Chapter 4

The n! garden

(3)

complements

and

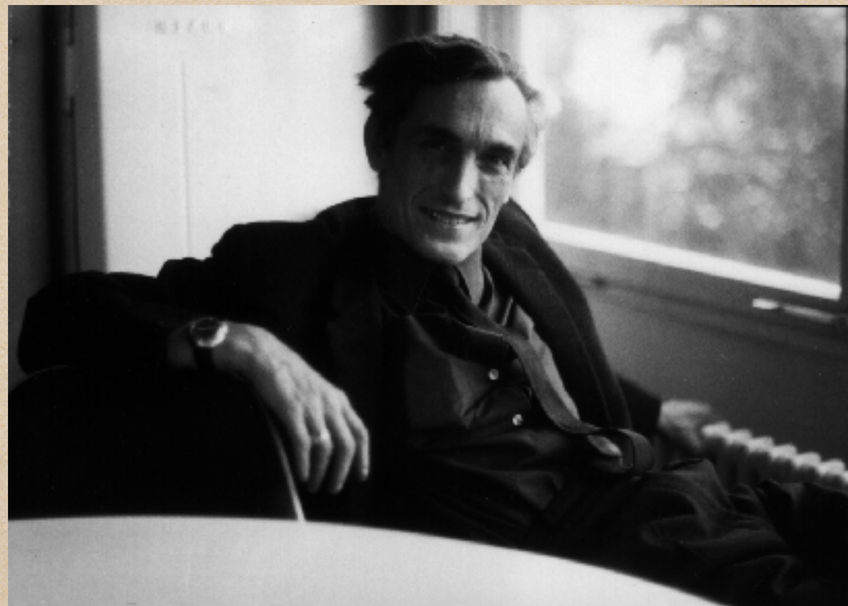
beautiful facts without proofs

IMSc

23 February 2016

Jeu de taquin

(without proof)



M.P. Schützenberger

$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$

$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$

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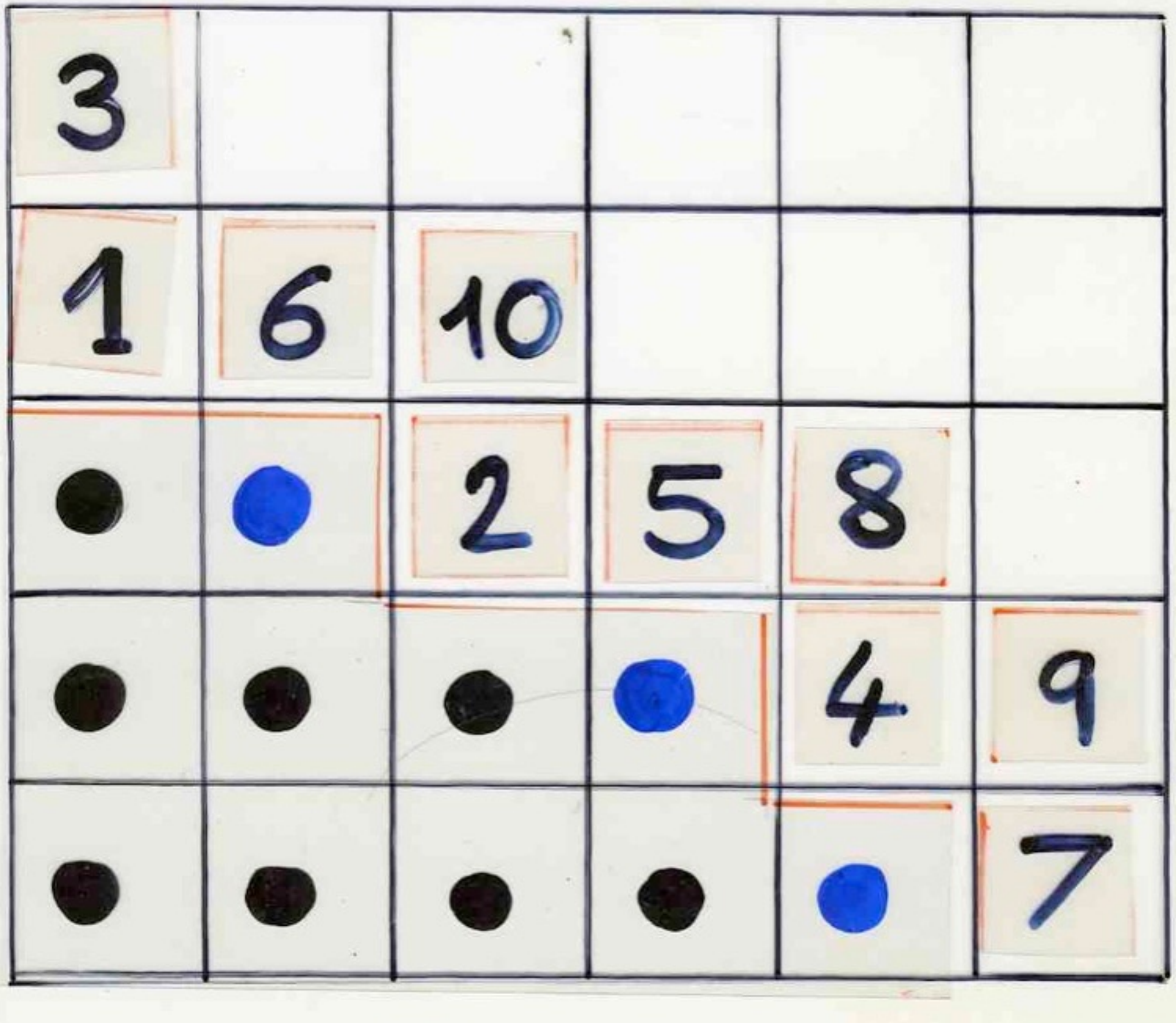
1	6	10			
		2	5	8	
				4	9
					7

$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$



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$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$

6	10				
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1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

