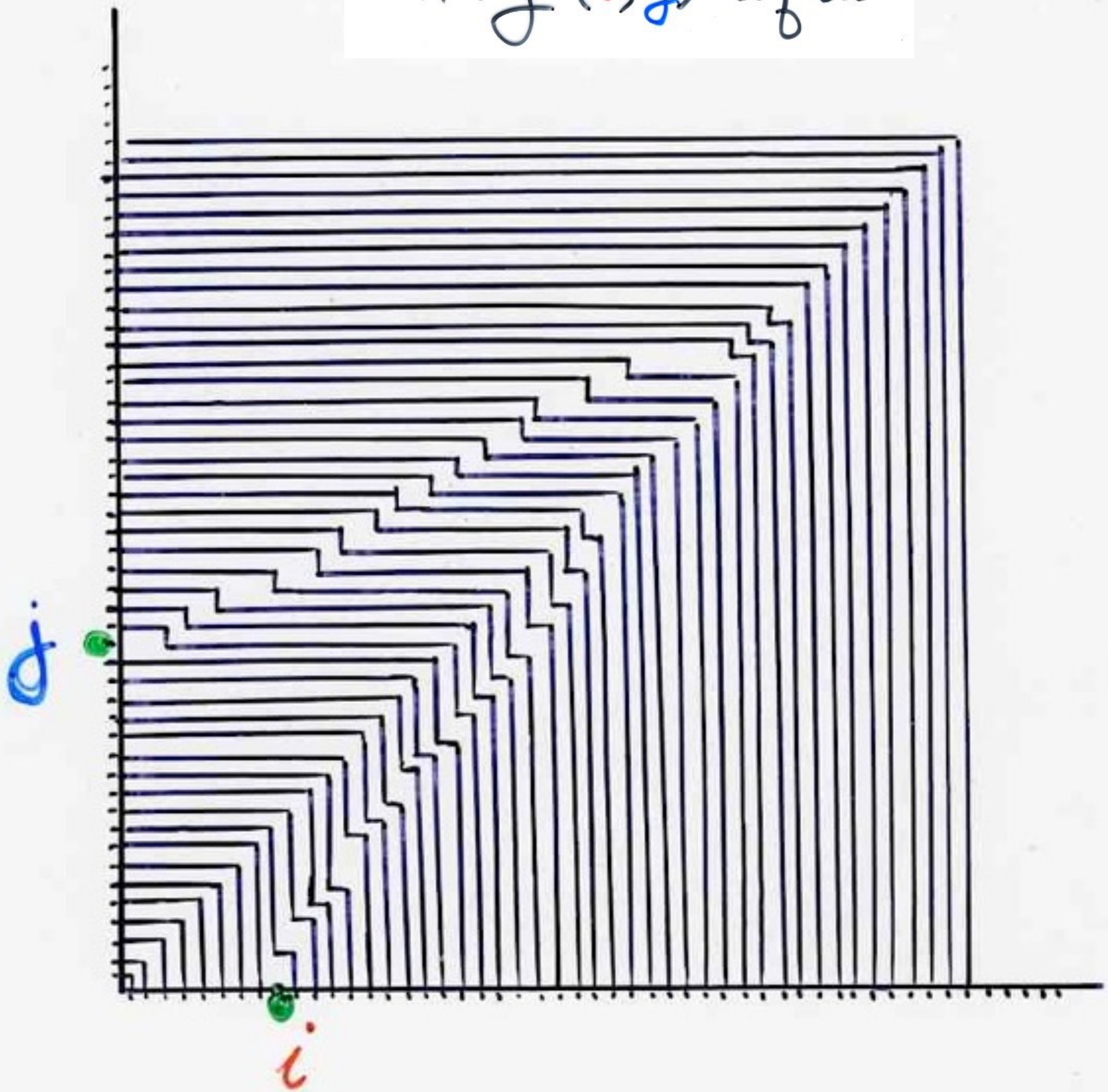
$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & & & \\ 1 & 5 & & & \binom{i+j}{i} & & \\ 1 & & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k \times k} = 1$$

<u>exercise</u>

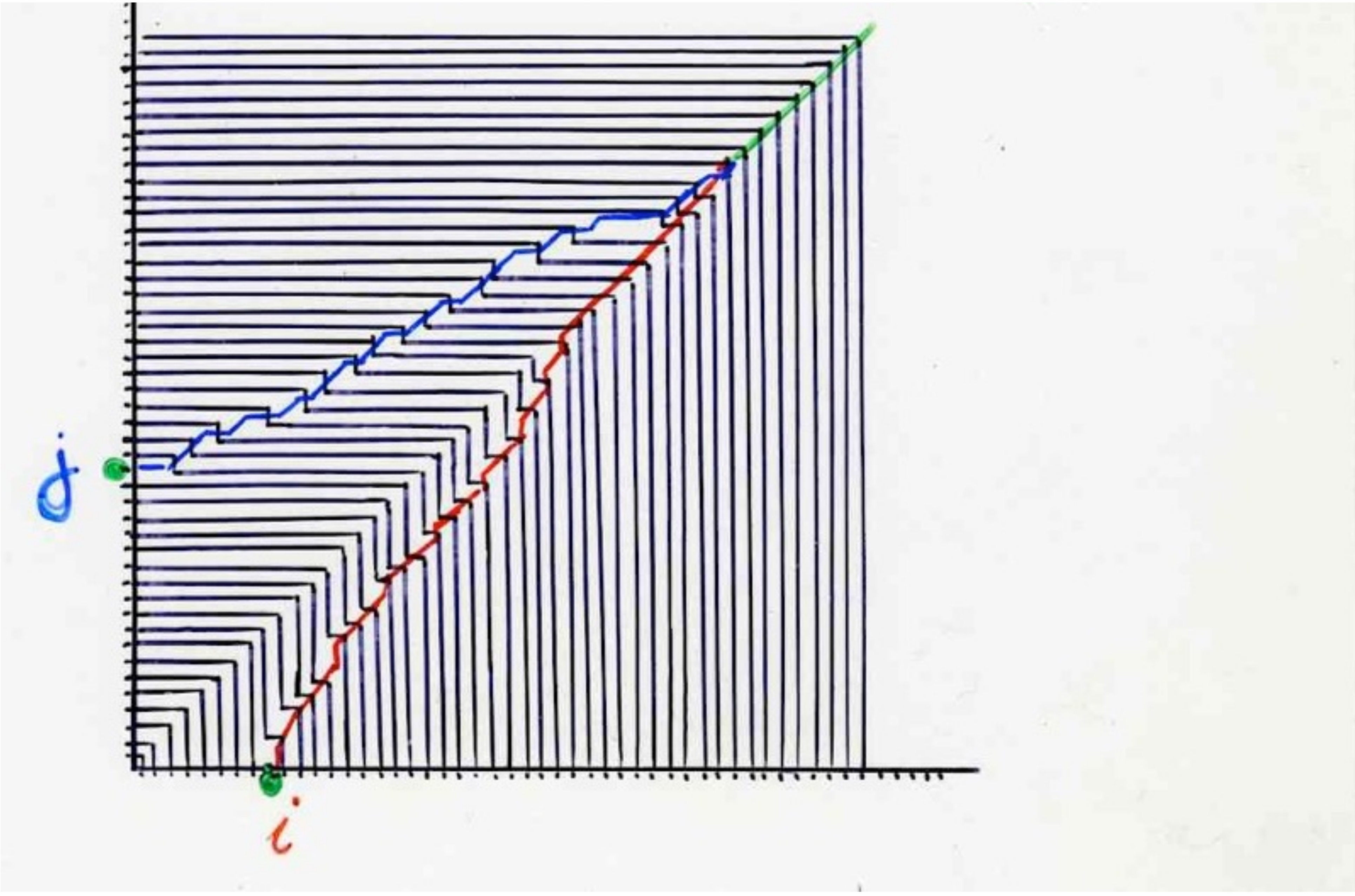
term (i, j) of the inverse matrix is

$$(-1)^{i+j} \sum_{k} \binom{k}{i}\binom{k}{j}$$

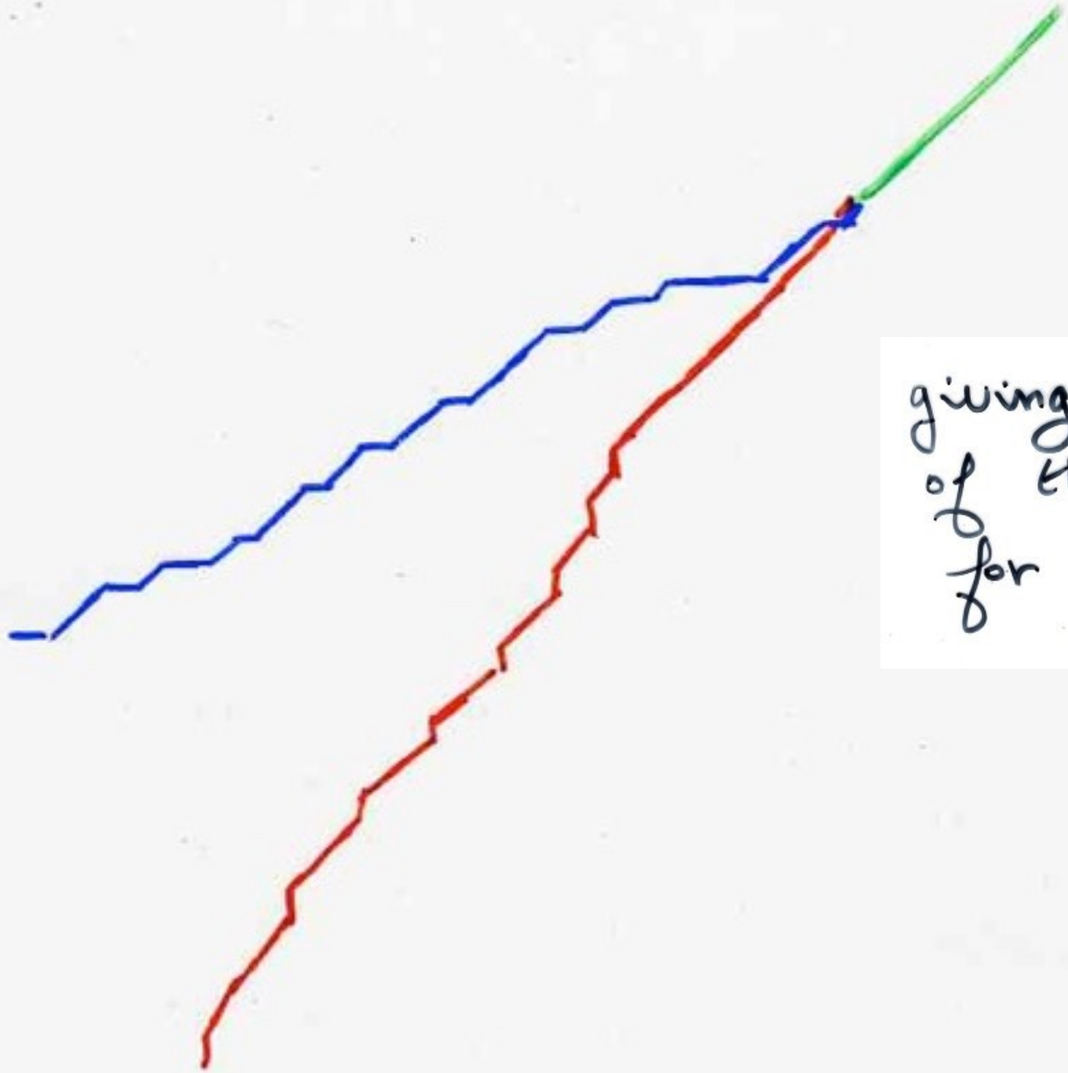
taking (i, j) cofactor



dual paths



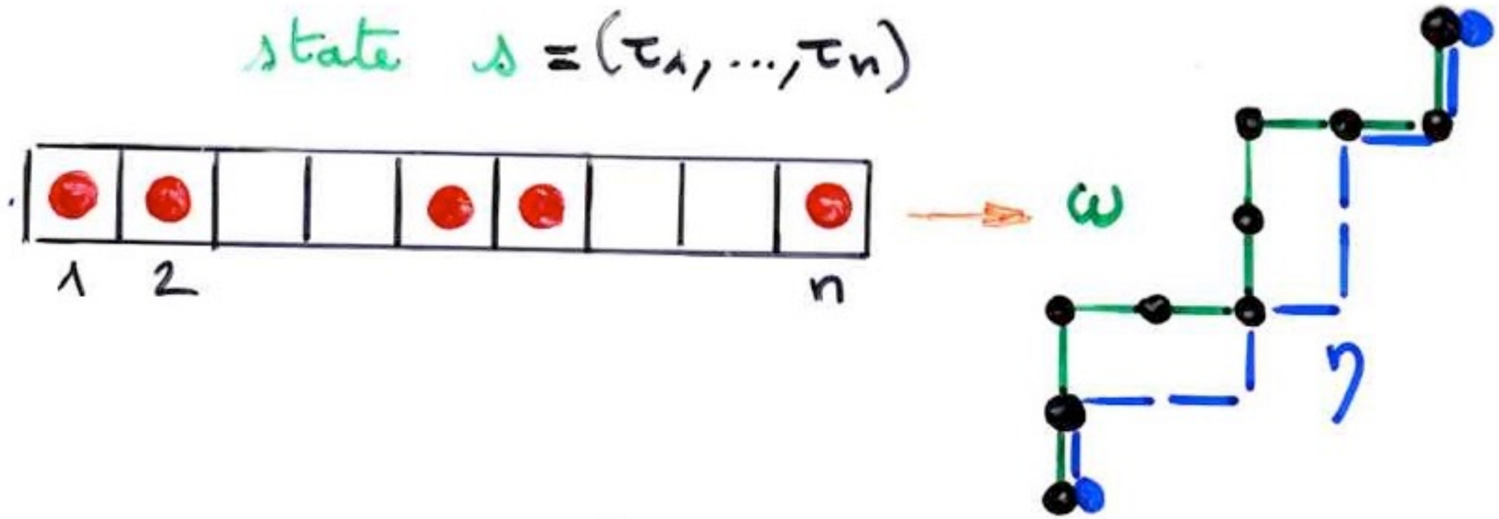
$$(-1)^{i+j}\sum_{k}\binom{k}{i}\binom{k}{j}$$



giving a proof
of the formula
for the (i,j) cofactor

TASEP
and
MacMahon-Narayana-Kreweras
determinants

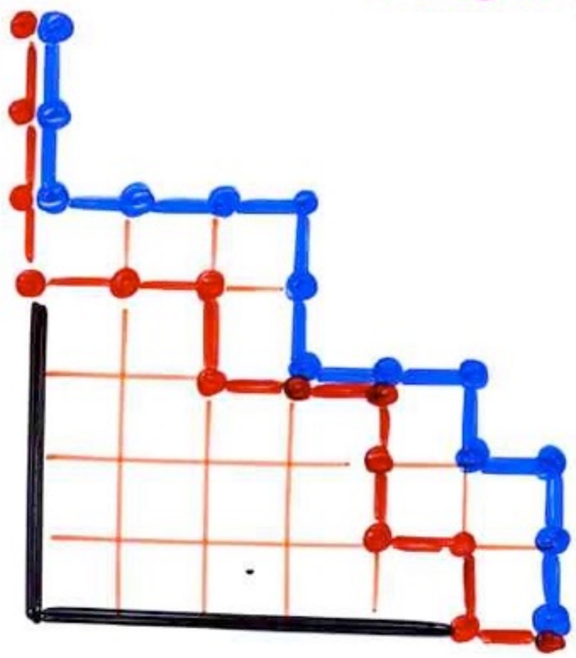
Paths duality



$$\mathbb{P}_n(s) = \frac{1}{C_{n+1}} \left(\begin{array}{c} \text{number of paths } \eta \\ \text{below the path } \omega \\ \text{associated to } s \end{array} \right)$$

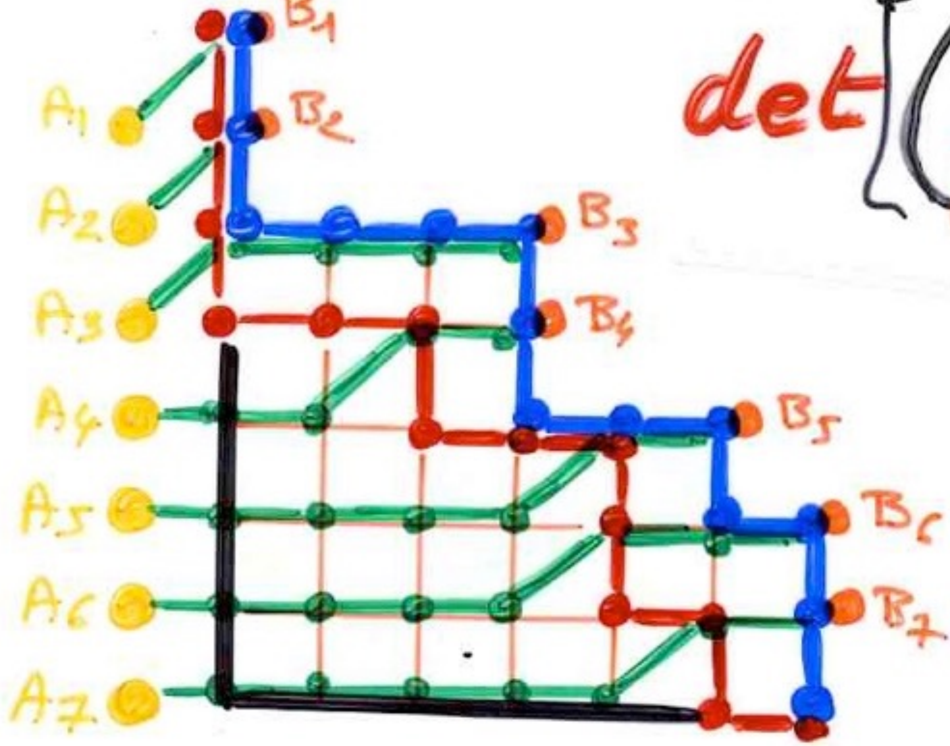
MacMahon
Narayana
Kreweras

determinant



$\lambda = (0,0,3,3,5,6,6)$

dual path

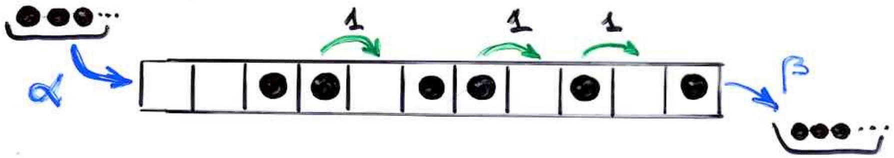


$$\det\left[\binom{\lambda_i+1}{j-i+1}\right]_{1\le i,j\le k}$$

$$\lambda = (0,0,3,3,5,6,6)$$
$$(\lambda_1, \ldots, \lambda_k)$$

TASEP

"totally asymmetric exclusion process"



stationary probabilities

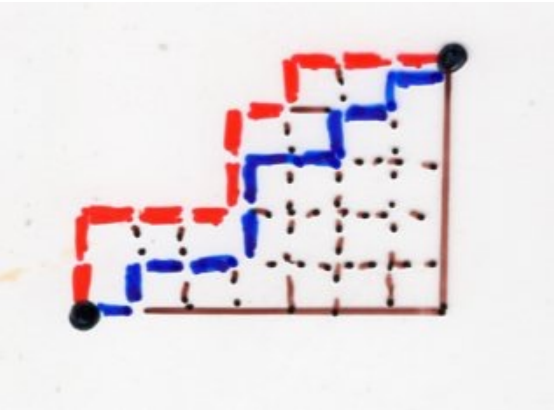
$$\frac{1}{Z_n} \sum_{\substack{\text{binary trees } T \\ c(T) = w \\ \text{canopy}}} \bar{\alpha}^{\,lb(T)} \, \bar{\beta}^{\,rb(T)}$$

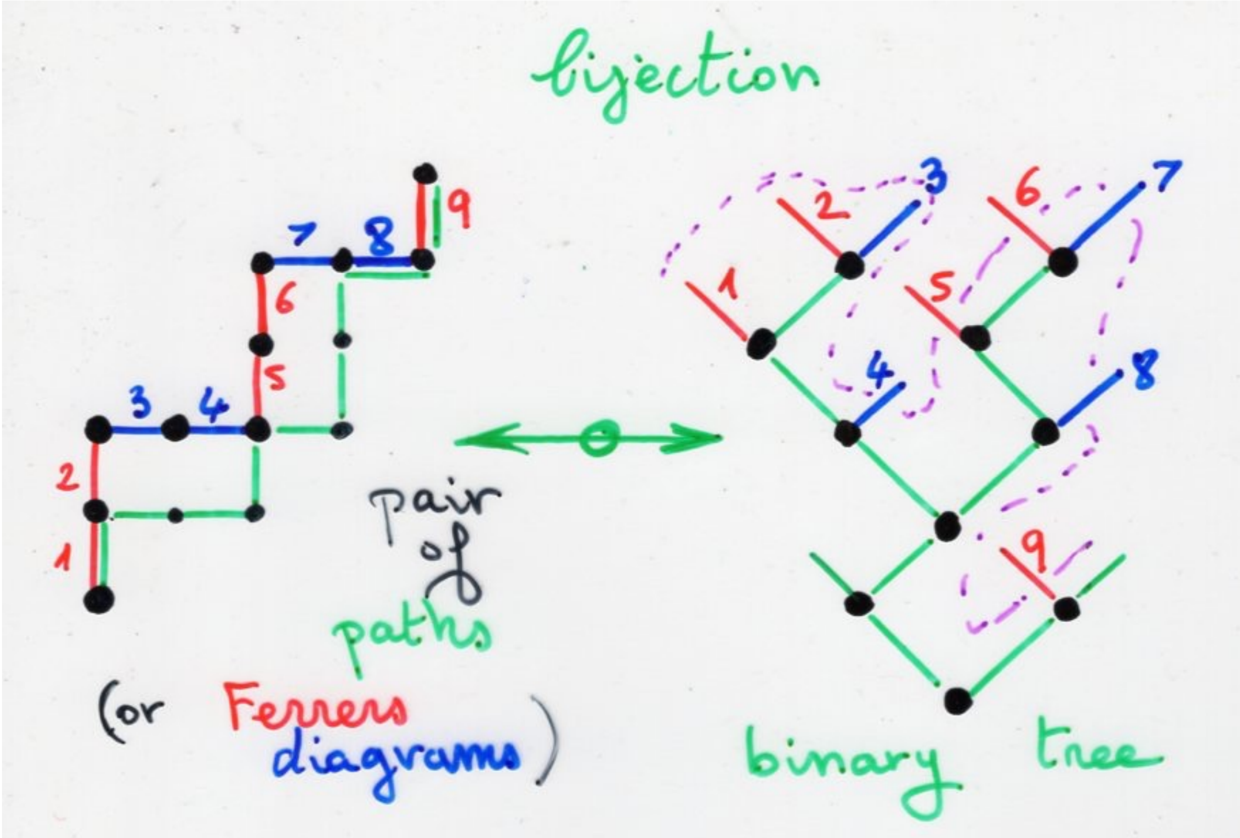
$$\bar{\alpha} = \alpha^{-1} \qquad \bar{\beta} = \beta^{-1}$$

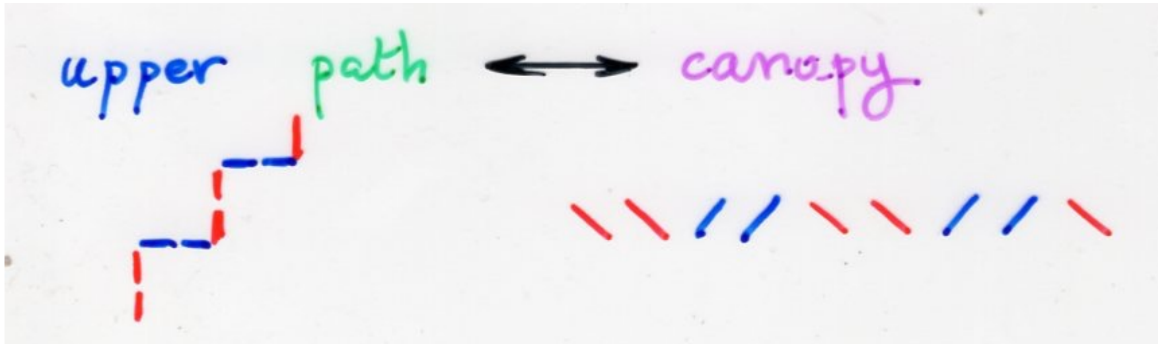
partition function

$$Z_n = \sum_{\substack{T \\ \text{binary trees} \\ n \text{ vertices}}} \bar{\alpha}^{\,lb(T)} \, \bar{\beta}^{\,rb(T)}$$

→ see course quadratic algebra in combinatorics




bijection
1 2 3 4 5 6 7 8 9
pair of paths
(or Ferrers diagrams)
1 2 3 4 5 6 7 8 9
binary tree


upper path ⟷ canopy

Olya Mandelstam
(2013)

(α, β) - analog of Narayana's determinant

TASEP with 2 parameters

$$P_{\{\lambda_1, \dots \lambda_k\}}(\alpha, \beta) = \det A_\lambda^{\alpha,\beta}$$

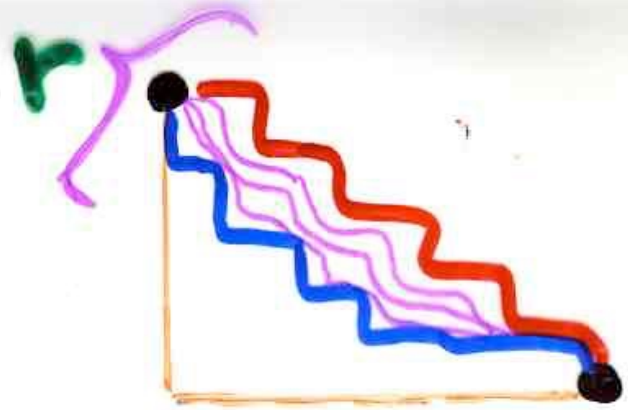
$$A_\lambda^{\alpha,\beta} = (A_{i,j})$$

$$A_{i,j} = \begin{cases} 0 & \text{for } j < i-1 \\ 1 & \text{for } j = i-1 \\ \beta^{j-i}\alpha^{\lambda_i - \lambda_{j+1}}\left(\binom{\lambda_j+1}{j-i} + \binom{\lambda_j+1}{j-i+1}\right) \\ +\beta^{j-i}\alpha^{\lambda_i-\lambda_j}\cdot\sum_{l=0}^{\lambda_j-\lambda_{j+1}-1}\alpha^{l}\left(\binom{\lambda_j-l-1}{j-i-1}+\binom{\lambda_j-l-1}{j-i}\right) & \text{for } j \geq i \end{cases}$$

Kreweras determinants

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$$

$$\mu = (\mu_1, \mu_2, \ldots, \mu_k)$$



Krewerás

$$\det\left(\binom{\lambda_i - \mu_j + r}{i - j + r}\right)_{1 \le i, j \le k}$$

orthogonal polynomials

computing the coefficients

λ_k $\qquad$ b_k

with Hankel determinants of moments

Orthogonal polynomials

Def. $\{P_n(x)\}_{n\geq 0}$ $\qquad P_n(x) \in \mathbb{K}[x]$

orthogonal iff $\exists\ f : \mathbb{K}[x] \to \mathbb{K}$

linear functional

$$\begin{cases} \text{(i)} & \deg(P_n(x)) = n \qquad (\forall n \geq 0) \\ \text{(ii)} & f(P_k P_l) = 0 \qquad \text{for } k \neq l \geq 0 \\ \text{(iii)} & f(P_k^2) \neq 0 \qquad \text{for } k \geq 0 \end{cases}$$

Thm. (Favard)

- $\{P_n(x)\}_{n\geqslant 0}$ sequence of monic polynomials, $\deg(P_n)=n$
- $\{b_k\}_{k\geqslant 0}$, $\{\lambda_k\}_{k\geqslant 1}$ coeff. in $\mathbb{K}$

orthogonality $\Longleftrightarrow$

$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

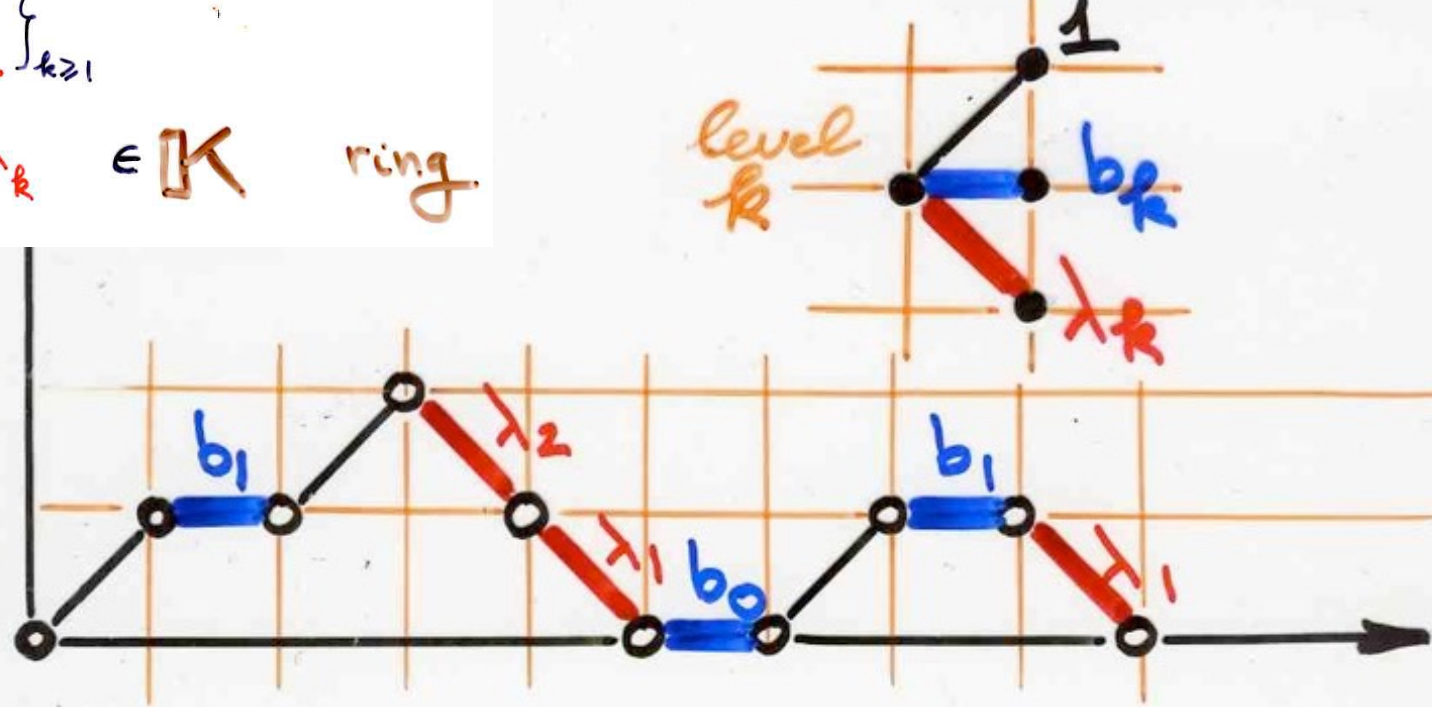
$(\forall k\geqslant 1)$

3 terms linear recurrence relation

$\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$

$b_k, \lambda_k \in \mathbb{K}$ ring

valuation v



ω Motzkin path

$$f(x^n) = \mu_n \quad (n \geq 0)$$

moments

$$\mu_n = \sum_{\omega} v(\omega)$$

ω Motzkin path $|\omega| = n$

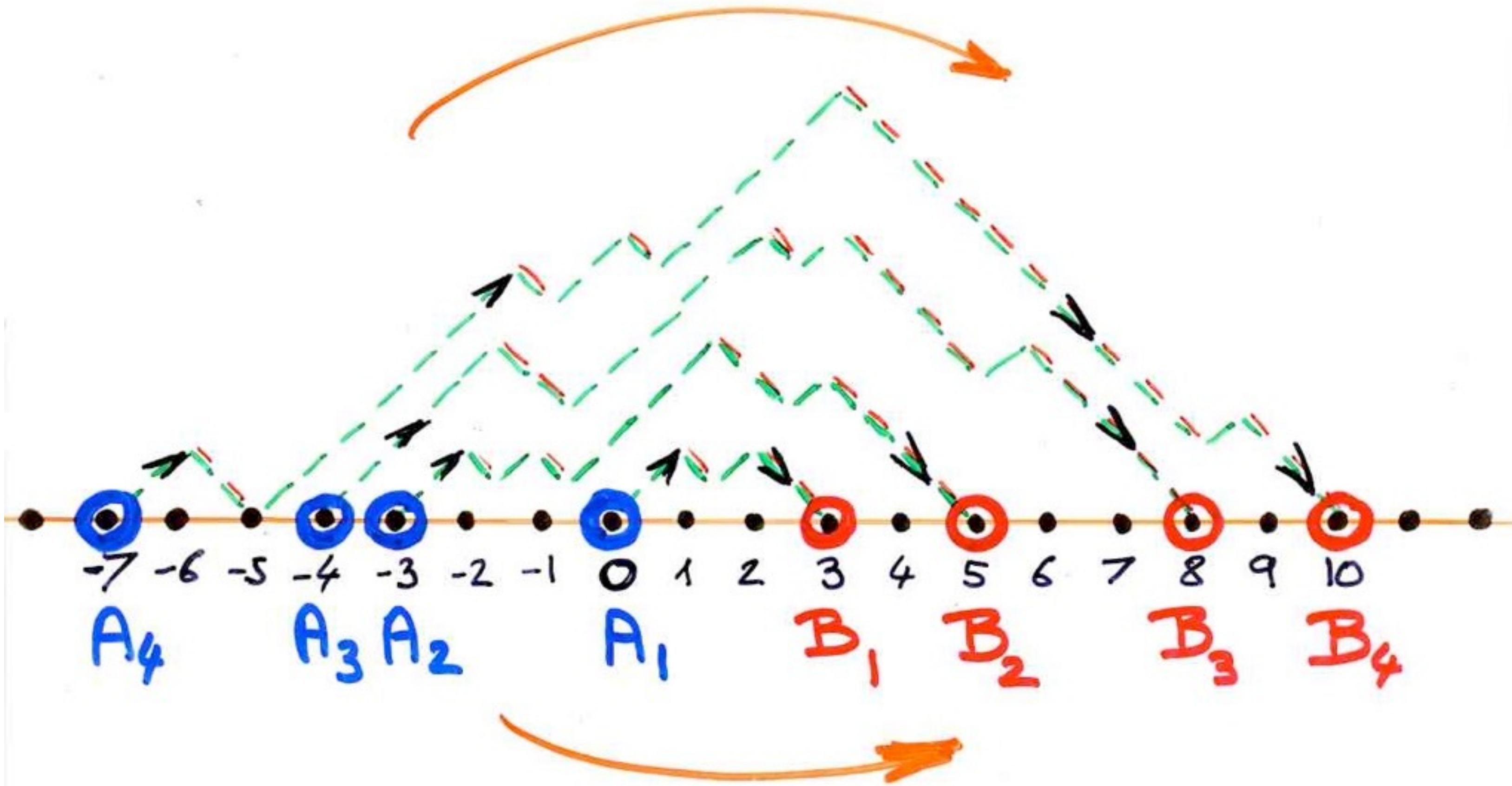
Hankel determinant

any minor of

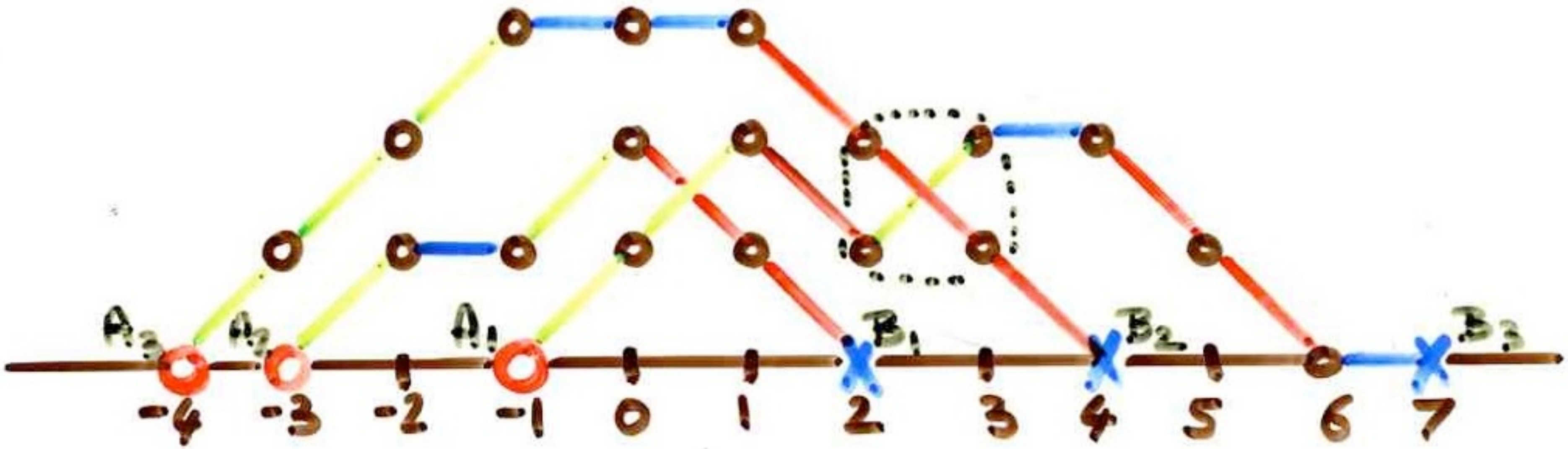
$$\begin{array}{c|cccccc}
 & & & & & & j \\
\hline
 & \mu_0 & \mu_1 & \mu_2 & \mu_3 & \cdots & \\
 & \mu_1 & \mu_2 & \mu_3 & \cdots & & \\
 & \mu_2 & \mu_3 & \cdots & & & \\
 & \mu_3 & & & & & \\
i & & & & & & \mu_{i+j}
\end{array}$$

$$\begin{vmatrix} \mu_3 & \mu_5 & \mu_8 & \mu_{10} \\ \mu_6 & \mu_8 & \mu_{11} & \mu_{13} \\ \mu_7 & \mu_9 & \mu_{12} & \mu_{14} \\ \mu_{10} & \mu_{12} & \mu_{15} & \mu_{17} \end{vmatrix}$$

