Chapter 4

The $n!$ garden
(4)

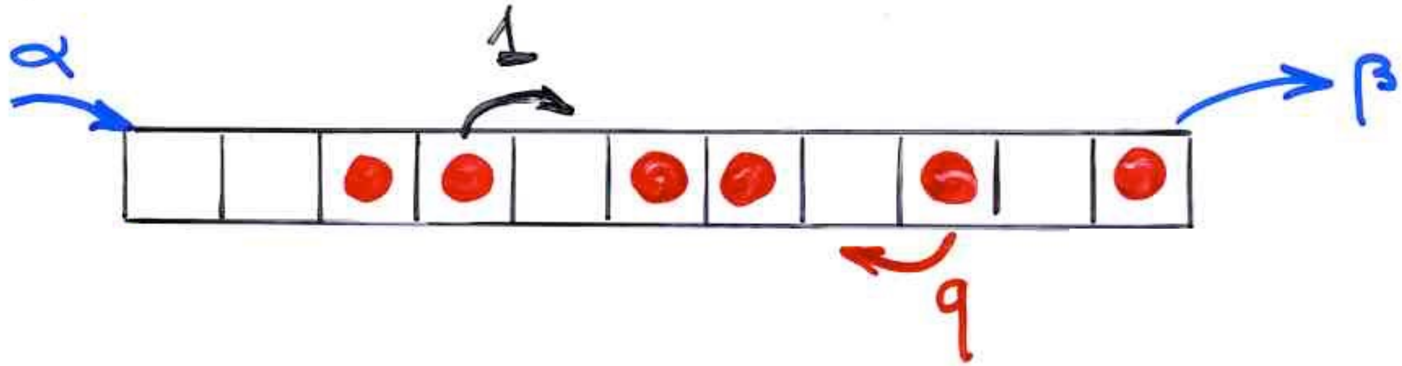
complements

IMSc
25 February 2016

The PASEP
(ASEP)

(Partially) ASymmetric Exclusion Process

ASEP
TASEP
PASEP

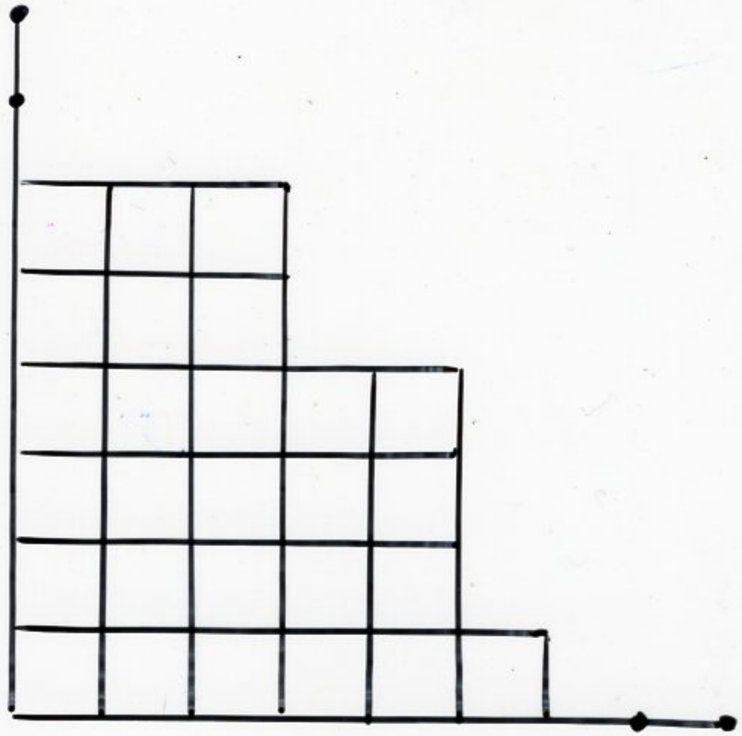


see Ch2d for TASEP $q=0$

Corollary. The stationary probability associated to the state $\tau = (\tau_1, \ldots, \tau_n)$ is

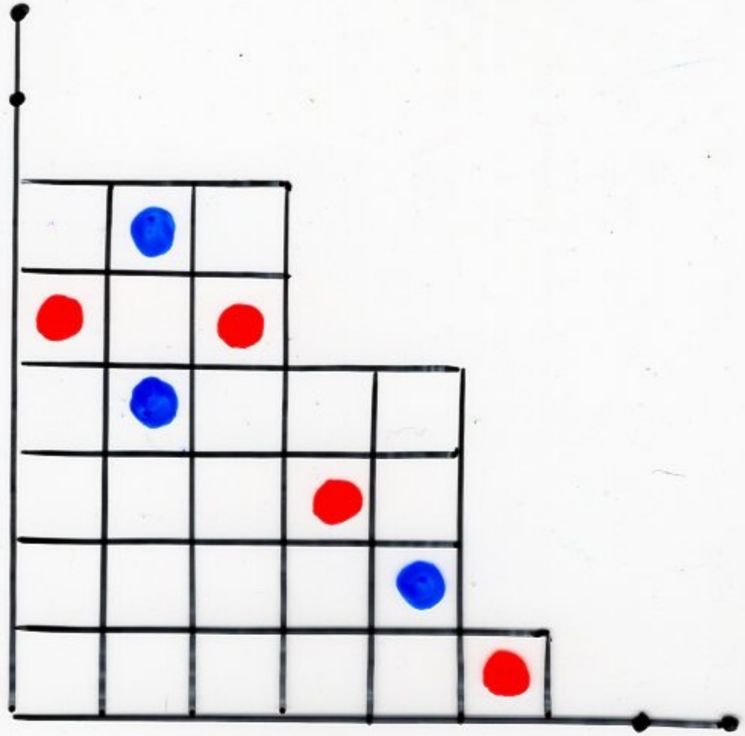
$$\mathrm{proba}_{\tau}(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$$

alternative
tableaux
profile τ



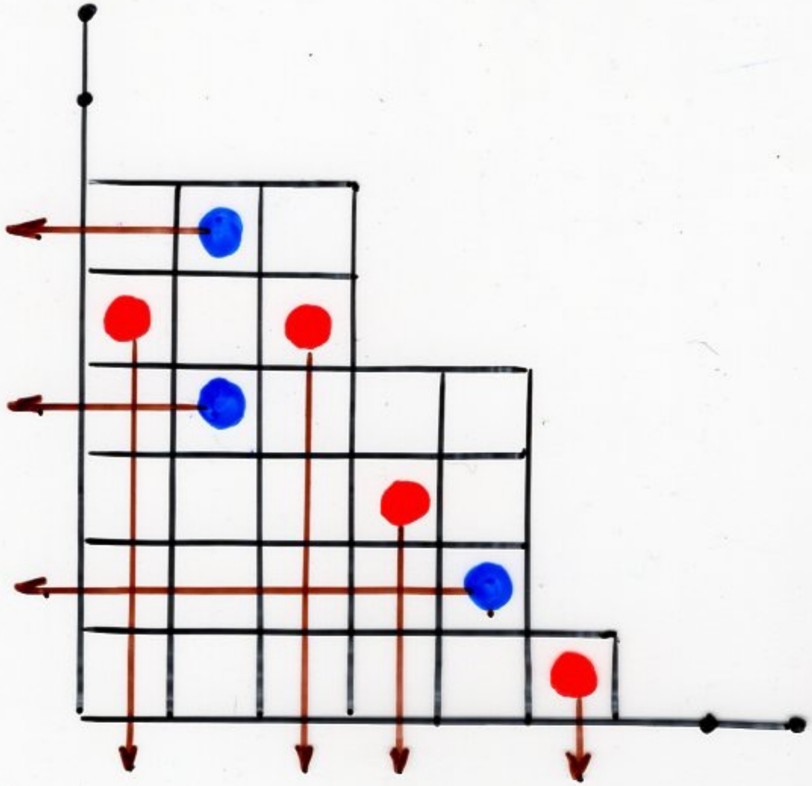
Ferrers
diagram
(possibly empty
rows and columns)

n = total nb of
rows and columns



alternative
tableau
T

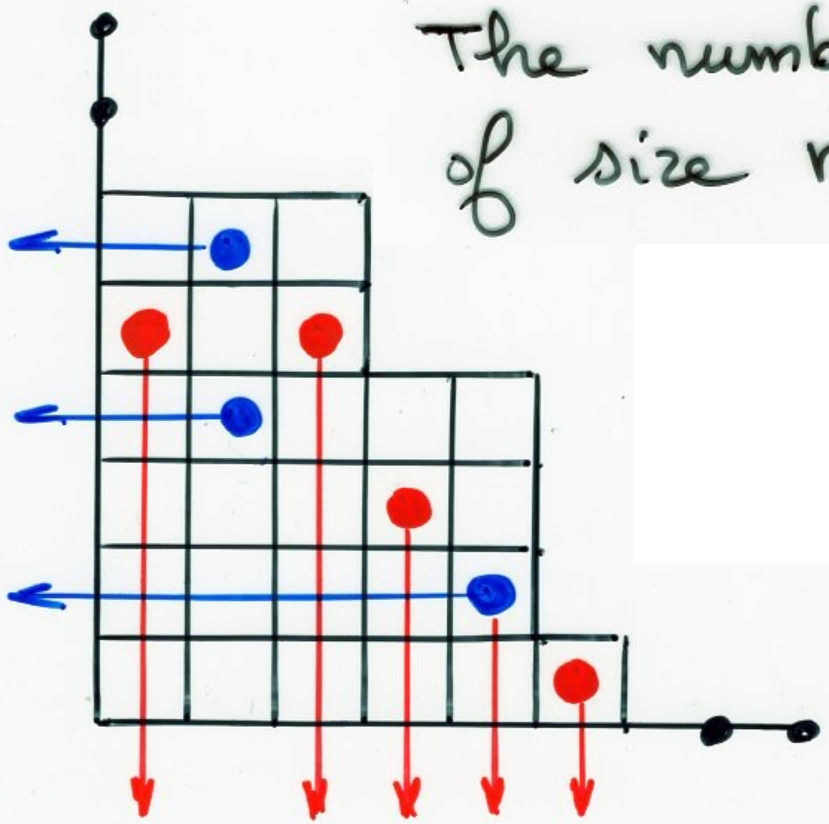
– some cells are coloured ● or ●



alternative
tableau
T

– some cells are coloured ● or ●

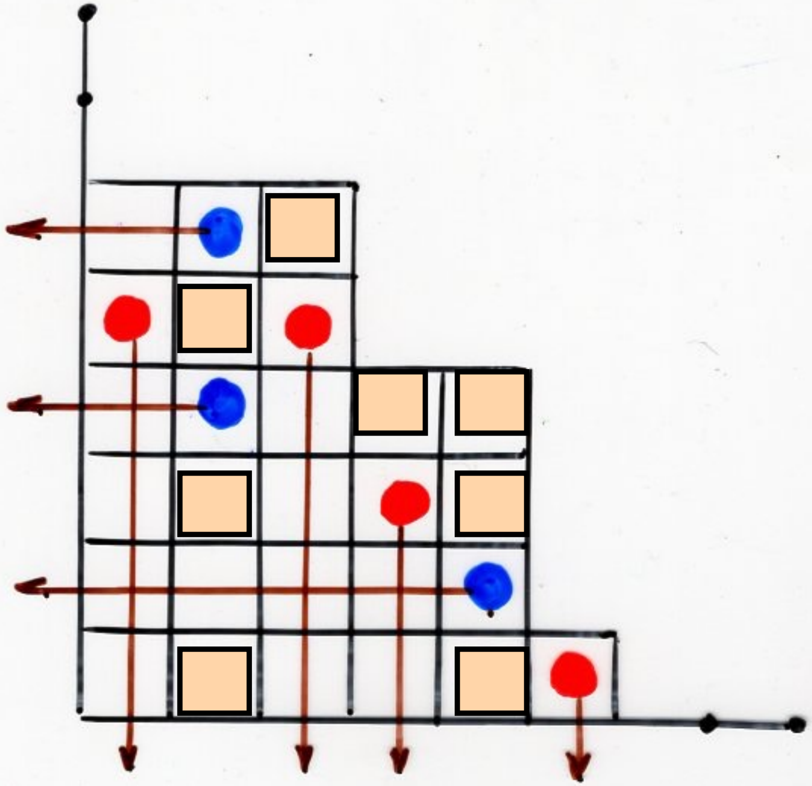
– { no coloured cell at the left of a $\boxed{●}$
   no coloured cell below a $\boxed{●}$



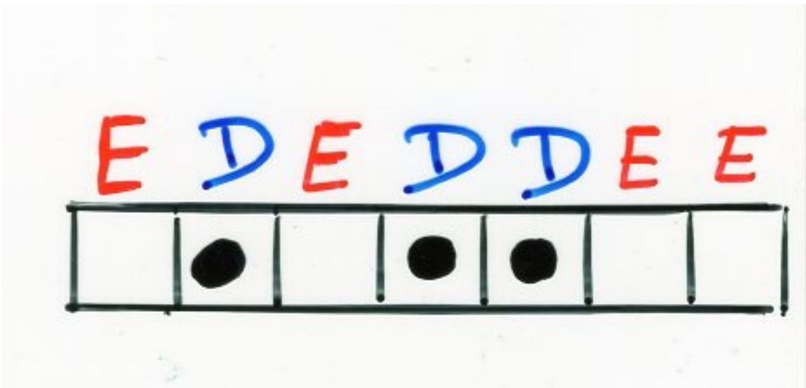
The number of alternative tableaux of size n is