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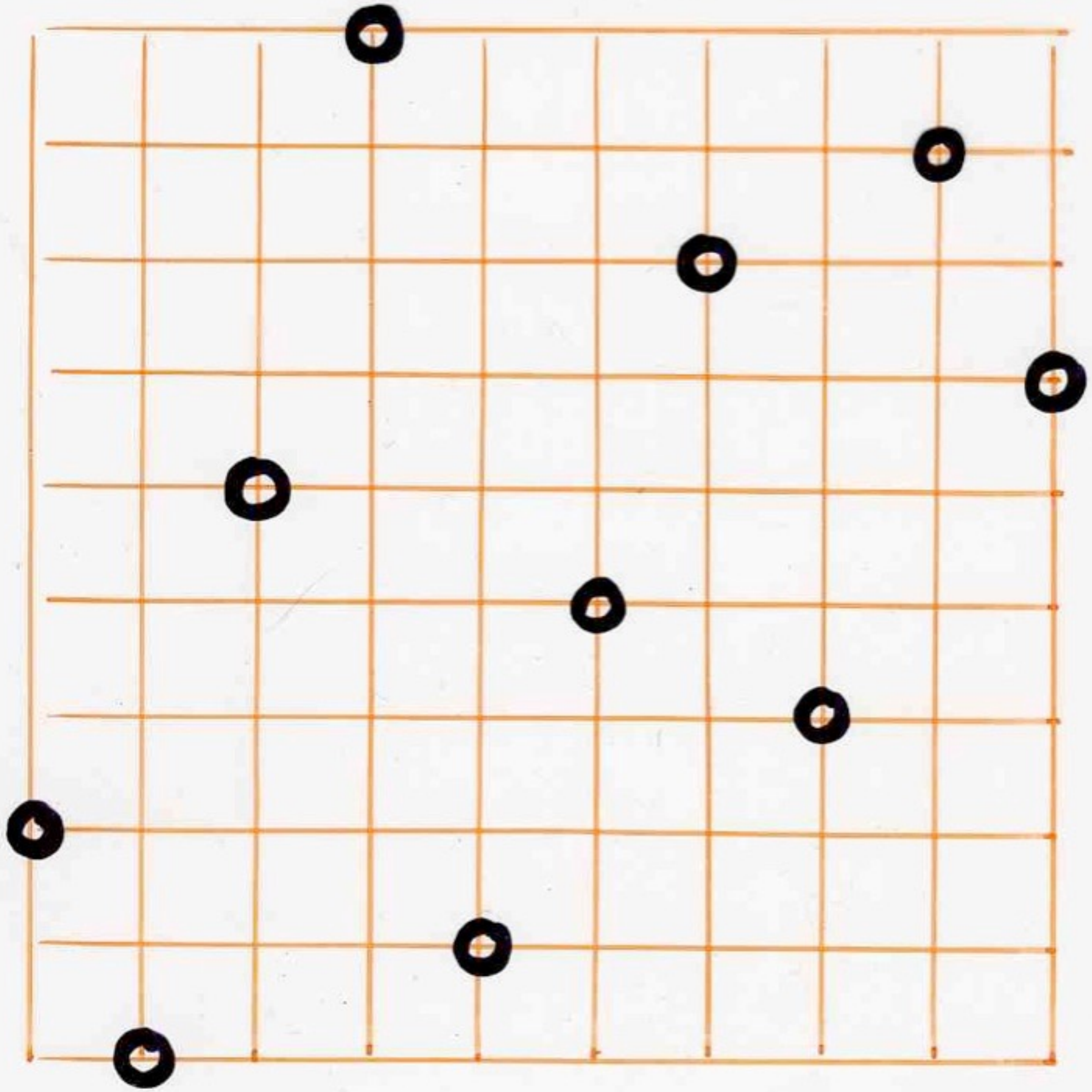