Dyck paths

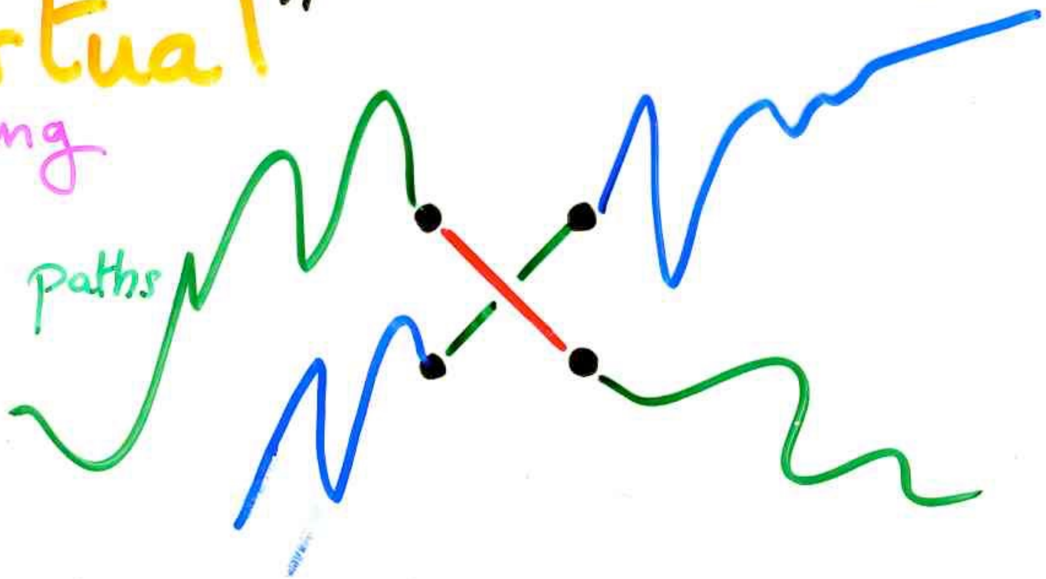


Motzkin paths



$$H\begin{pmatrix} 1, & 3, & 4 \\ 2, & 4, & 7 \end{pmatrix}$$

"virtual"
crossing
of
Motzkin paths



LGV Lemma. general form

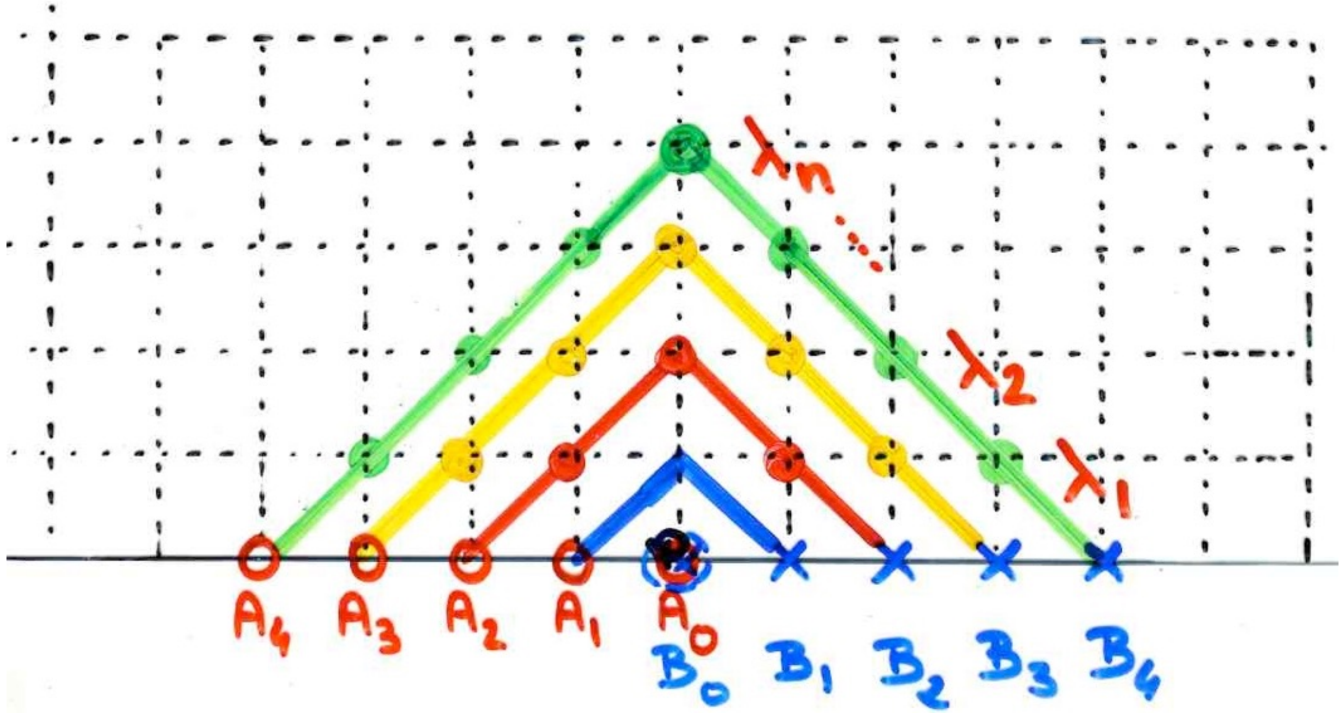
$$\det(a_{ij}) = \sum_{(\sigma;\, \omega_1, \ldots, \omega_k)} (-1)^{\mathrm{inv}(\sigma)}\, v(\omega_1) \ldots (\omega_k)$$

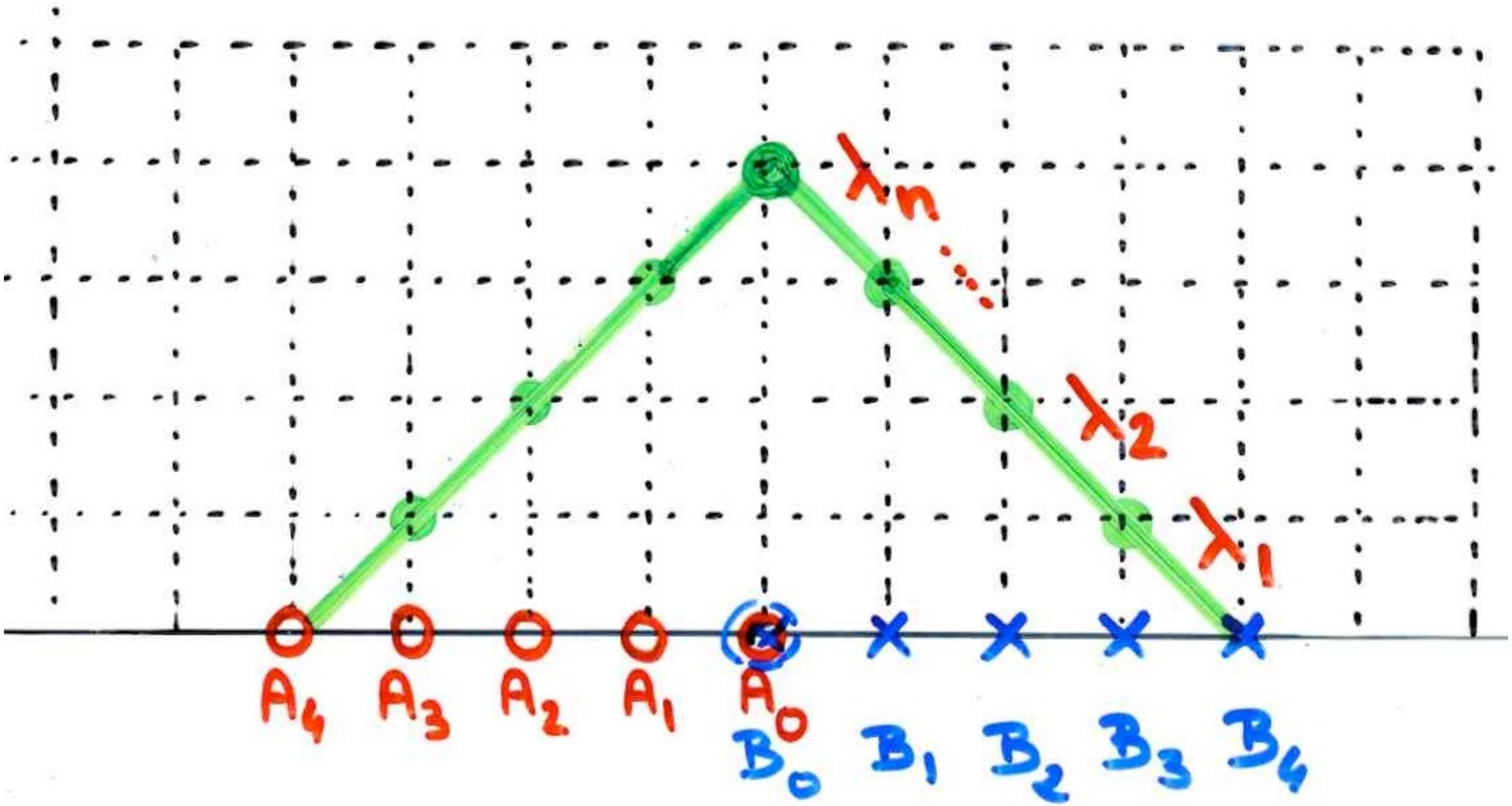
$$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$$

paths non-intersecting

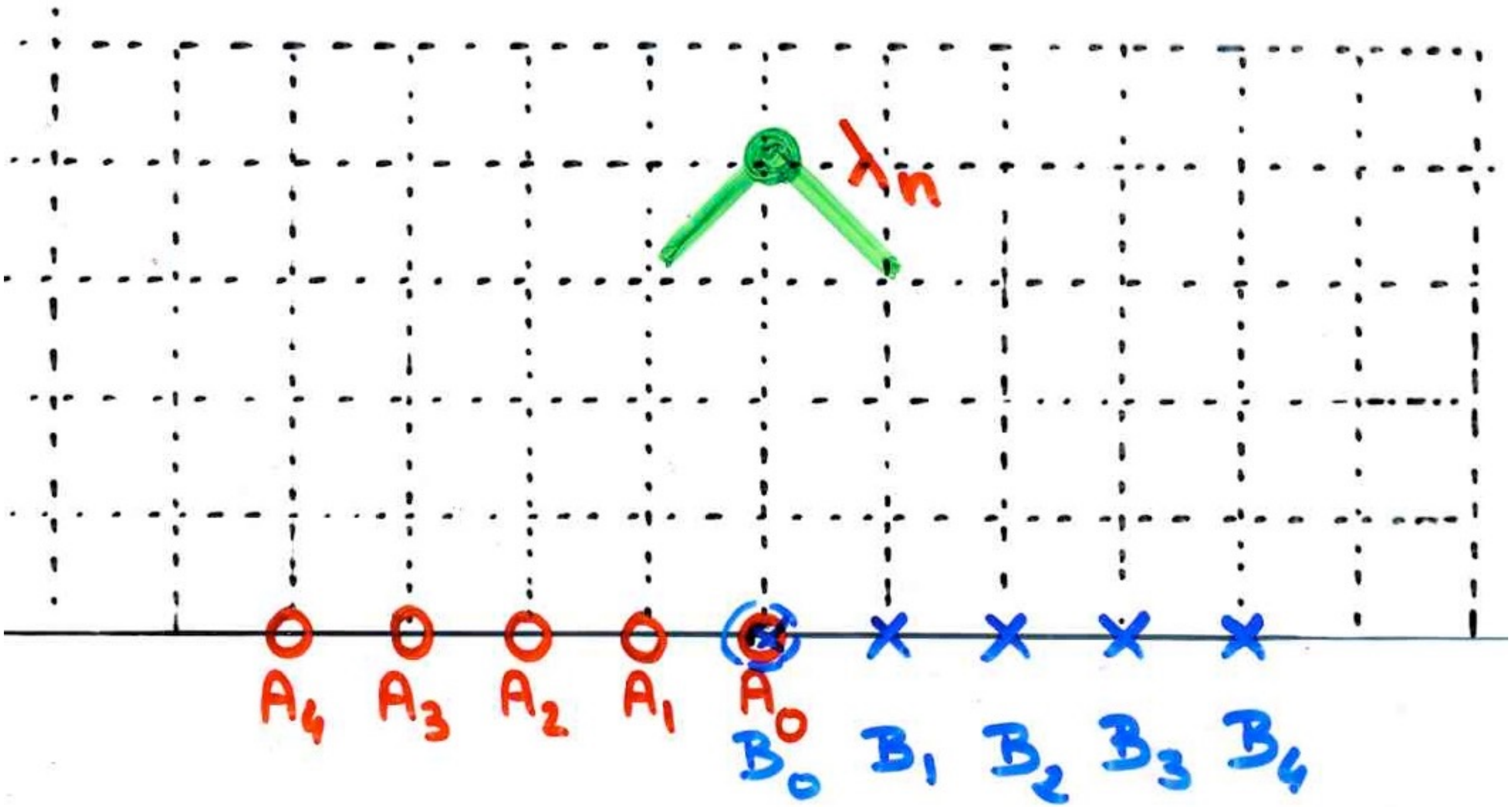
Hankel

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & & & \\ \mu_n & - & - & \mu_{n+n} \end{vmatrix}$$

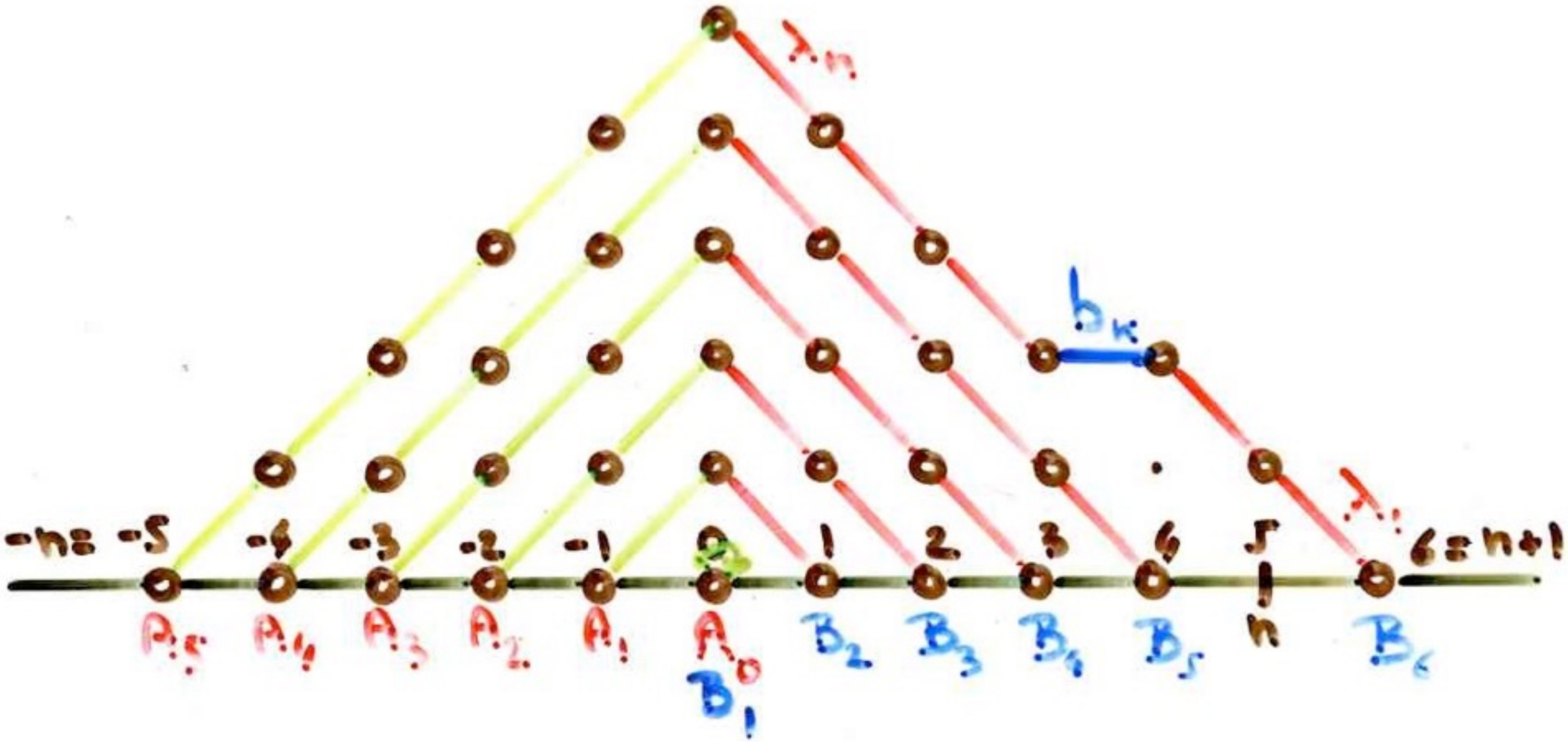




$$\frac{\Delta_n}{\Delta_{n-1}}$$



$$\frac{\Delta_n}{\Delta_{n-1}} : \frac{\Delta_{n-1}}{\Delta_{n-2}} = \lambda_n$$

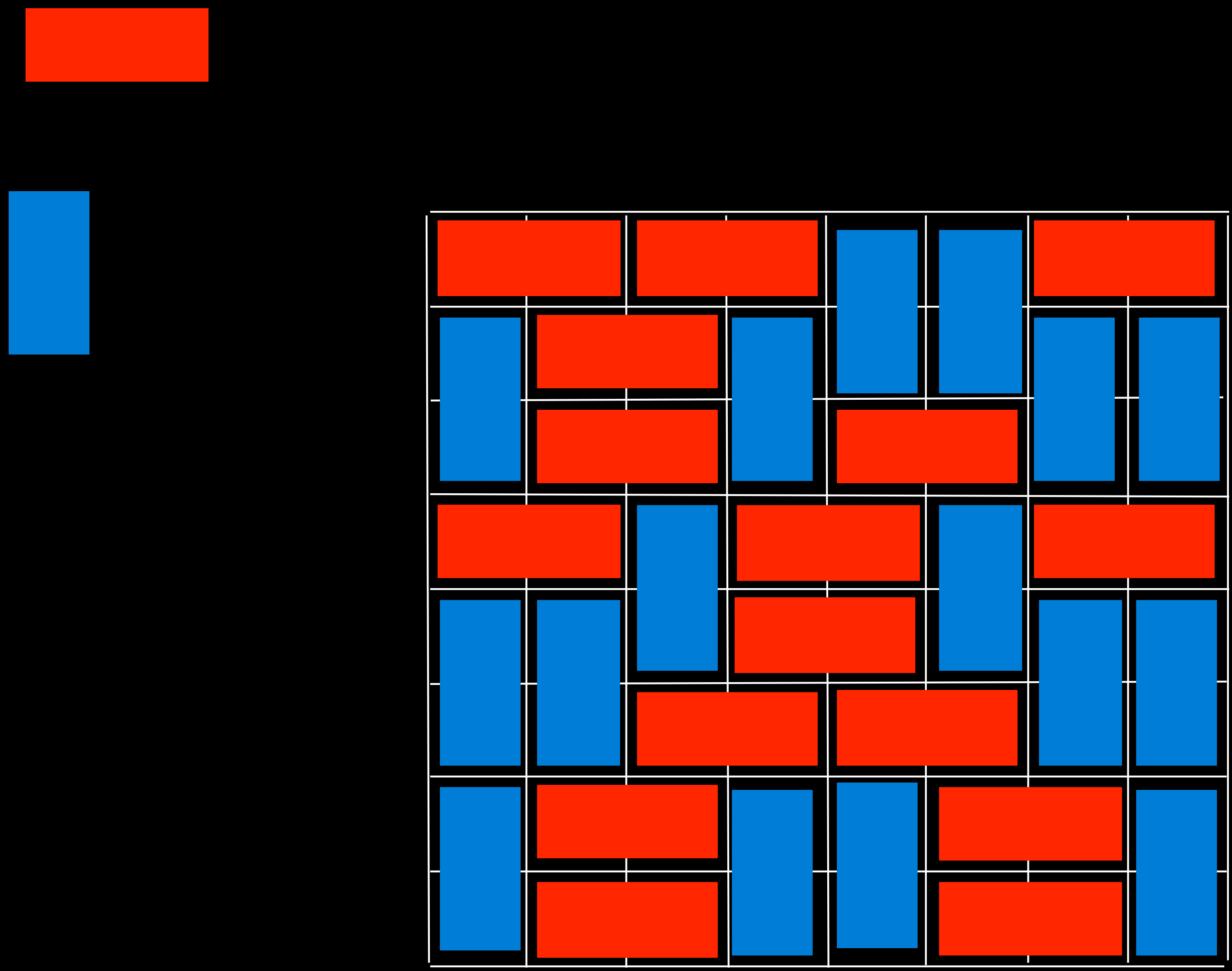


χ_n

$$\chi_n = \sum_k b_k \Delta_n \qquad b_n = \frac{\chi_n}{\Delta_n} - \frac{\chi_{n-1}}{\Delta_{n-1}}$$

Tilings

tiling in Kuperberg’ s bathroom



number of tilings on a 8 x 8 chessboard

= 12 988 816

number of tilings with dimers of a $m \times n$ rectangle

$$4^{mn} \prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \left(4\cos^2 \frac{i\pi}{m+1} + 4\cos^2 \frac{j\pi}{n+1} \right)$$

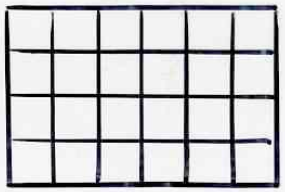
Kasteleyn (1961)

it is an integer !!

for the chessboard $m=n=8$: 12 988 816

rectangle $2m \times 2n$

$m = 2$
$n = 3$



$2m = 4$

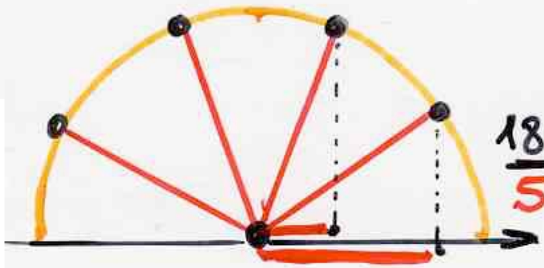
$2n = 6$

$6 = mn = 2 \times 3$

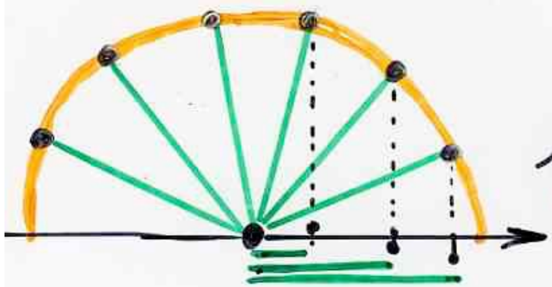
4^6

$\cos^2\left(\frac{180°}{5}\right) + \cos^2\left(\frac{180°}{7}\right)$	$\cos^2\left(\frac{180°}{5}\right) + \cos^2\left(2 \times \frac{180°}{7}\right)$	$\cos^2\left(\frac{180°}{5}\right) + \cos^2\left(3 \times \frac{180°}{7}\right)$
$\cos^2\left(2 \times \frac{180°}{5}\right) + \cos^2\left(\frac{180°}{7}\right)$	$\cos^2\left(2 \times \frac{180°}{5}\right) + \cos^2\left(2 \times \frac{180°}{7}\right)$	$\cos^2\left(2 \times \frac{180°}{5}\right) + \cos^2\left(3 \times \frac{180°}{7}\right)$

281
pavages



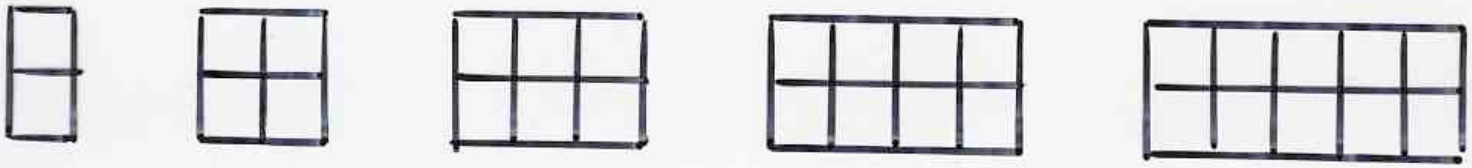
$\frac{180°}{5} = 36°$



$\frac{180°}{7} = 25{,}71...°$

exercise :

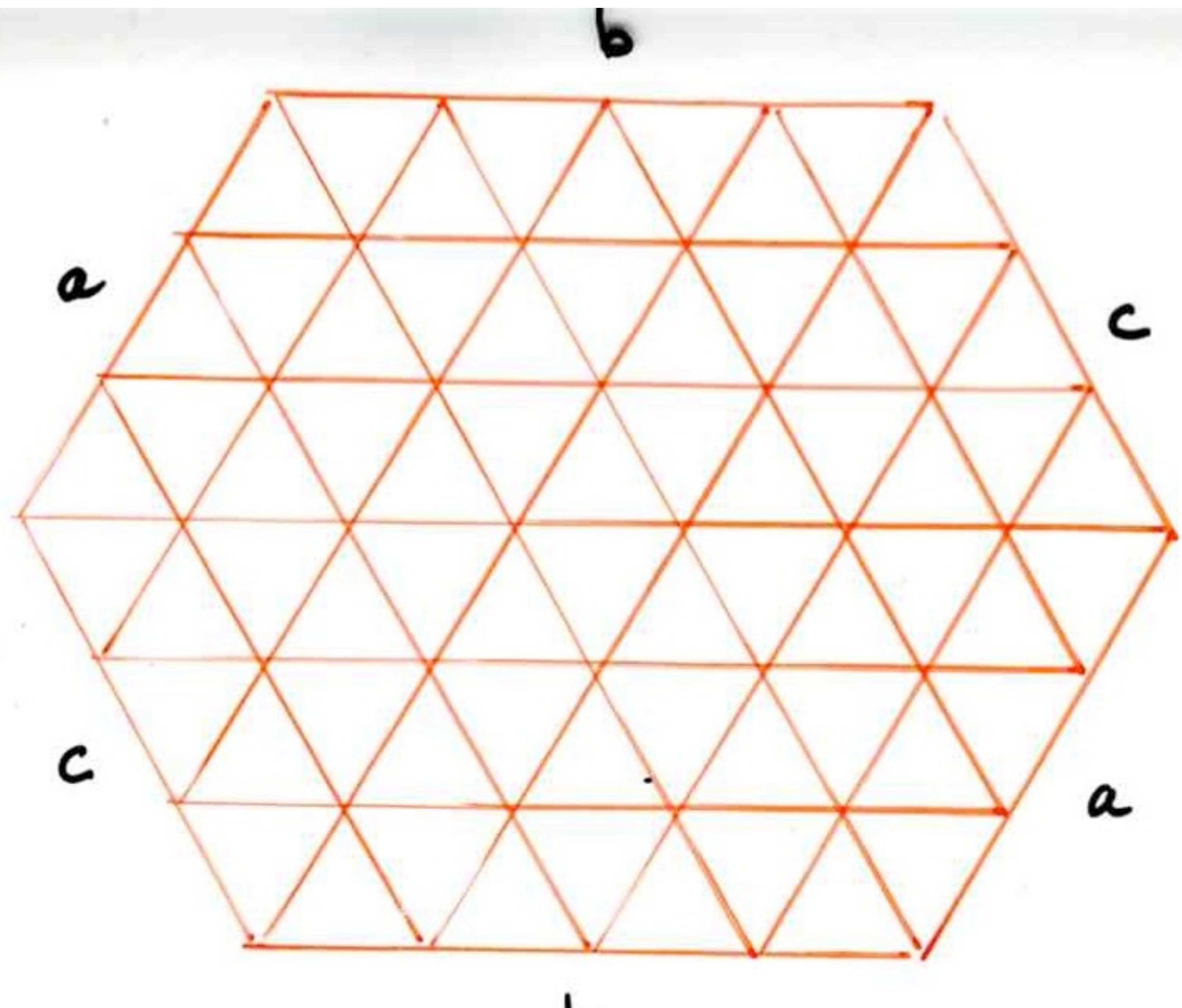
nombre de pavages rectangle $2 \times n$