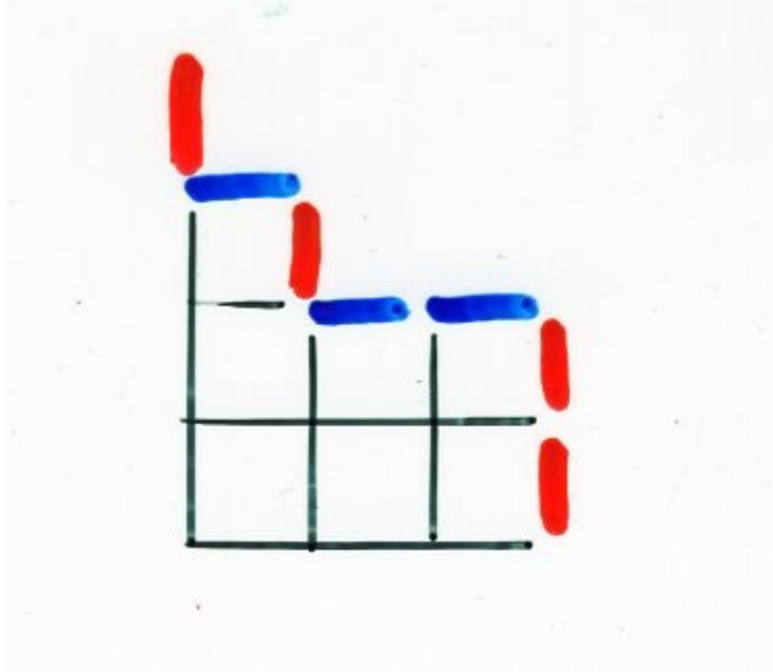
$$(n+1)!$$

ex: $n=2$



$k(T)$ = nb of cells ▢
$i(T)$ = nb of rows without ●
$j(T)$ = nb of columns without ●





Def - profile of an alternative tableau

word $w \in \{E, D\}^*$

Corollary. The stationary probability associated to the state $\tau = (\tau_1, \ldots, \tau_n)$ is

$$\mathrm{proba}_{\tau}(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} q^{k(T)} \alpha^{-i(T)} \beta^{-j(T)}$$

alternative tableaux profile τ

$k(T)$ = nb of cells

$i(T)$ = nb of rows without

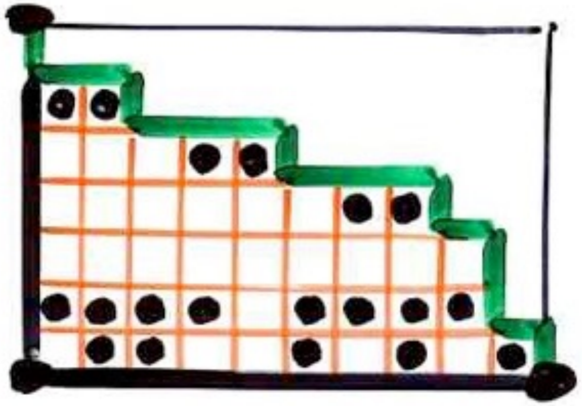
$j(T)$ = nb of columns without

permutation tableau

S. Corteel, L. Williams (2007) (2008) (2009)

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



□ = 0   ⊡ = 1

filling of the cells with 0 and 1

(i) in each column : at least one 1

(ii) 

forbidden

exercise • Give a bijection between permutation tableaux (size $(n+1)$) and alternating tableaux (size n)

• Via this bijection, give an interpretation of the 3 parameters $k(T)$, $i(T)$, $j(T)$ involved in the expression of the stationary probabilities

permutations tableaux A. Postnikov (2001)
E. Steingrímsson (2005)
+ L. W.

S. Corteel, L. Williams (2007)

alternative tableaux

X. V. (2008)

tree-like tableaux

J.-C. Aval, A. Boussicault, P. Nadeau
(2013)

staircase tableaux

S. Corteel, L. Williams (2011)

see Ch4b

q-Laguerre polynomials

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

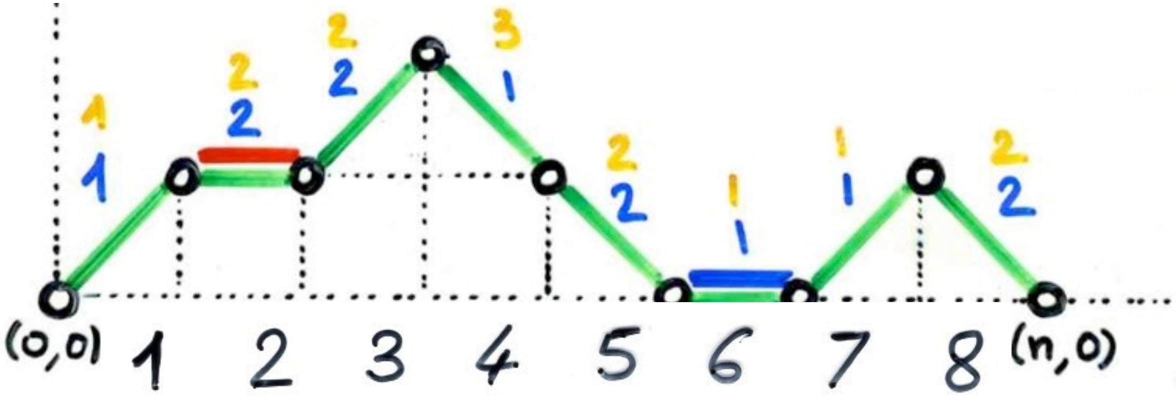
$$\begin{cases} b'_k = [k+1]_q \\ b''_k = [k+1]_q \\ a_k = [k+1]_q \\ c_k = [k+1]_q \end{cases}$$

weighted q-Laguerre histories

$$q^{\left[\sum_{i=1}^{n} (p_i - 1)\right]}$$

choice function

"q-analogue" of Laguerre histories



choice function

$i =$	1	2	3	4	5	6	7	8
$p_i =$	1	2	2	1	2	1	1	2
$p_i - 1 =$	0	1	1	0	1	0	0	1

weighted q-Laguerre histories

$$q^4$$

␣

␣ 1 ␣

␣ 1 ␣ 2

␣ 1 ␣ 3 ␣ 2

4 1 ␣ 3 ␣ 2

4 1 ␣ 3 5 2

4 1 6 ␣ 3 5 2

4 1 6 ␣ 7 ␣ 3 5 2

4 1 6 ␣ 7 8 3 5 2

4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$

Prop. The distribution of this parameter q among q-Laguerre histories (length n) is the same as the distribution of the parameter q among alternative tableaux. (size n)

Cor The moments of the polynomials "q-Laguerre I" are equal to the partition function $Z_n(q)$ of the model (for $\alpha=\beta=1$)

$\longrightarrow$ can be extended with (q, α, β)

q-Laguerre polynomials

q-Laguerre I

then
$$\begin{cases} b_k = ([k]_q + [k+1]_q) \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

$$\mu_n = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \left(\sum_{i=0}^{k} q^{i(k+1-i)} \right)$$

Corteel, Josuat-Vergès, Prellberg, Rubey (2008)

"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

representation
by operators

Physics

bijections

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

RSK

permutations $\longleftrightarrow$ pairs of Tableaux Young

$DE = qED + E + D$

alternative tableaux $\longleftrightarrow$ permutations

PASEP

dynamical systems in physics stationary probabilities

quadratic algebra Q

Q-tableaux

commutations
rewriting rules

planarization

$\longrightarrow$ see the course
« Quadratic algebra
and combinatorics »

complements (for Ch 2 and Ch 4)

posets, lattices

2^n Catalan n!

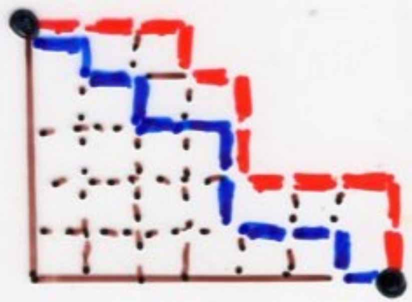
poset $\preccurlyeq$

partially ordered set

lattice

every two elements
have a unique
least upper bound (join)

and a unique
greatest lower bound
(meet)



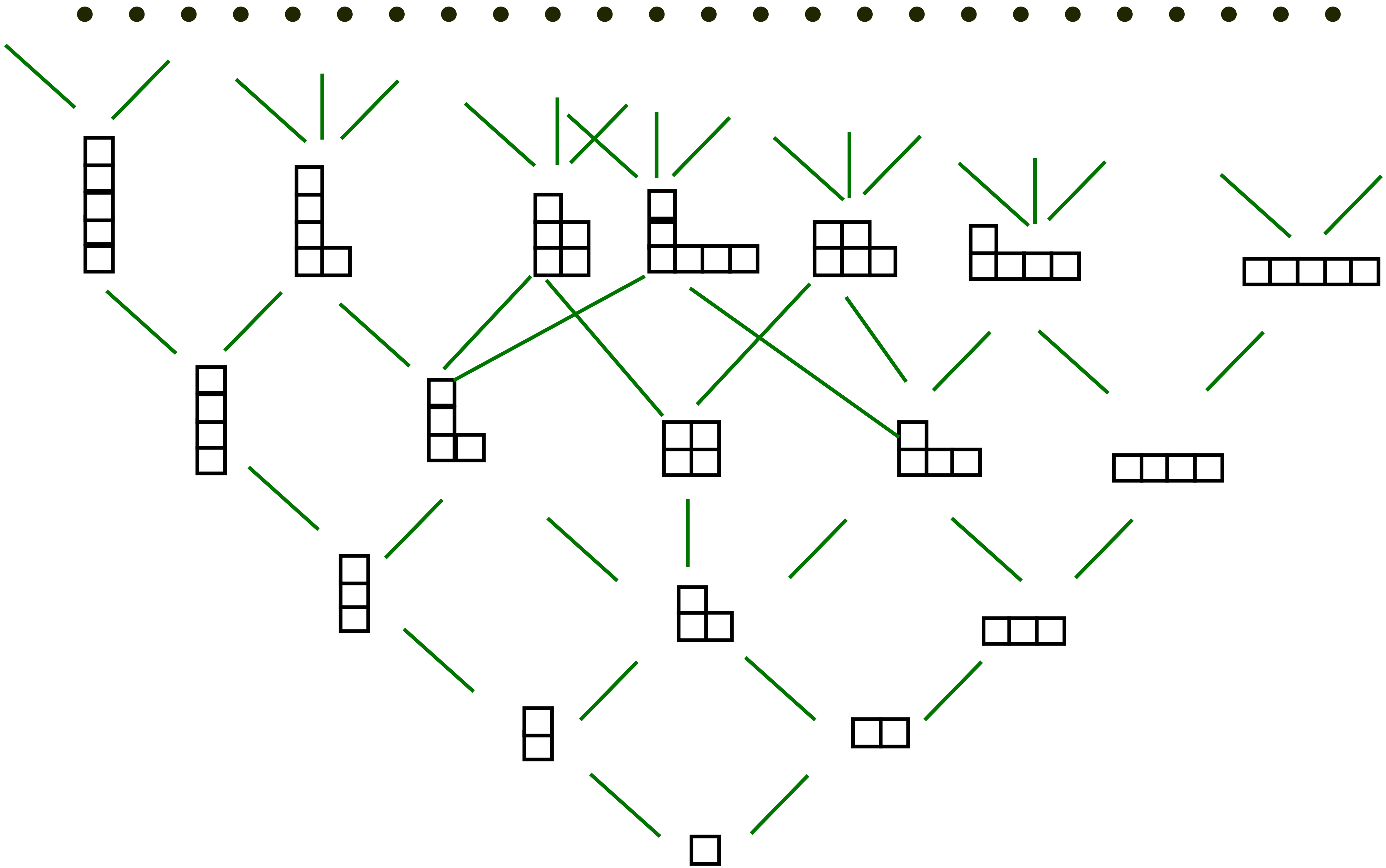
Young lattice

Hasse diagram

covering
relation

$\alpha \preccurlyeq \beta$
no γ between
α and β

Young lattice



Boolean lattice

inclusion

set $\mathcal{P}(X)$ subsets of X

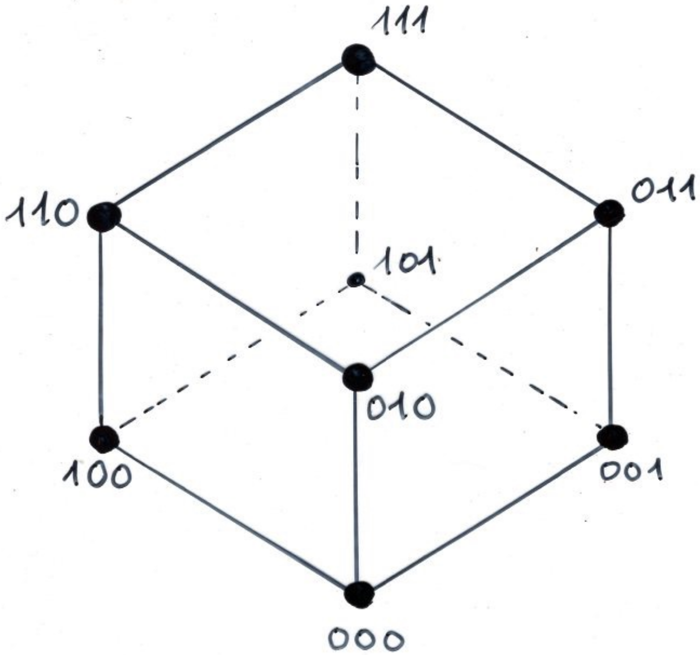
$A \subseteq B$

order relation

$A, B \subseteq X$

sup $(A, B) = A \cup B$

inf $(A, B) = A \cap B$



$|X| = n \qquad X = \{1, 2, .., n\}$

$A = \{2, 3, 6\} \subseteq \{1, 2, .., 8\}$

$$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline w = & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}$$

$$
\begin{tikzcd}
\sigma \arrow[r] \arrow[dd] & T(\sigma) \arrow[d] \\
 & B \arrow[d] \\
UD(\sigma) \arrow[r, phantom, "="] & c(B)
\end{tikzcd}
$$

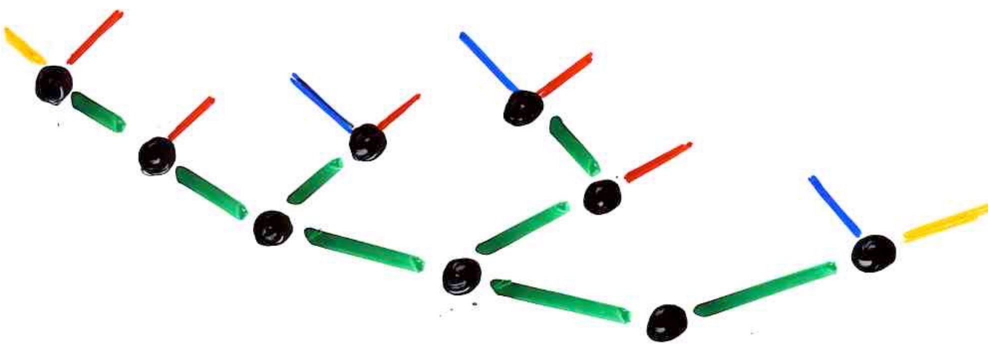
σ: permutation

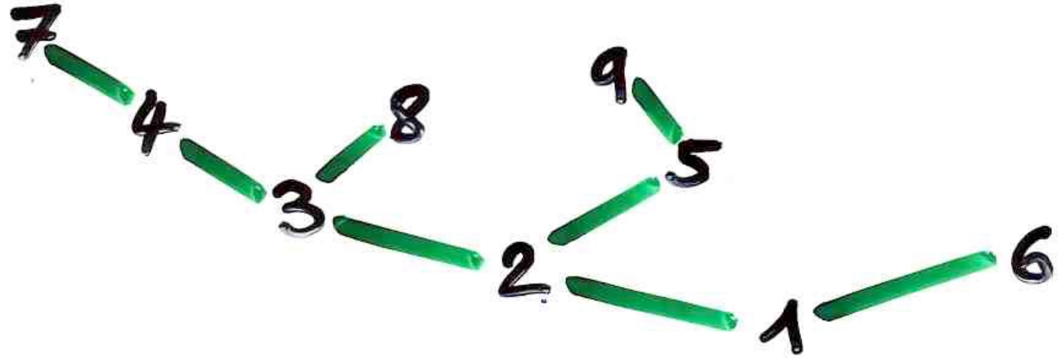
T(σ): increasing binary tree

B: binary tree

UD(σ): up-down sequence

c(B): canopy





σ = 7 4 3 8 2 9 5 1 6 ...