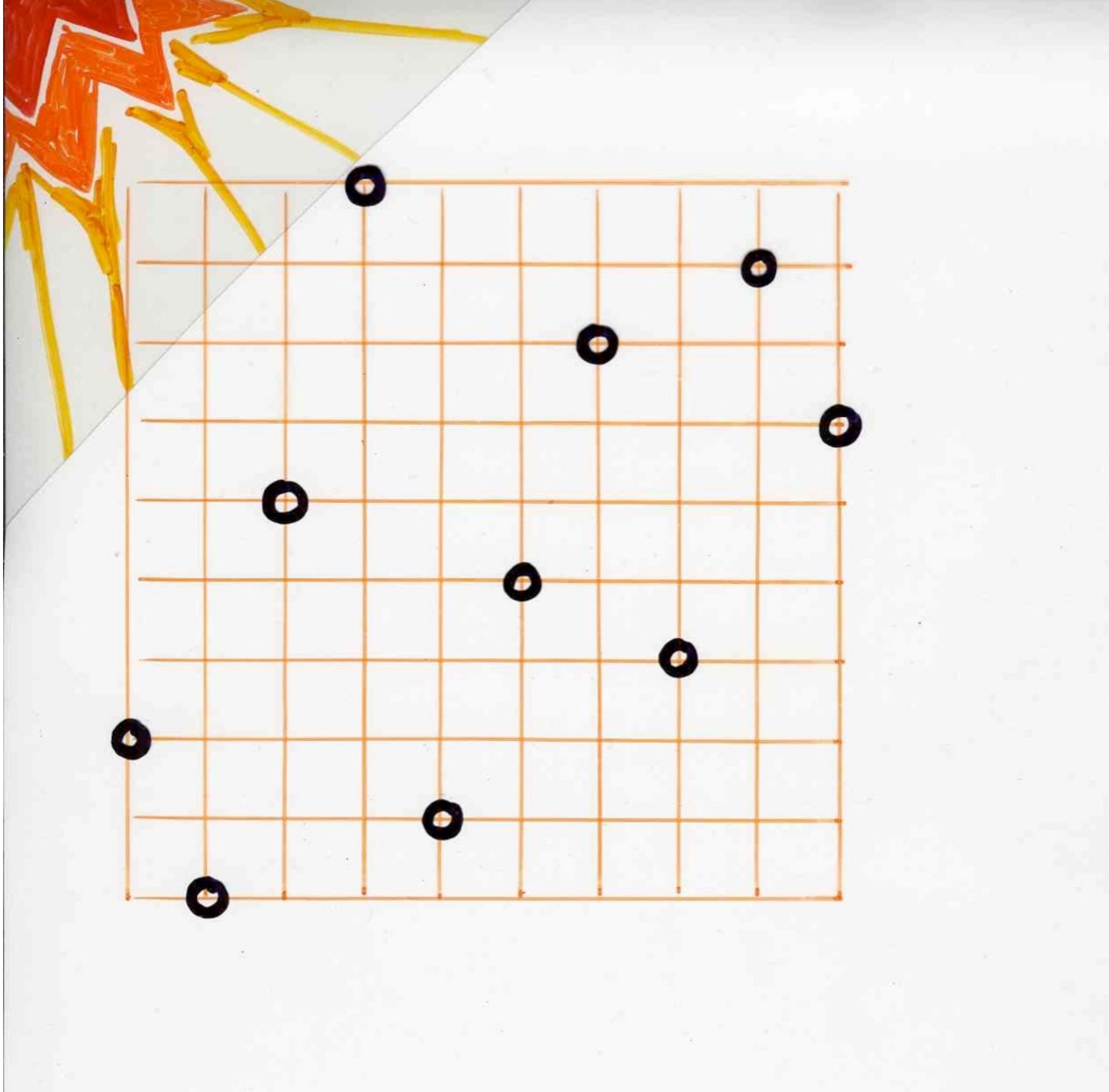
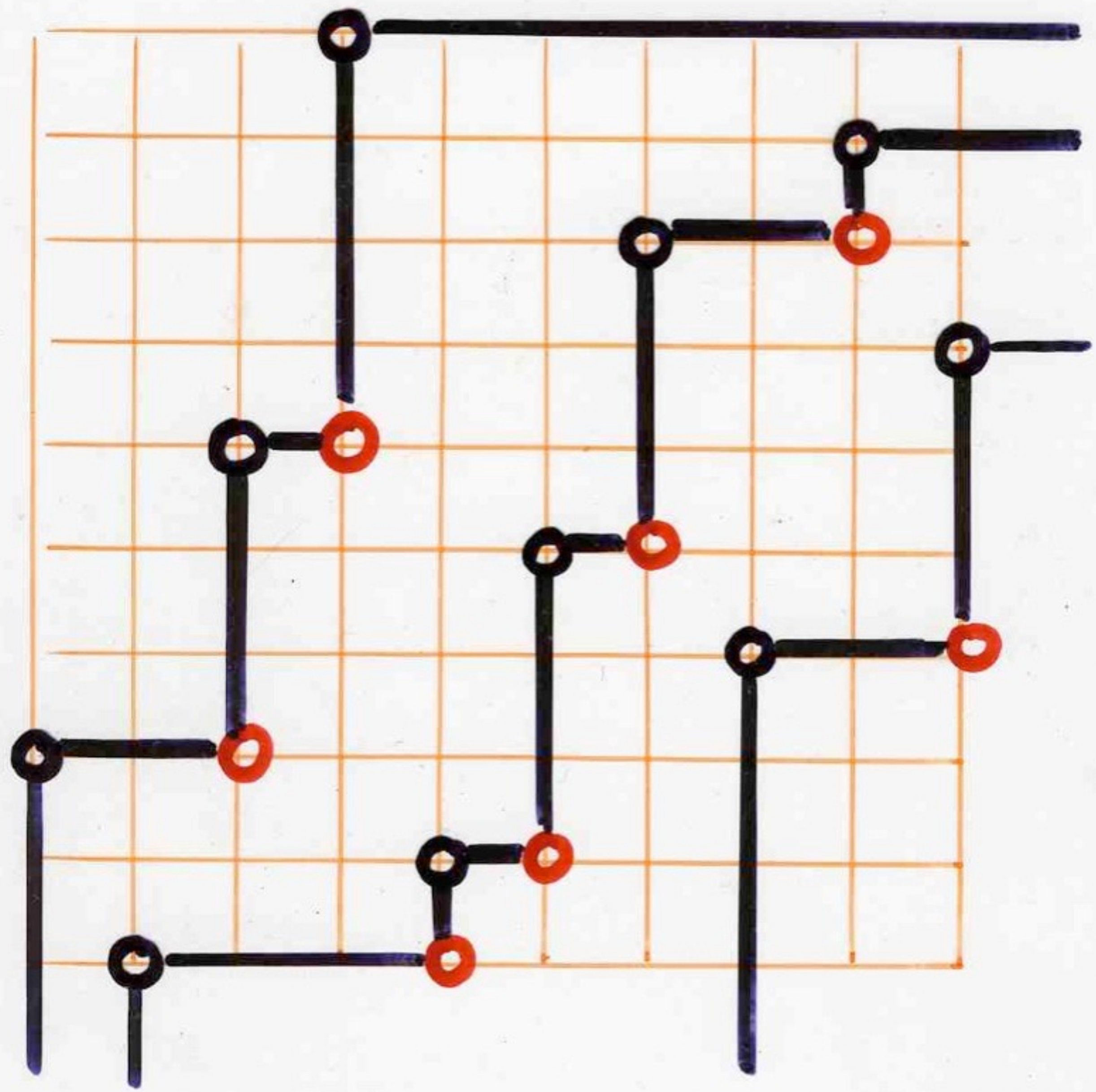
duality