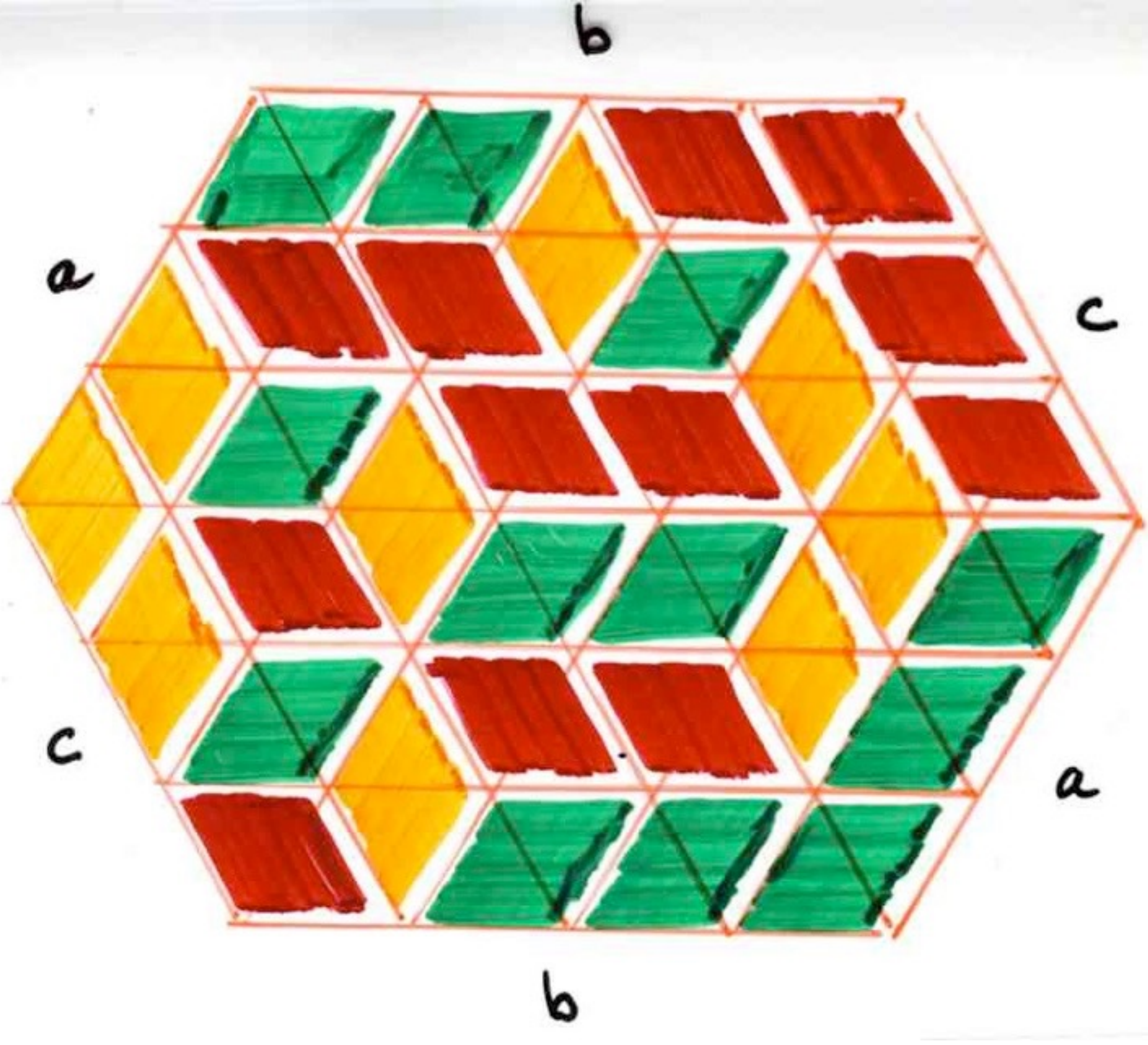


1 2 3 5 8

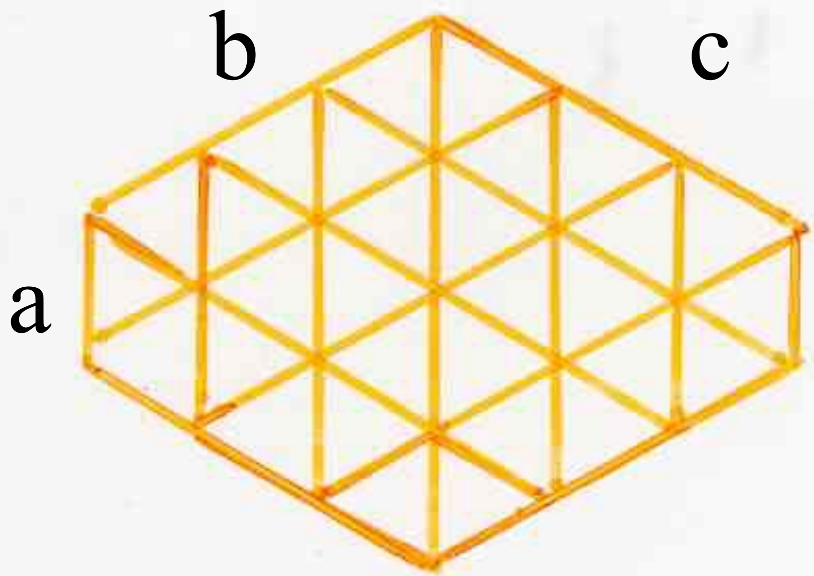
Fibonacci numbers

Tilings on triangular lattice


b
a
c
c
a
b


b
a
c
c
a
b

exercise :



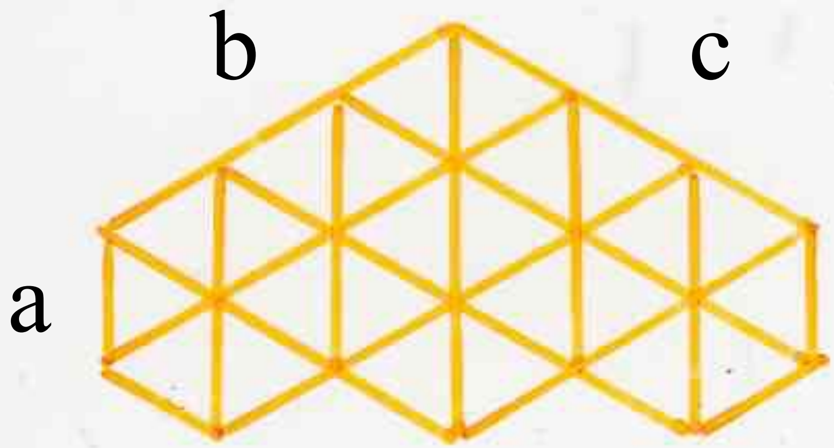
$a=1 \quad b=c=n$

number of tilings

$$= \binom{2n}{n}$$

(bijection with bilateral Dyck paths)

exercise :

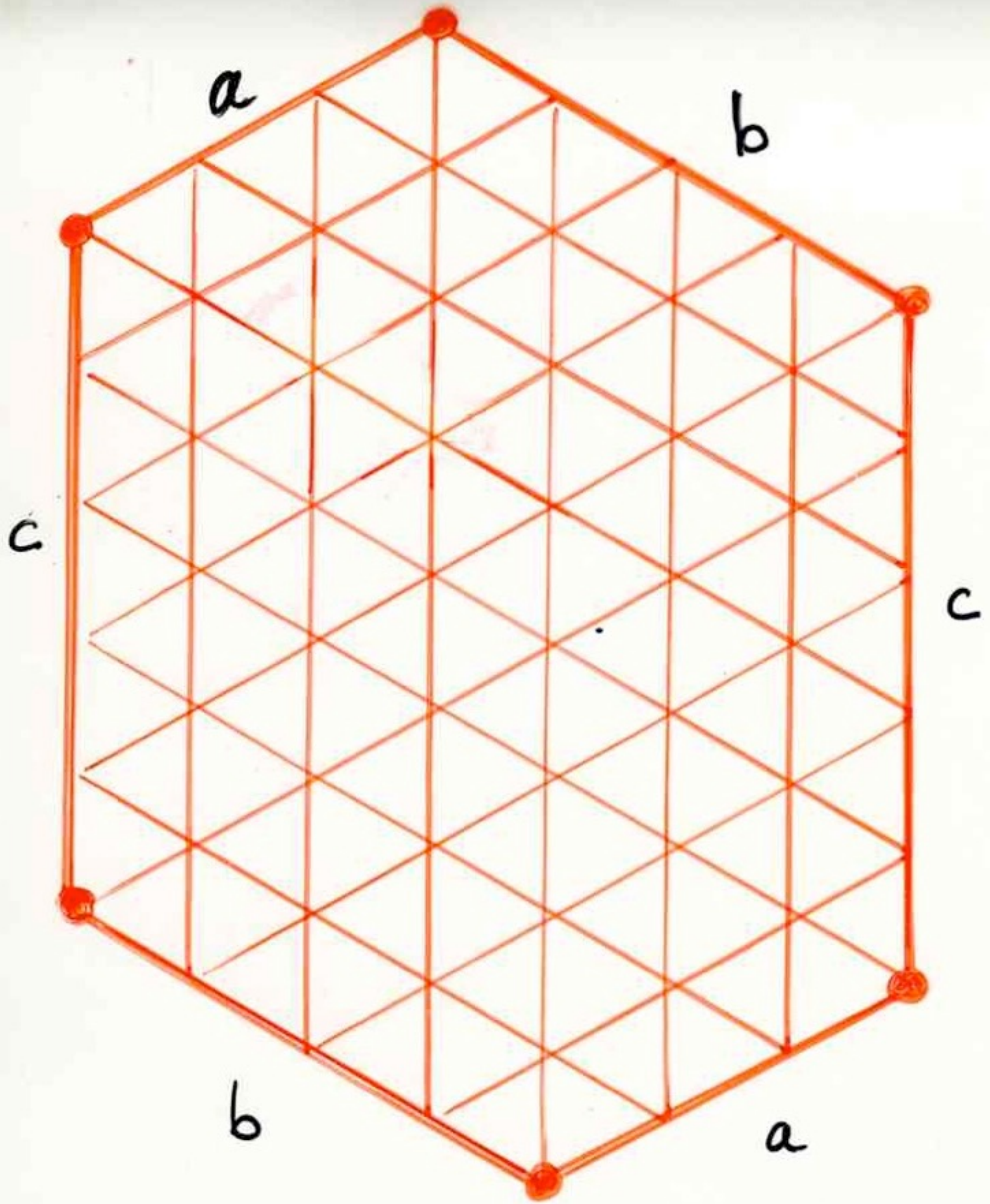


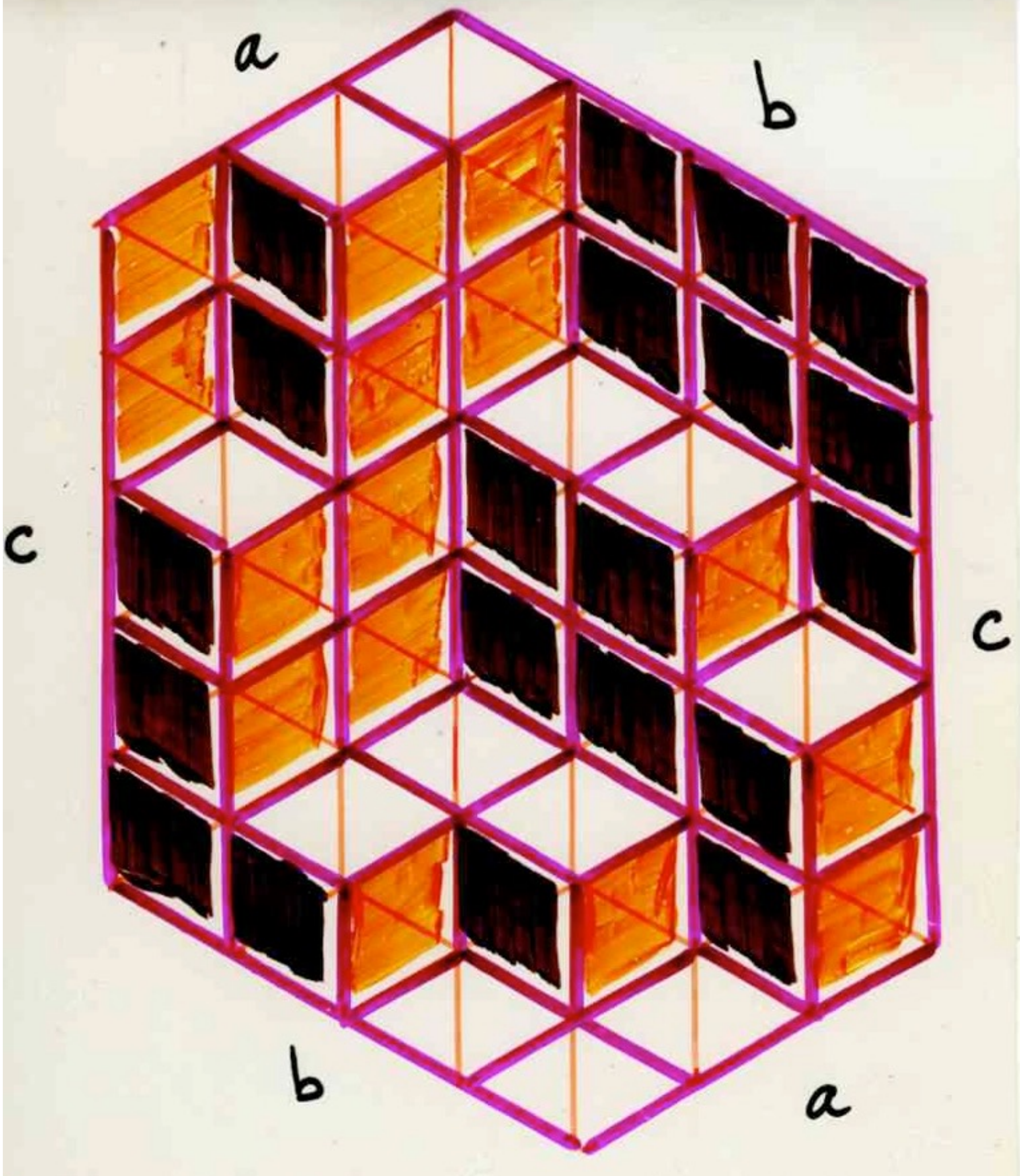
$a = 1 \qquad b = c = n$

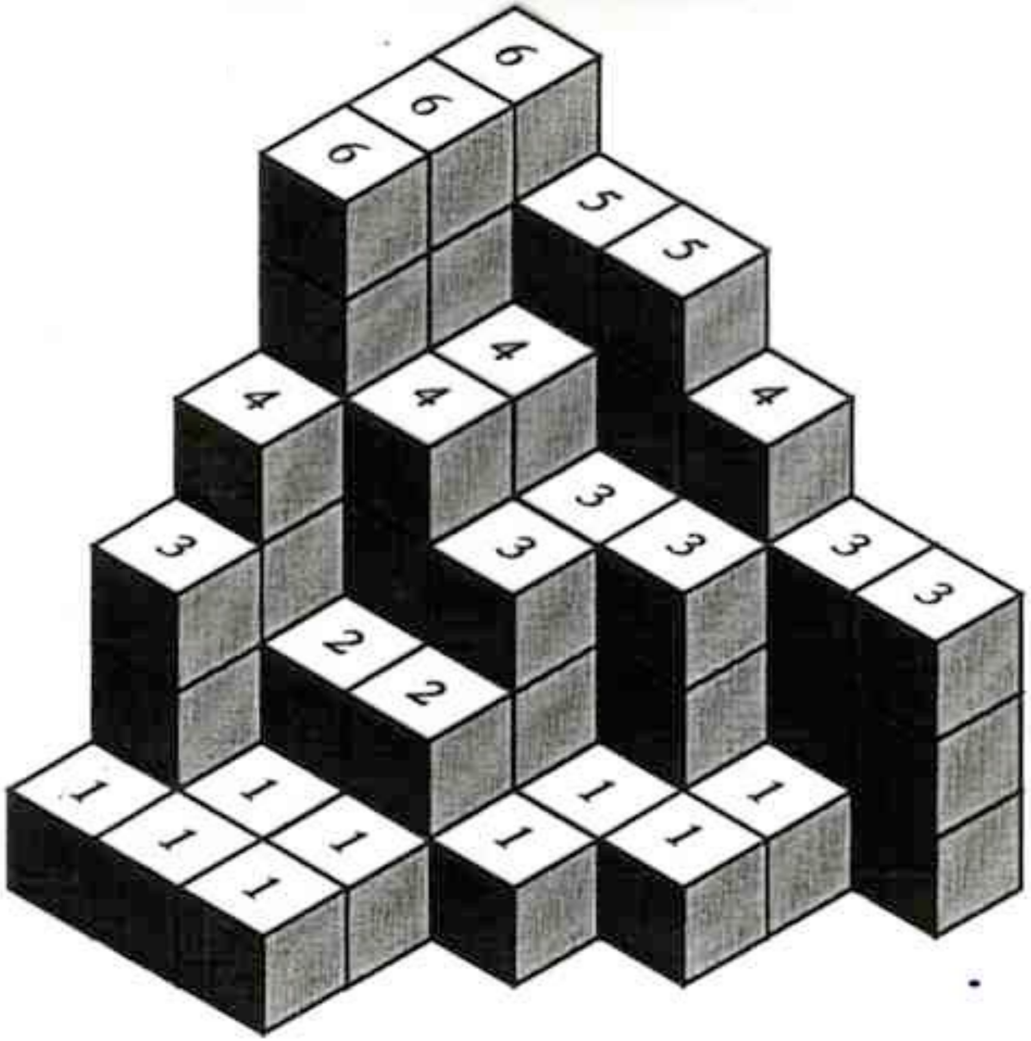
number of tilings

Catalan number $\frac{1}{n+1}\binom{2n}{n}$

(bijection with Dyck paths)


a
b
c
c
b
a


a
b
c
c
b
a



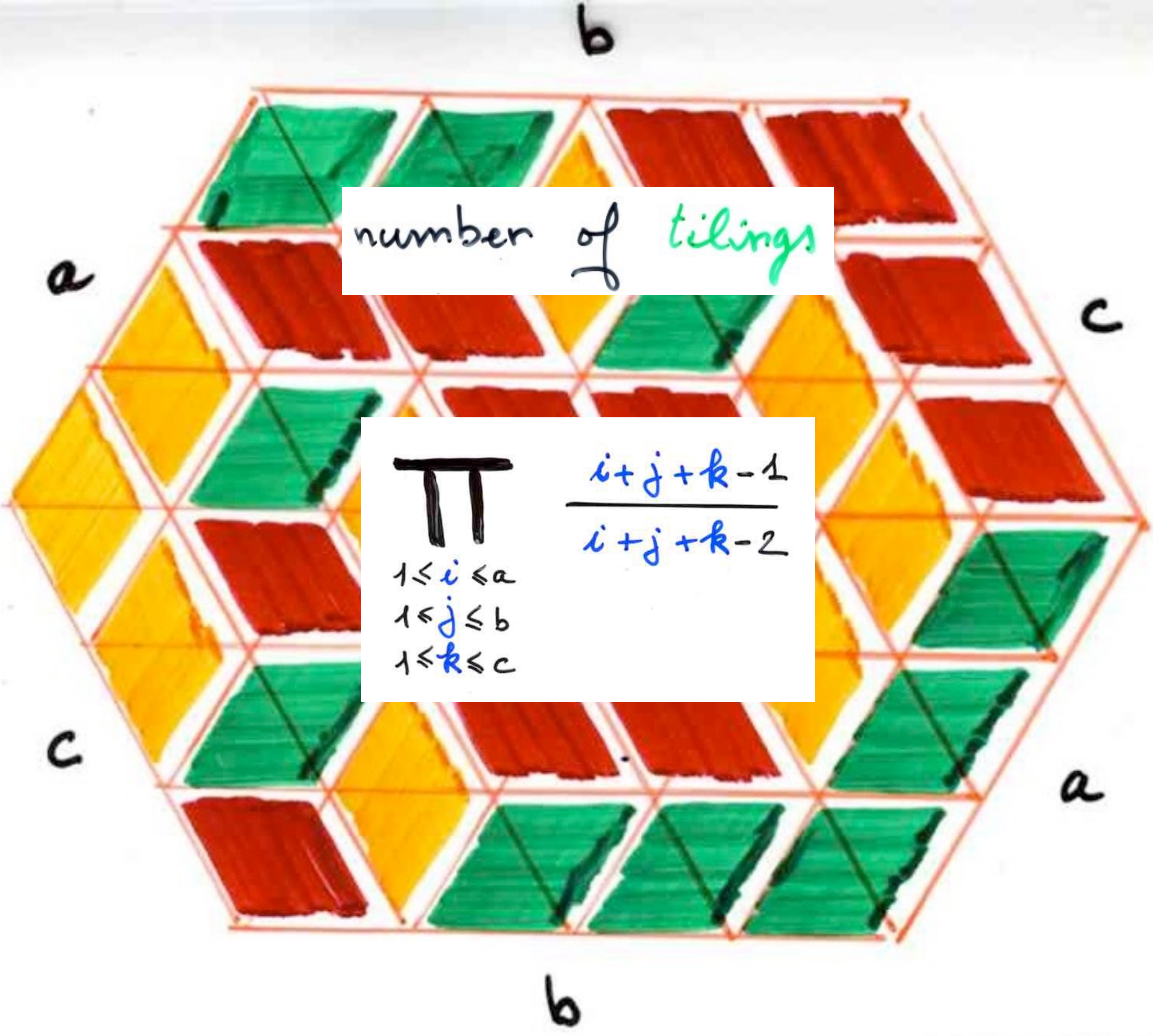
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

plane partitions

3D Ferrers diagrams

in a box $\mathcal{B}(a,b,c)$

$$\prod_{\substack{1 \le i \le a \\ 1 \le j \le b \\ 1 \le k \le c}} \frac{i+j+k-1}{i+j+k-2}$$



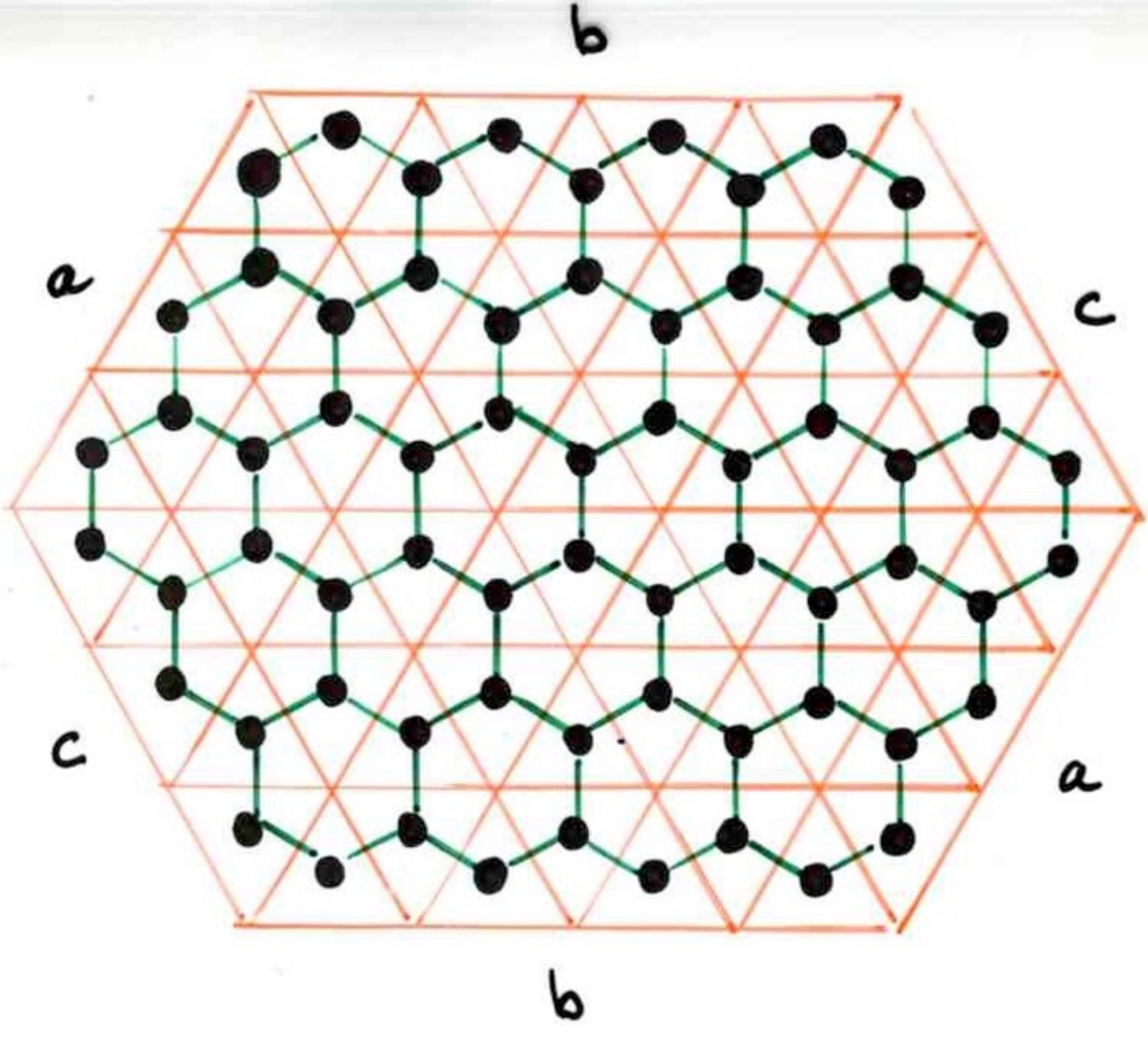
number of tilings

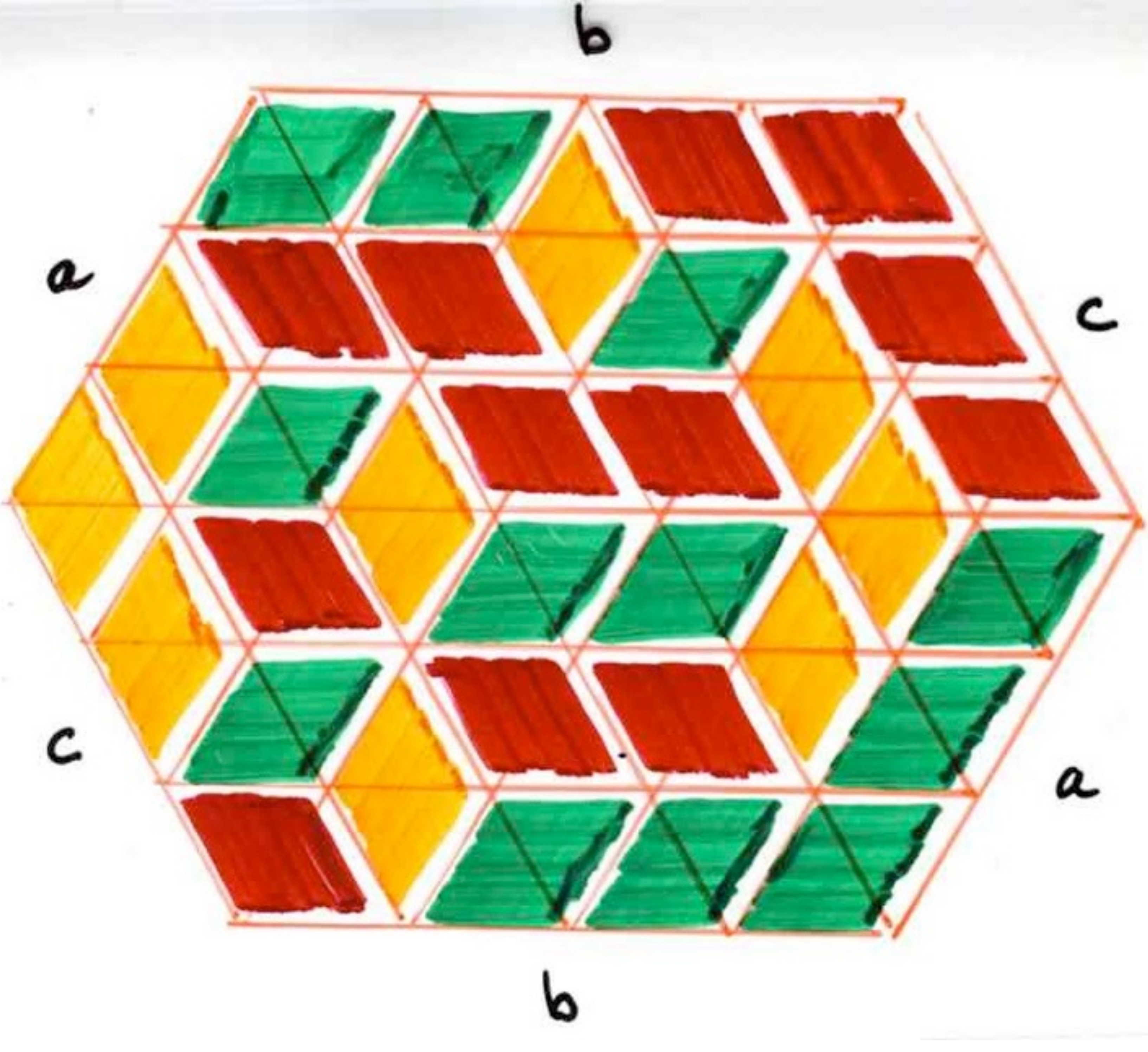
$$\prod_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b \\ 1 \leq k \leq c}} \frac{i+j+k-1}{i+j+k-2}$$