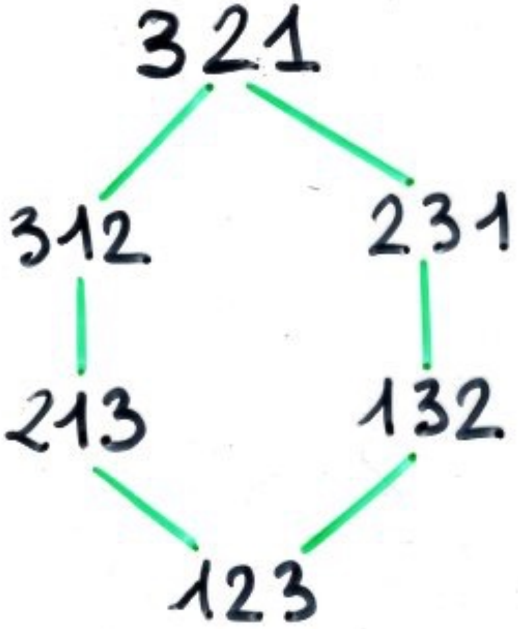
up-down sequence − − + − + − − +

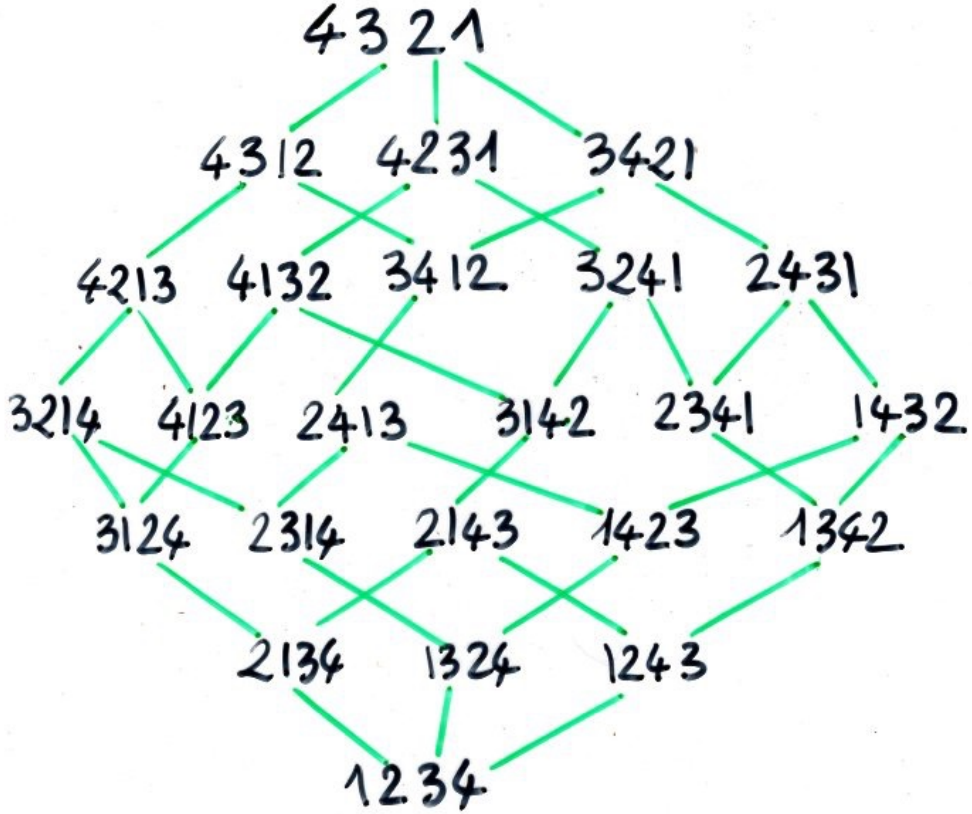
right

weak Bruhat order

$$\sigma = \sigma(1)\cdots\sigma(i)\,\sigma(i+1)\cdots\sigma(n)$$

$$\sigma \qquad \cdots\sigma(i+1)\,\sigma(i)\cdots\sigma$$





left

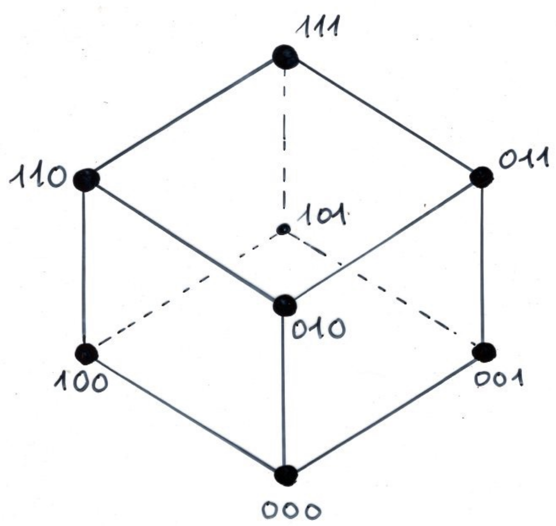
weak Bruhat order

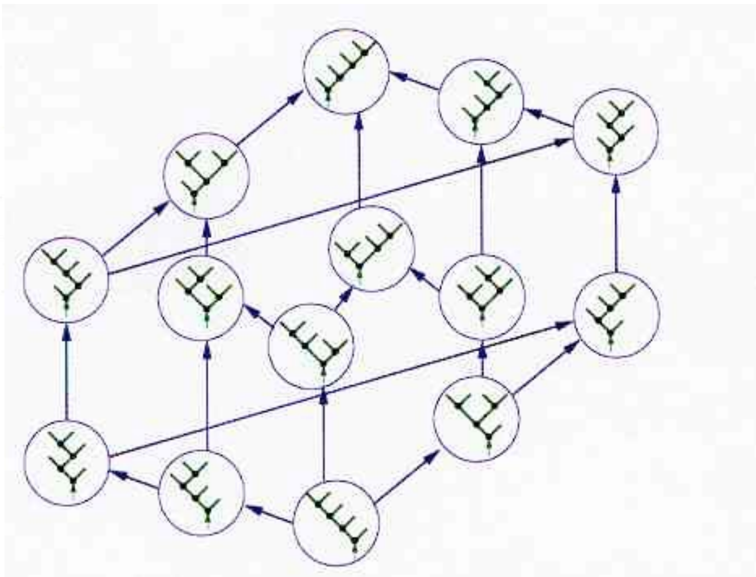
$$\sigma = \sigma(1)\cdots(x)\cdots\cdots(x+1)\cdots\sigma(n)$$

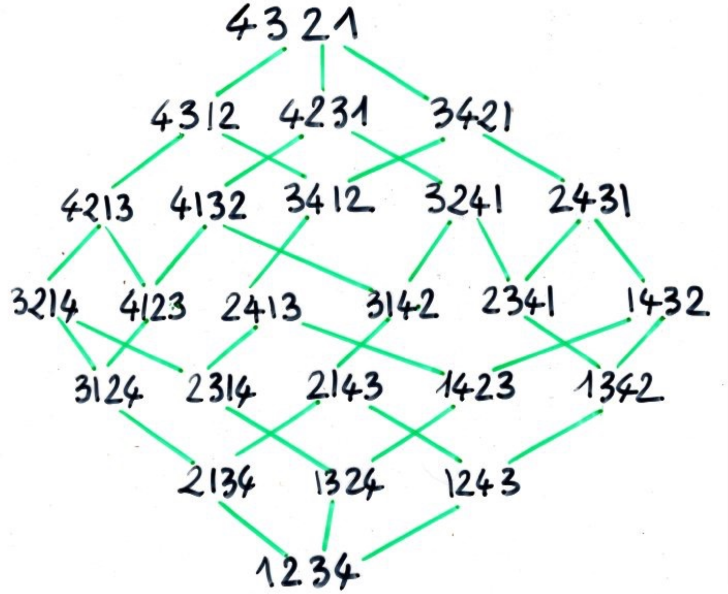
$$\sigma(1)\cdots(x+1)\cdots\cdots(x)\cdots\sigma(n)$$

combinatorial structures

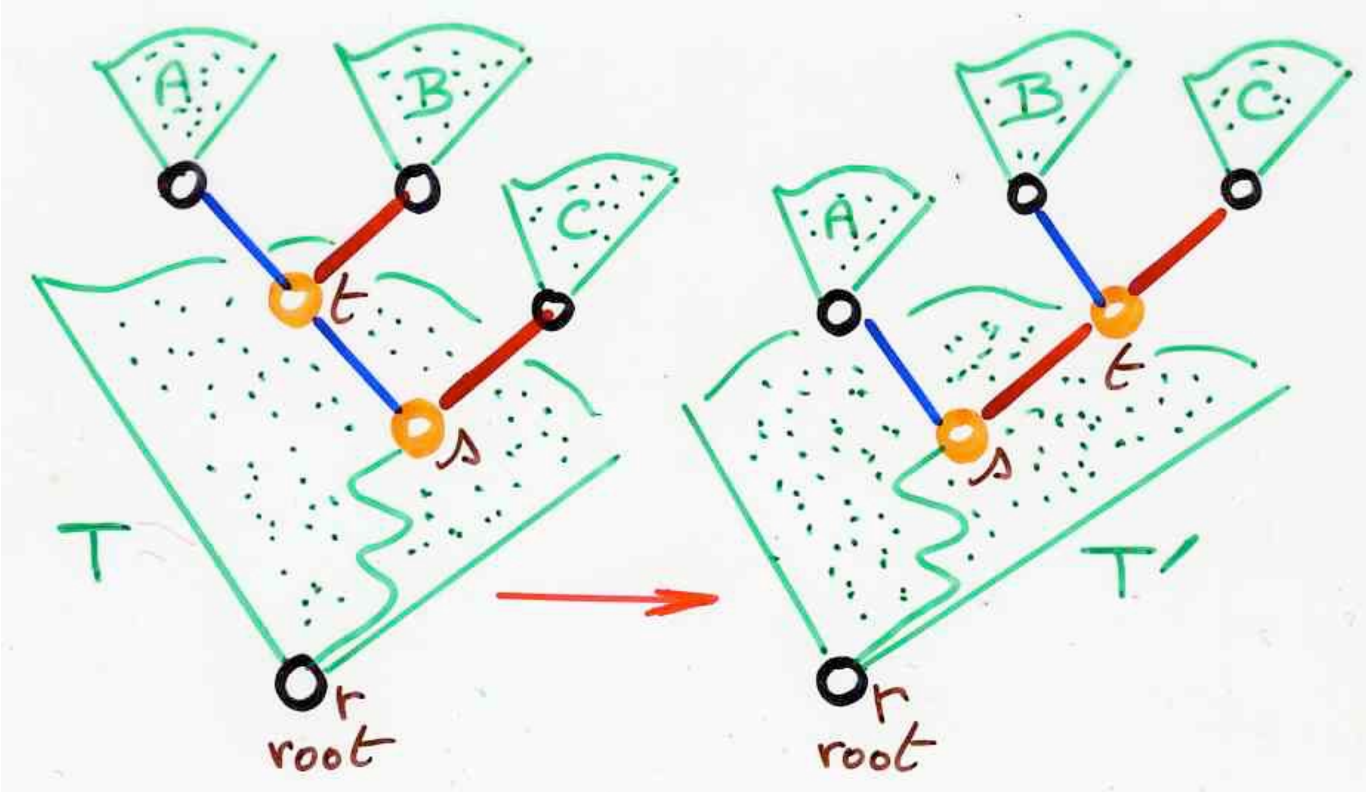
hypercube	associahedron	permutahedron
Boolean lattice inclusion ↔	Tamari order ↔	weak Bruhat order
dim 2^{n-1}	C_n Catalan	$n!$