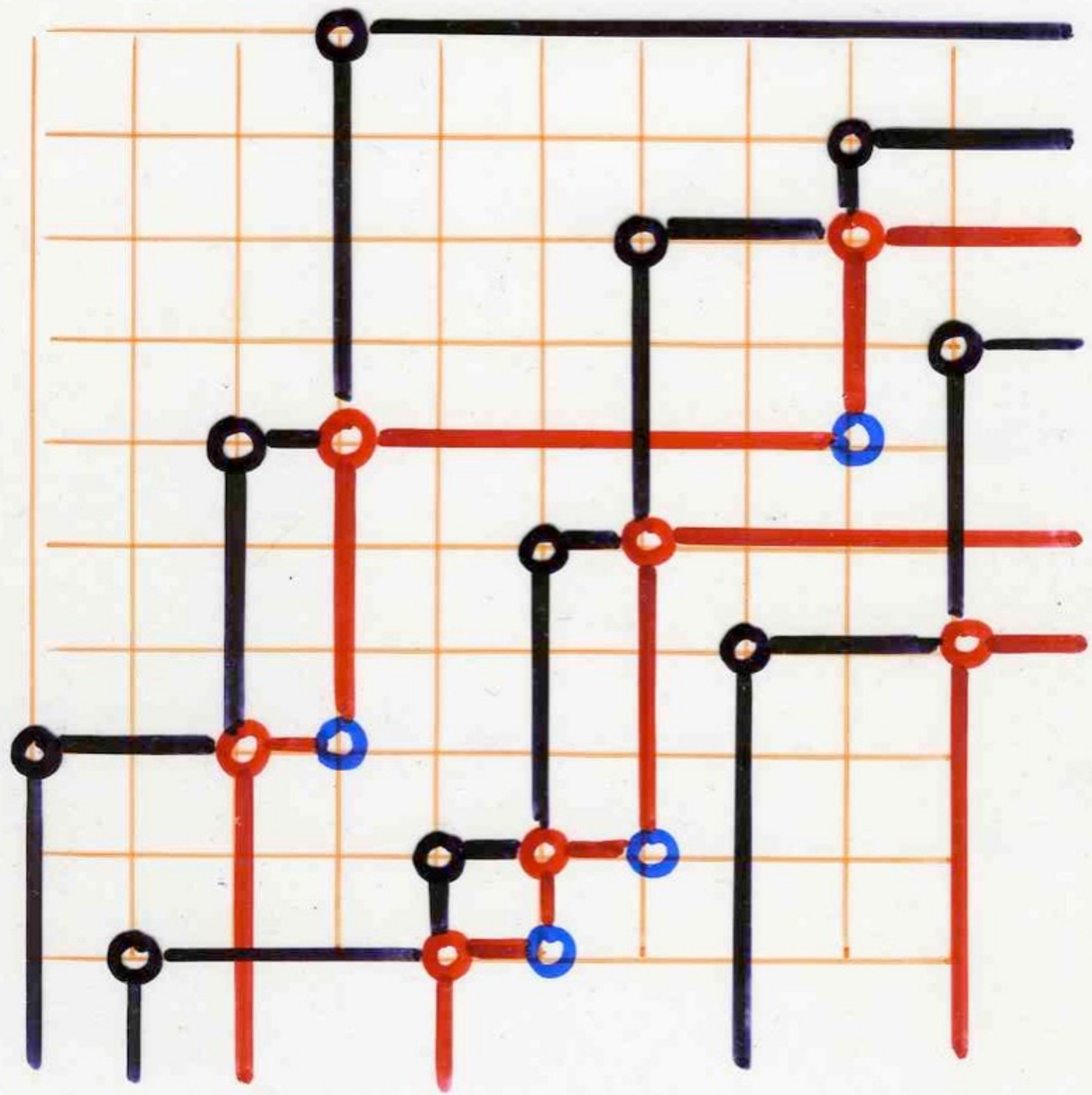
M.P. Schützenberger, 1963, 1972

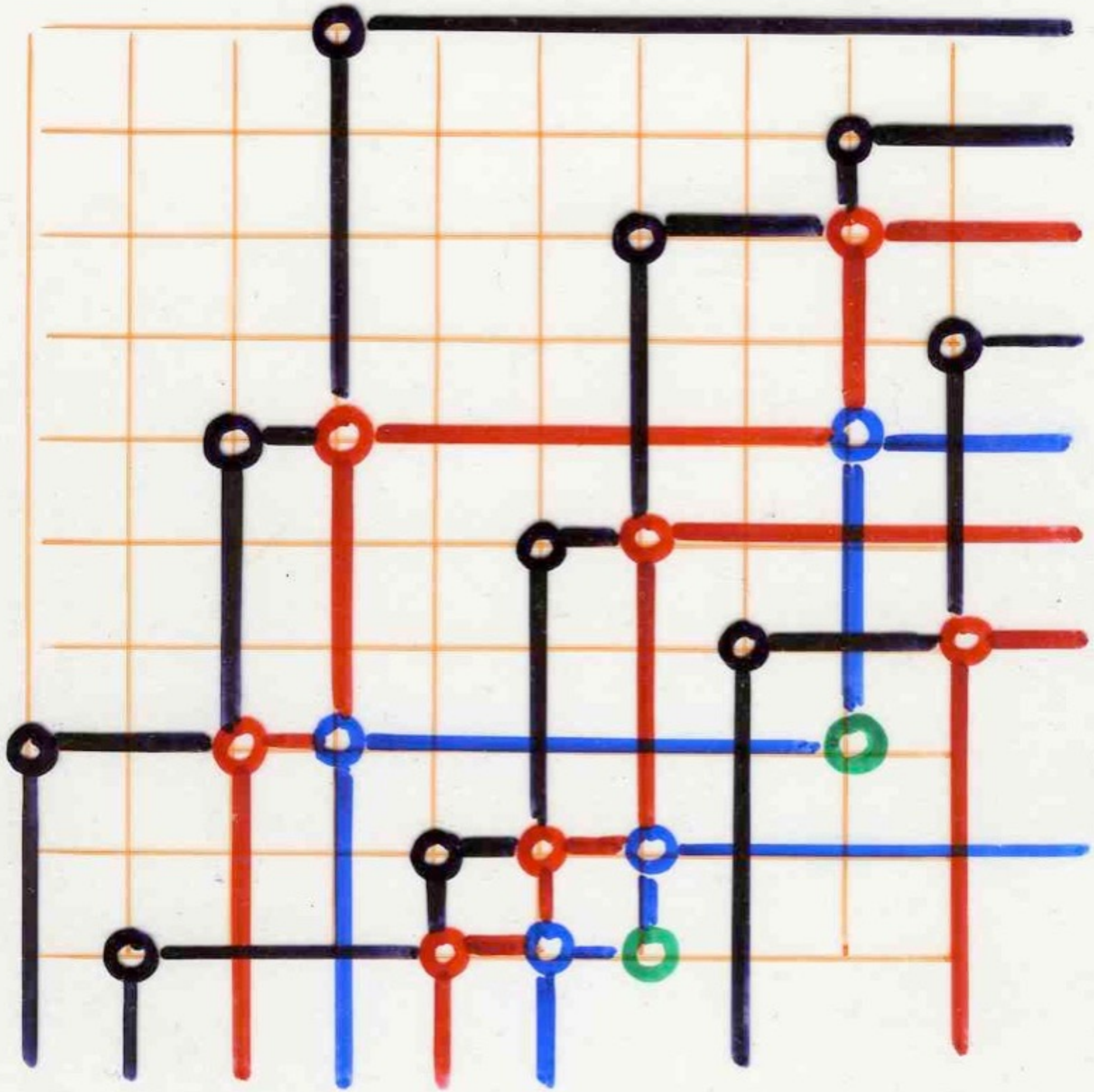
(without proof)