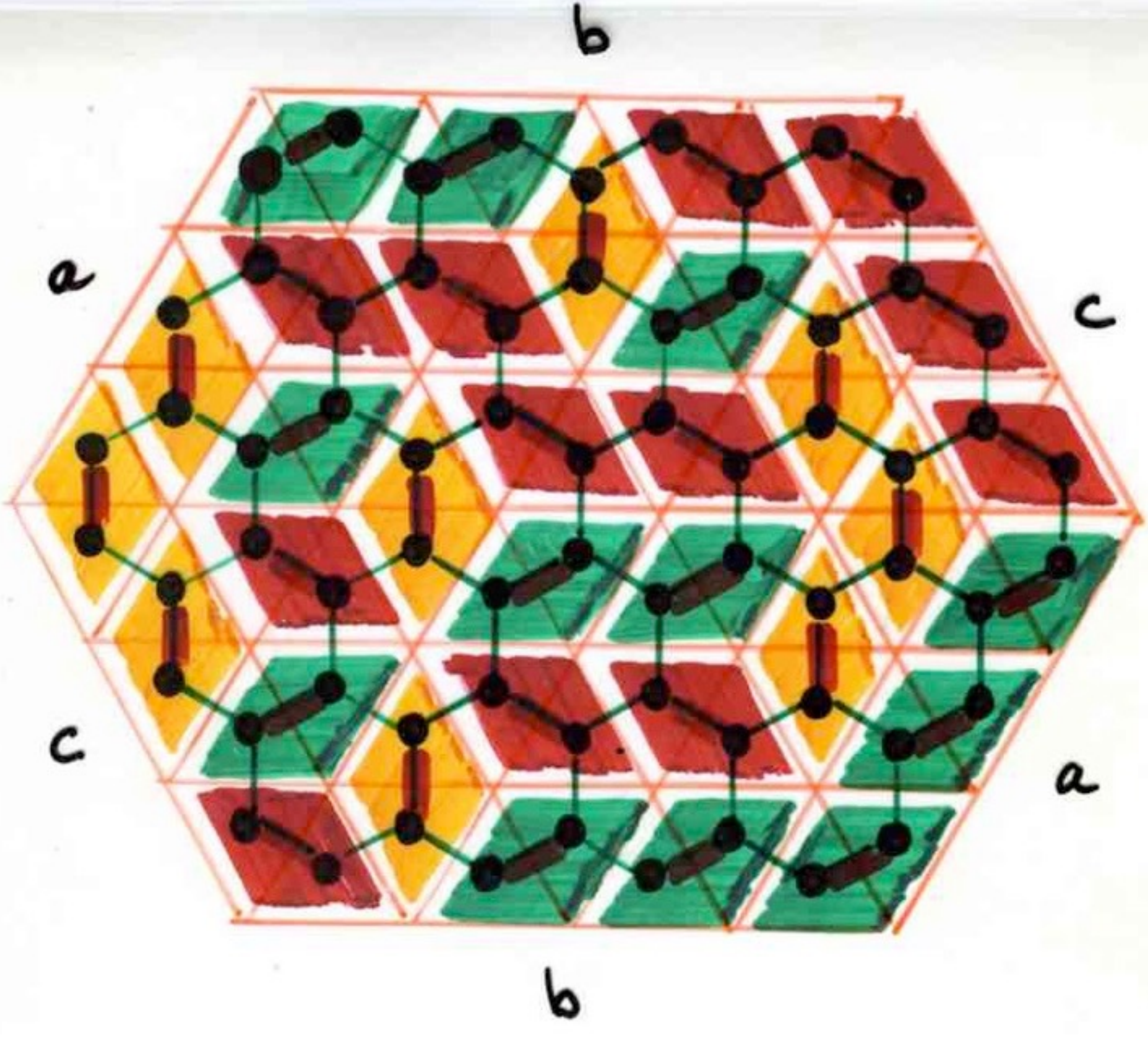
Tilings

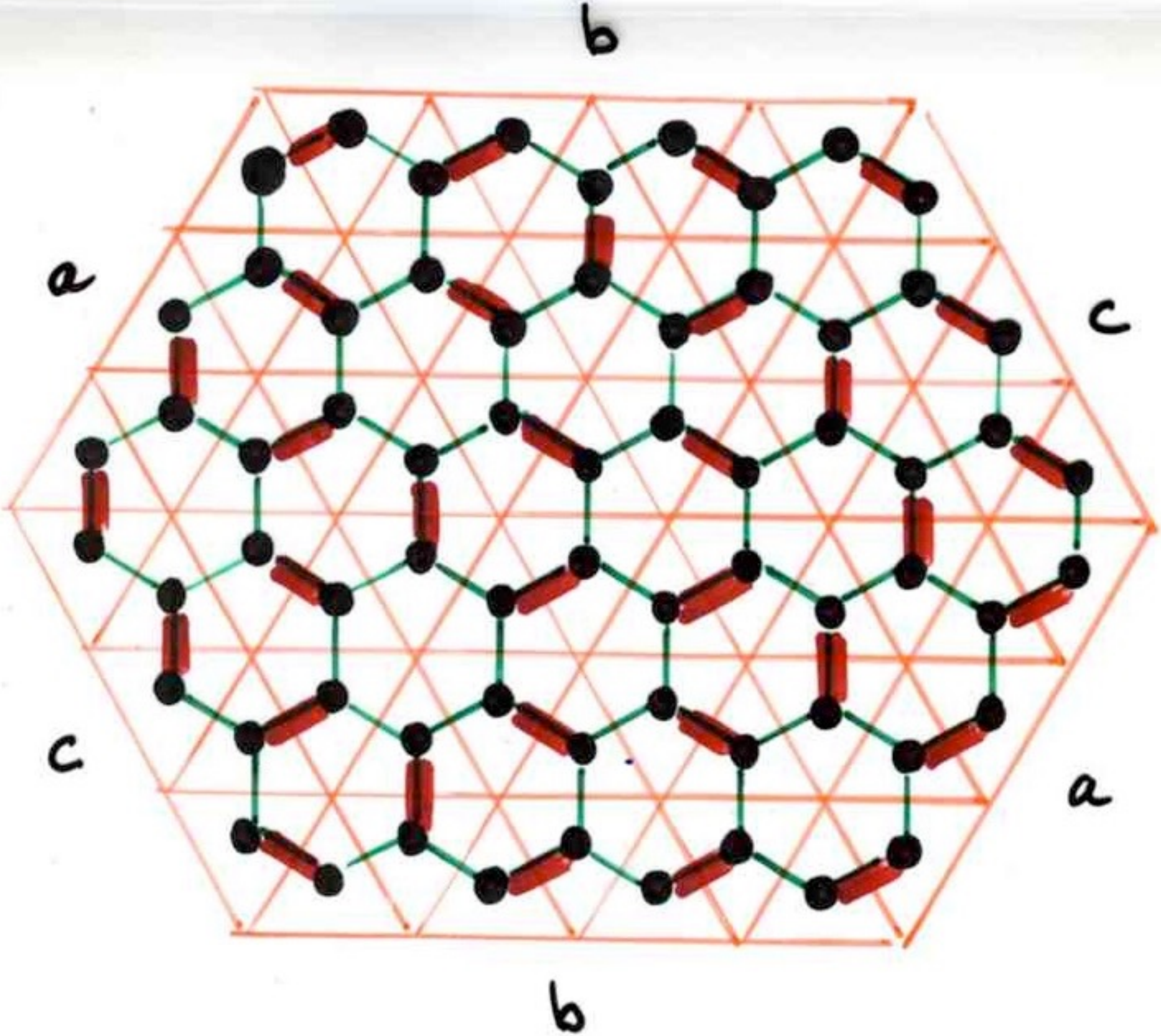
and

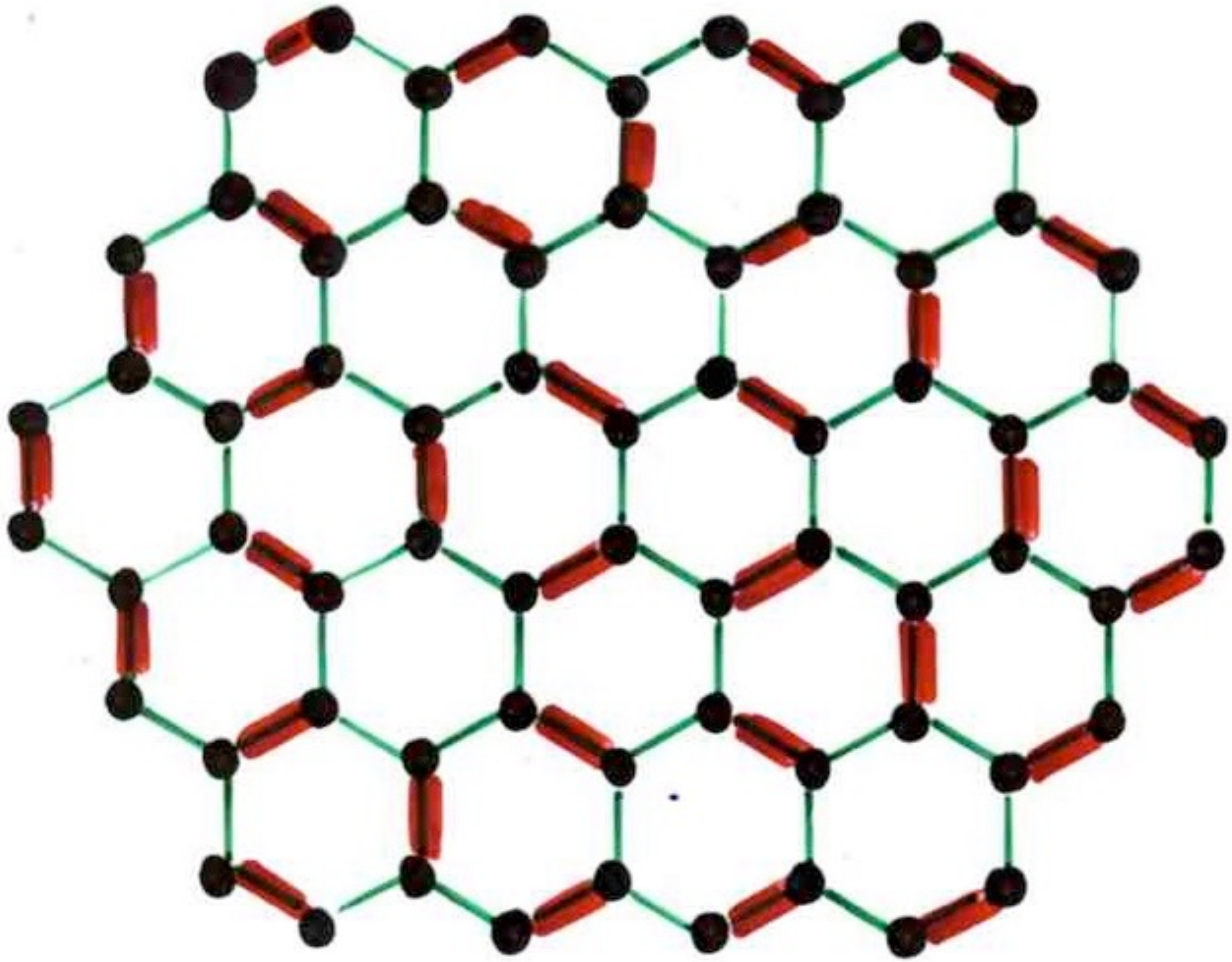
Perfect matchings

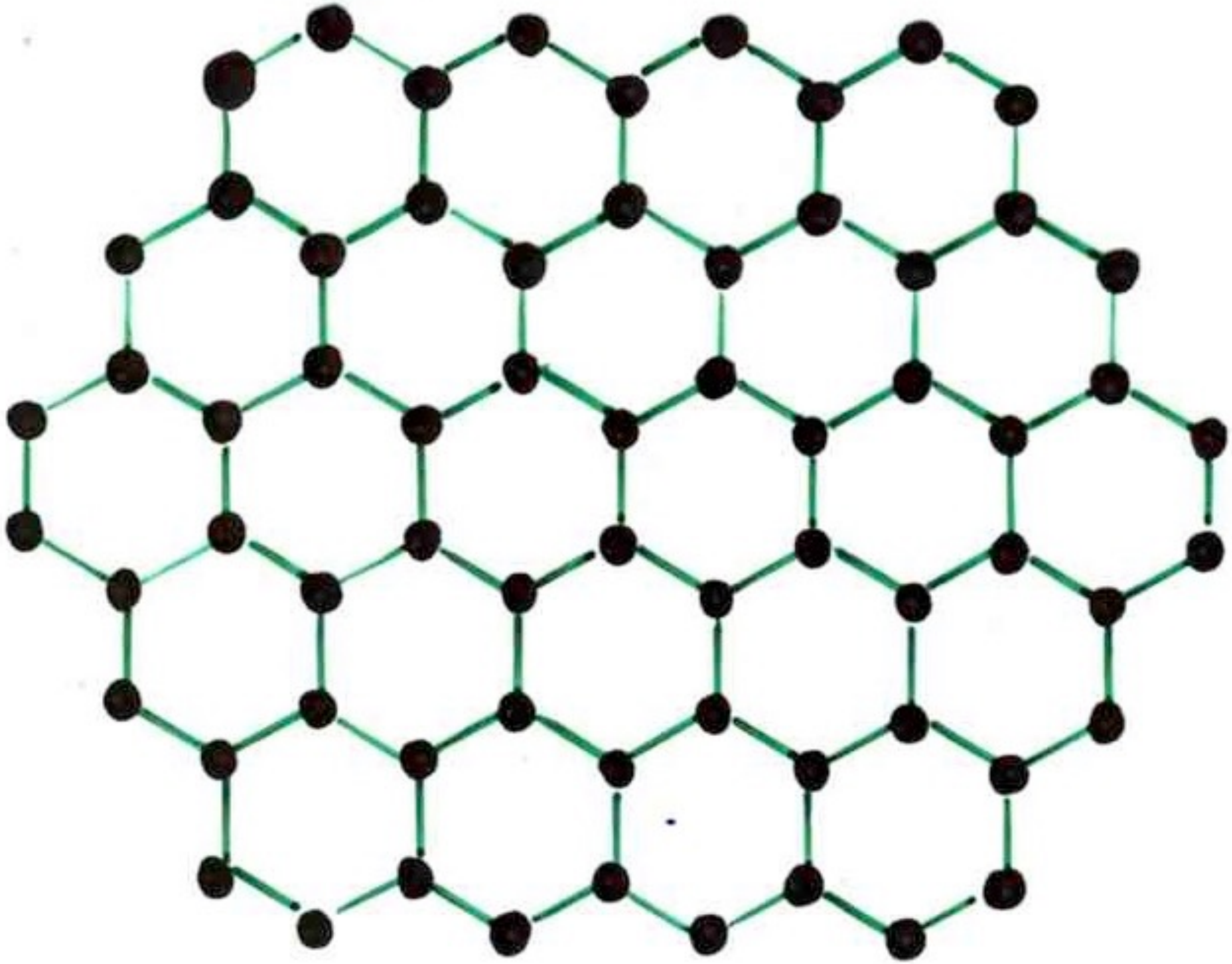

b
a
c
c
a
b

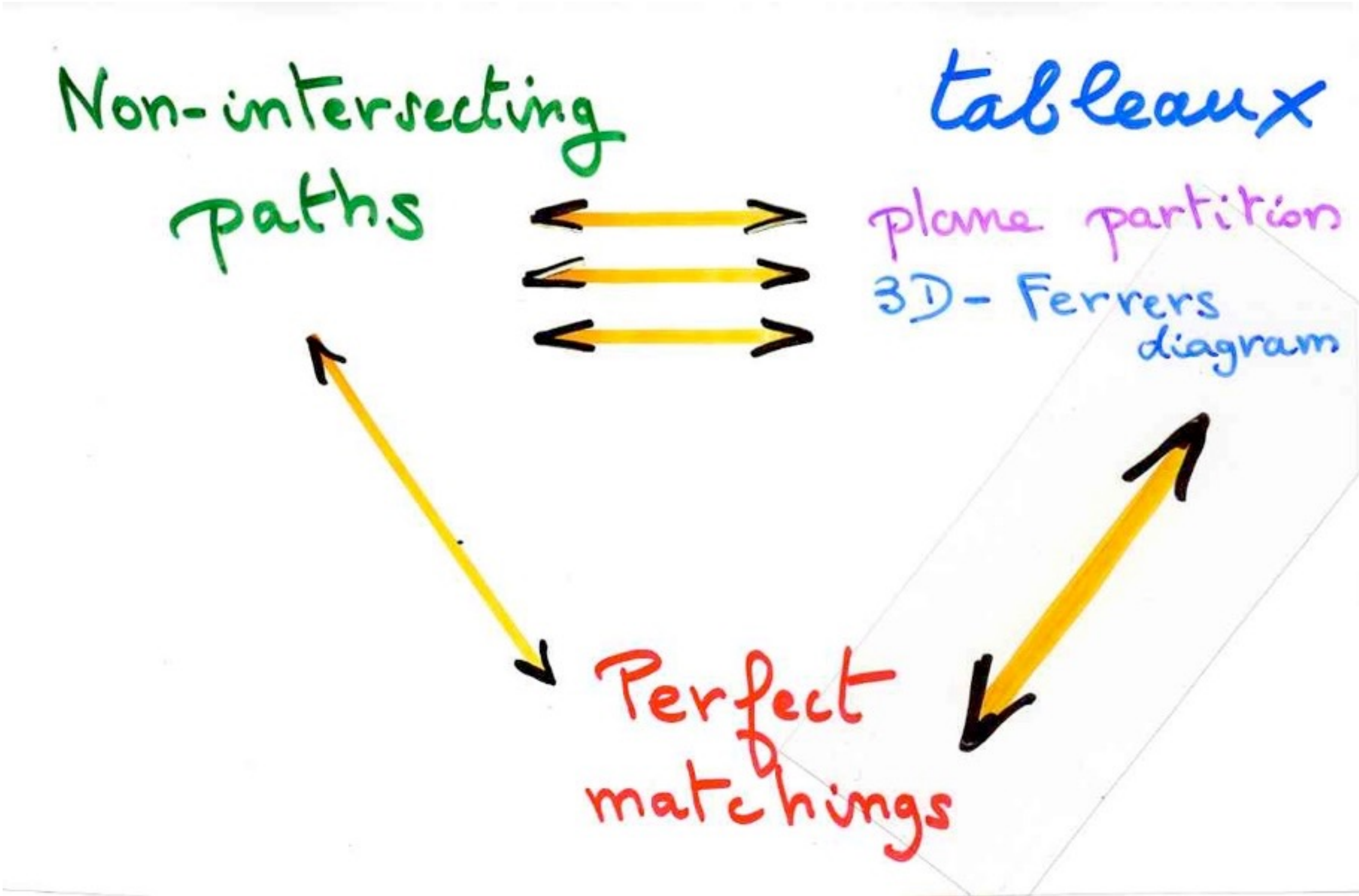

b
a
c
c
a
b


b
a
c
c
a
b

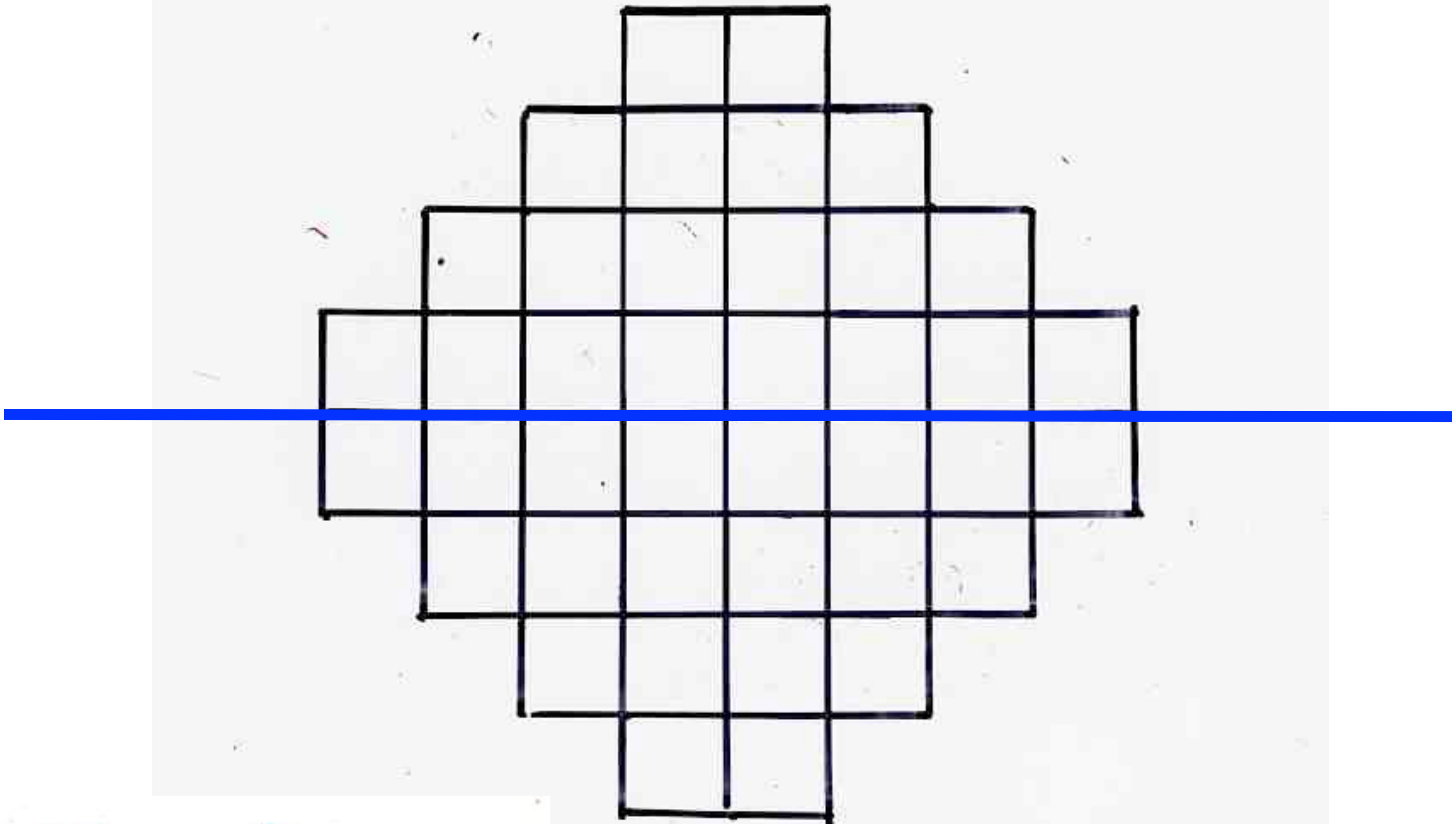

b
a
c
c
a
b





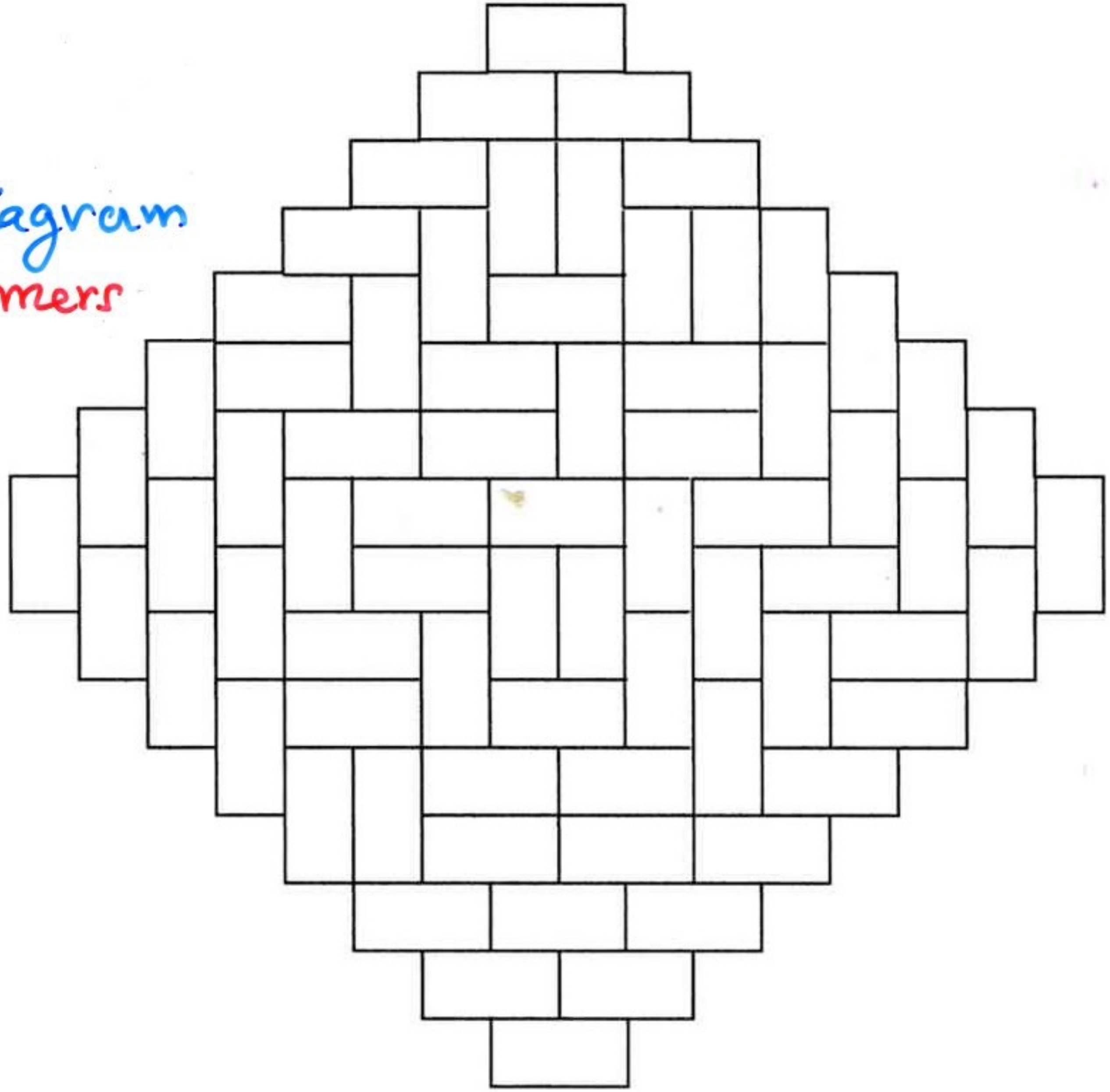

Non-intersecting paths
tableaux
plane partition
3D- Ferrers diagram
Perfect matchings

Aztec tilings

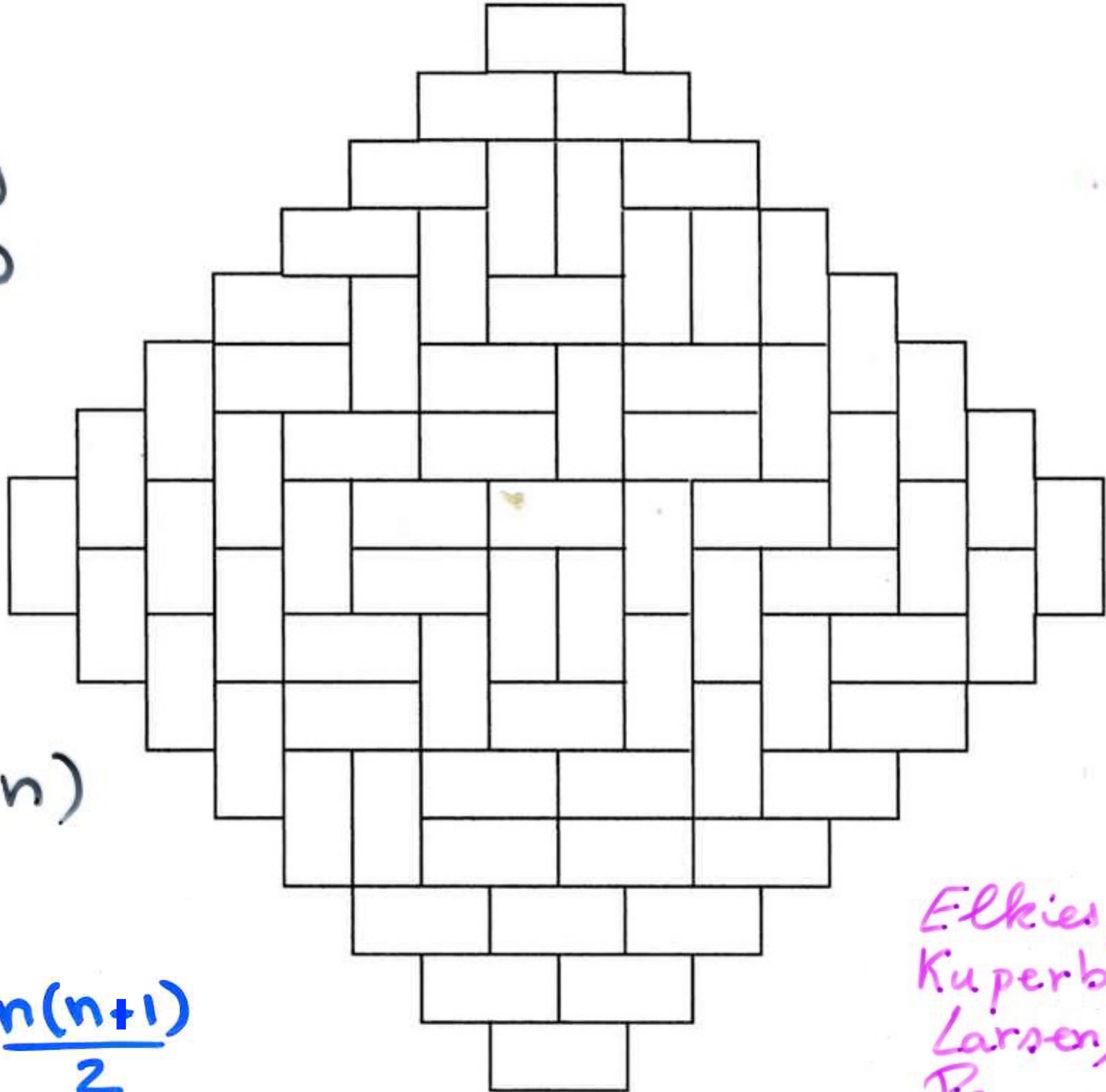


Aztec diagram

tilings
of the
Aztec diagram
with dimers



number of tilings

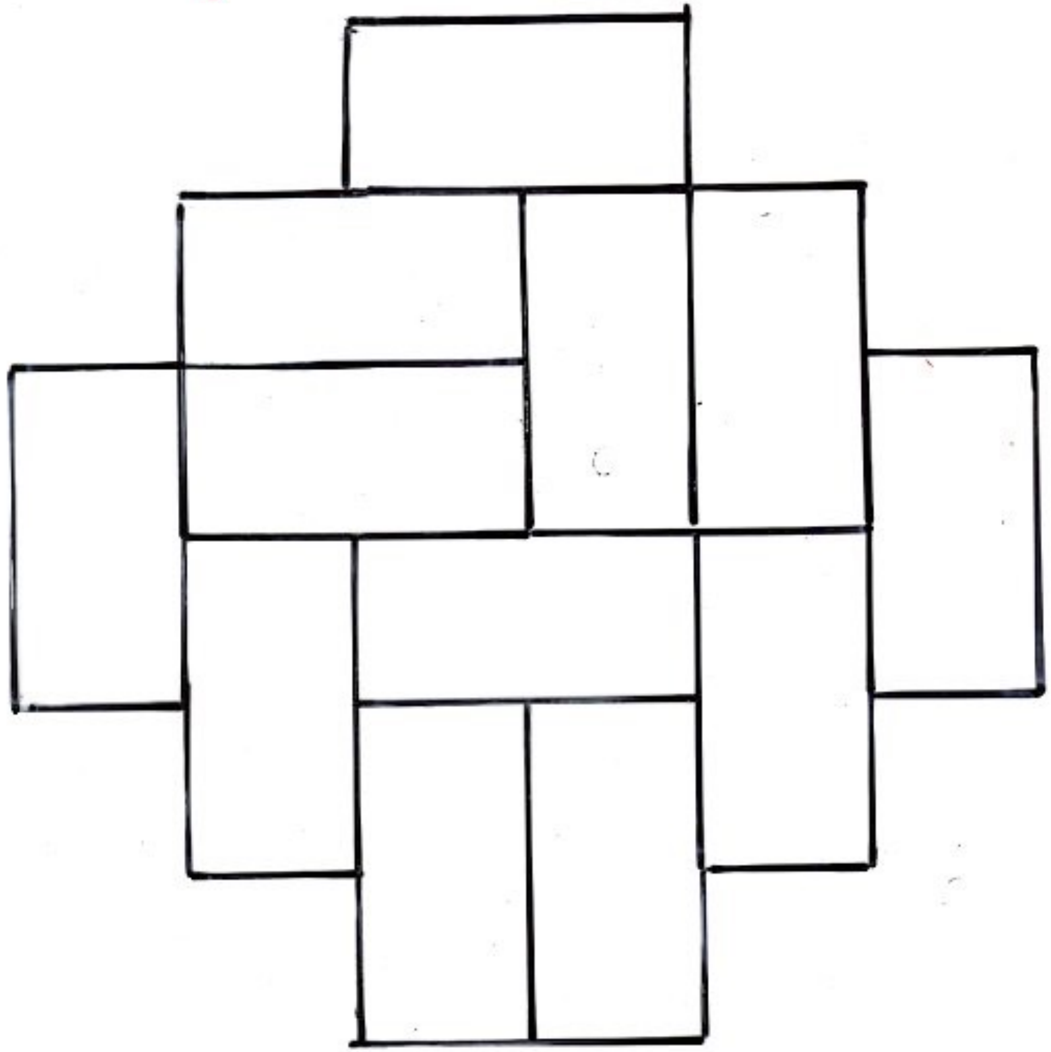


$$2^{(1+2+\dots+n)}$$

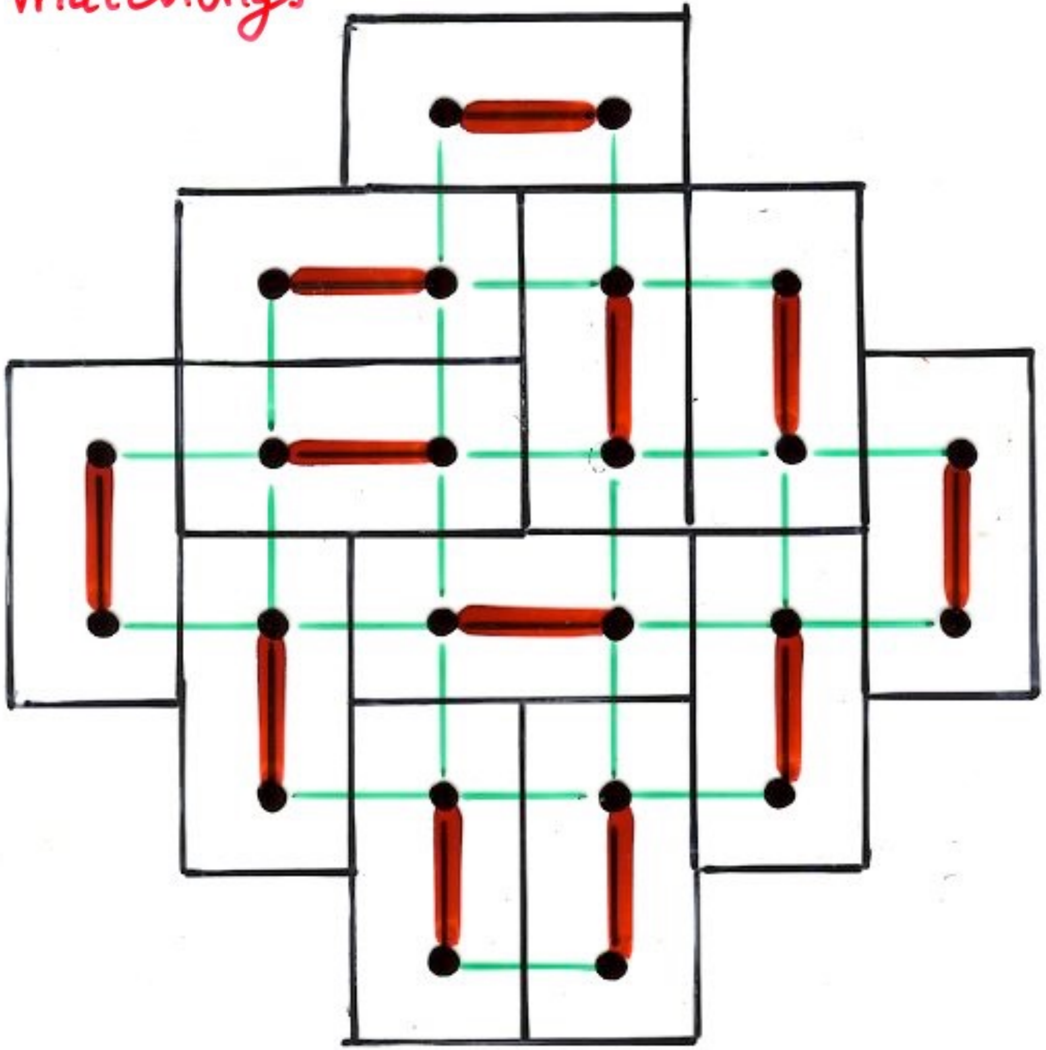
$$2^{\frac{n(n+1)}{2}}$$

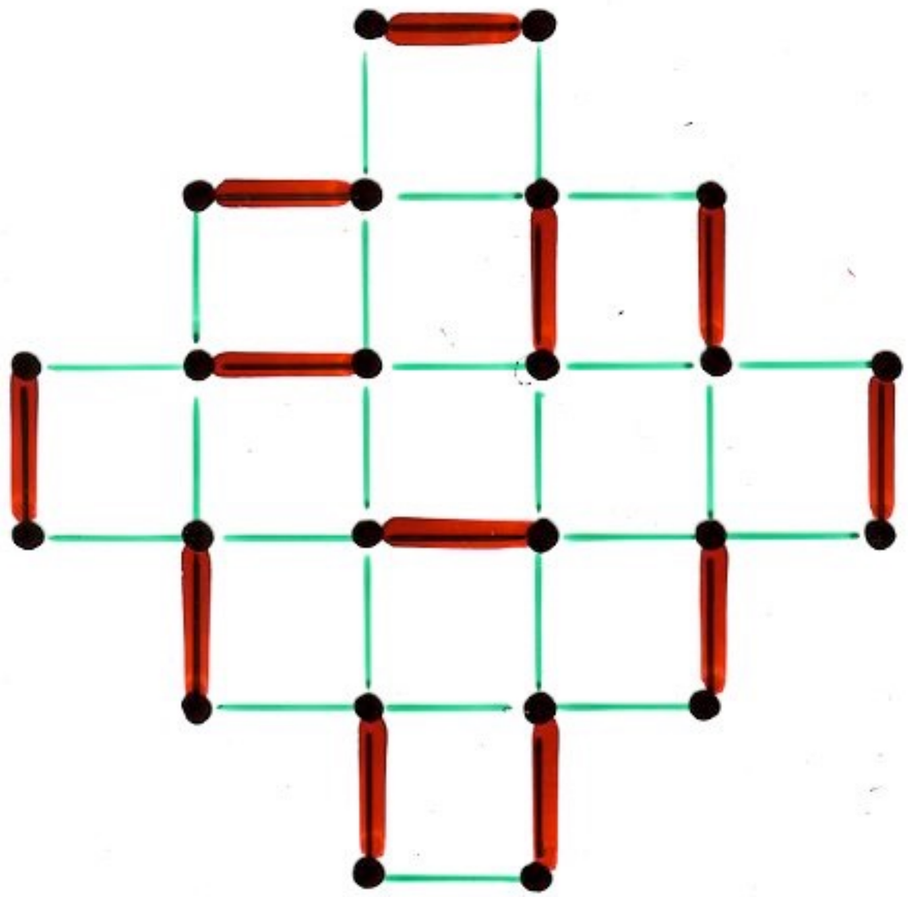
Elkies, Kuperberg, Larsen, Propp (1992)