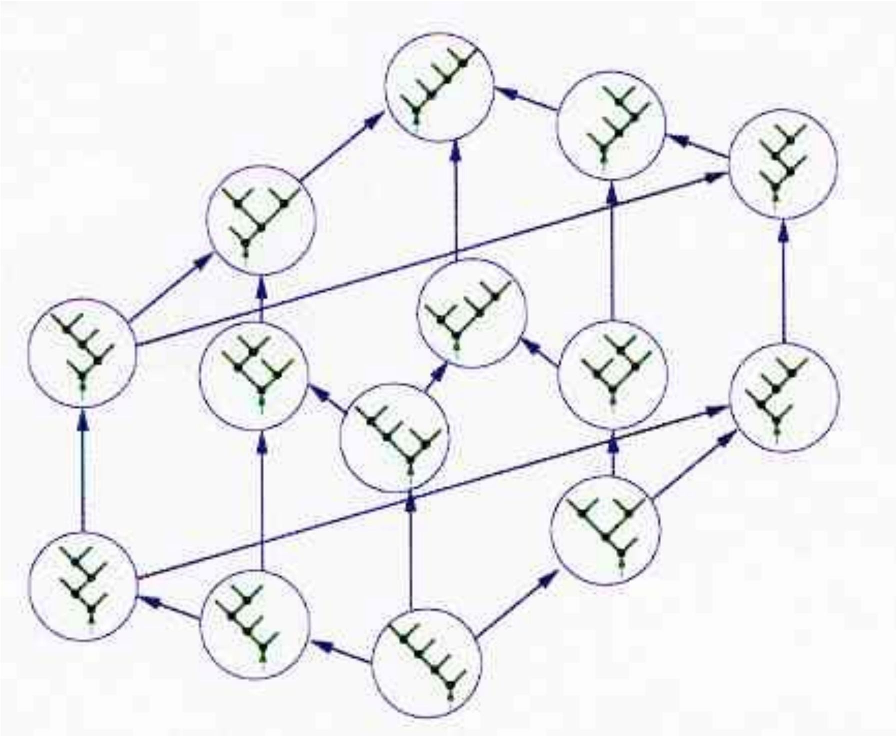
Tamari lattice



Rotation in a binary tree:
the covering relation in the
Tamari lattice

order relation

Tamari lattice

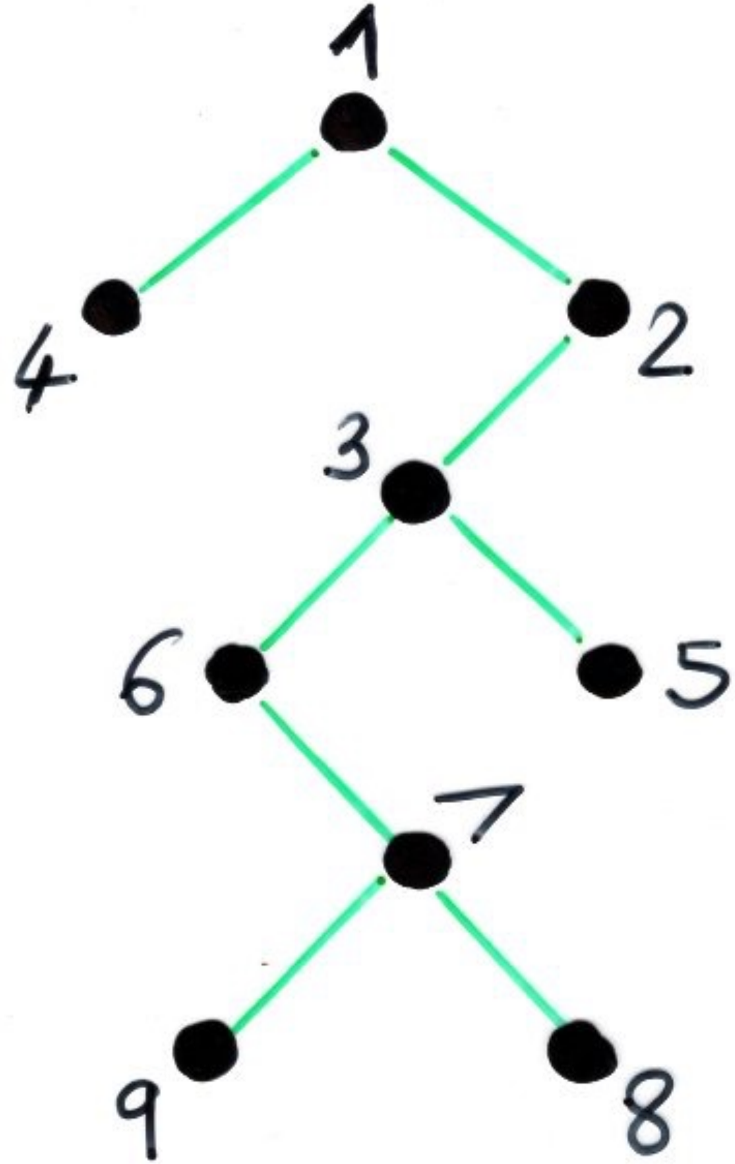


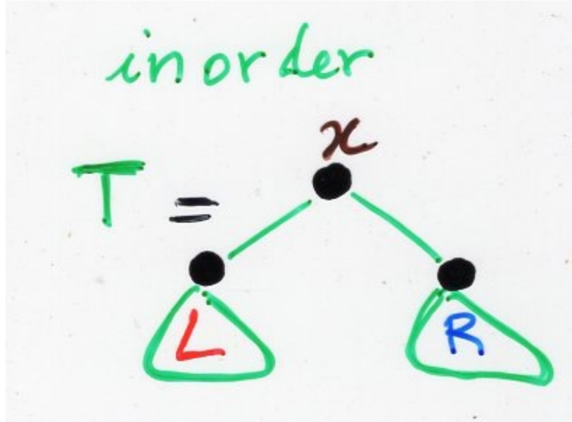
Tamari lattice

$C_4 = 14$

Catalan

Dov Tamari (1951) thèse Sorbone
"Monoides preórdonnés et chaînes de Malcev"





$$\Pi(T) = \underbrace{\Pi(L)}\, x\, \underbrace{\Pi(R)}$$

projection of $T \in \mathcal{T}_n$

$$\Pi(T) = 4\,1\,6\,9\,7\,8\,3\,5\,2$$

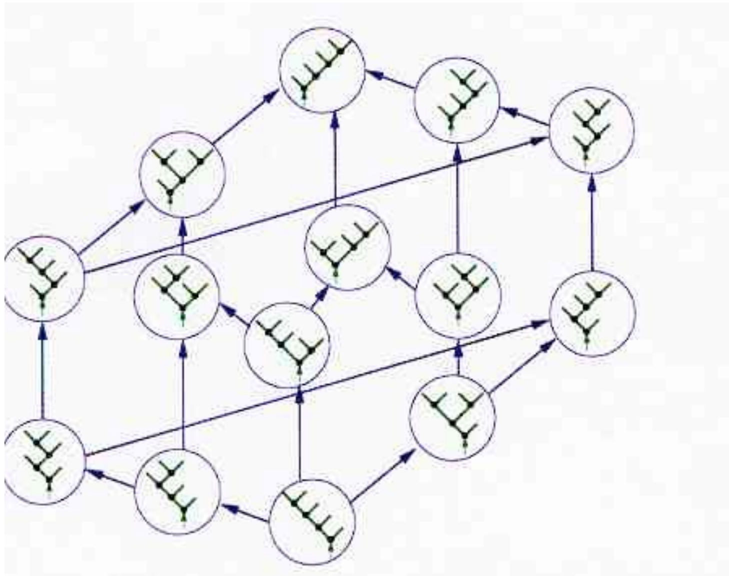
combinatorial structures

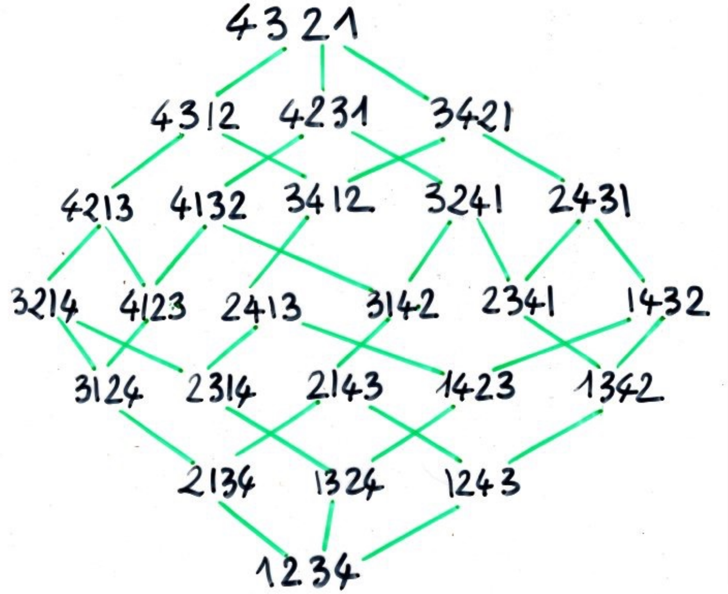
associahedron permutahedron

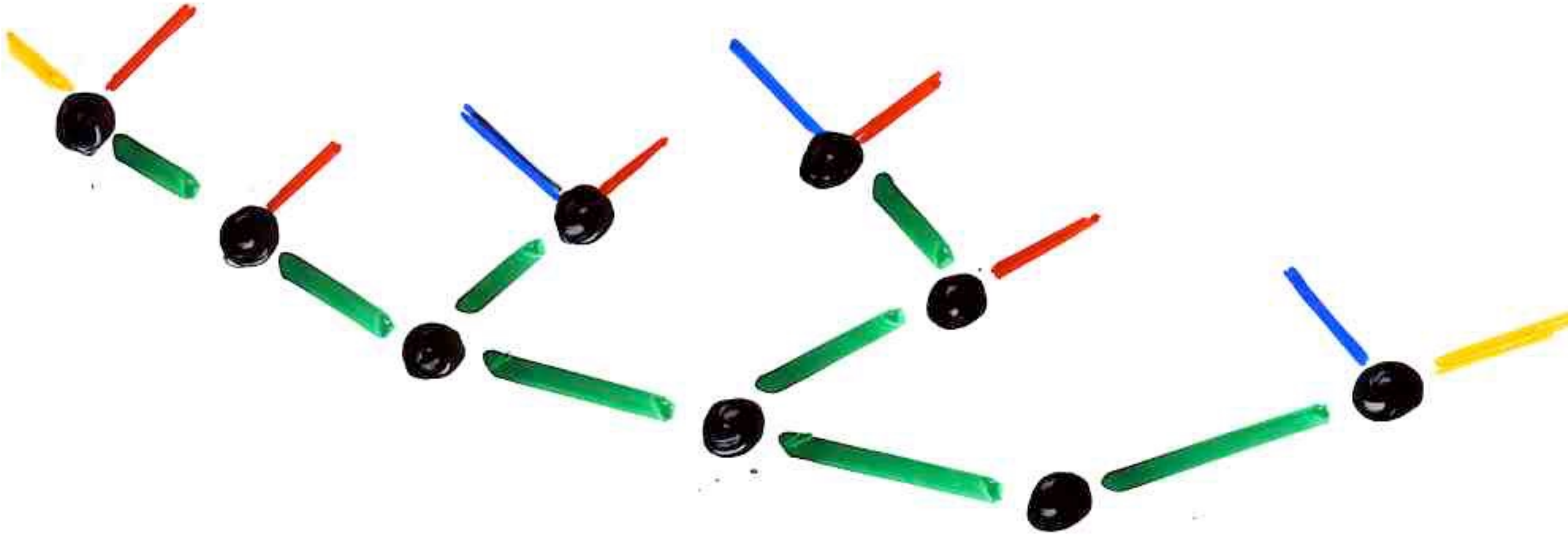
Tamari order ⟷ weak Bruhat order

C_n $n!$

Catalan







canopy of a binary tree

c(B) = / / \ / \ / / \

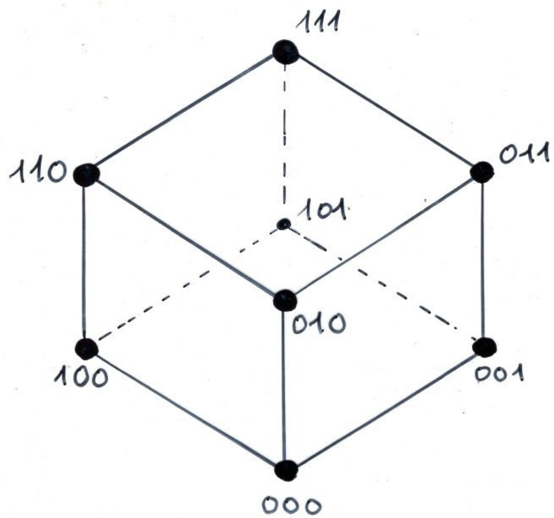
Loday, Ronco (1998, 2012)

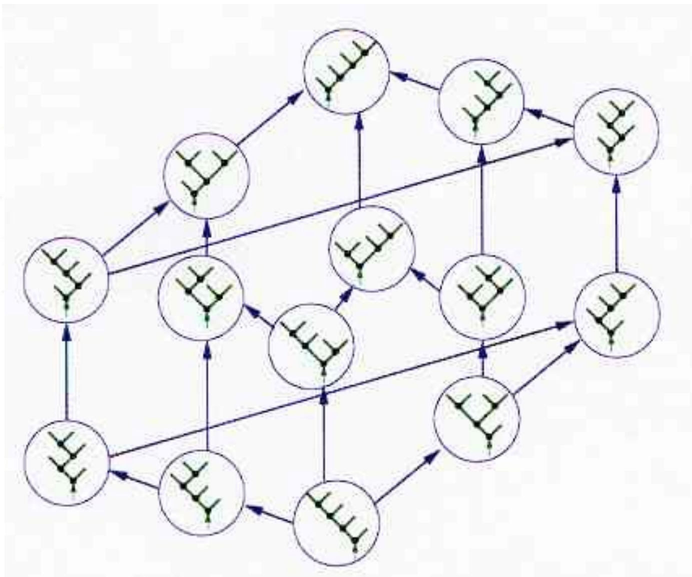
combinatorial structures

hypercube — associahedron

Boolean lattice inclusion ⟷ Tamari order

dim 2^{n-1} — C_n Catalan





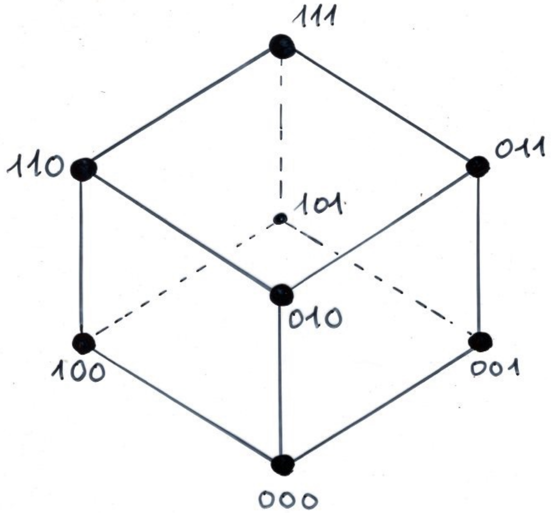
combinatorial structures

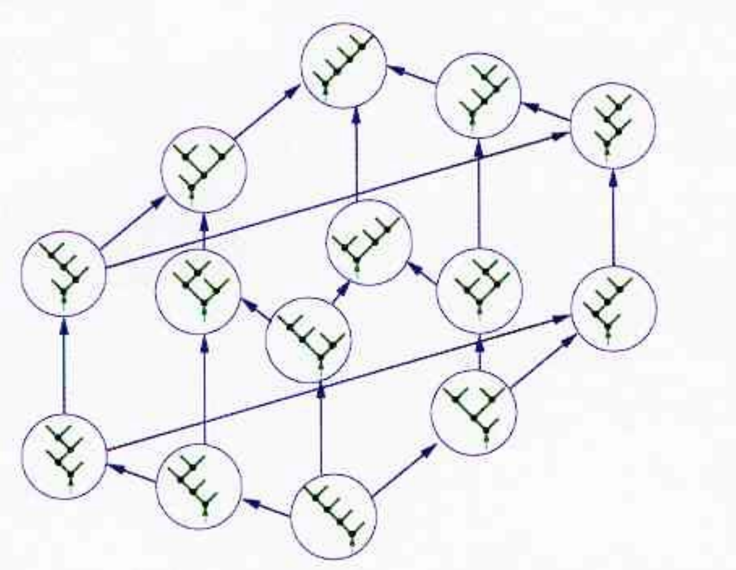
hypercube — associahedron — permutahedron