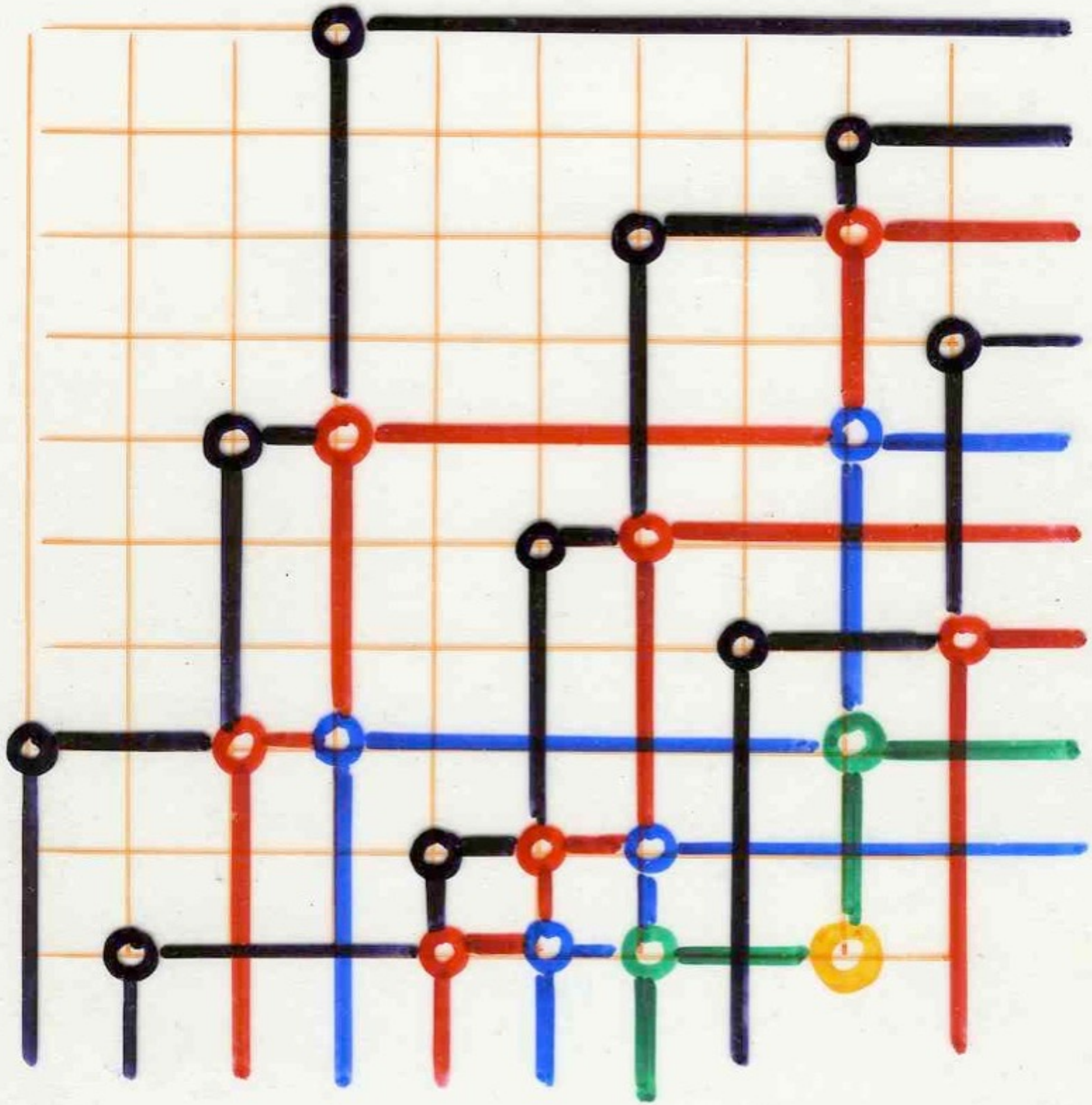


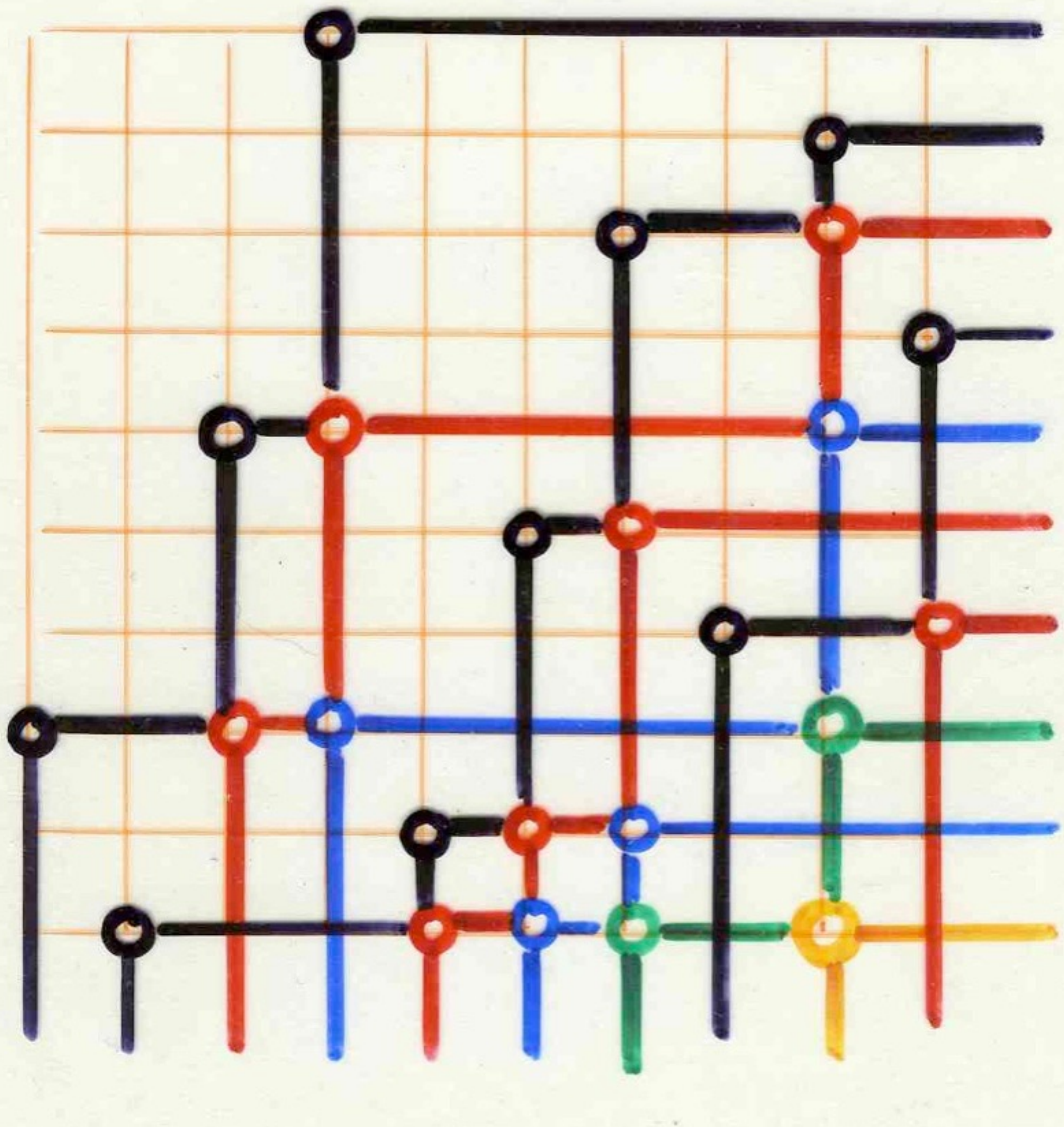


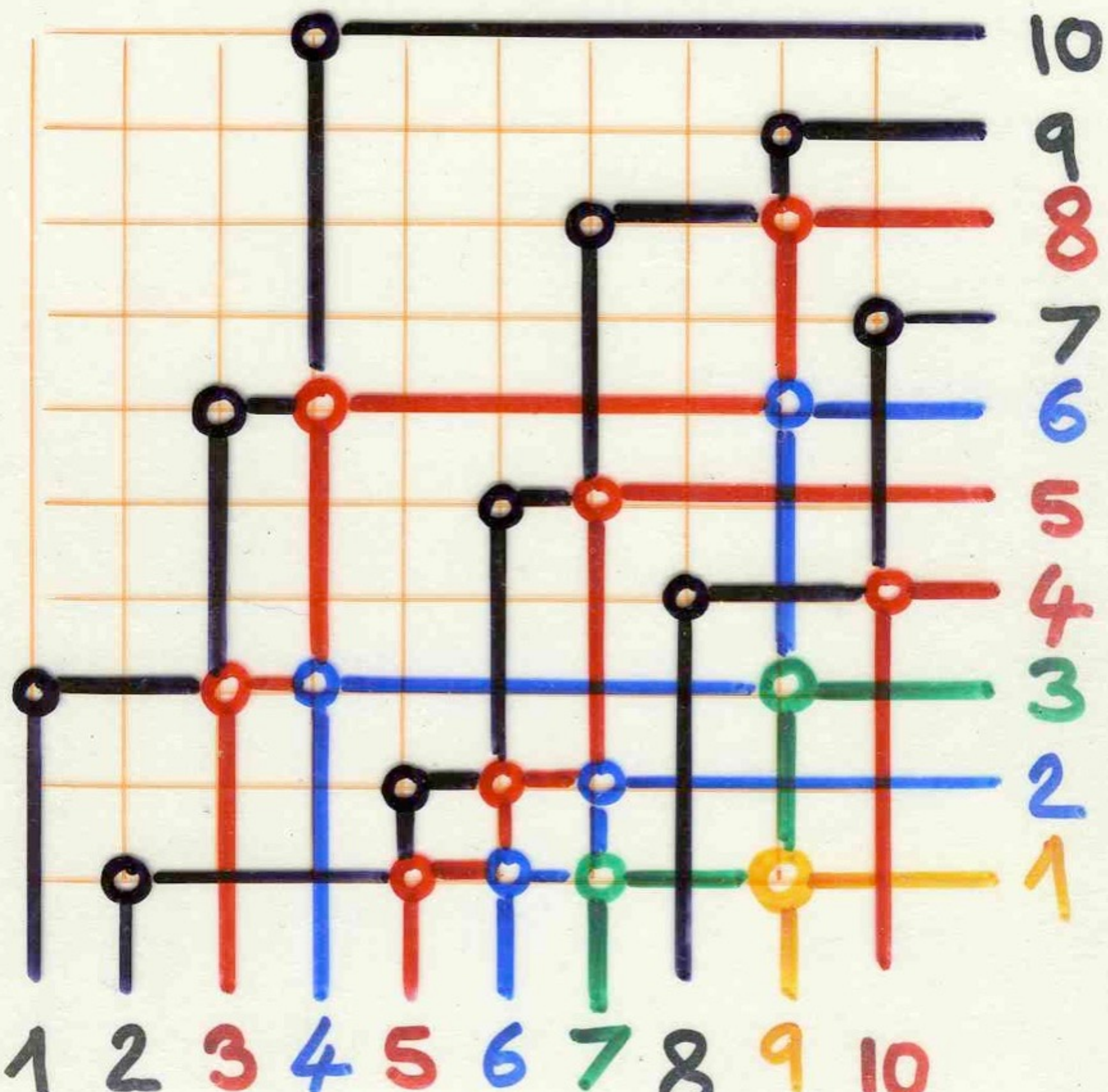