from dimers tilings
to perfect matchings



from dimers tilings
to perfect matchings



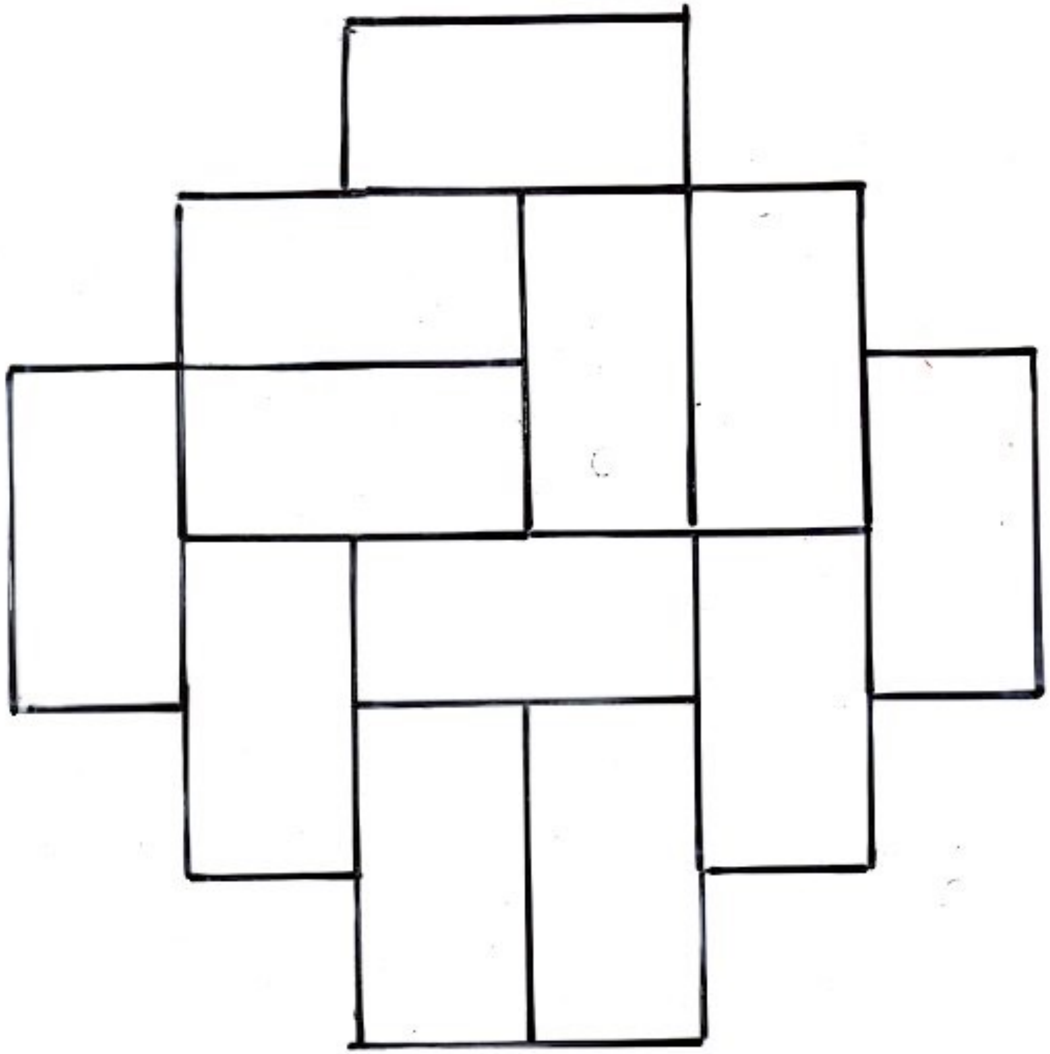


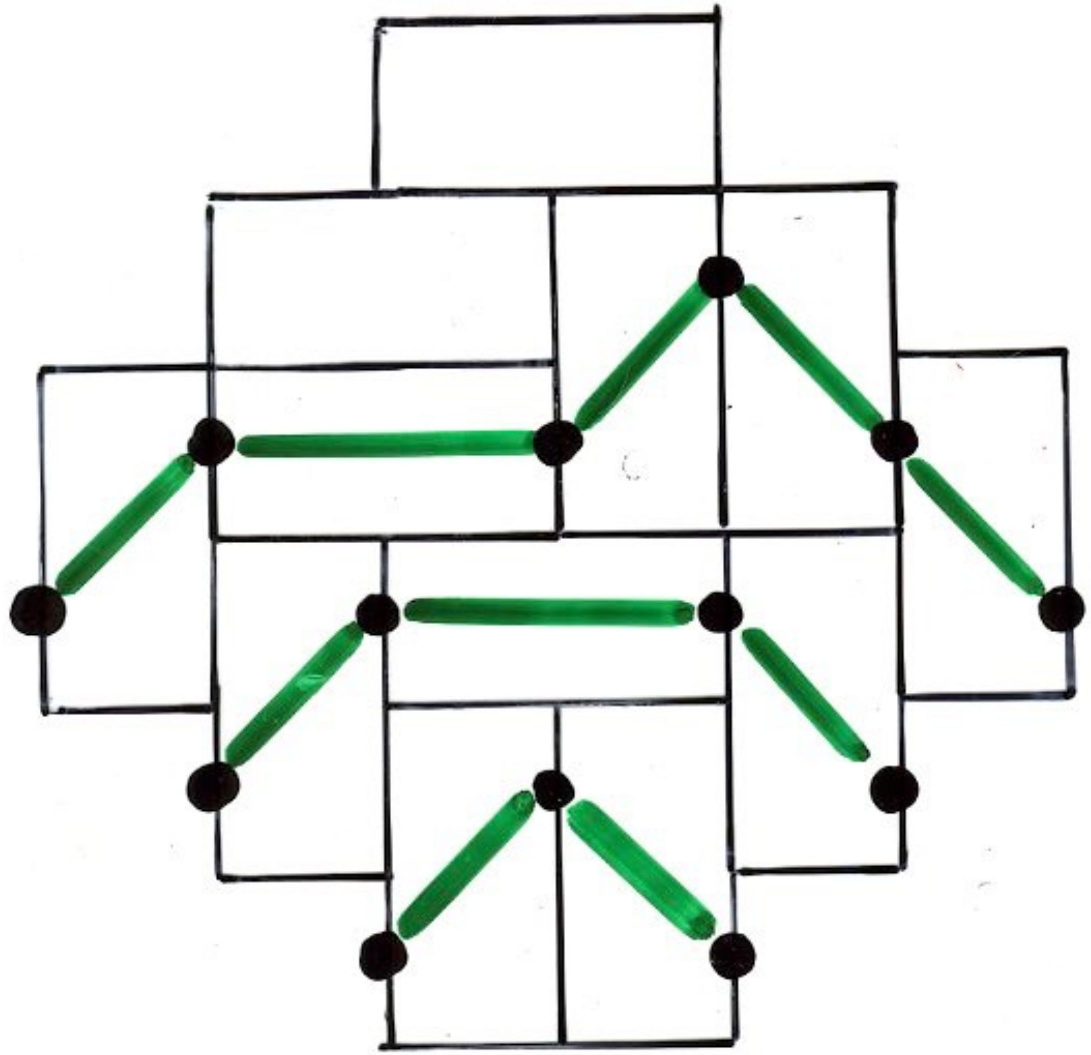
bijection

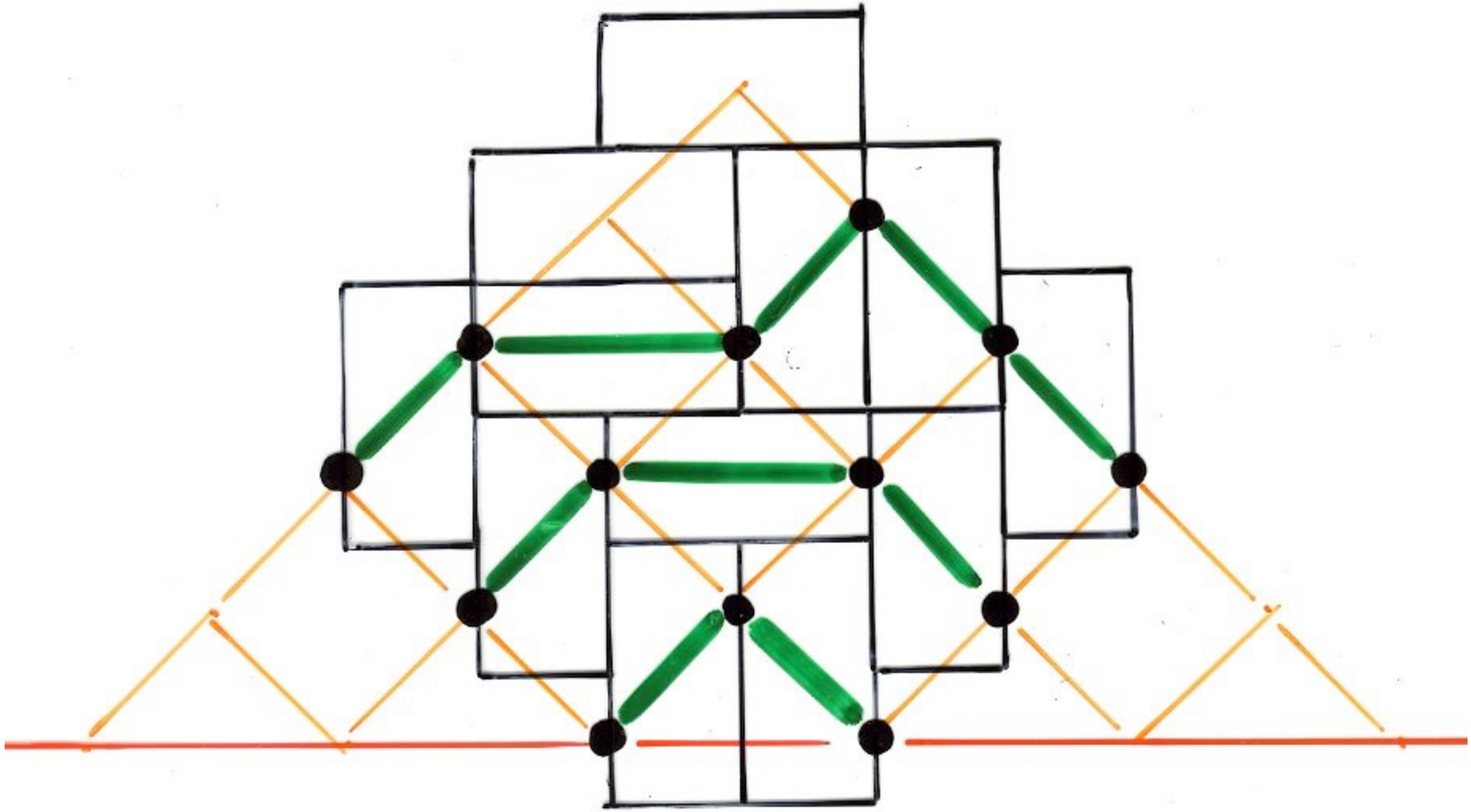
Aztec tilings

$$\updownarrow$$

non-intersecting paths
related to a Hankel determinant

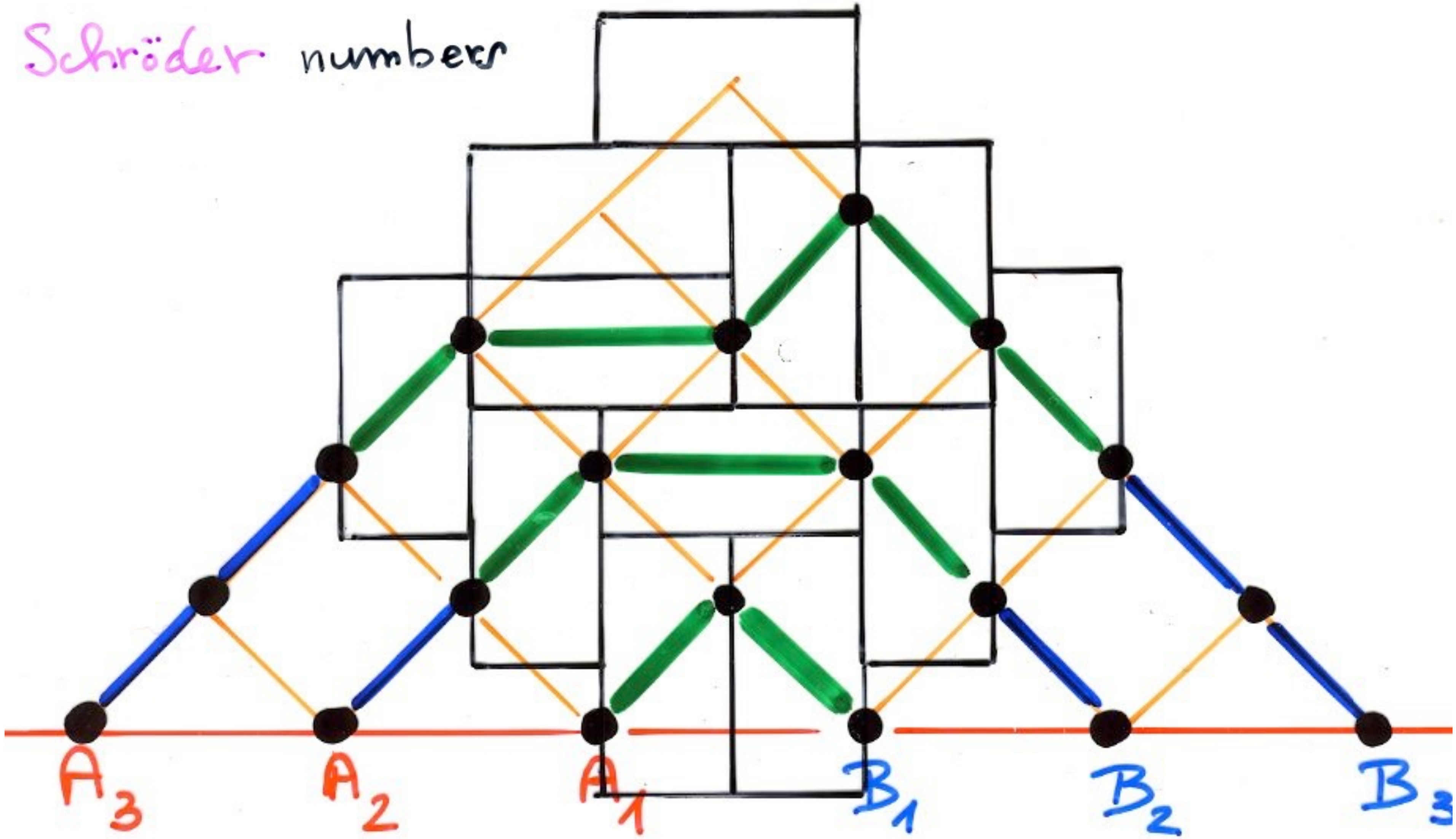






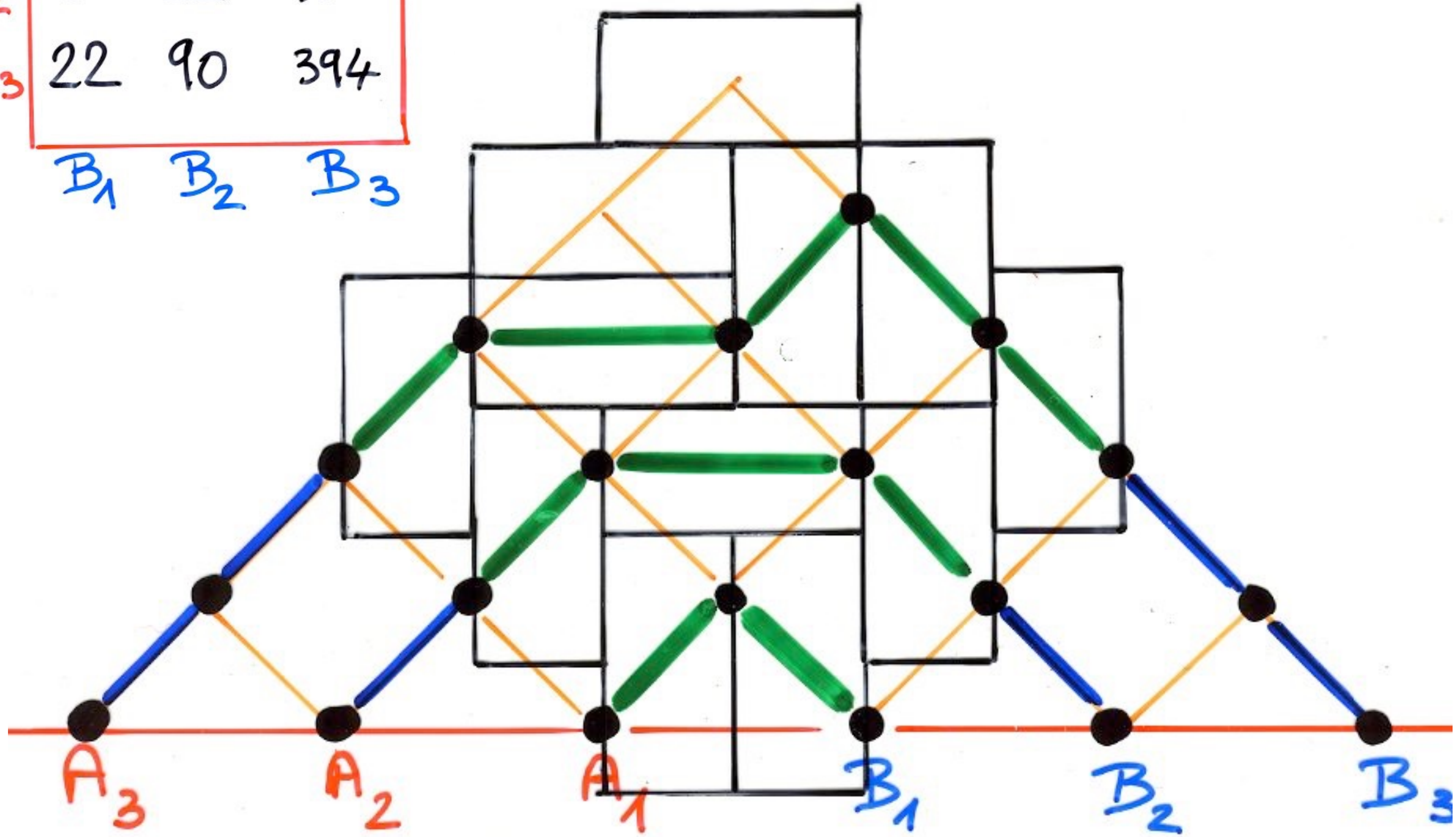
Schröder paths

Schröder numbers



Hankel determinant

$$\begin{array}{c|ccc|}
A_1 & 2 & 6 & 22 \\
A_2 & 6 & 22 & 90 \\
A_3 & 22 & 90 & 394 \\
\hline
 & B_1 & B_2 & B_3
\end{array}$$



$$\det\begin{pmatrix} 2 & 6 \\ 6 & 22 \end{pmatrix} = (2 \times 22) - (6 \times 6)$$

$$= 44 - 36$$

$$\det\begin{pmatrix} 2 & 6 \\ 6 & 22 \end{pmatrix} = (2 \times 22) - (6 \times 6)$$

$$= 44 - 36$$

$$= 8 = 2^3$$

(!)

$$\det\begin{pmatrix} 2 & 6 & 22 \\ 6 & 22 & 90 \\ 22 & 90 & 394 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & 22 & \cdot \\ \cdot & \cdot & 394 \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & 22 \\ 6 & \cdot & \cdot \\ \cdot & 90 & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & 6 & \cdot \\ \cdot & \cdot & 90 \\ 22 & \cdot & \cdot \end{pmatrix}$$

$$+\,17336 \qquad +\,11880 \qquad +\,11880 \;\rightarrow\; 41096$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & \cdot & 90 \\ \cdot & 90 & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & 6 & \cdot \\ 6 & \cdot & \cdot \\ \cdot & \cdot & 394 \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & 22 \\ \cdot & 22 & \cdot \\ 22 & \cdot & \cdot \end{pmatrix}$$

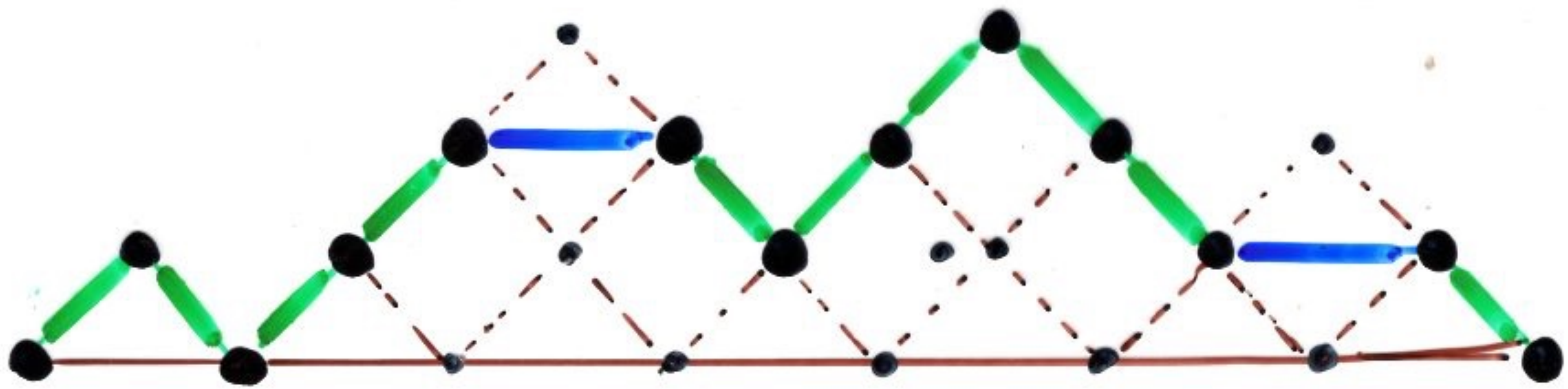
$$-\,16200 \qquad -\,14184 \qquad -\,10648 \;\rightarrow\; -41032$$

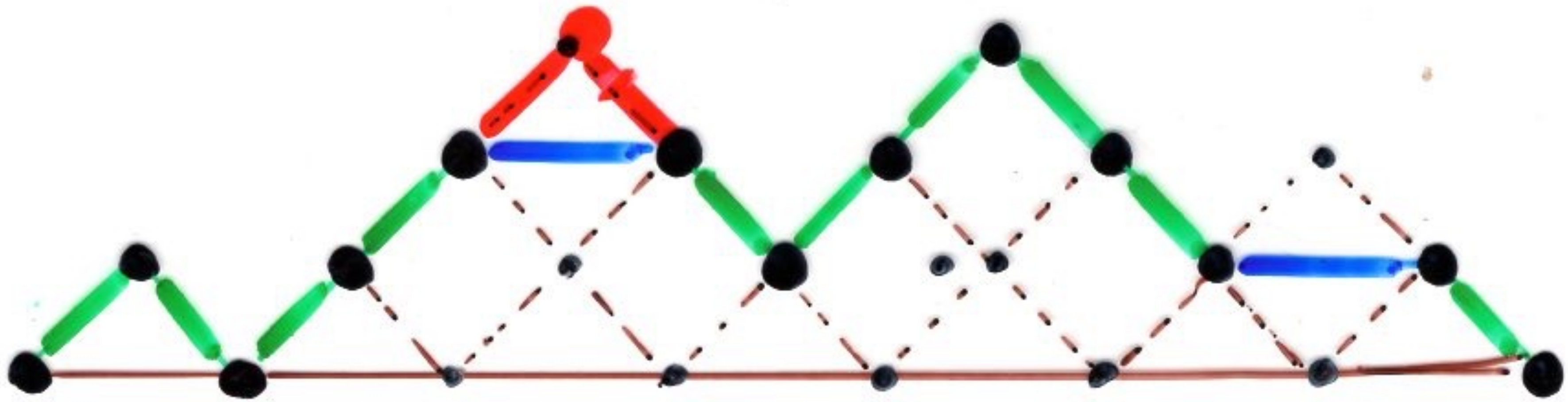
$$64$$

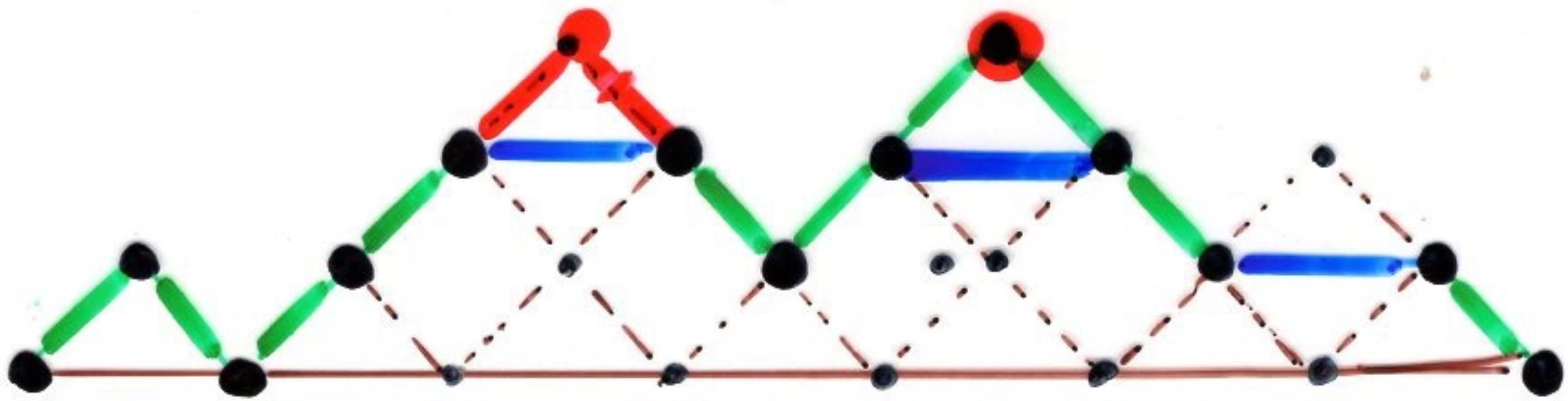
$$= 2^6 \quad (!!)$$

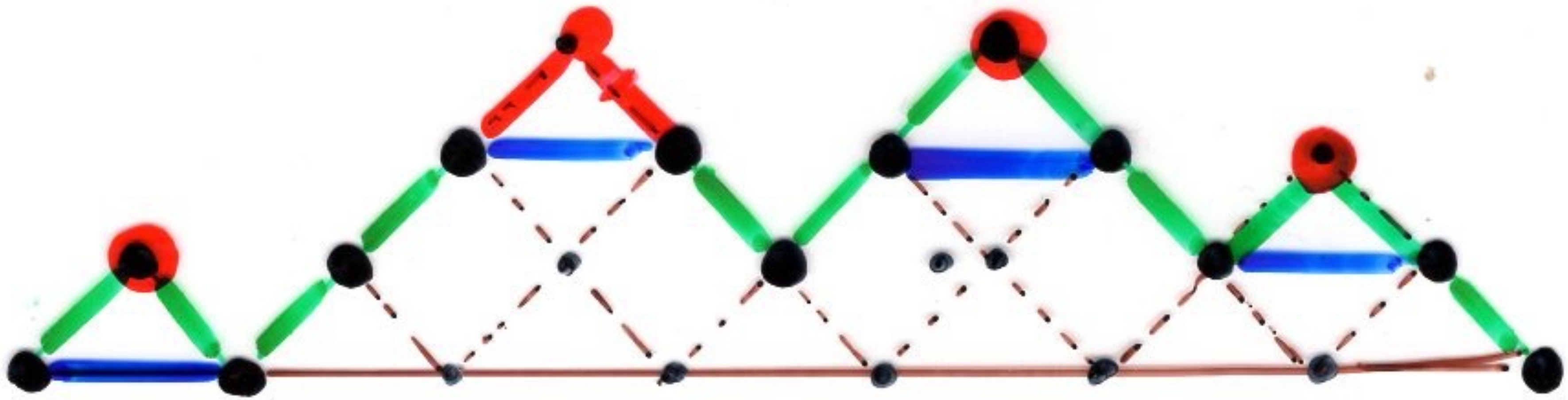
« bijective computation »
of the Hankel determinant

of Schröder numbers giving
the number of tilings of the Aztec diagram



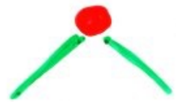






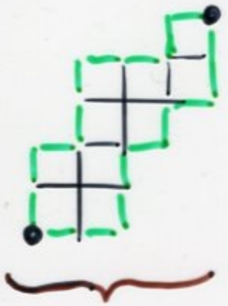
$$S_n = \sum_{\substack{\omega \\ \text{Dyck path} \\ |\omega| = 2n}} 2^{\mathrm{peak}(\omega)}$$

$\mathrm{peak}(\omega) =$ number of peaks of the path ω



(β)- distribution

$\rightarrow$ Ch 2c the Catalan garden



k

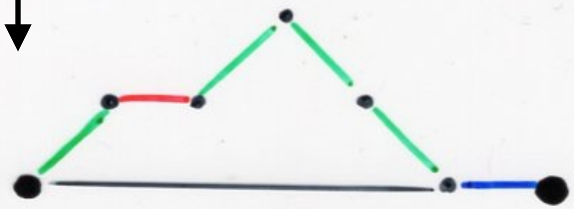
staircase polygons

number of columns

Dyck path

number of peaks

(β)- distribution $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$



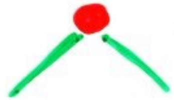
number of / or — in a 2-colored Motzkin path

Dyck path

number of SE steps from odd to even level

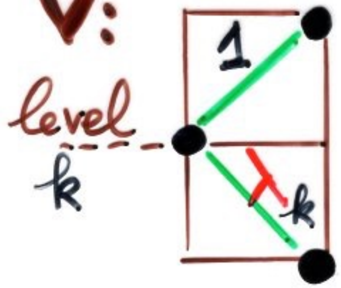
$$S_n = \sum_{\substack{\omega \\ \text{Dyck path} \\ |\omega| = 2n}} 2^{\mathrm{peak}(\omega)}$$

$\mathrm{peak}(\omega) =$ number of peaks of the path ω



$$S_n = \sum_{\substack{\omega \\ \text{Dyck path} \\ |\omega| = 2n}} v(\omega)$$

v:



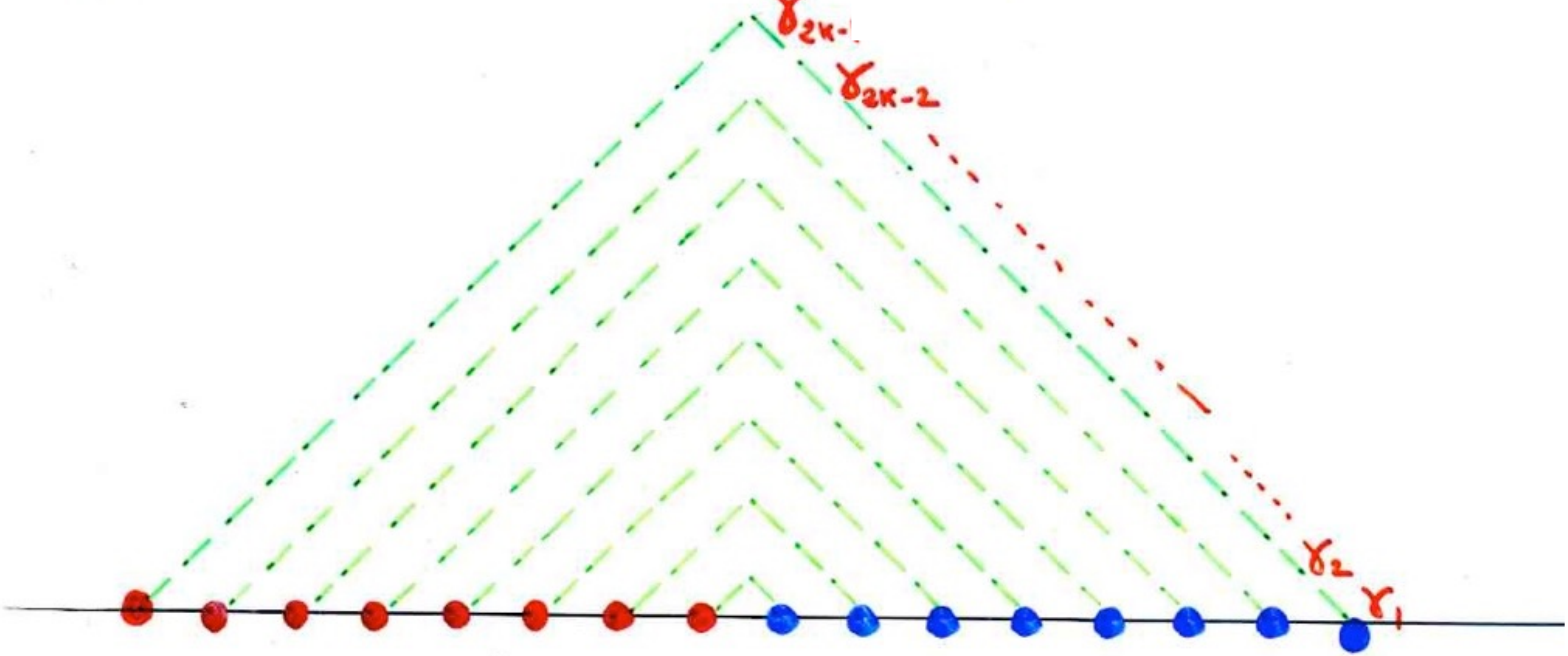
$$\lambda_k = \begin{cases} 1, & k \text{ even} \\ 2, & k \text{ odd} \end{cases}$$

(β)- distribution

$\rightarrow$ Ch 2c the Catalan garden

$$H_k^{(1)} = \begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_{k+1} \\ \mu_2 & \cdots & \cdots & \vdots \\ \vdots & & & \vdots \\ \mu_{k+1} & \cdots & \cdots & \mu_{2k+1} \end{vmatrix}$$