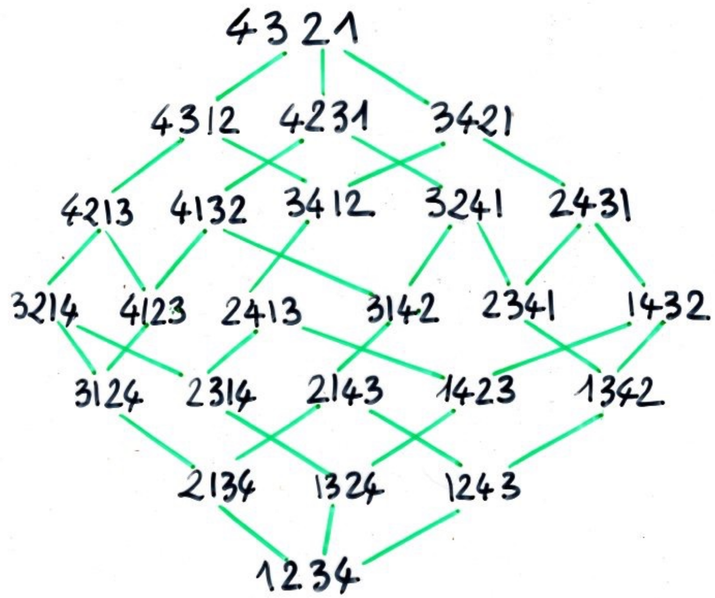
Boolean lattice inclusion ⟷ Tamari order ⟷ weak Bruhat order

dim 2^{n-1} $\qquad C_n$ $\qquad n!$

Catalan



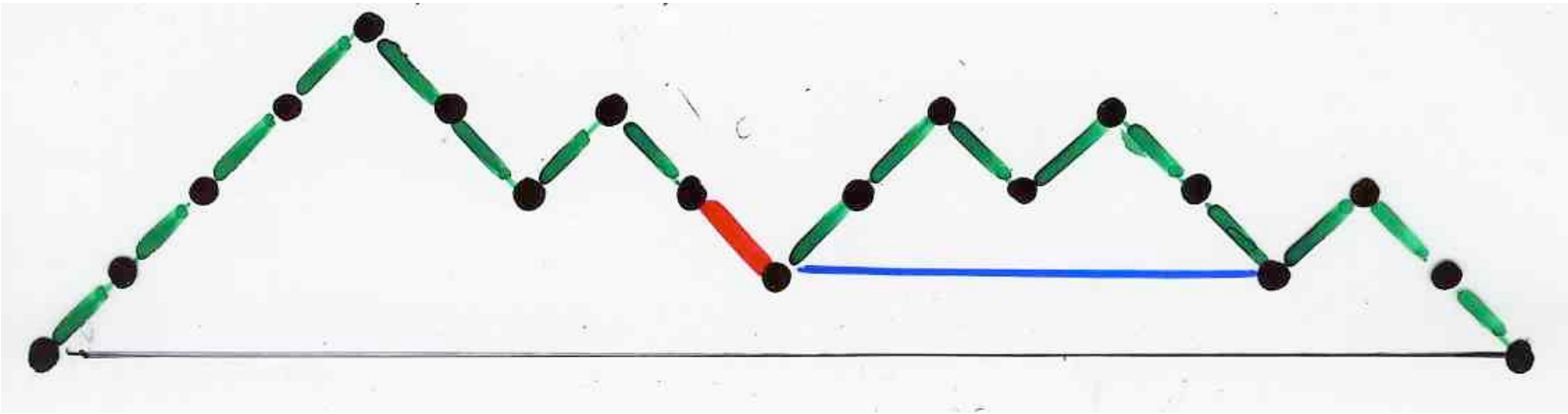


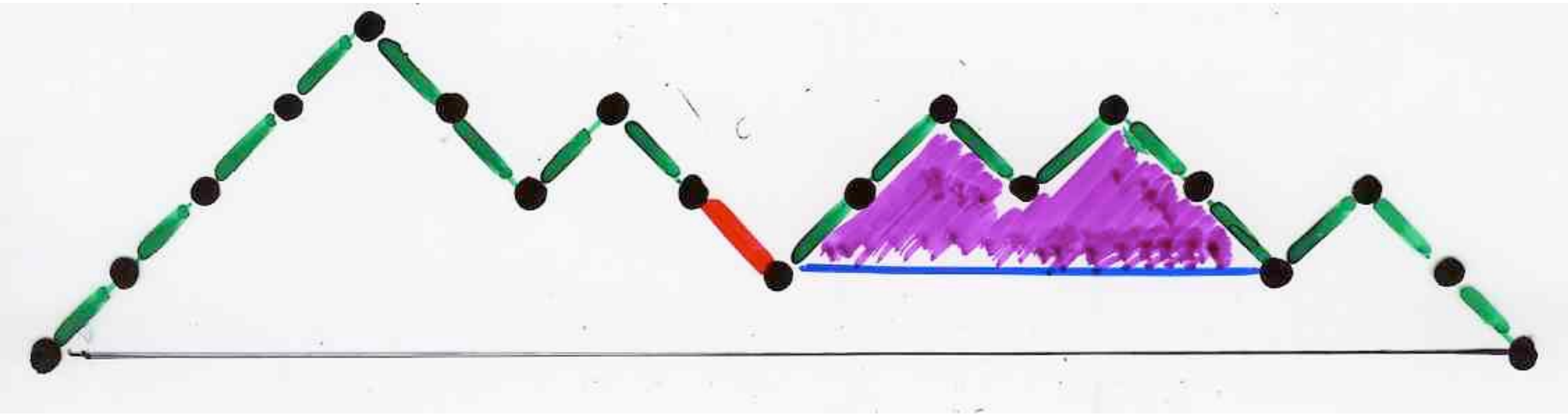


Loday, Ronco (1998, 2012)

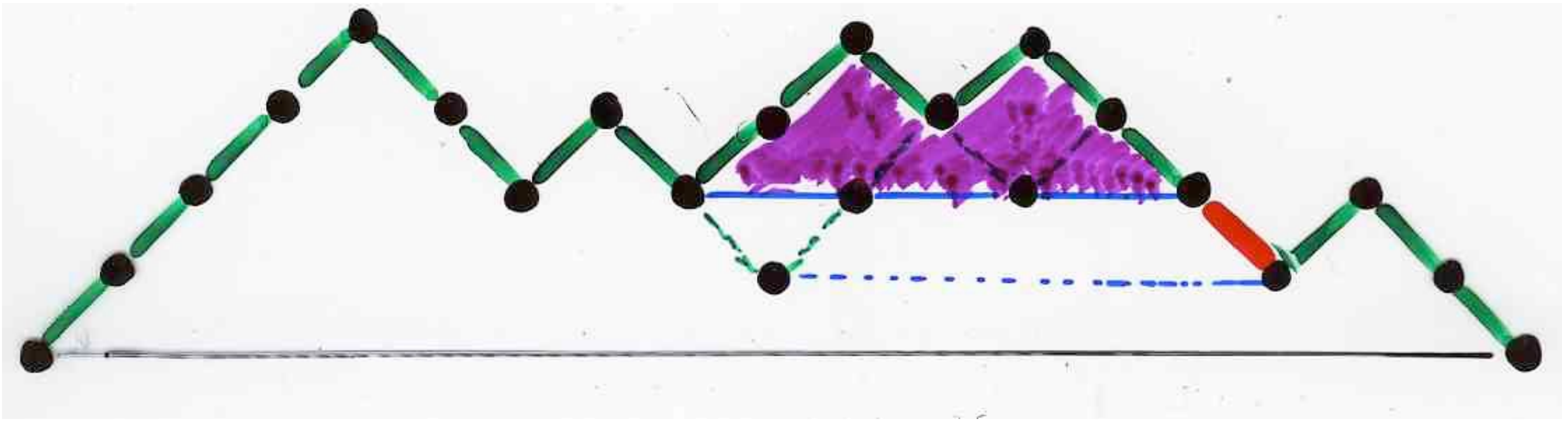
A. Björner, M. Wachs (1991)

the Tamari lattice
in terms
of Dyck paths





facteur Dyck primitif



factor Dyck primitif

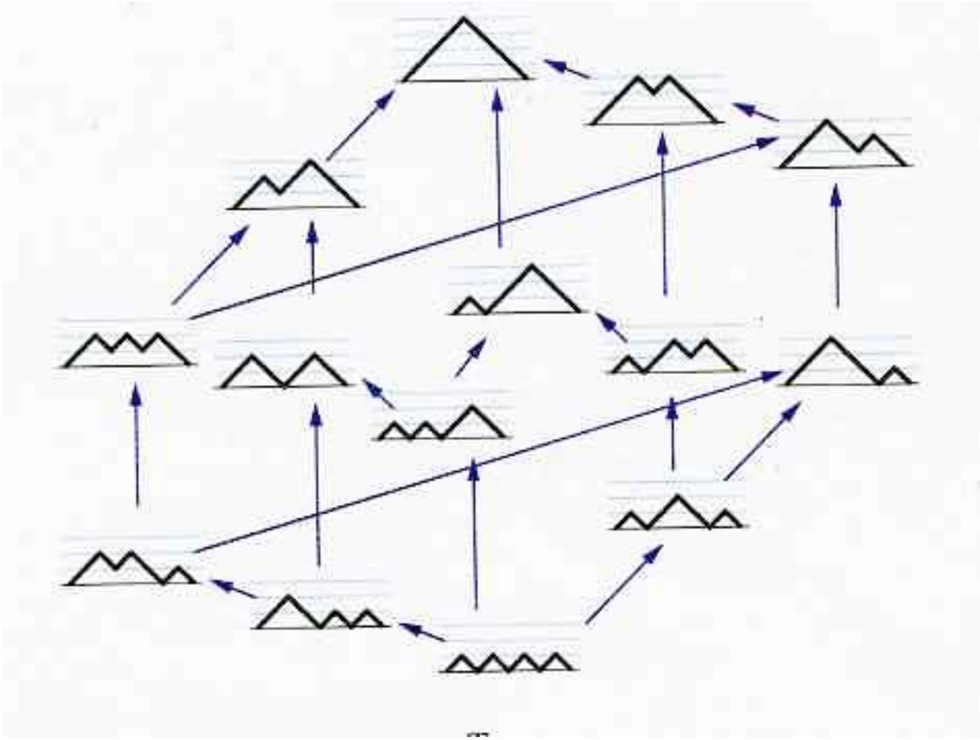
If $T \leqslant T'$ in $(\text{Tamari})_n$ lattice

then $T \leqslant T'$ in $(\text{Dyck})_n$ lattice

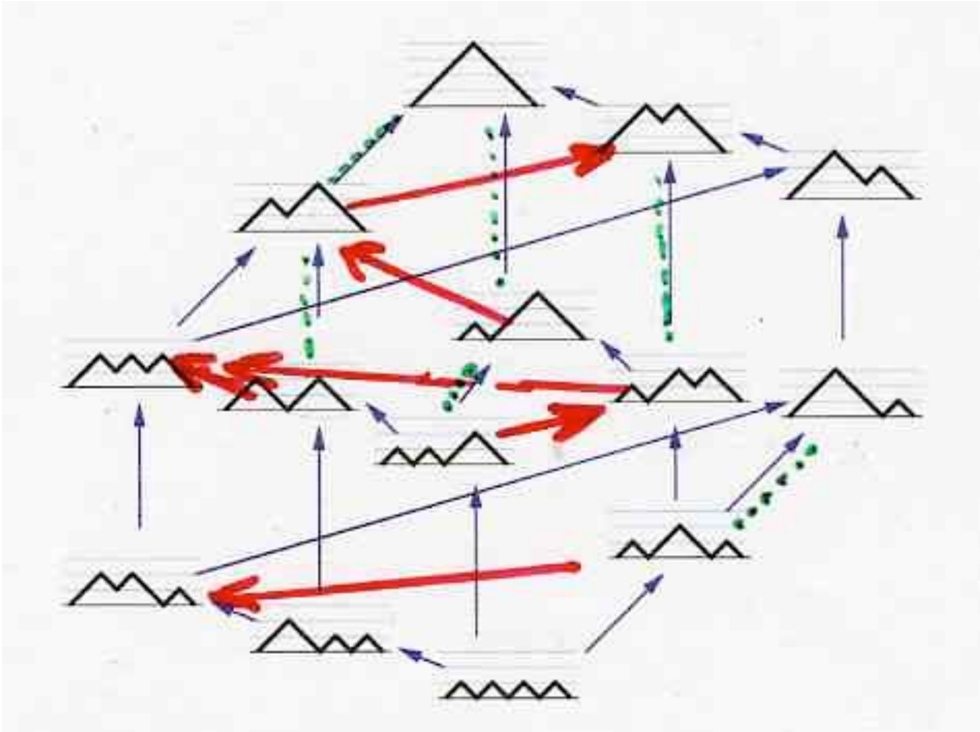
[i.e. T below T']