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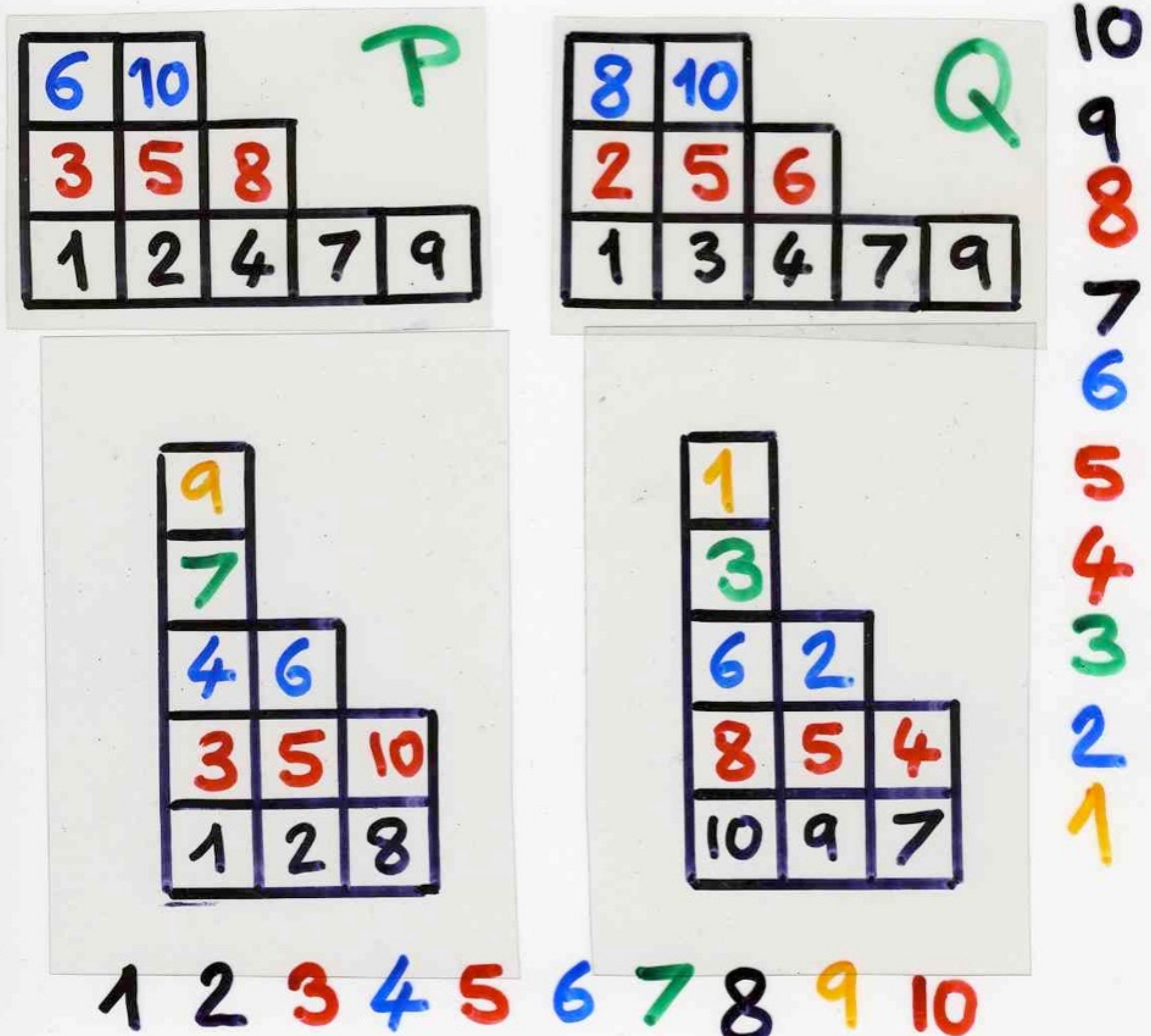