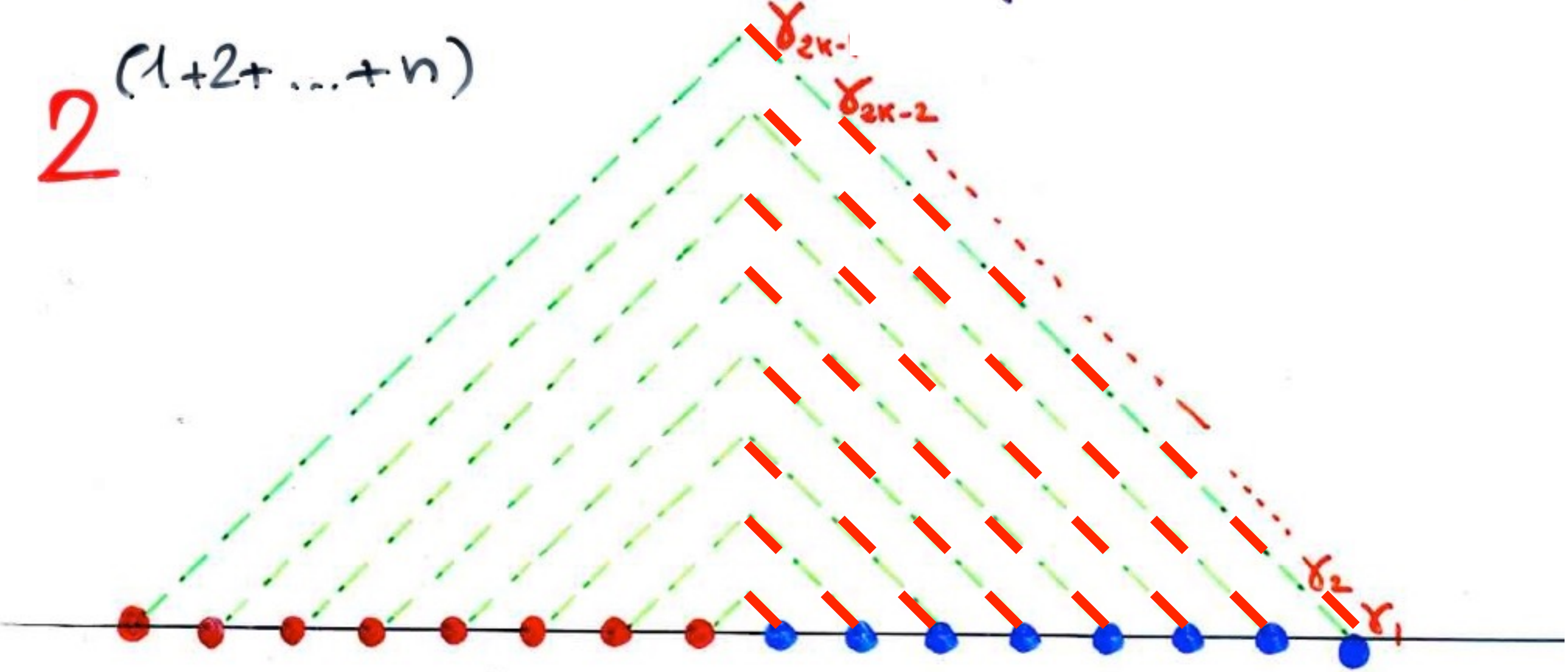
$H_k^{(1)}$



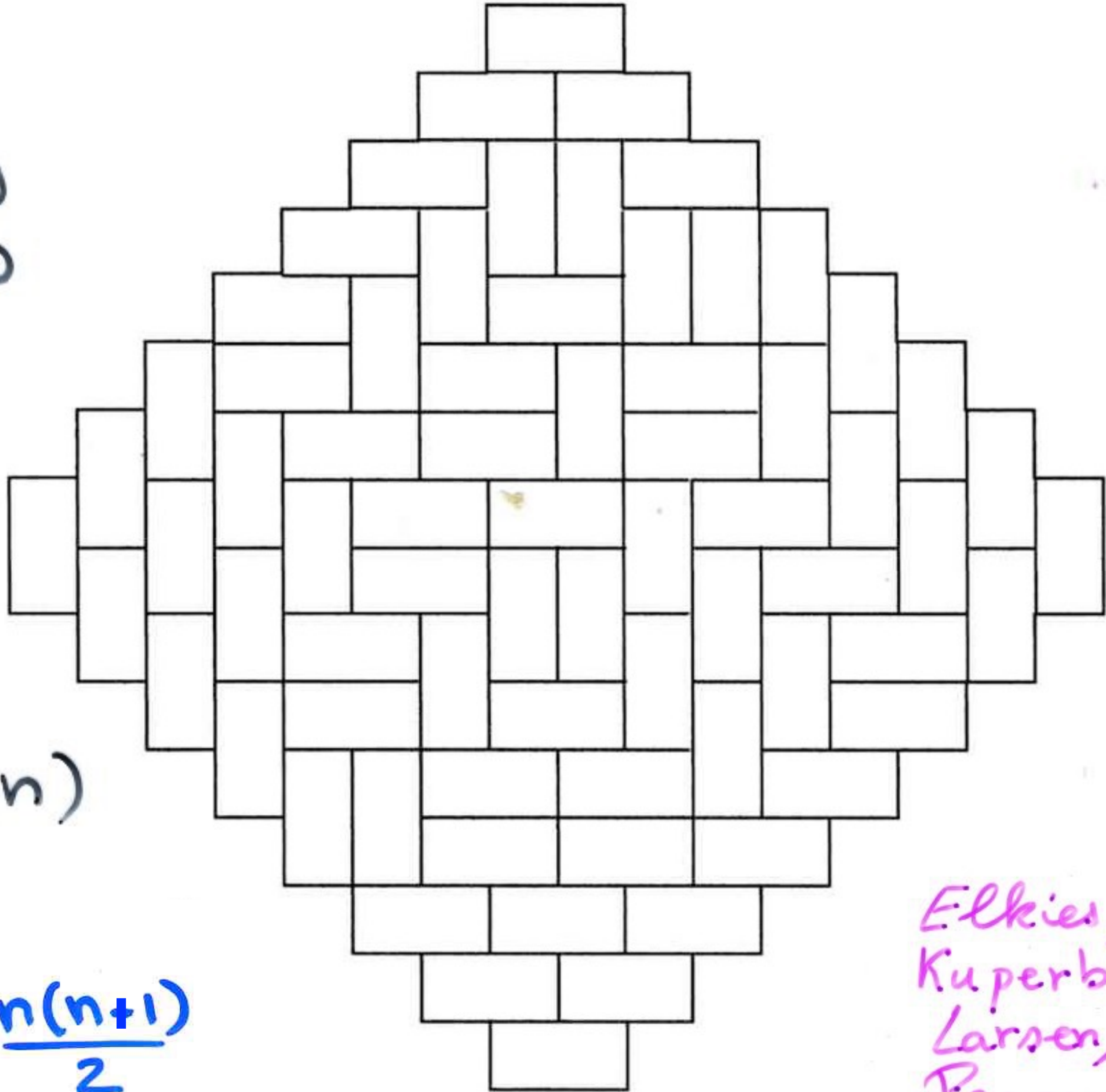
$$H_k^{(1)} = \begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_{k+1} \\ \mu_2 & \cdots & \cdots & \vdots \\ \vdots & & & \vdots \\ \mu_{k+1} & \cdots & \cdots & \mu_{2k+1} \end{vmatrix}$$

$H_k^{(1)}$

$2^{(1+2+\dots+n)}$



number of tilings



$2^{(1+2+\dots+n)}$

$2^{\frac{n(n+1)}{2}}$

Elkies, Kuperberg, Larsen, Propp (1992)

complements:

Schröder numbers

and

the associahedron

S_n number of Schröder paths

$1, 2, 6, 22, 90, 394, \ldots$

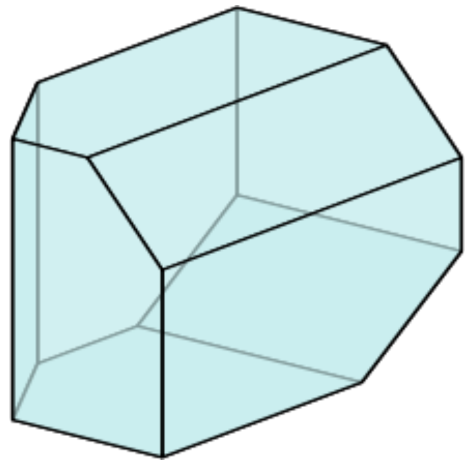
little Schröder $\frac{1}{2} S_n$

$1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049$

Total number of cells

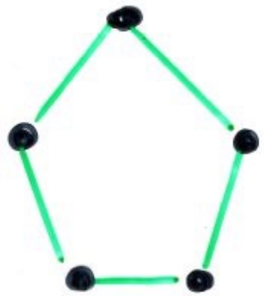
$$14 + 21 + 9 + 1 = 45$$

vertices edges faces associahedron



$$5 + 5 + 1 = 11$$

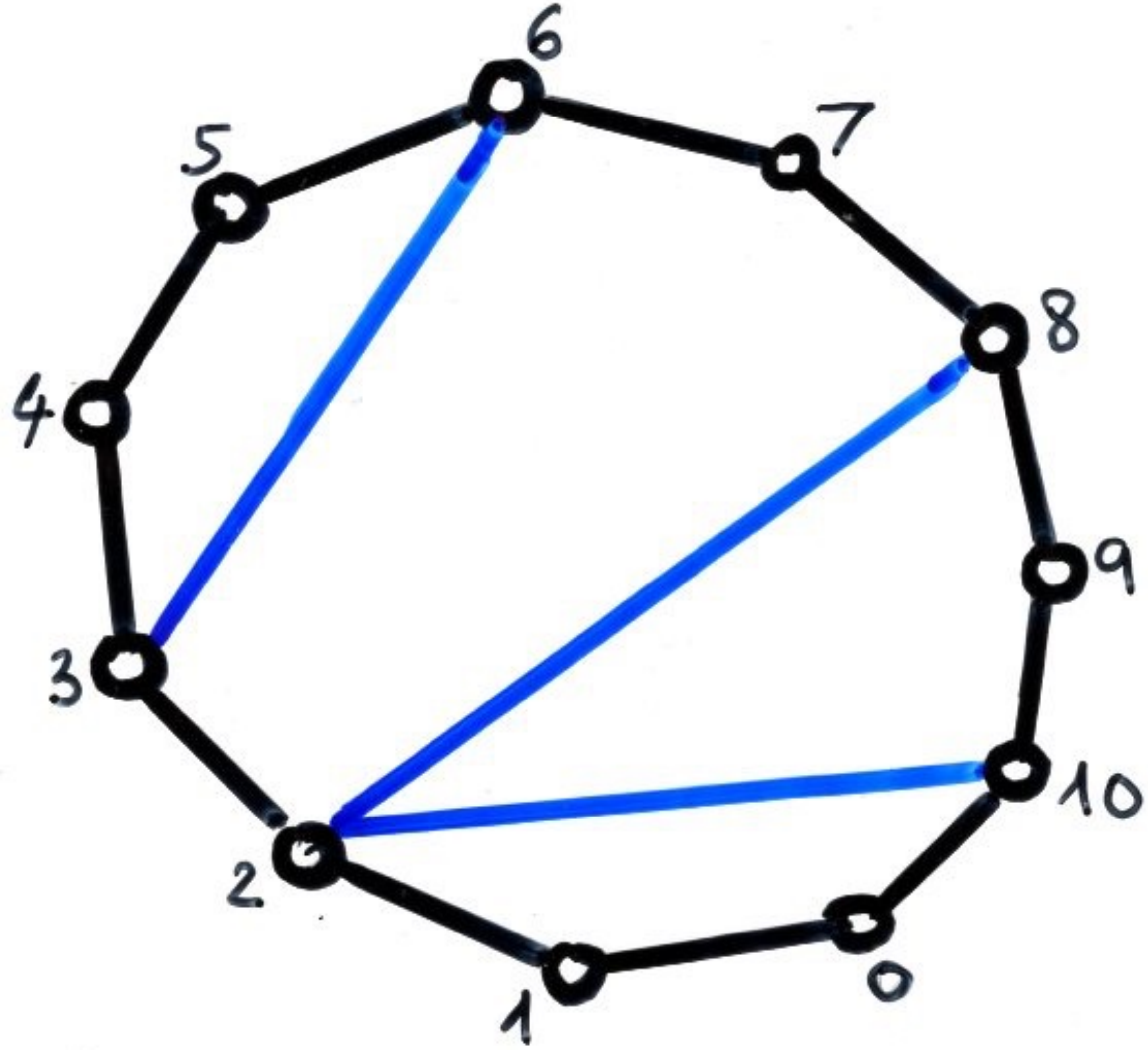
vertices edges



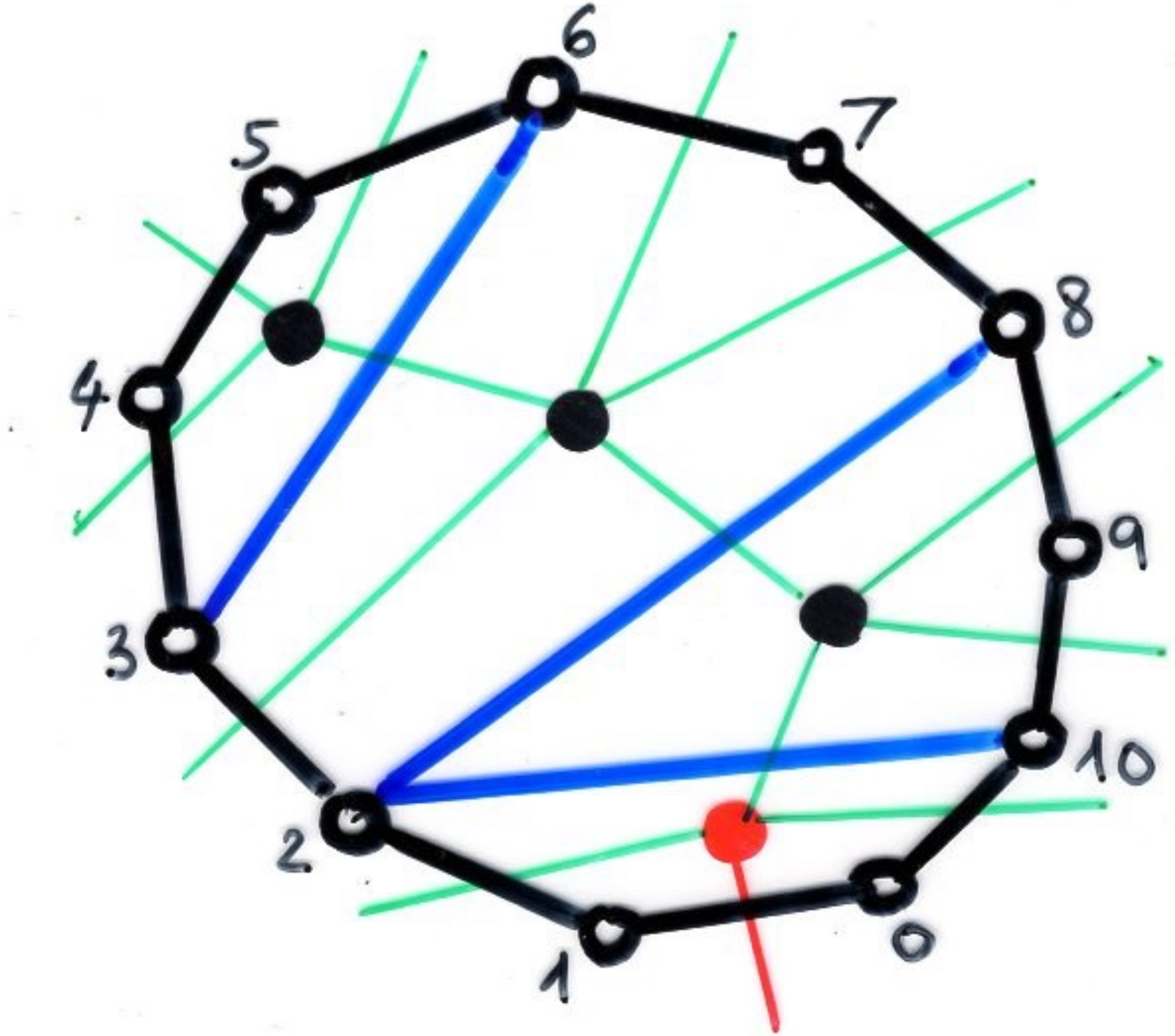
$$2 + 1 = 3$$

1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, ..

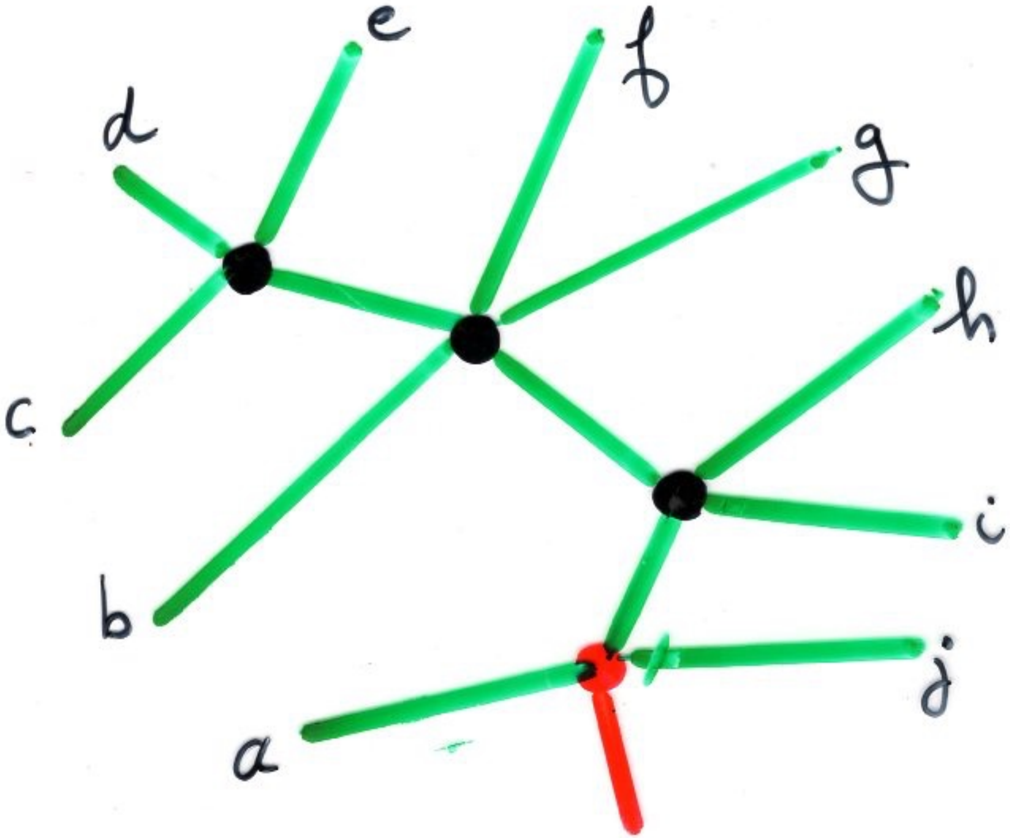
little Schröder $\frac{1}{2} S_n$



cells of the associahedron $\longleftrightarrow$ partial triangulations



partial triangulations $\longleftrightarrow$ Schröder trees



Def. Schröder tree

- planar tree (→ch2a)
- no vertex with a single child

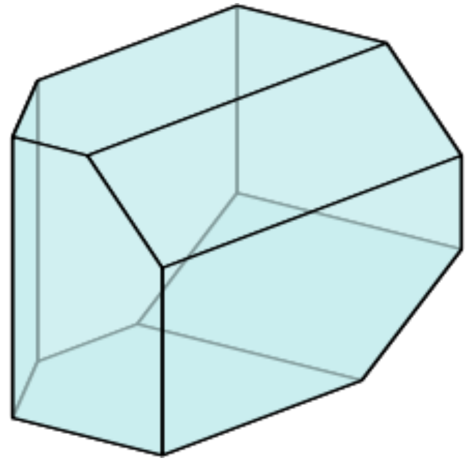
- The number of Schröder trees with n leaves is $\frac{1}{2} S_n$ (little Schröder) where S_n is the number of Schröder paths going from $(0,0)$ to $(2n,0)$

<u>exercise</u> prove this fact, possilly with a bijection.

Total number of cells

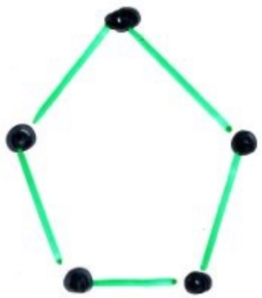
$14 + 21 + 9 + 1 = 45$

vertices edges faces associahedron



$5 + 5 + 1 = 11$

vertices edges



$2 + 1 = 3$

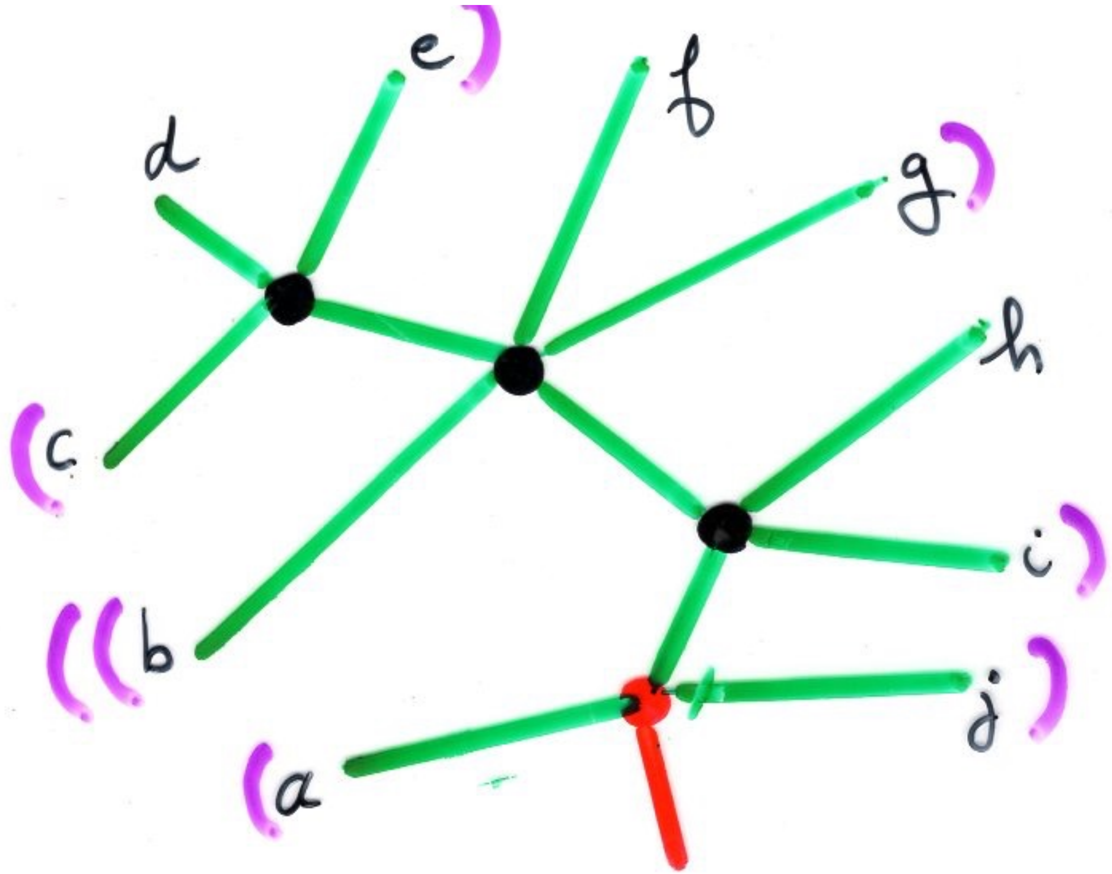
1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, ..

Schröder - Hipparchus numbers

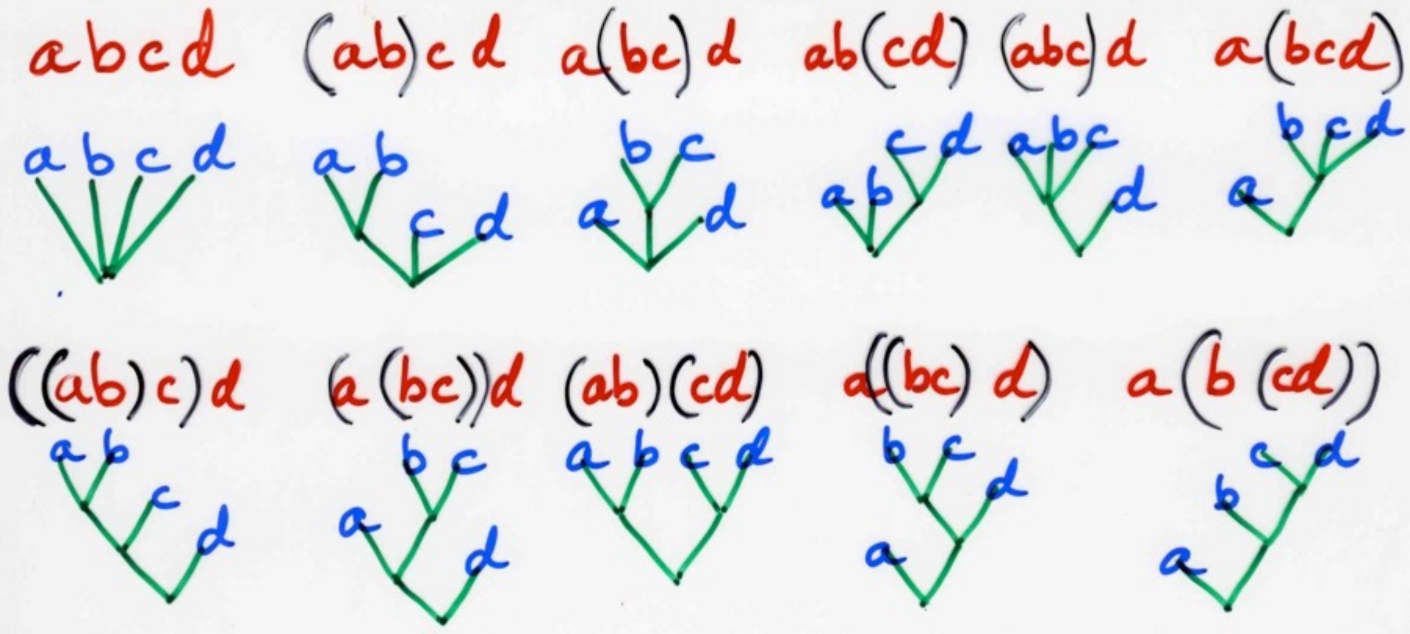
Plutarch :

Chrysippus says that the number of compound propositions that can be made from only ten simple propositions exceeds a million. Hipparchus, to be sure, refuted this by showing that this number is 103 049.

D. Hough (1994)


e)
f
d
g)
h
(c
i)
((b
j)
(a


(a ((b (c d e) f g) h i) j)



$\frac{1}{2} S_4 = 11$

Schröder trees with Hipparchus parenthesis expressions

1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, ...

little Schröder $\frac{1}{2} S_n$

complements:

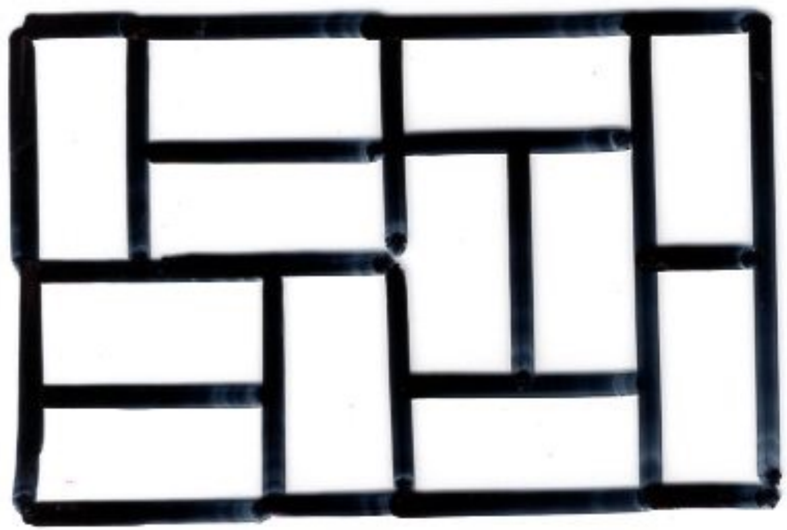
Tiling a rectangle

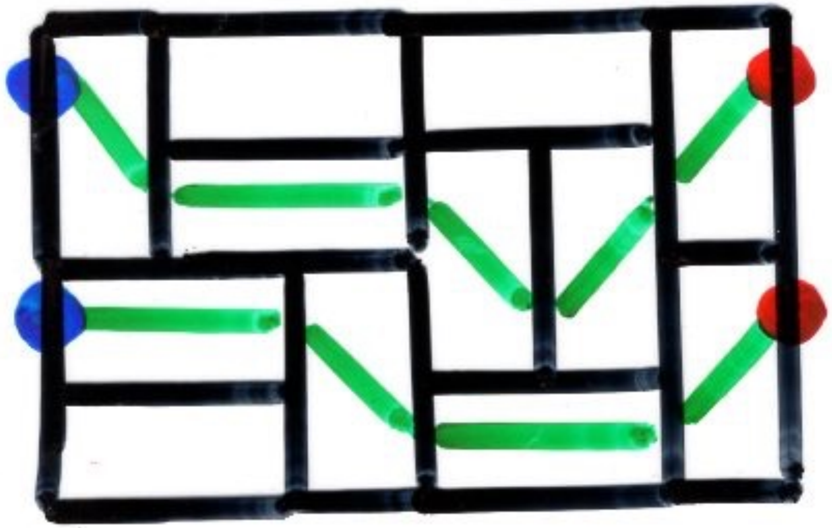
number of tilings with dimers of a $m \times n$ rectangle

$$4^{mn} \prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \left(4 \cos^2 \frac{i\pi}{m+1} + 4 \cos^2 \frac{j\pi}{n+1} \right)$$

Kasteleyn (1961)

it is an integer !!