converse not true

$(\text{Dyck})_n$ extension of $(\text{Tamari})_n$



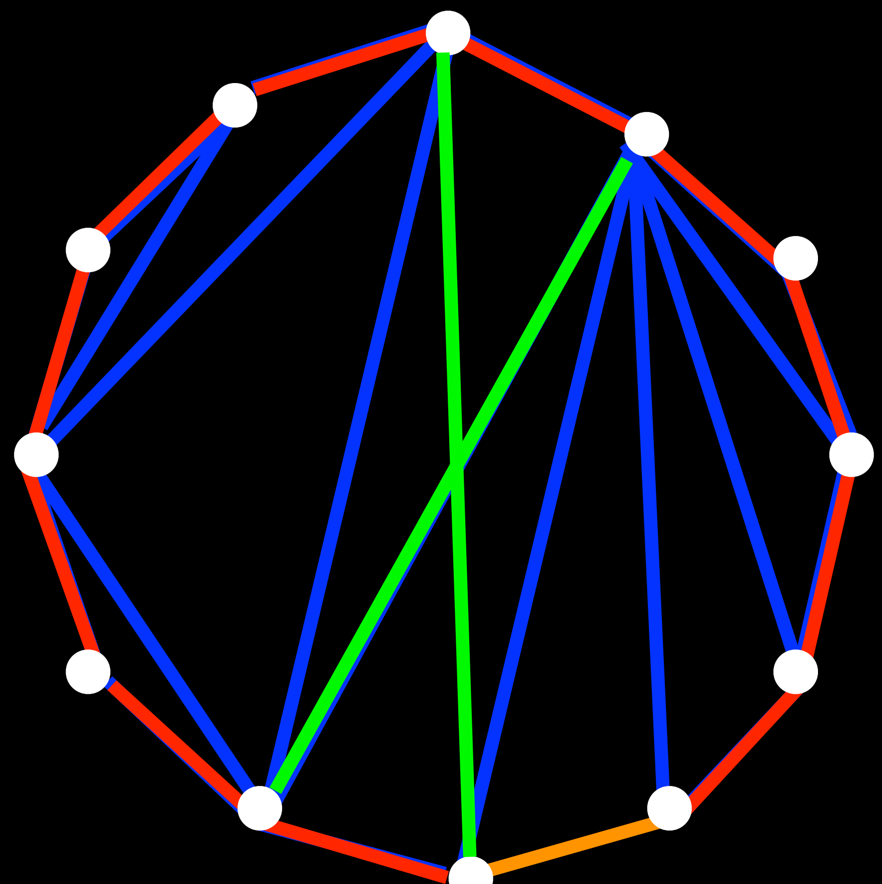
$\left(\text{Tamari}\right)_4$
lattice

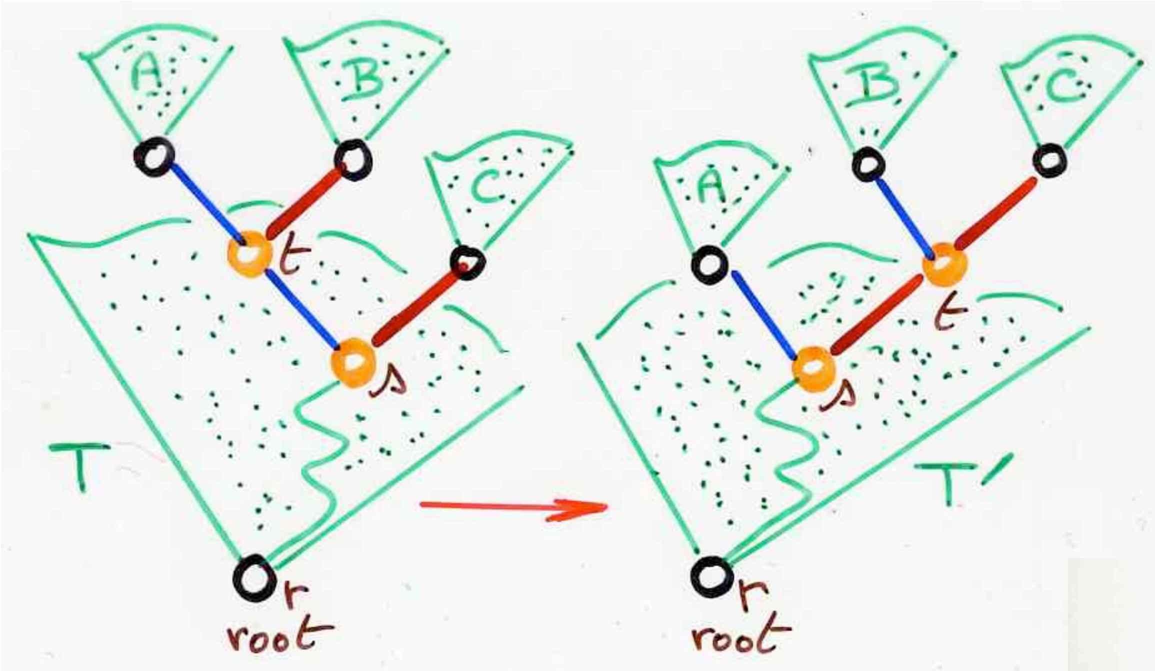


$\left(\text{Dyck lattice}\right)_4$

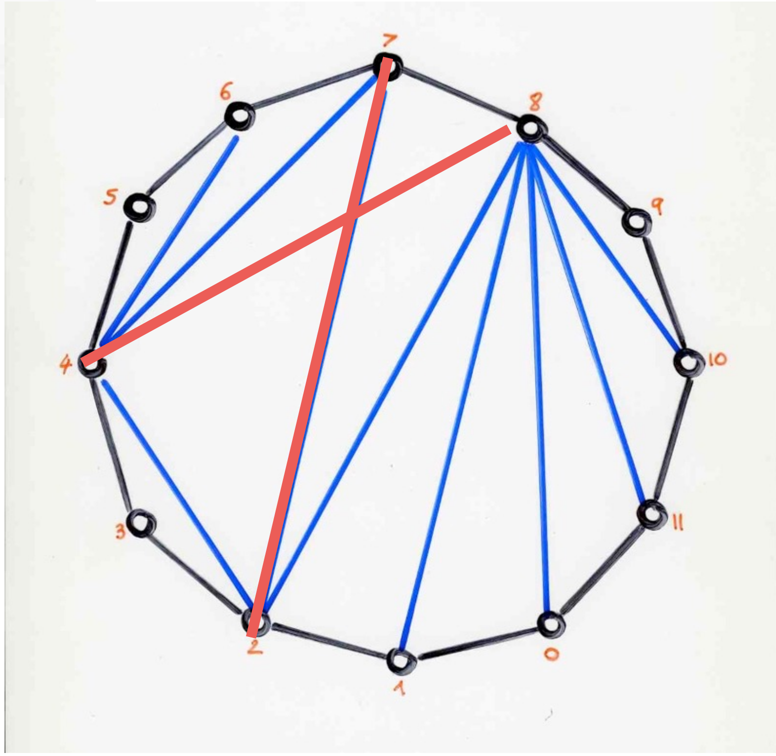
the Tamari lattice

with triangulations



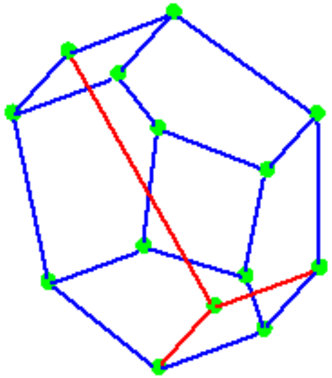


Rotation in a binary tree:
the covering relation in the
Tamari lattice



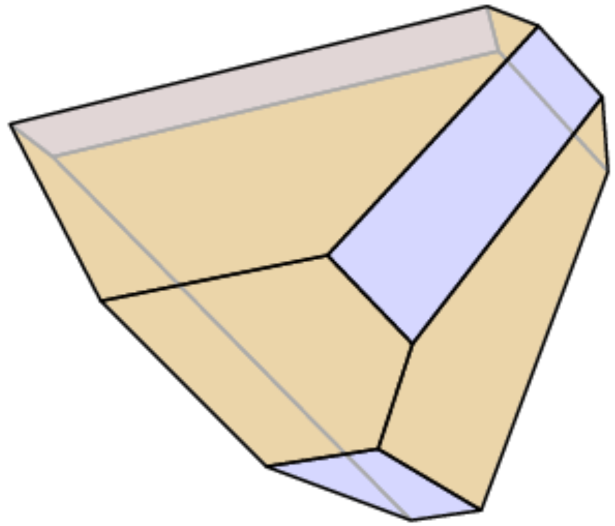
realisation of the associahedron

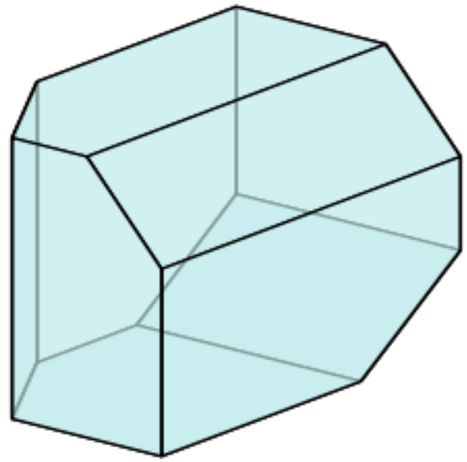
Is it possible to realize the cells structure of the associahedron as the cells of a convex polytope ?

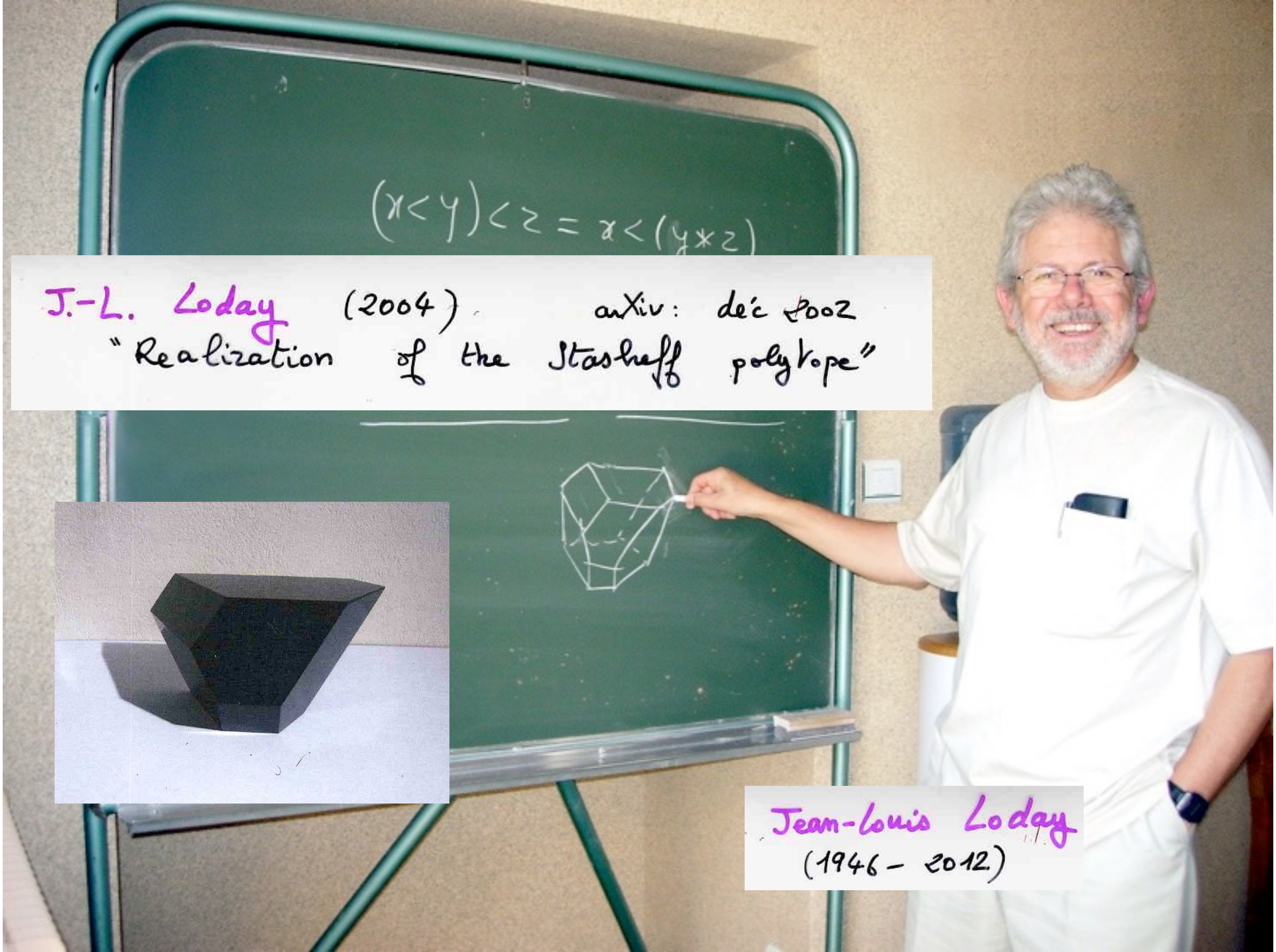


14 vertices

21 edges





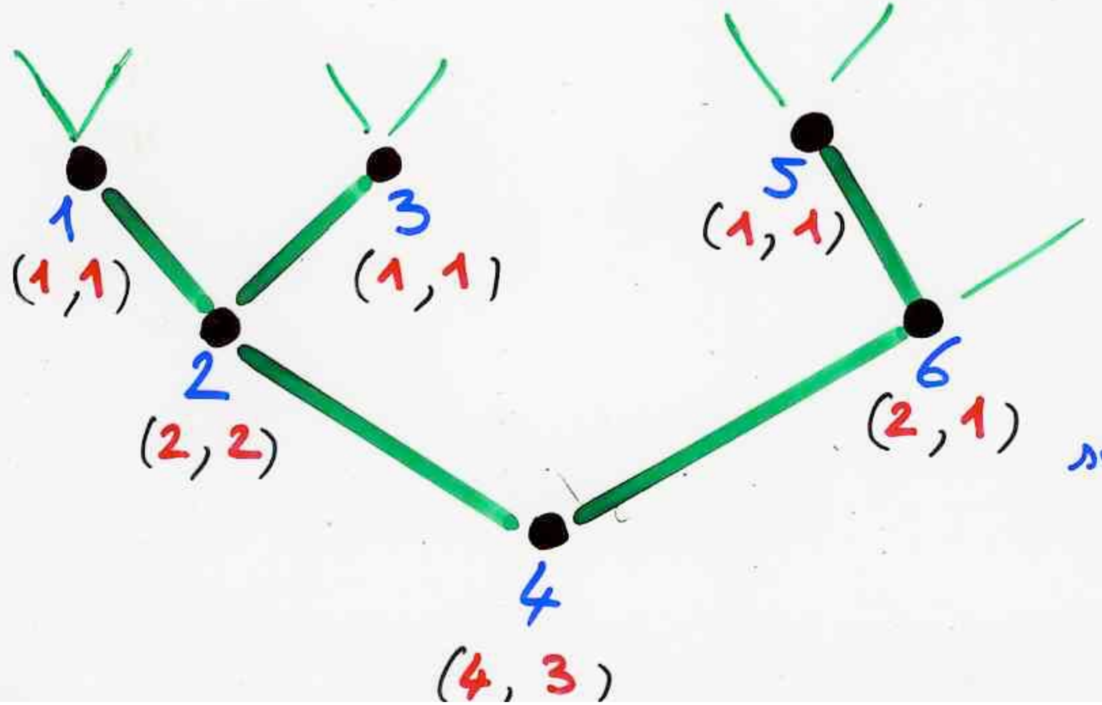


J.-L. Loday (2004) arXiv: déc 2002
"Realization of the Stasheff polytope"