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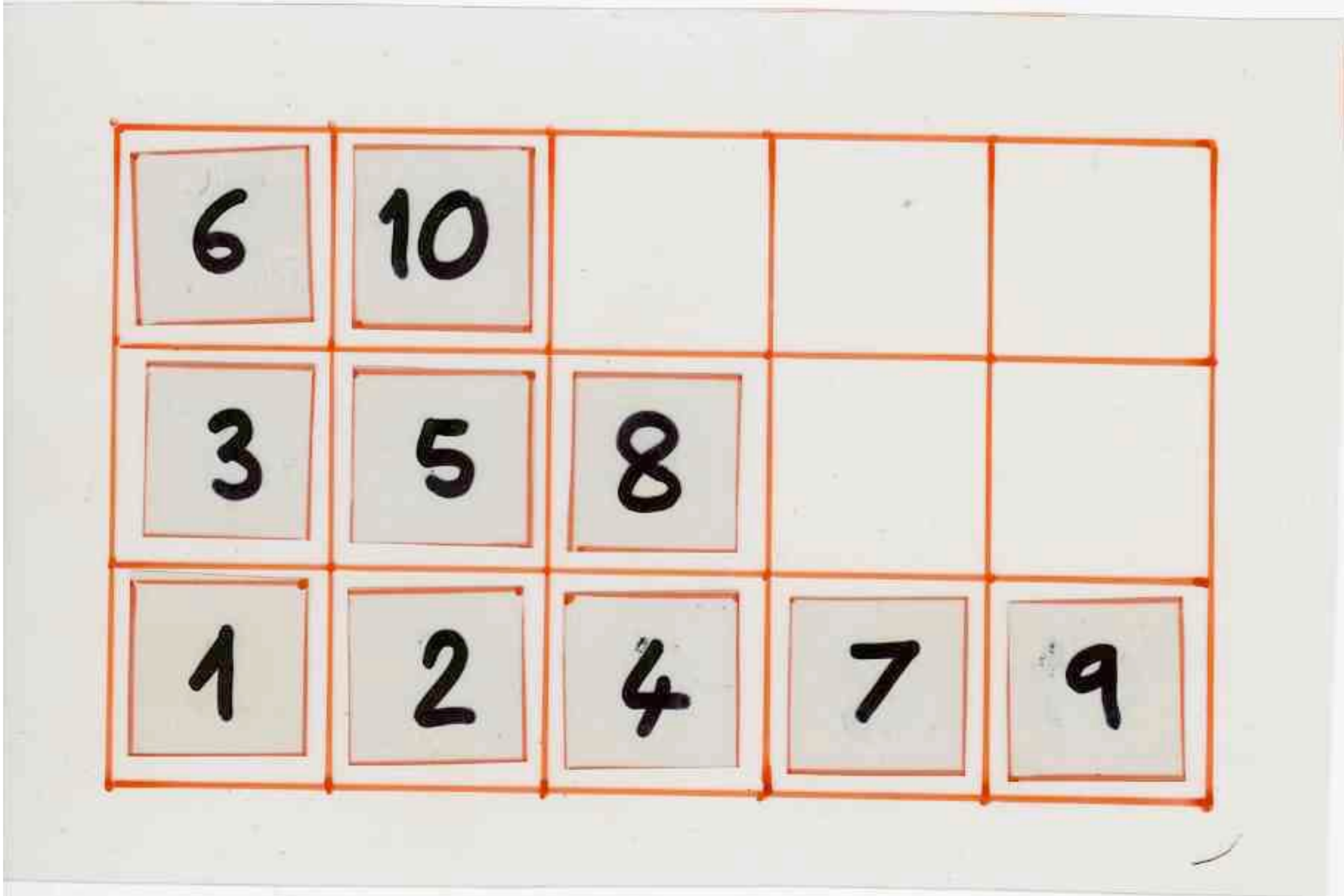
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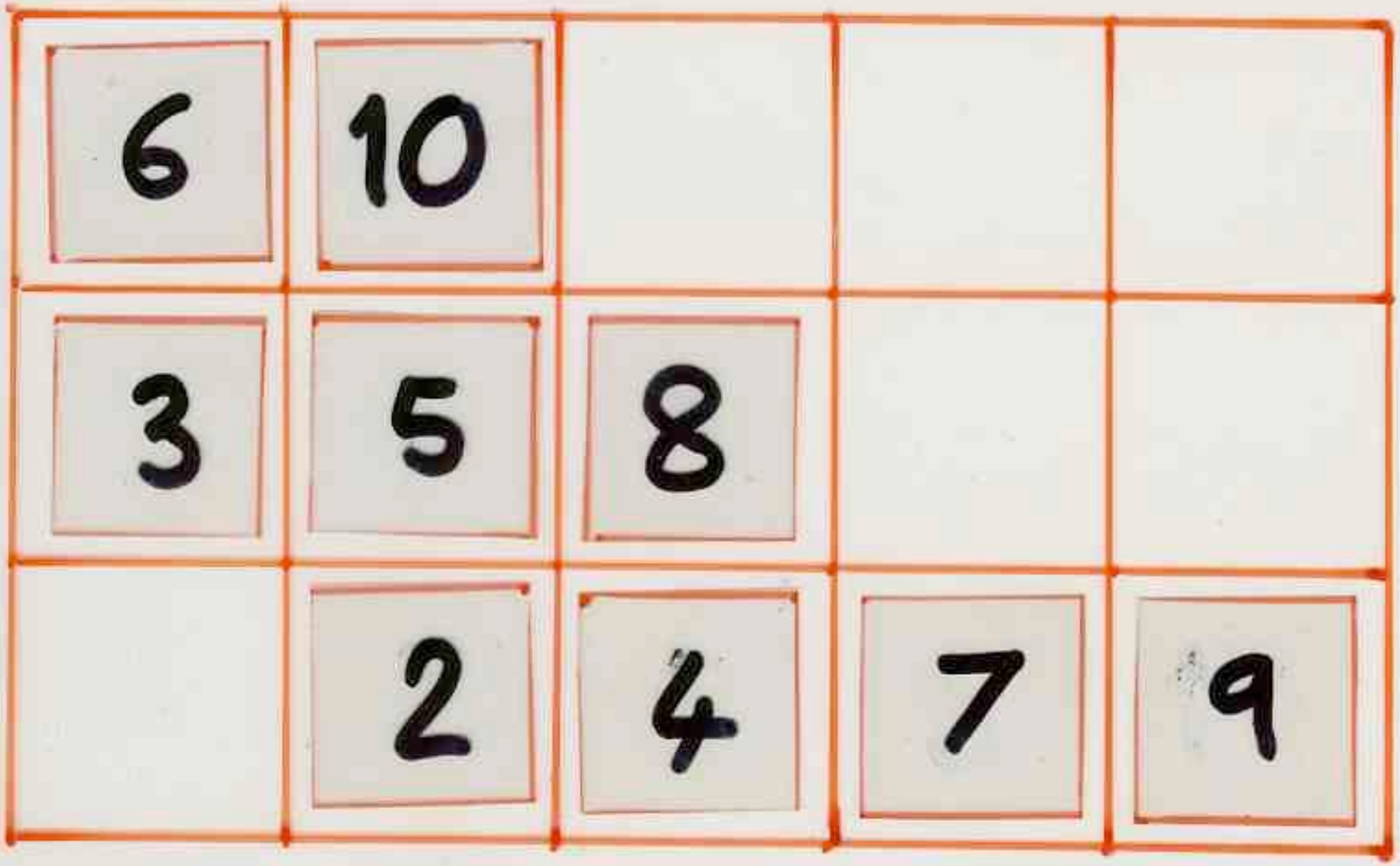