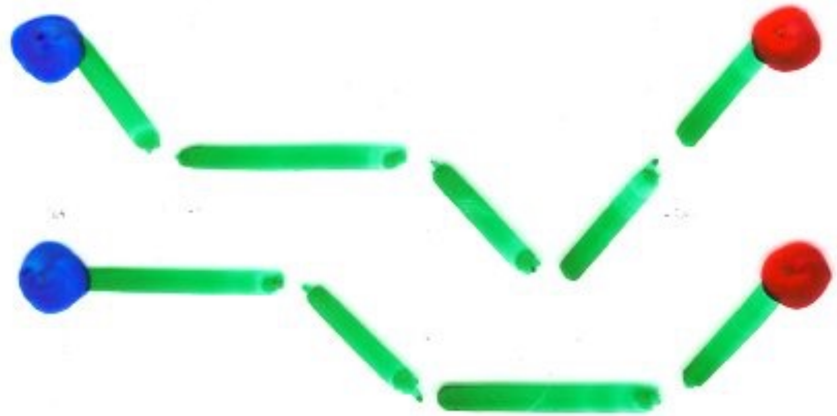




V. Strehl bijective proof
resultant of 2 Tchebychef polynomials
(Filonacci) 2nd kind

$U_m(x), U_n(x)$

→ Ch 1c



V. Strehl bijective proof
resultant of 2 Tchebychef polynomials
(Fibonacci) 2nd kind

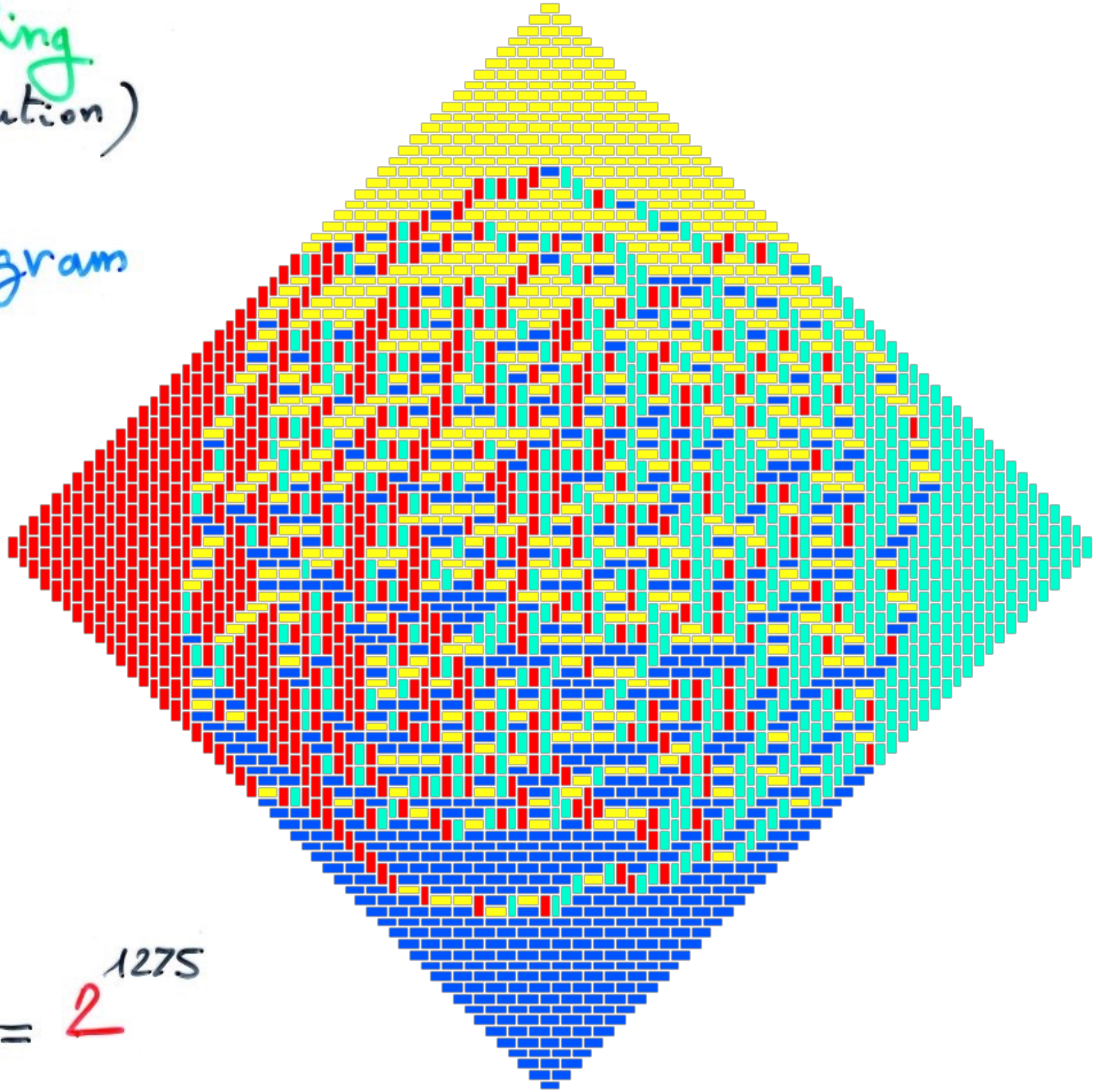
$U_m(x), U_n(x)$

→ Ch 1c

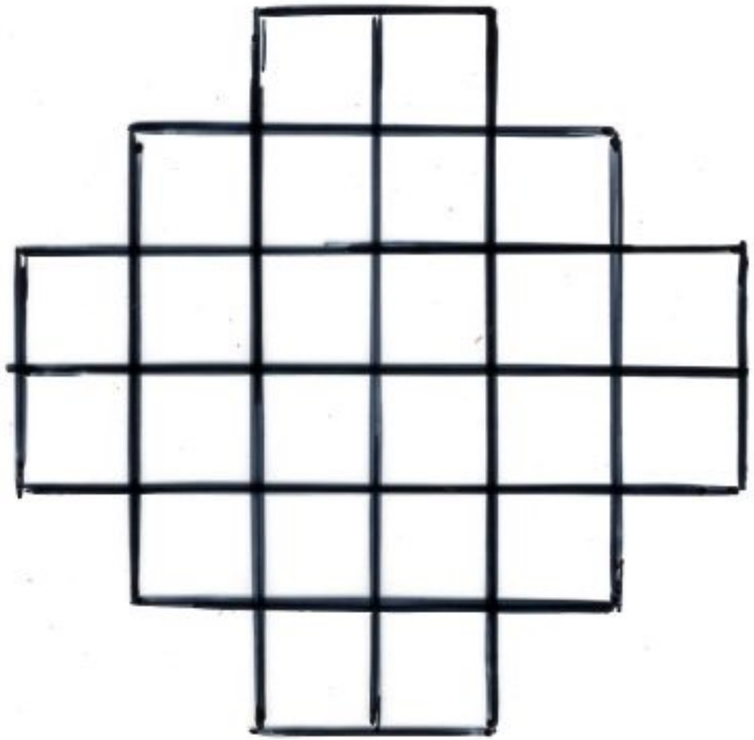
complements

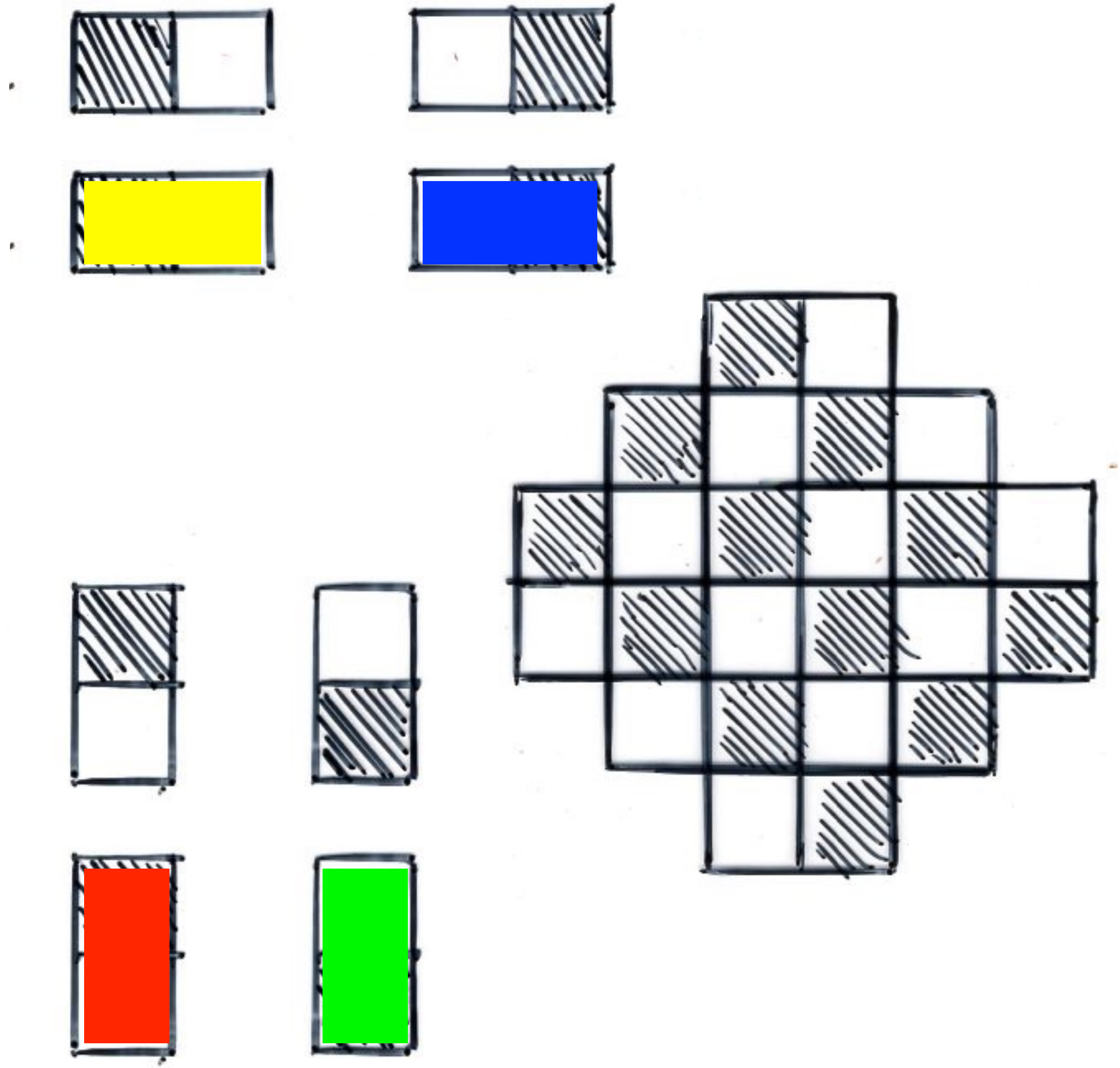
the arctic circle theorem

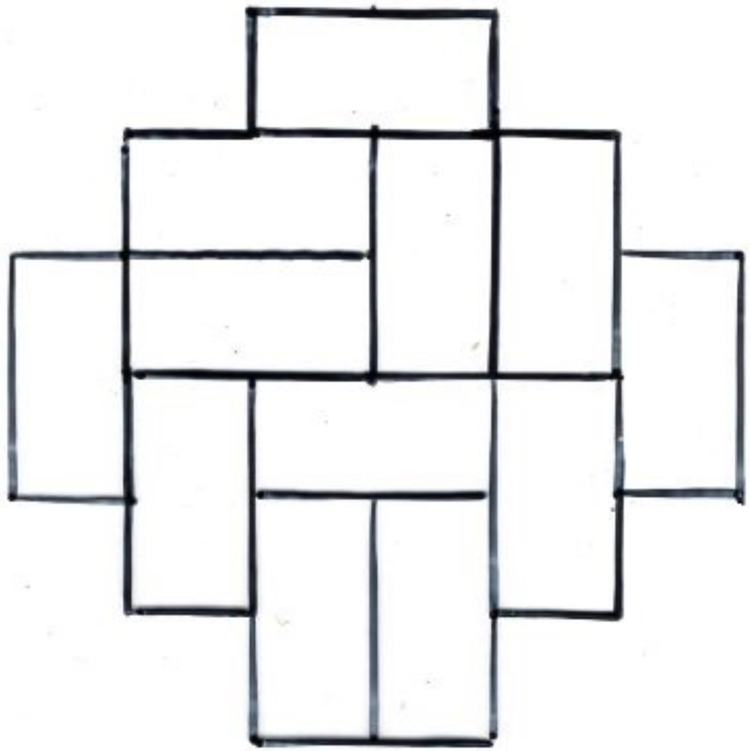
random tiling
(equidistribution)
of the
Aztec diagram

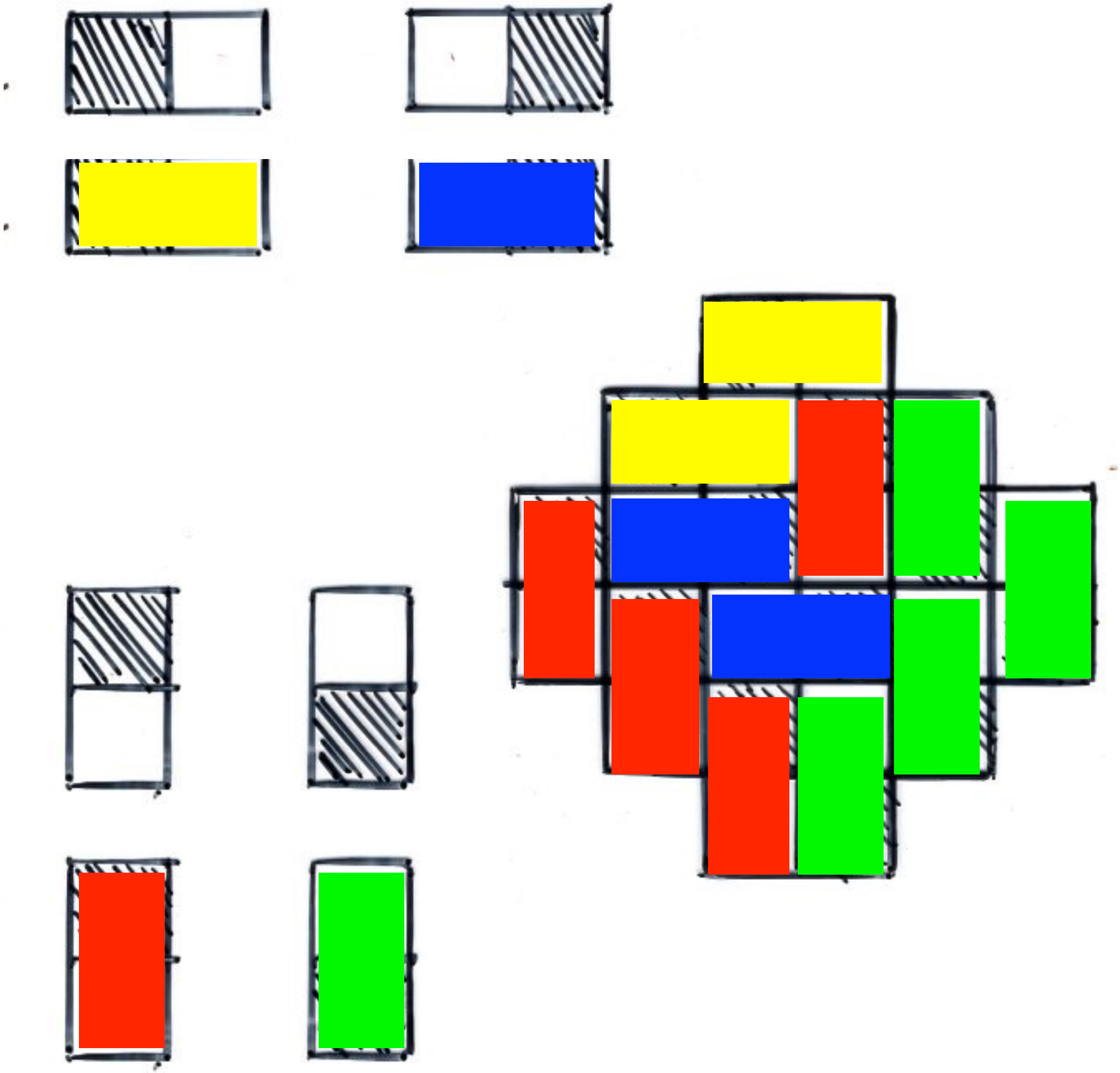


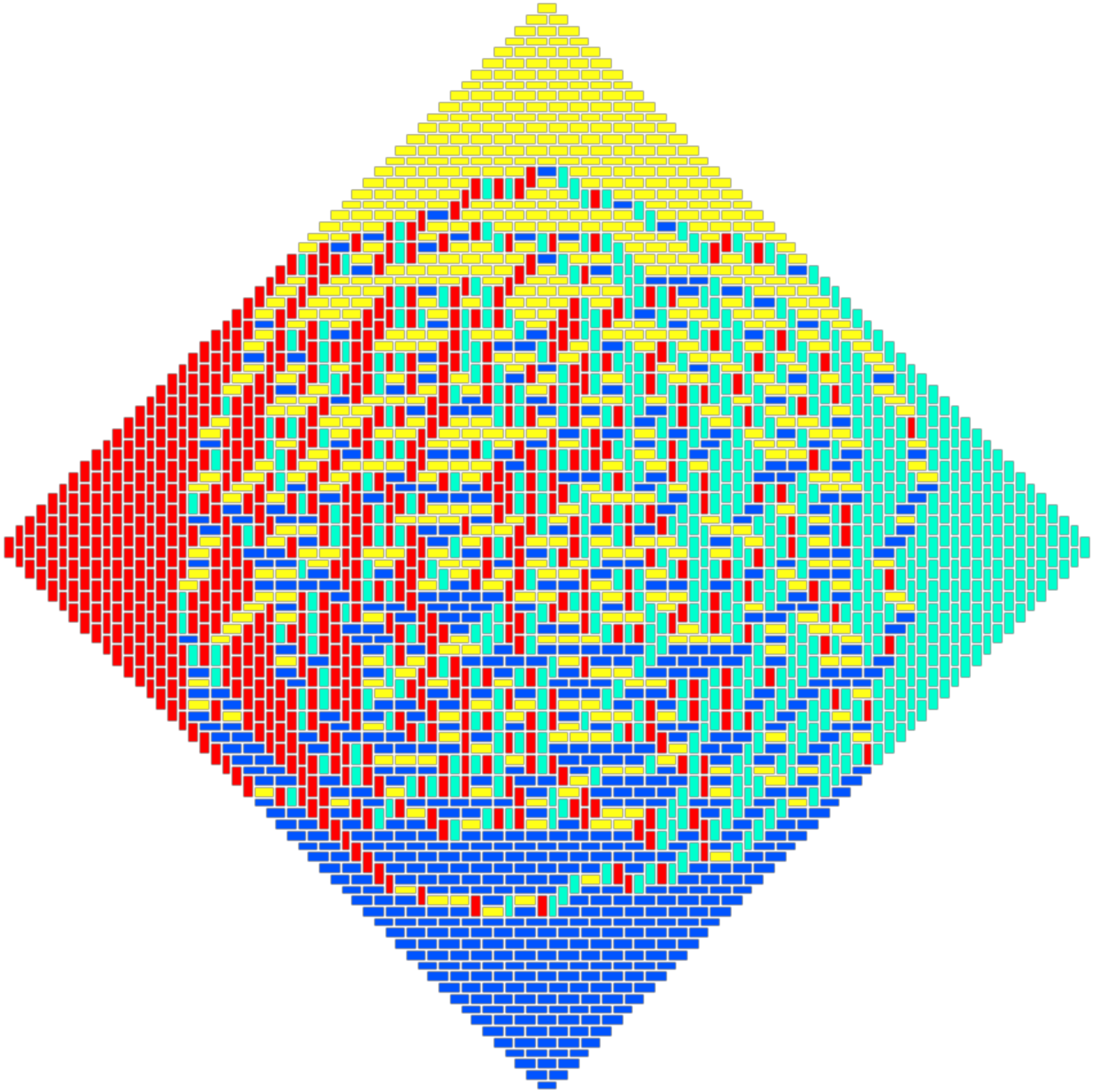
$$2^{(50\times 51)/2} = 2^{1275}$$





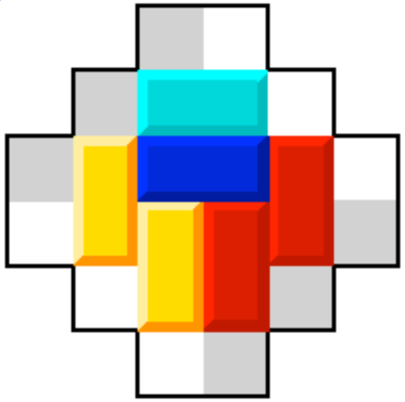






$2^{(1+2)}$

$2^3 = 8$



$w = w_1 w_2 w_3$

$w_i = 0$

$w_i = 1$

$2^{(1+2+3)}$

$2^6 = 64$

bijection

random tilings
with
domino shufflings

random tilings
domino shufflings

Elise Janvresse et **Thierry de la Rue** ,

« **Pavages aléatoires par touillage de dominos** » —

Images des Mathématiques, CNRS, 2009.

http://images.math.cnrs.fr/Pavages-aleatoires-par-touillage.html

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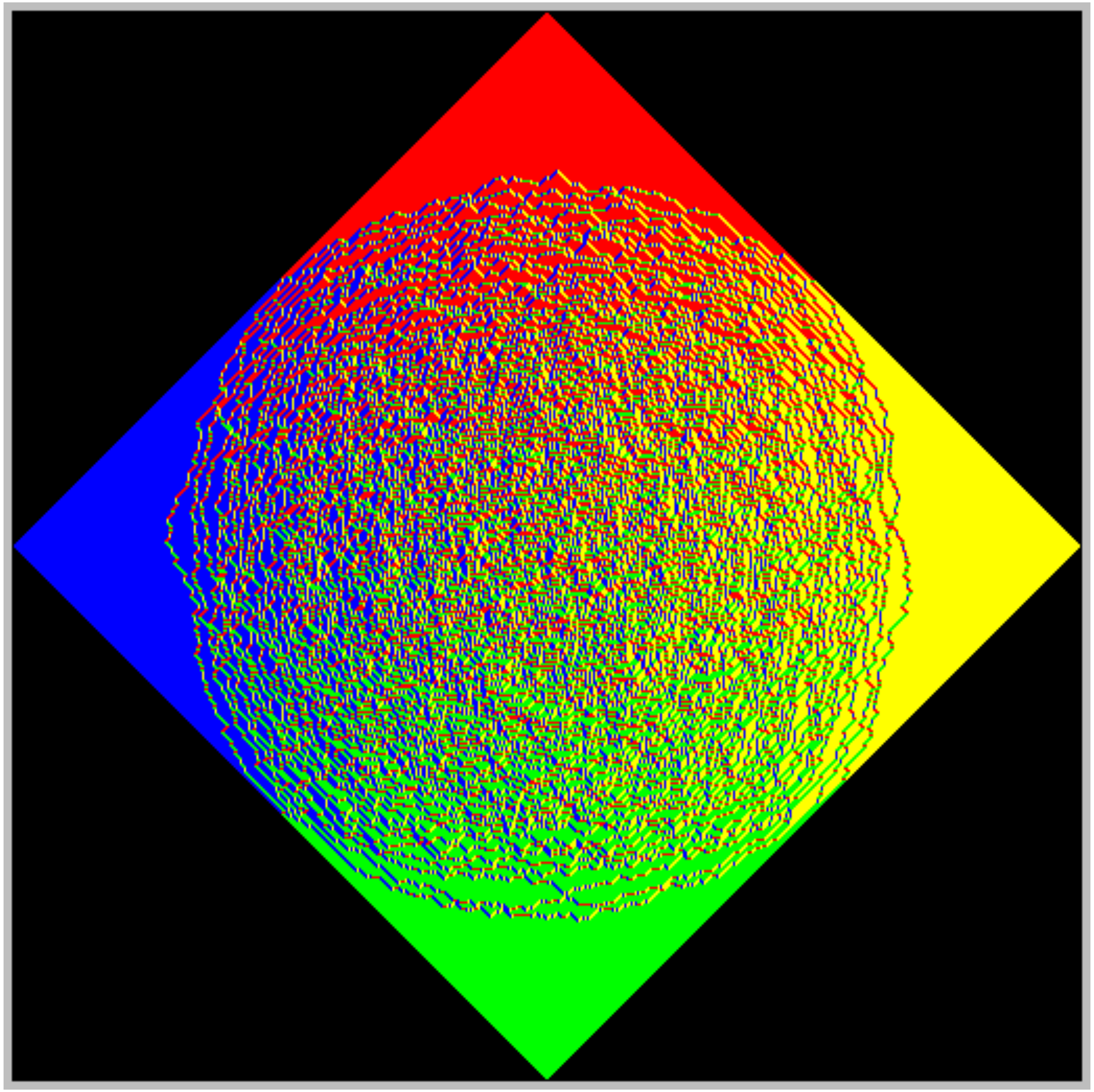
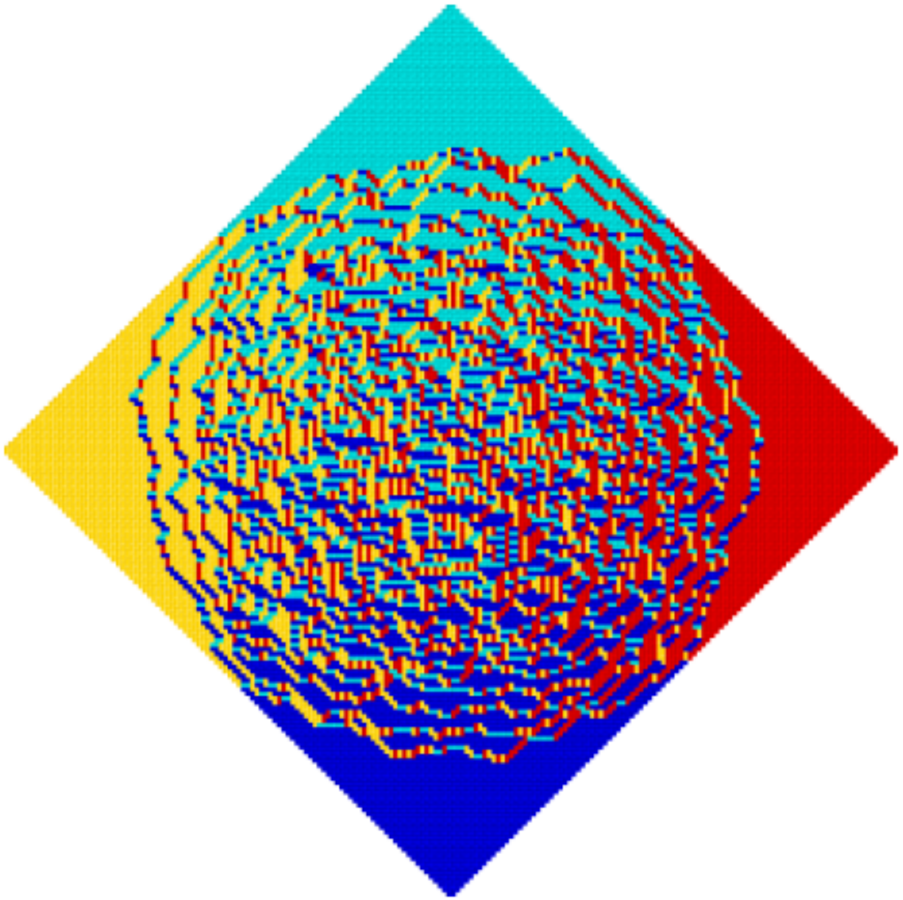
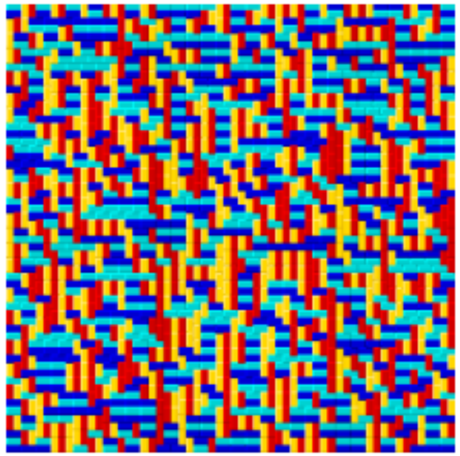
the
arctic
circle
theorem







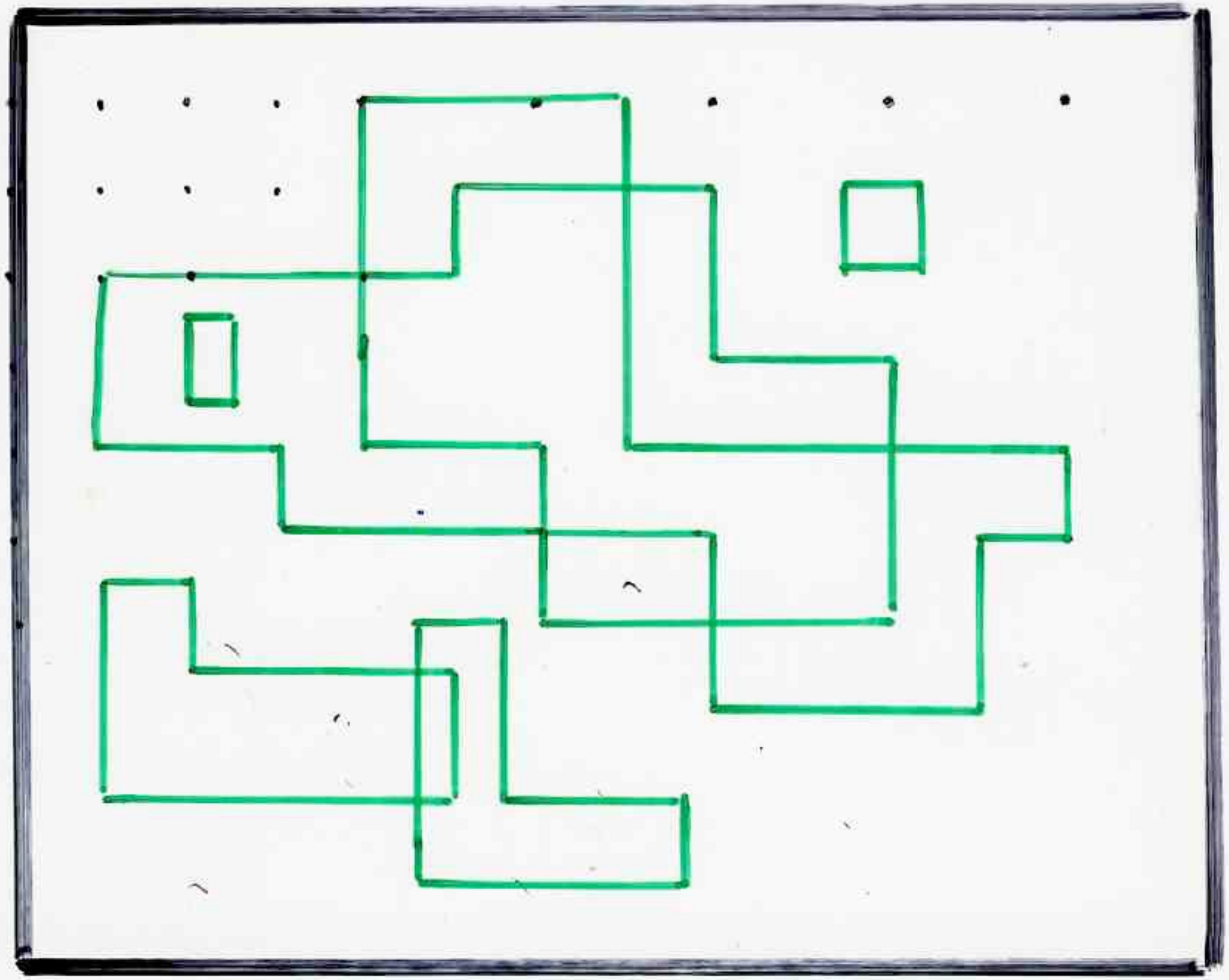
Perfect matchings

Pfaffian methodology

- enumeration of perfect matchings on a graph

- Pfaffian methodology for planar graphs

- Ising model (1925)
  Kasteleyn, Fisher, Temperley (1961, ...)
  Onsager (1944)



"closed" graph

Ising model

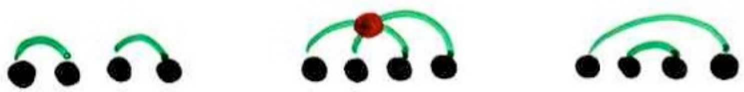
Pfaffian

Pfaf (1815)
Caianiello (1953, 59)
Wick

$$T = (a_{ij}) \quad 1 \le i < j \le 2k$$

ex:

$$\left| \begin{array}{ccc} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{array} \right| = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$$



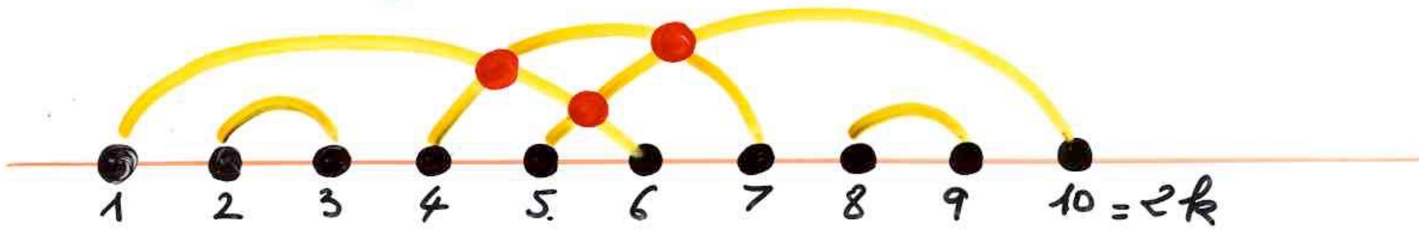
$$\mathrm{Pf}(T) = \sum_{J} \prod_{i < J(i)} a_{i, J(i)} \quad (-1)^{cr(J)}$$

involutions
on $[1, 2n]$
no fixed points

Involutions α

with no fixed points

crossing number $\qquad cr(\alpha) = 3$



skew-symmetric matrix

$$A = (a_{ij})_{1 \le i,j \le 2k}$$

$$\begin{cases} a_{ij} = -a_{ji} & i \ne j \\ a_{ii} = 0 & 1 \le i,j \le 2k \end{cases}$$

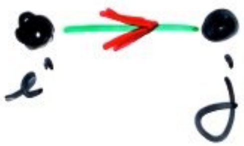
Cayley (1847)

$$\det(A) = \left(\mathrm{Pf}(T)\right)^2$$

Pfaffian methodology

graph G vertices $\{1,2,..,k\}$

admissible orientation of an edge



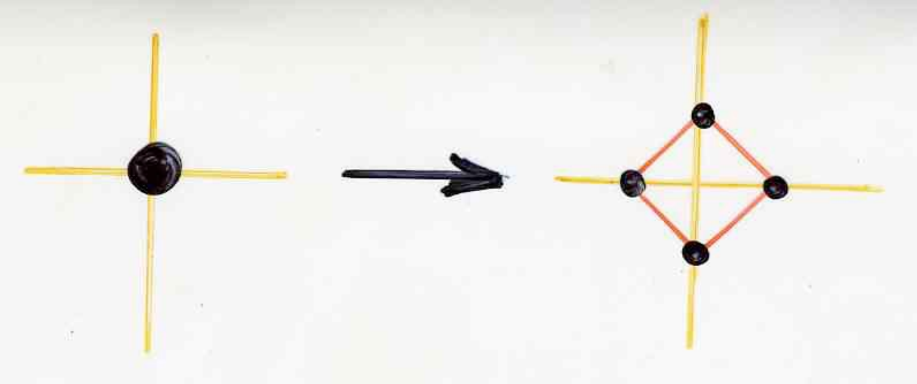
$$a_{ij} = \begin{cases} \pm 1 \\ 0 \end{cases}$$

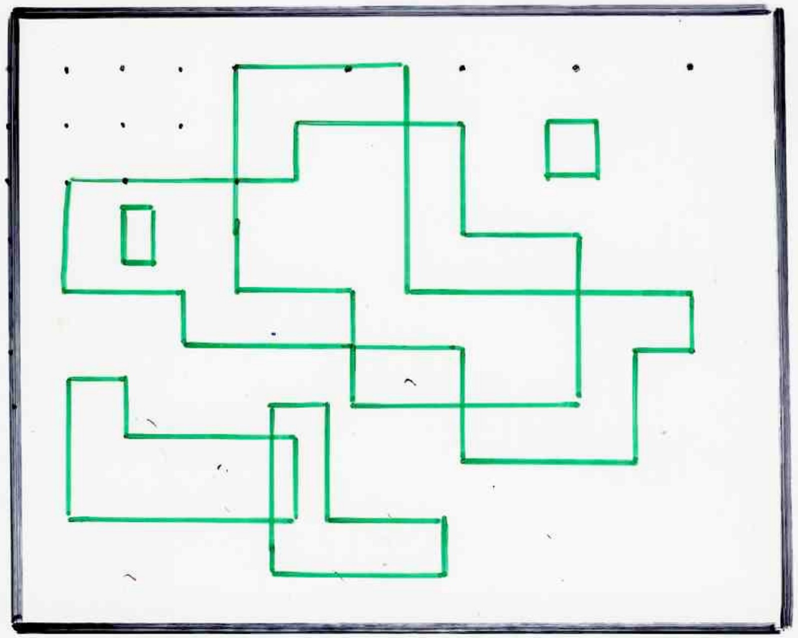
number of perfect matchings $= \mathrm{Pf}\,(a_{ij})_{1 \leq i < j \leq 2k}$

$$= \left(\det\,(a_{ij})_{1 \leq i,j \leq 2k}\right)^{1/2}$$

Proposition Kasteleyn (1967)

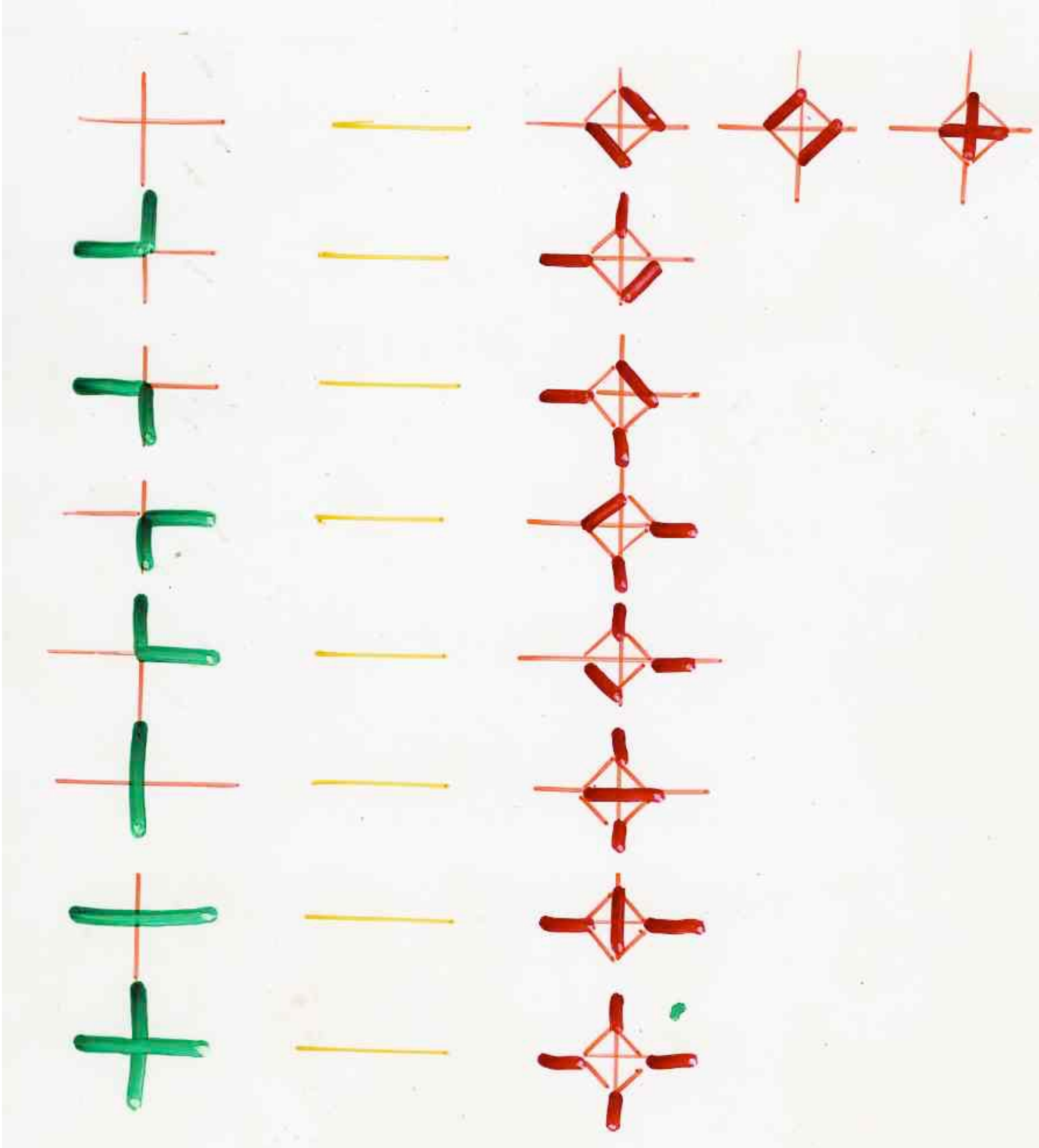
Every planar graph has an admissible orientation