Jean-Louis Loday
(1946 – 2012)

J.-L. Loday (2004) arXiv: déc 2002

"Realization of the Stasheff polytope"



symmetric order

1 2 3 4 5 6

(1 , 4 , 1 , 12 , 1 , 2)

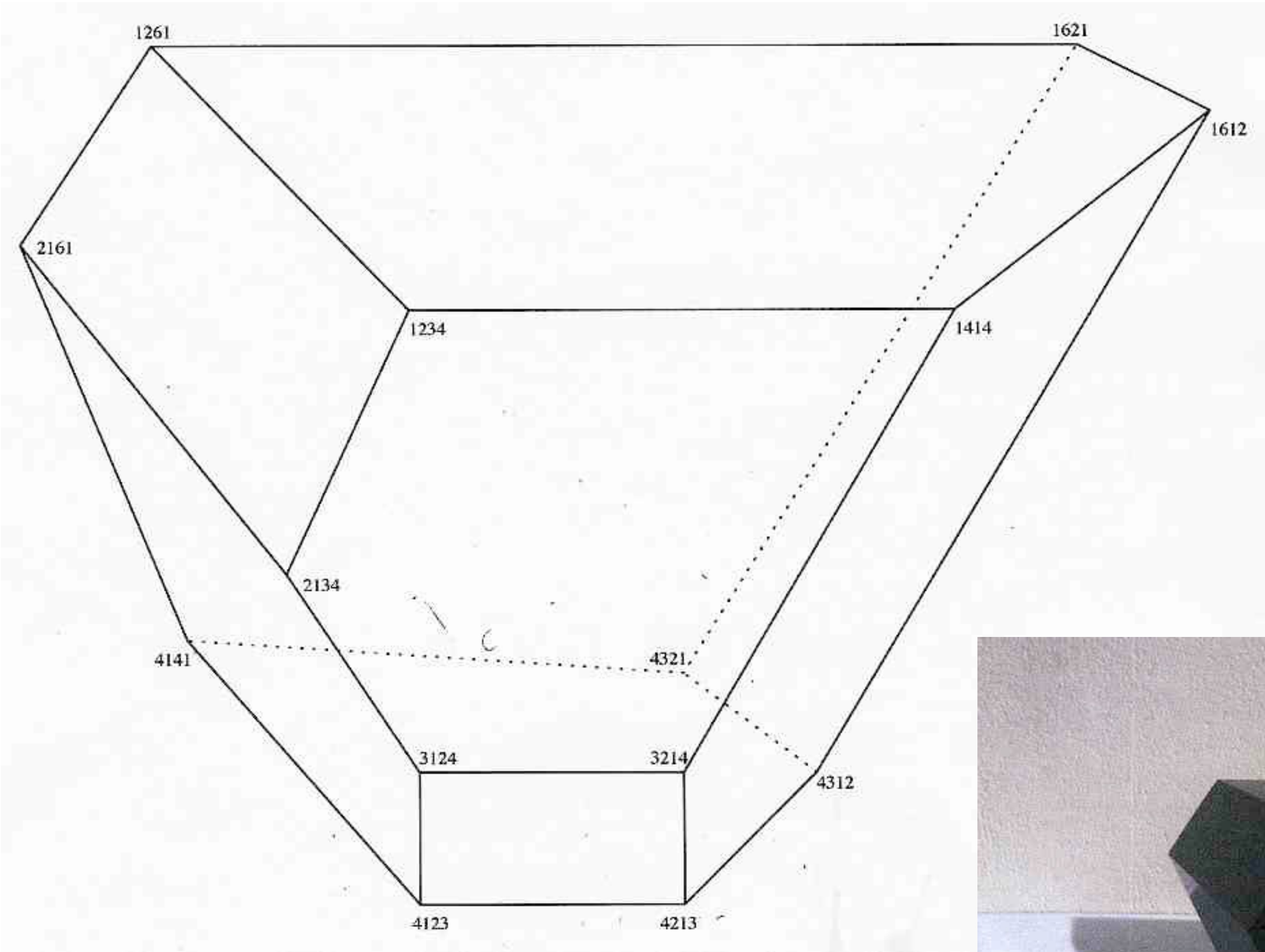
$\xrightarrow[\text{sum}]{} 21$

$$\frac{n(n+1)}{2}$$

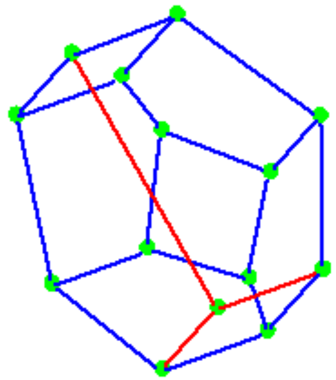
convex hull of the points

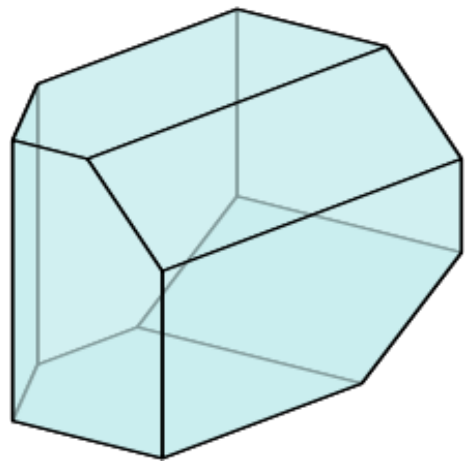
hyperplan

$$x_1 + \dots + x_n = \frac{n(n+1)}{2}$$


1261
1621
1612
2161
1234
1414
2134
4141
4321
3124
3214
4312
4123
4213


4000 BASEL
e-k+f=2
Leonhard Euler
1707-1783
AUSGABETAG 6. MÄRZ 2007
LEONHARD EULER
1707–1783
130
300
300. Geburtstag
300ème anniversaire
300° anniversario
300th anniversary





14 vertices
21 edges
9 faces

$$S - A + F$$
$$14 - 21 + 9 = 2$$

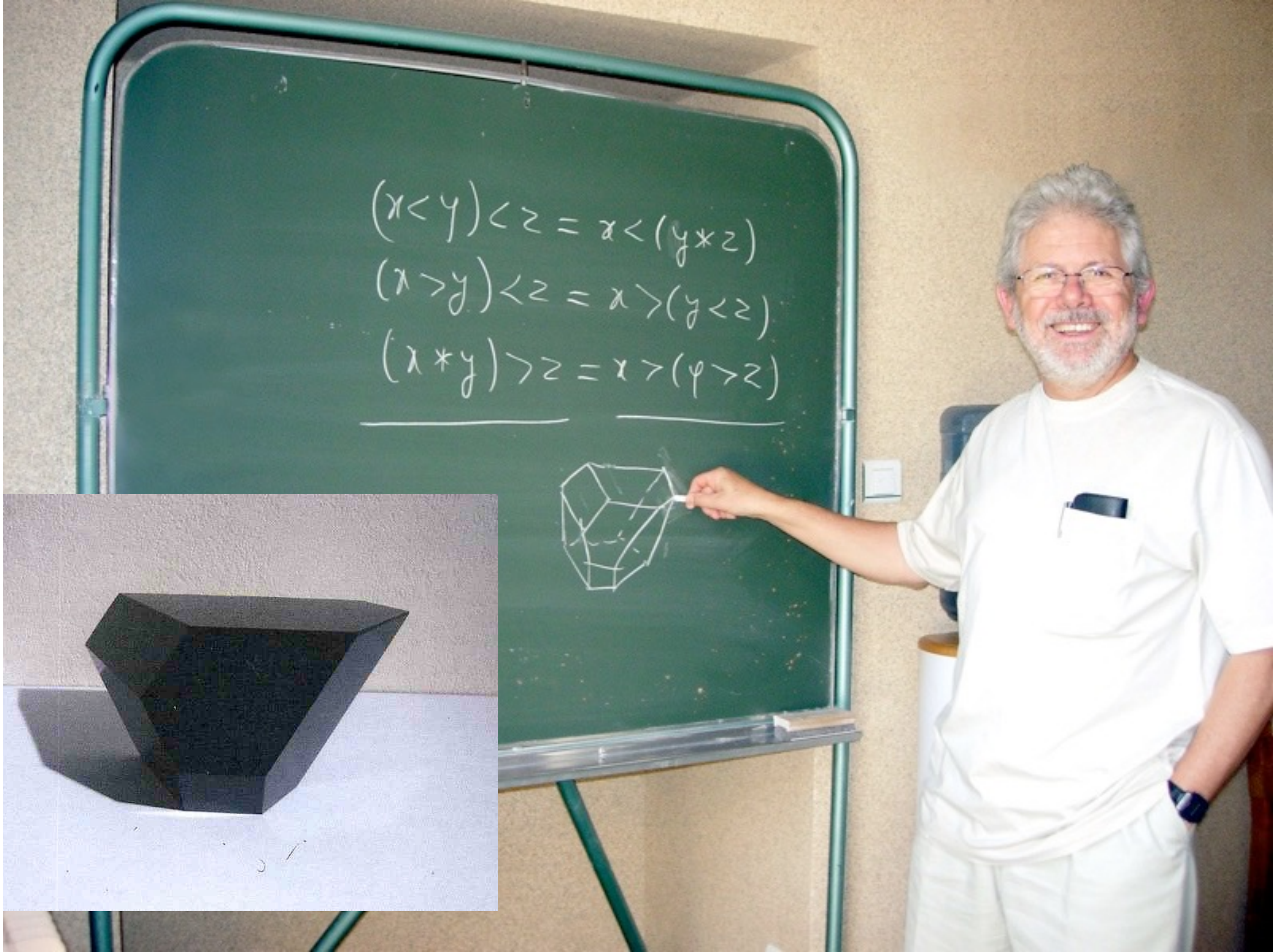
$\begin{cases} 6 \\ 3 \end{cases}$ pentagons
rectangles

3 geometric structures

hypercube

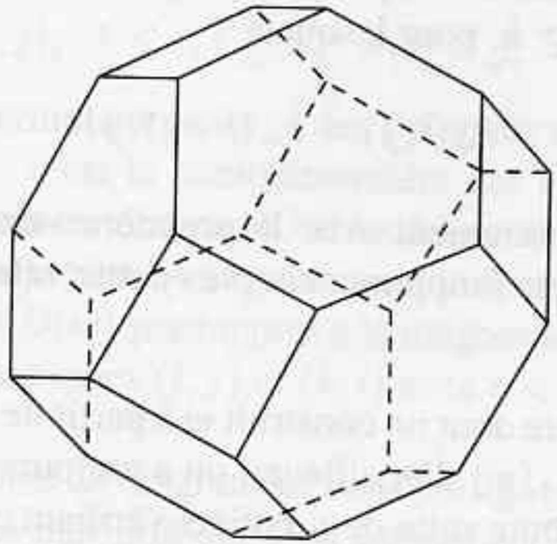
associahedron

permutohedron

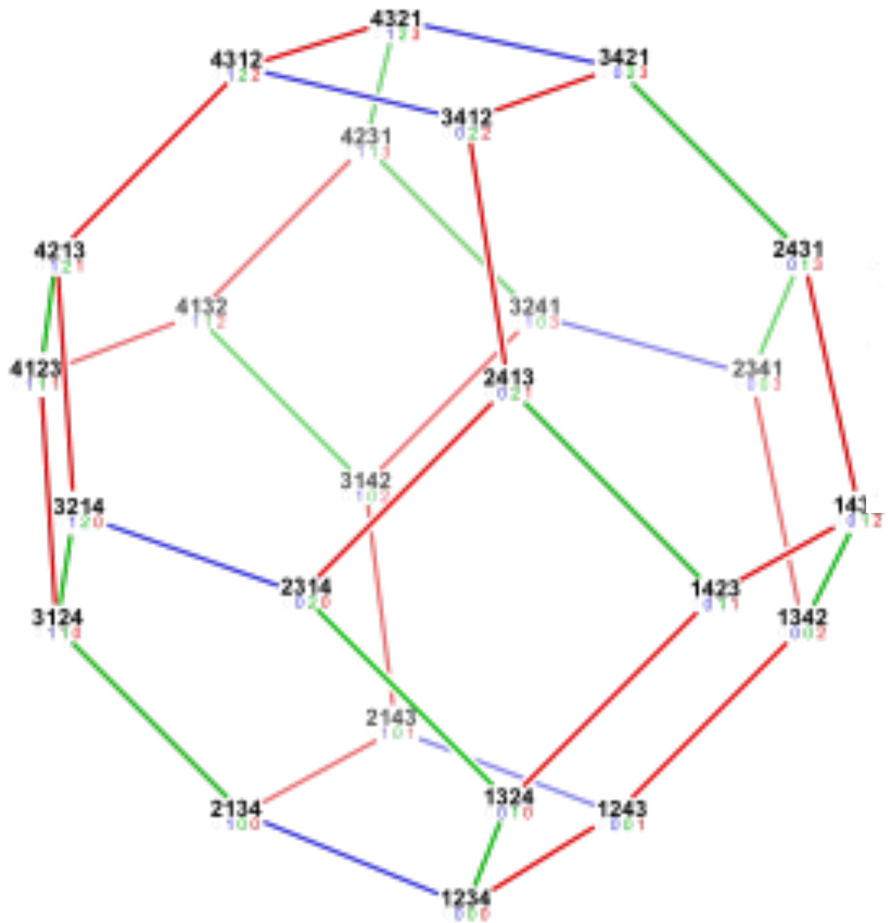

(x<y)<z = x<(y*z)
(x>y)<z = x>(y<z)
(x*y)>z = x>(y>z)

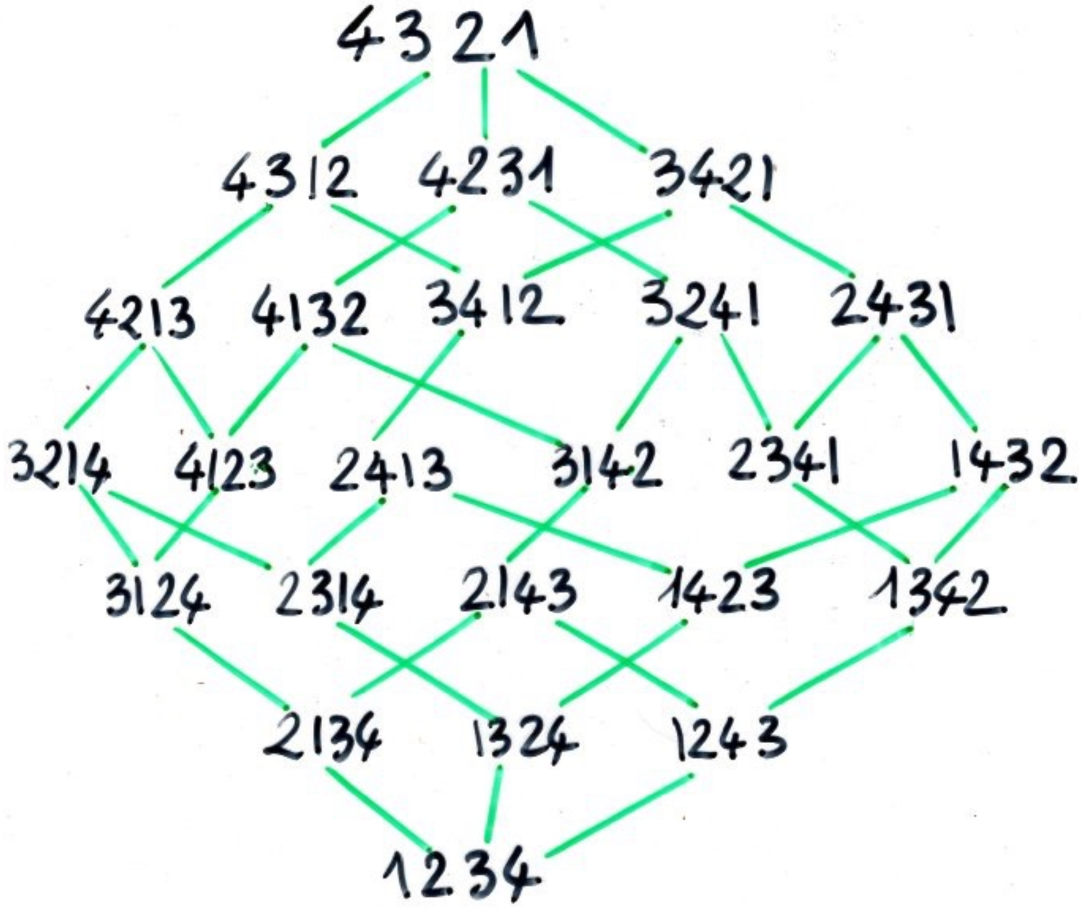
Alain Lascoux
(1944–2013)

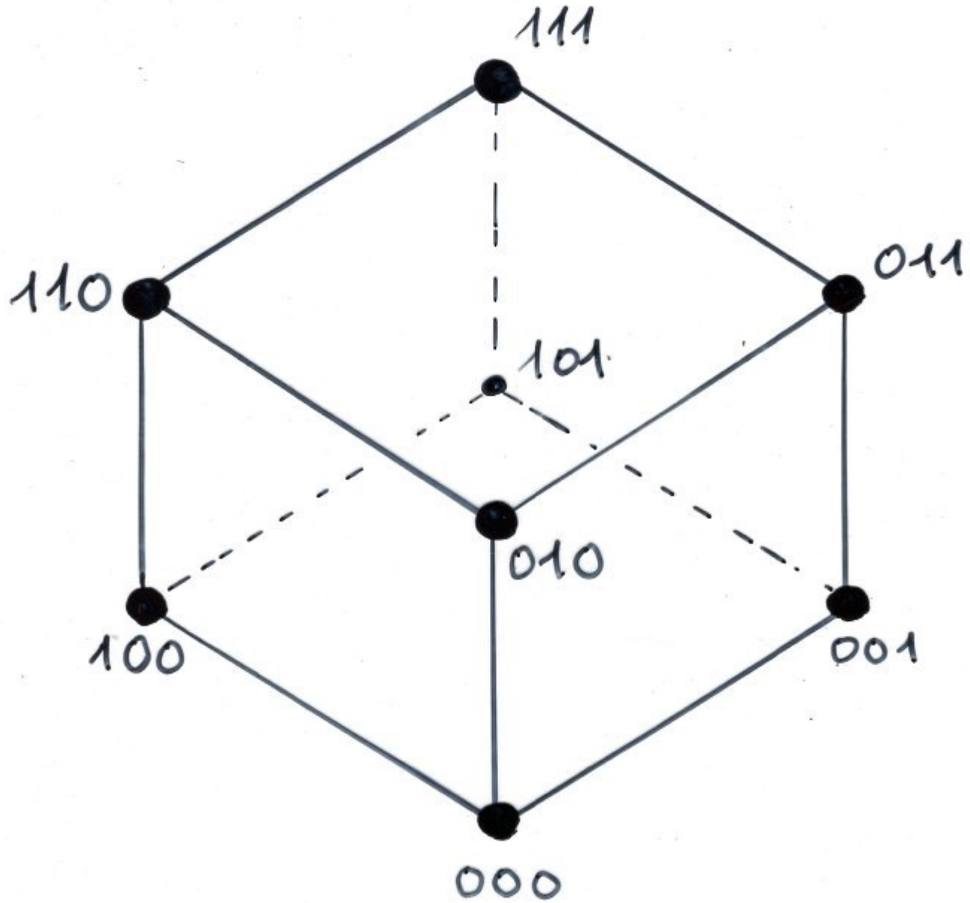
permutohedron



2. Le permutoèdre Π_3.


4321
4312
3421
3412
4231
4213
2431
4132
3241
4123
2413
2341
3142
3214
2314
1423
3124
1342
2143
2134
1324
1243
1234


4321
4312 4231 3421
4213 4132 3412 3241 2431
3214 4123 2413 3142 2341 1432
3124 2314 2143 1423 1342
2134 1324 1243
1234



Boolean lattice
inclusion

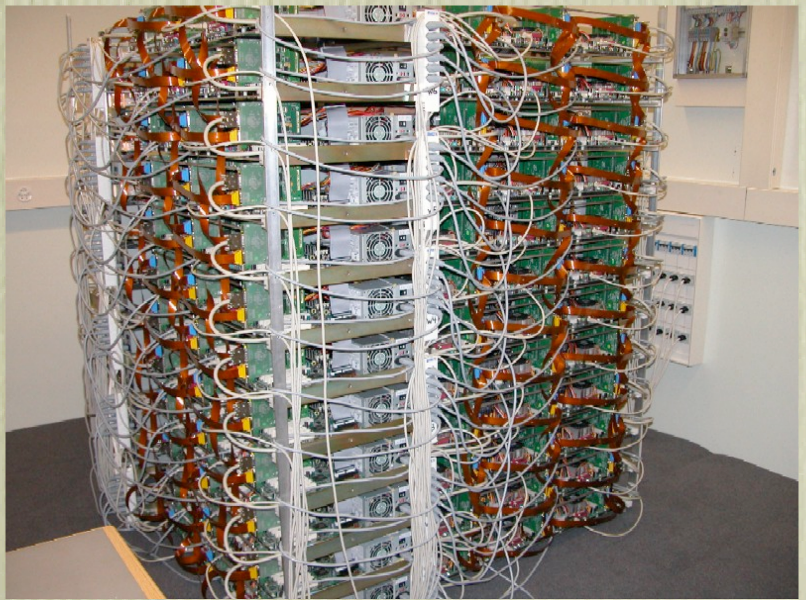
$A \subseteq B$
order relation

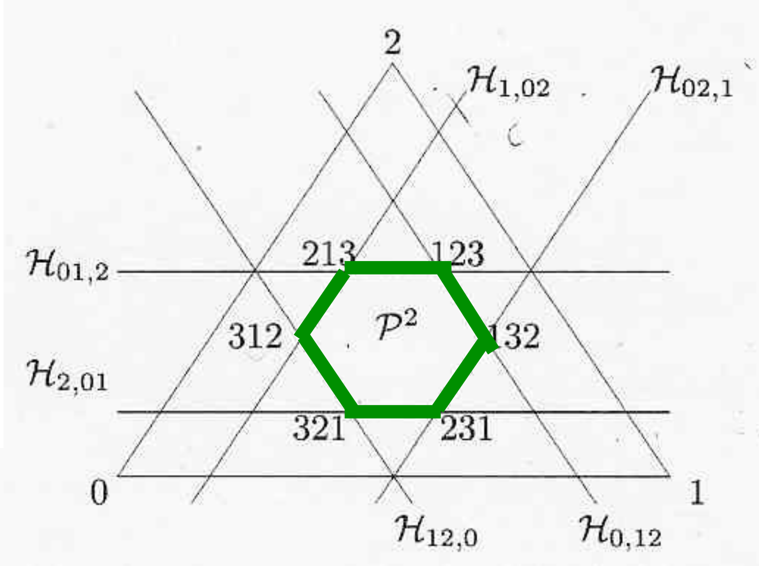
$|X| = n \qquad X = \{1, 2, .., n\}$

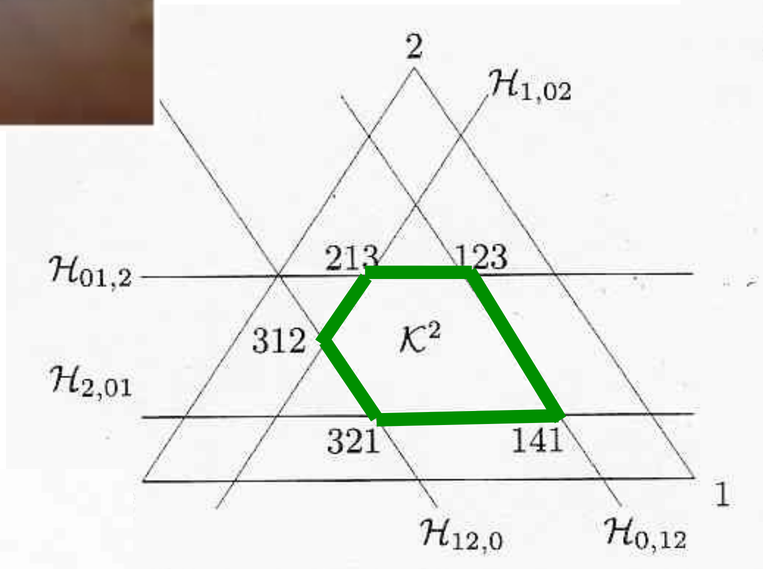
$A = \{2, 3, 6\} \subseteq \{1, 2, .., 8\}$

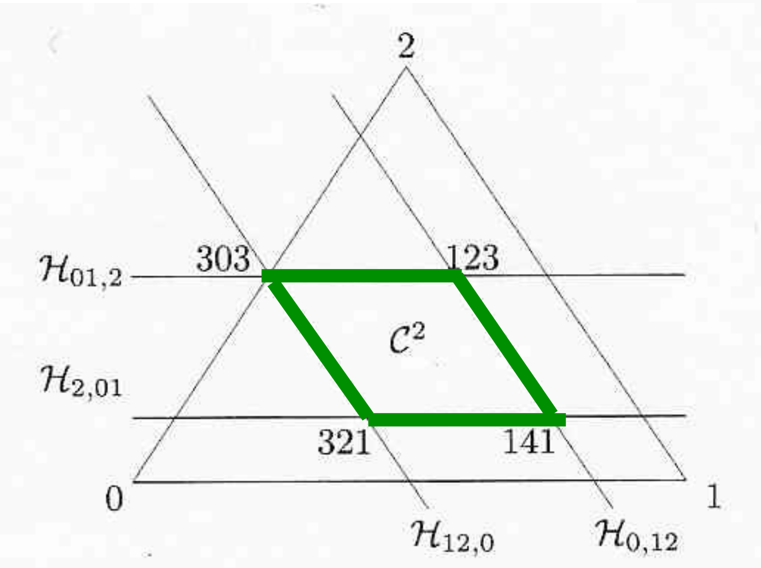
$$\begin{array}{ccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline w = & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}$$

Binohedron	associahedron	permutahedron
00110100110001100000 ↔	Tamari order ↔	weak Bruhat order
dim 2^{n-1}	C_n Catalan	$n!$




2
$\mathcal{H}_{1,02}$
$\mathcal{H}_{02,1}$
$\mathcal{H}_{01,2}$
213
123
312
$\mathcal{P}^2$
132
$\mathcal{H}_{2,01}$
321
231
0
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$


2
$\mathcal{H}_{1,02}$
$\mathcal{H}_{01,2}$
213
123
312
$\mathcal{K}^2$
$\mathcal{H}_{2,01}$
321
141
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$


2
$\mathcal{H}_{01,2}$
303
123
$\mathcal{C}^2$
$\mathcal{H}_{2,01}$
321
141
0
1
$\mathcal{H}_{12,0}$
$\mathcal{H}_{0,12}$