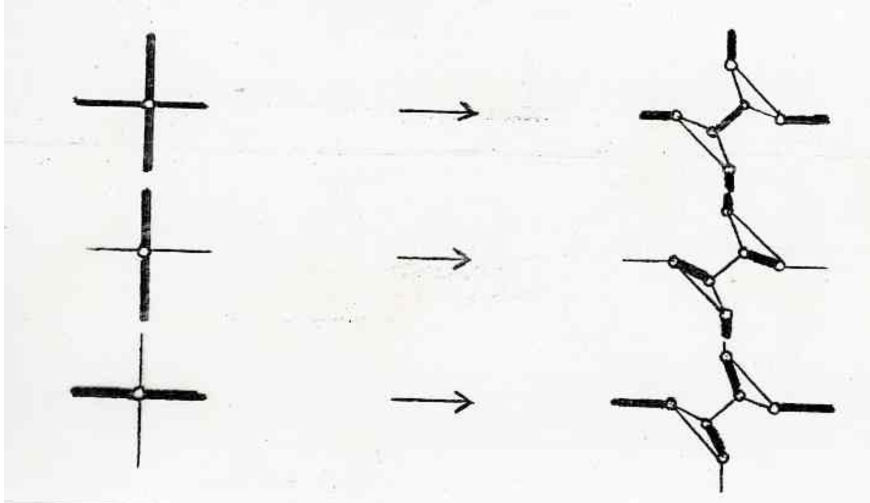
"closed" graph

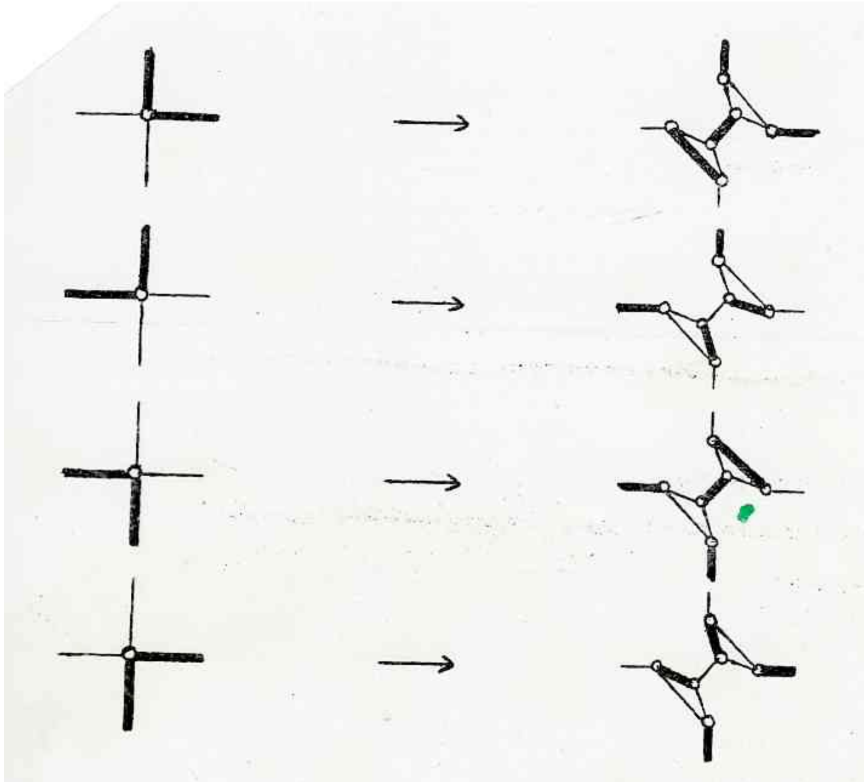
Ising model

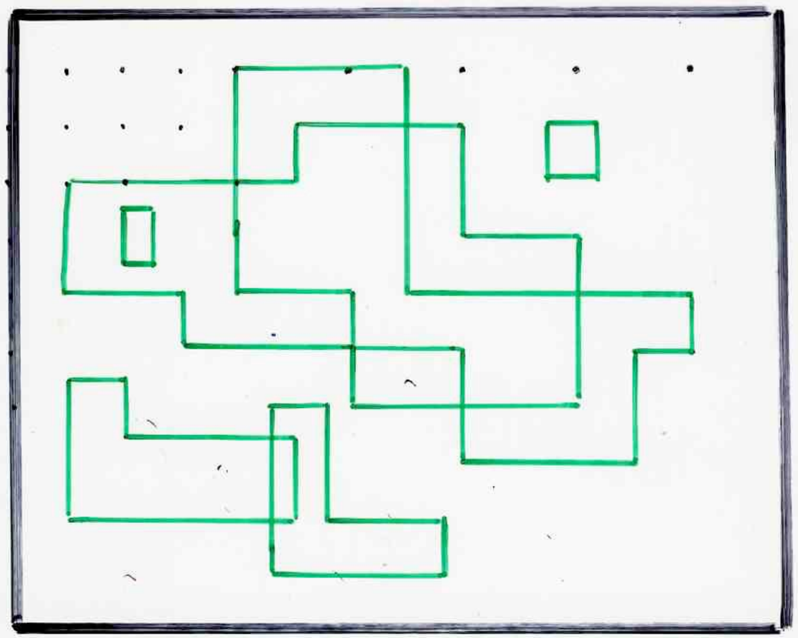


Fisher (1967)

- one-to-one
- planar graph







"closed" graph

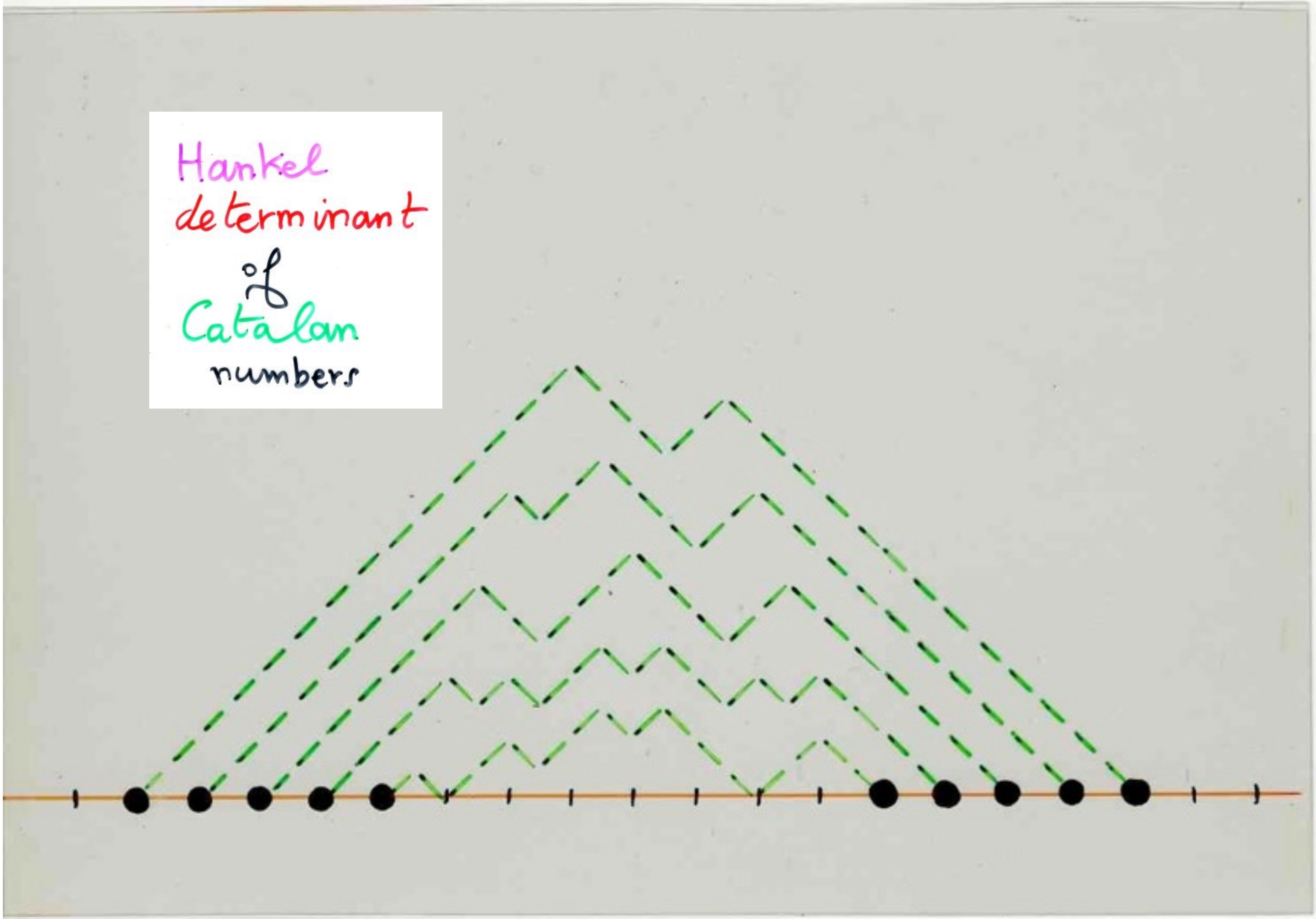
Ising model

in conclusion

a nice formula ....

$$\begin{vmatrix} C_n & C_{n+1} & \cdots & C_{n+k-1} \\ C_{n+1} & \cdots & \cdots & \vdots \\ \vdots & & & \vdots \\ C_{n+k-1} & \cdots & \cdots & C_{n+2k-2} \end{vmatrix} = \prod_{1\leq i\leq j\leq n} \frac{(i+j+2k)}{(i+j)}$$

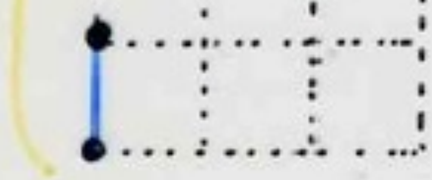
Hankel
determinant
of
Catalan
numbers


Hankel
determinant
of
Catalan
numbers

in conclusion

a nice formula ....

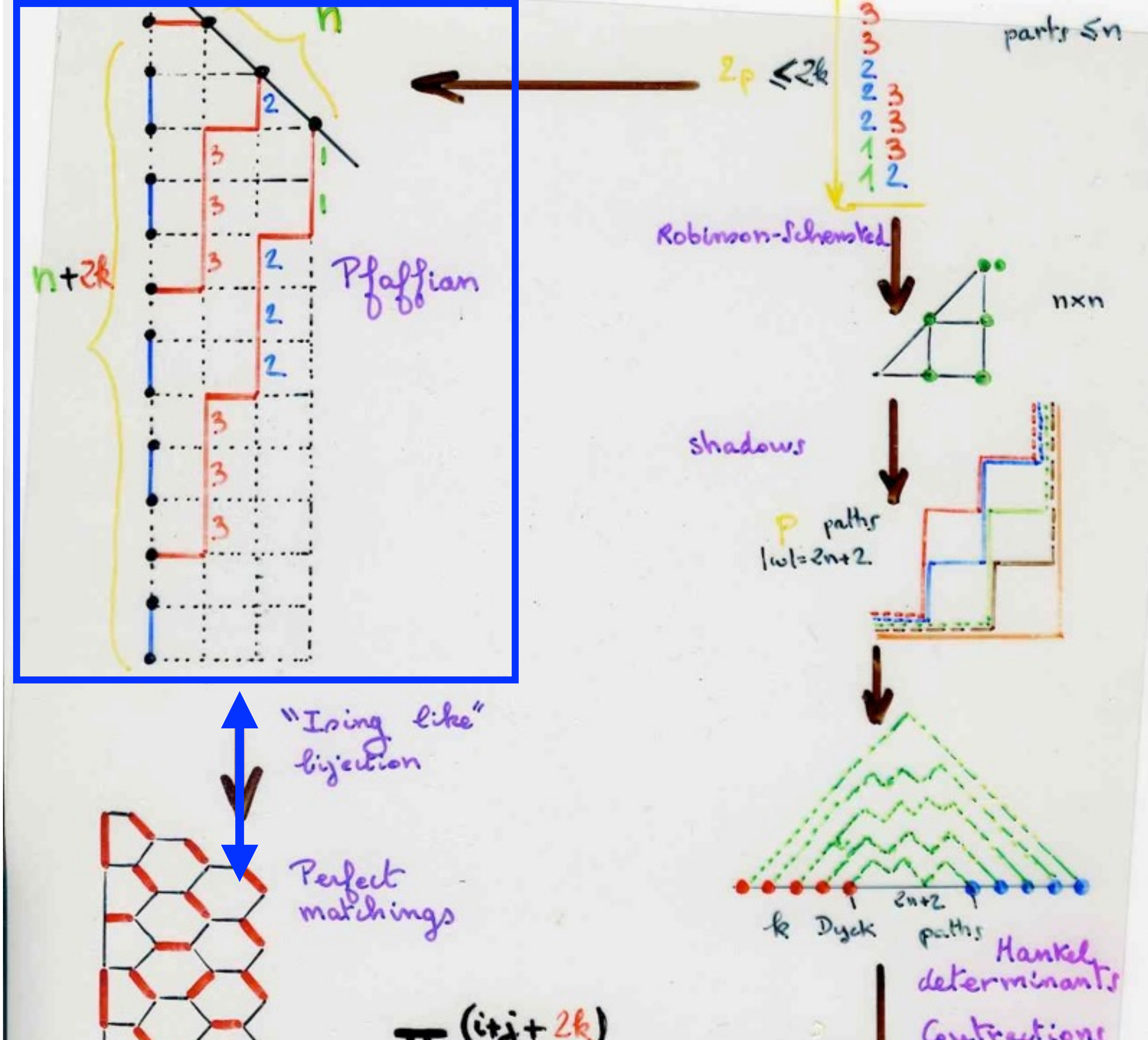
with a festival of bijections



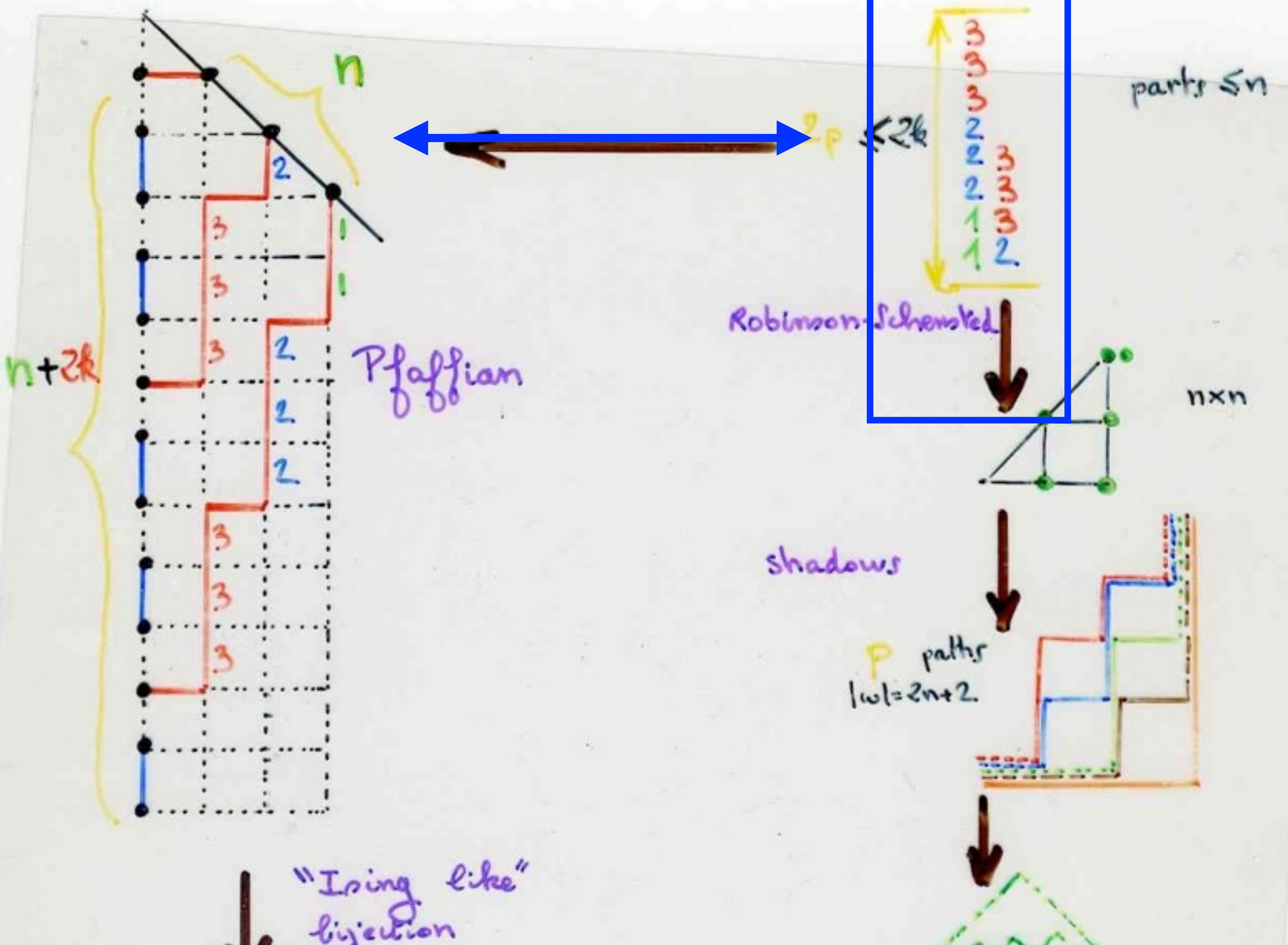
"Ising like" bijection

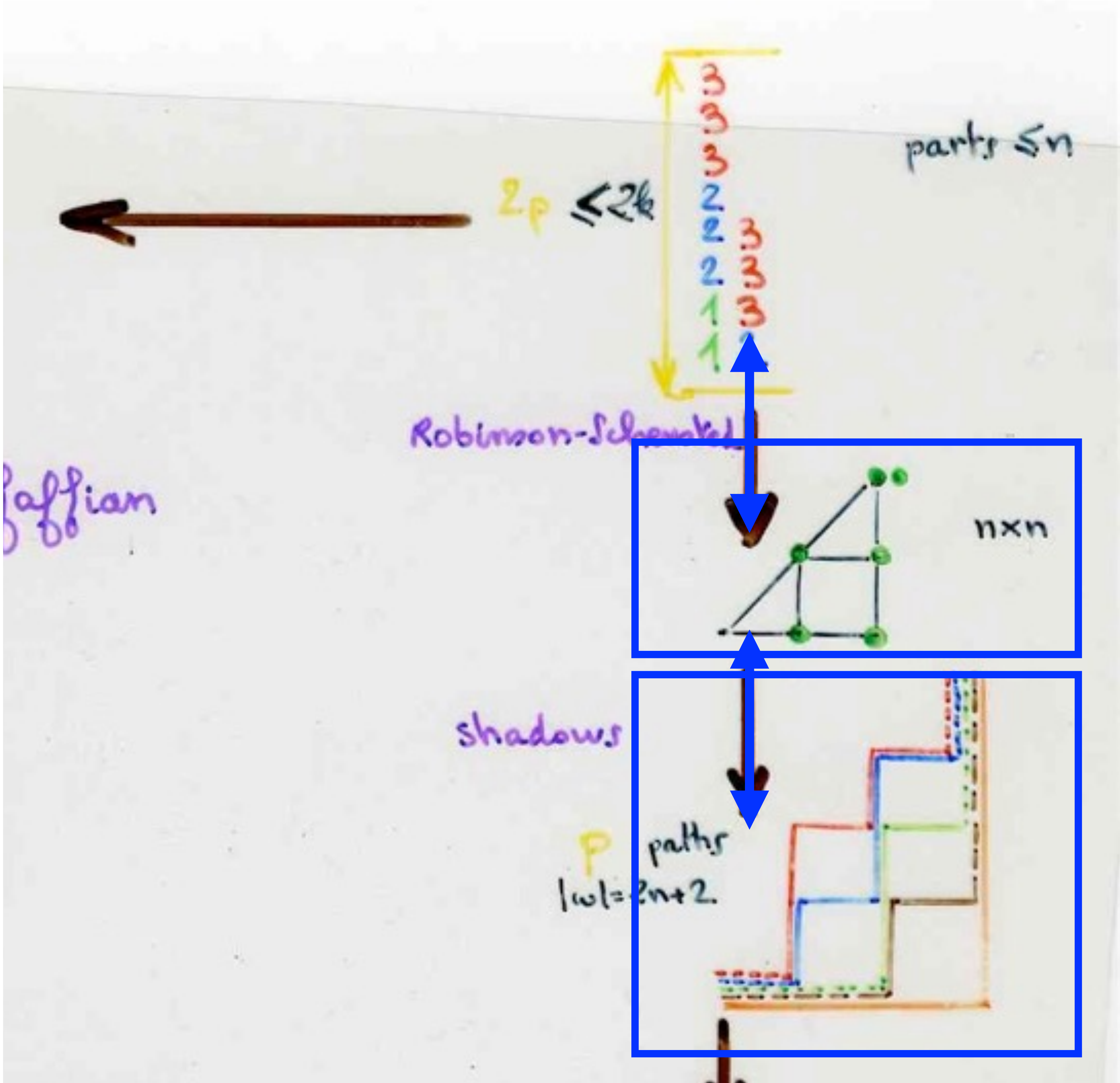
Perfect matchings

$$\prod_{1\leq i\leq j\leq n} \frac{(i+j+2k)}{(i+j)}$$



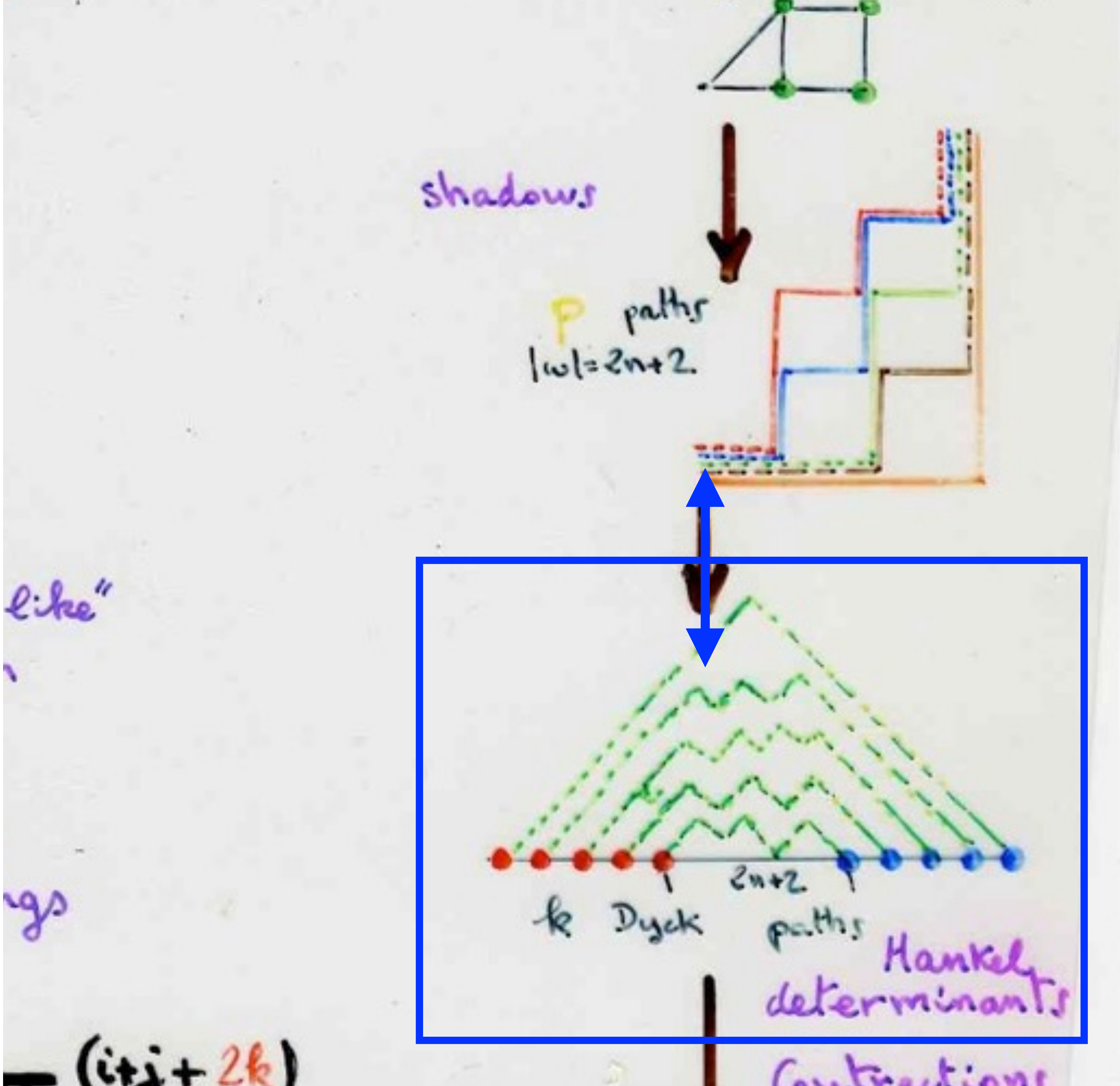
parts ≤n
n
$2_p \leq 2k$
n+2k
Pfaffian
Robinson-Schensted
n×n
shadows
p paths
|w|=2n+2
"Ising like"
bijection
Perfect
matchings
2n+2
k Dyck paths
Hankel
determinants
(i+j+2k)


n
n+2k
Pfaffian
2p ≤2k
parts ≤n
Robinson-Schensted
n×n
shadows
p paths
|ω|=2n+2
"Ising like"
bijection

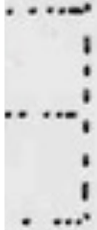


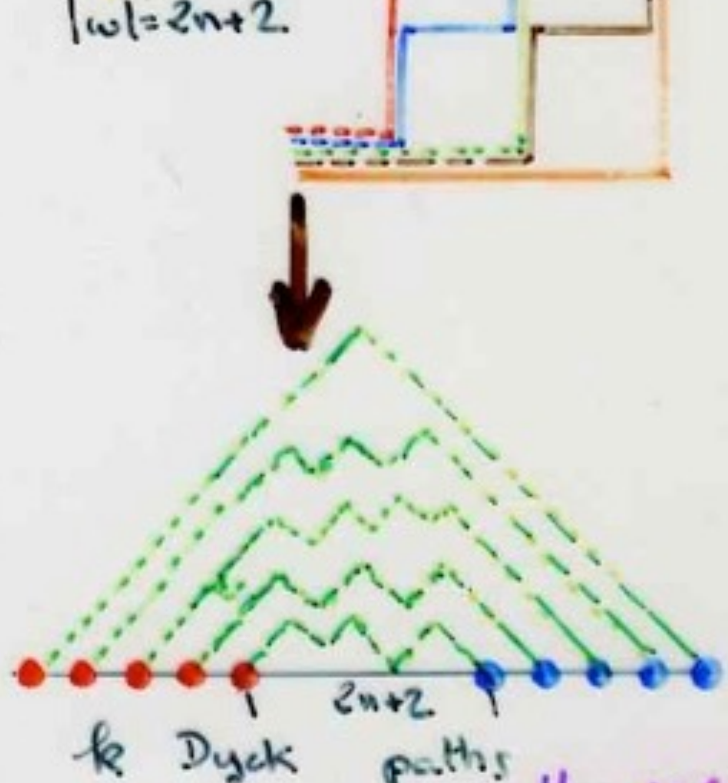
parts ≤n
2p ≤2k
3 3 3 2 2 2 1 1
3 3 3
Robinson-Sch
faffian
n×n
shadows
P paths
|ω|=2n+2

2019


shadows
P paths
|ω|=2n+2
2n+2
k Dyck paths
Hankel determinants
like"
(i+j+2k)

2020





"Ising like" bijection

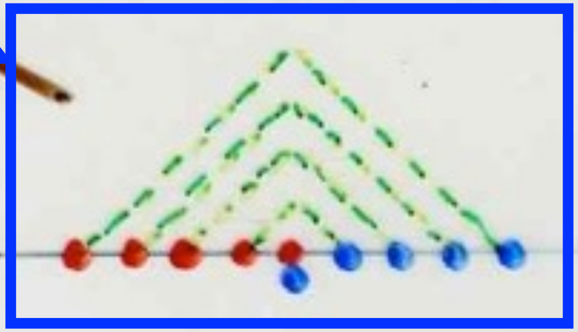
Perfect matchings

Hankel determinants

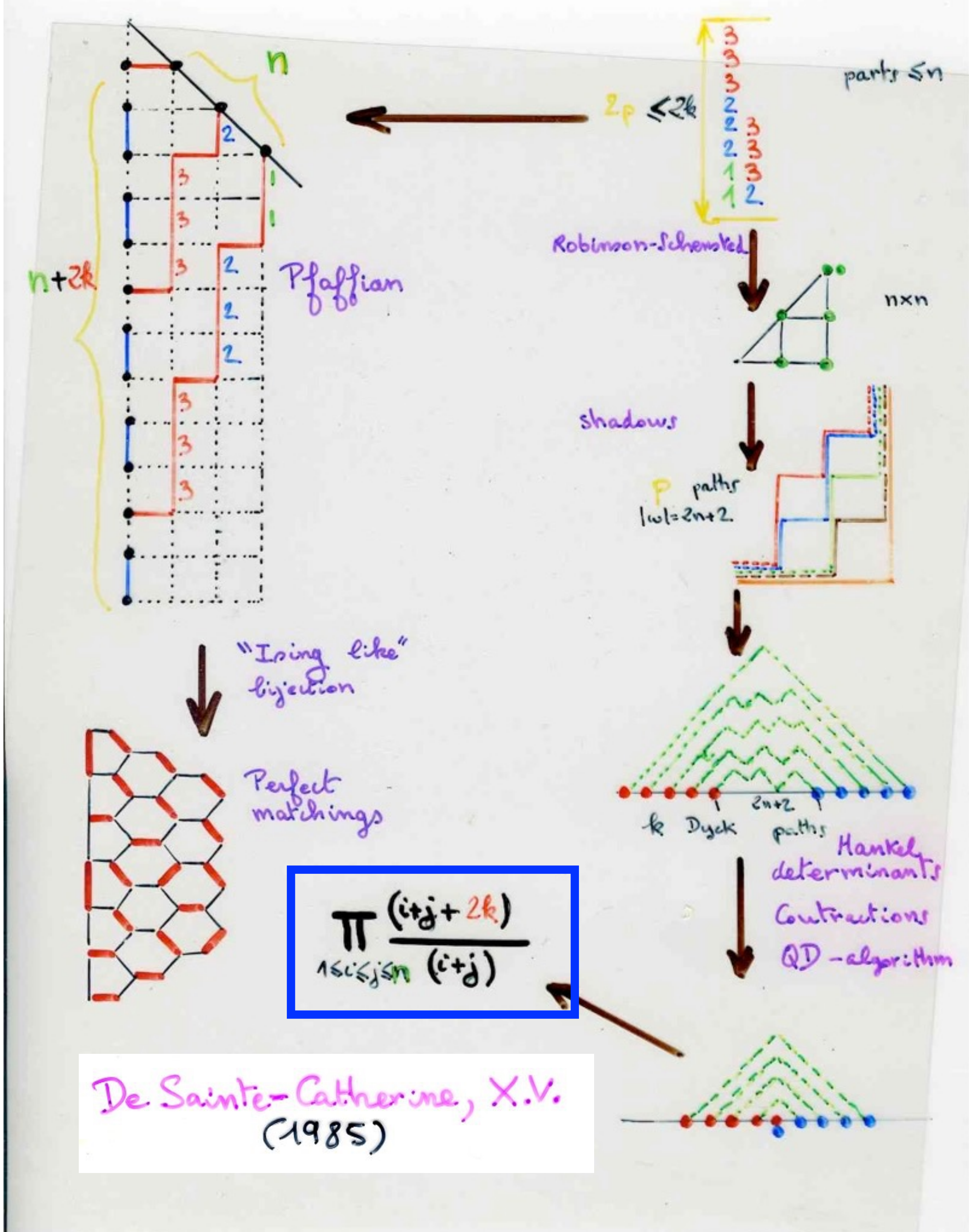
Contractions

QD-algorithm

$$\prod_{1\le i\le j\le n} \frac{(i+j+2k)}{(i+j)}$$



2021


n
parts ≤ n
2p ≤ 2k
n+2k
Pfaffian
Robinson-Schensted
n×n
shadows
p paths
|w|=2n+2
"Ising like" bijection
Perfect matchings
k Dyck paths
2n+2
Hankel determinants
Contractions
QD-algorithm
$\prod_{1\le i\le j\le n} \frac{(i+j+2k)}{(i+j)}$
De Sainte-Catherine, X.V.
(1985)

2022

website of the course:

coursimsc2016.xavierviennot.org

lecture notes of the course: coming

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2024

ॐ सरस्वत्यै नमः।

Om Sarasvatyai Namaḥ