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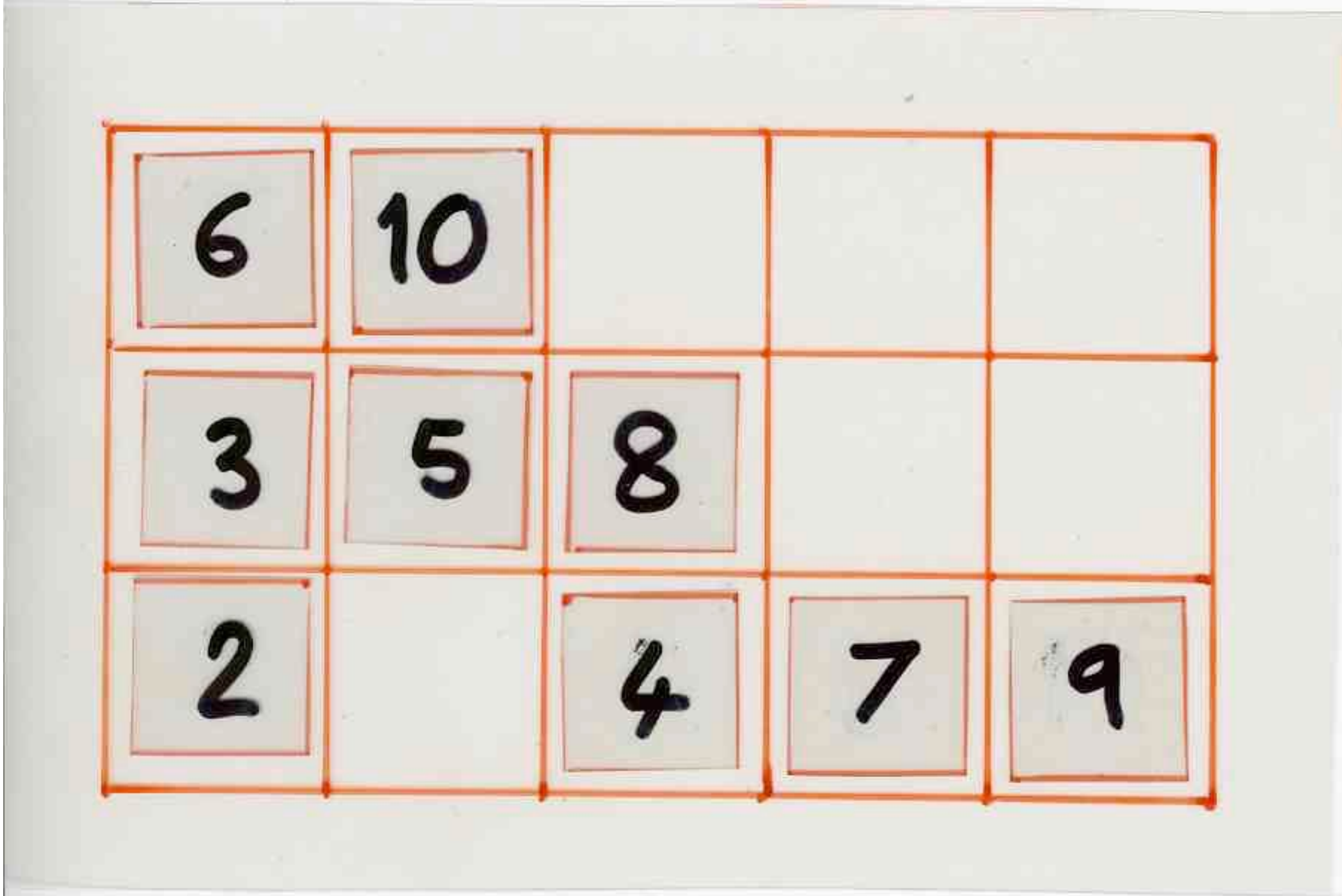
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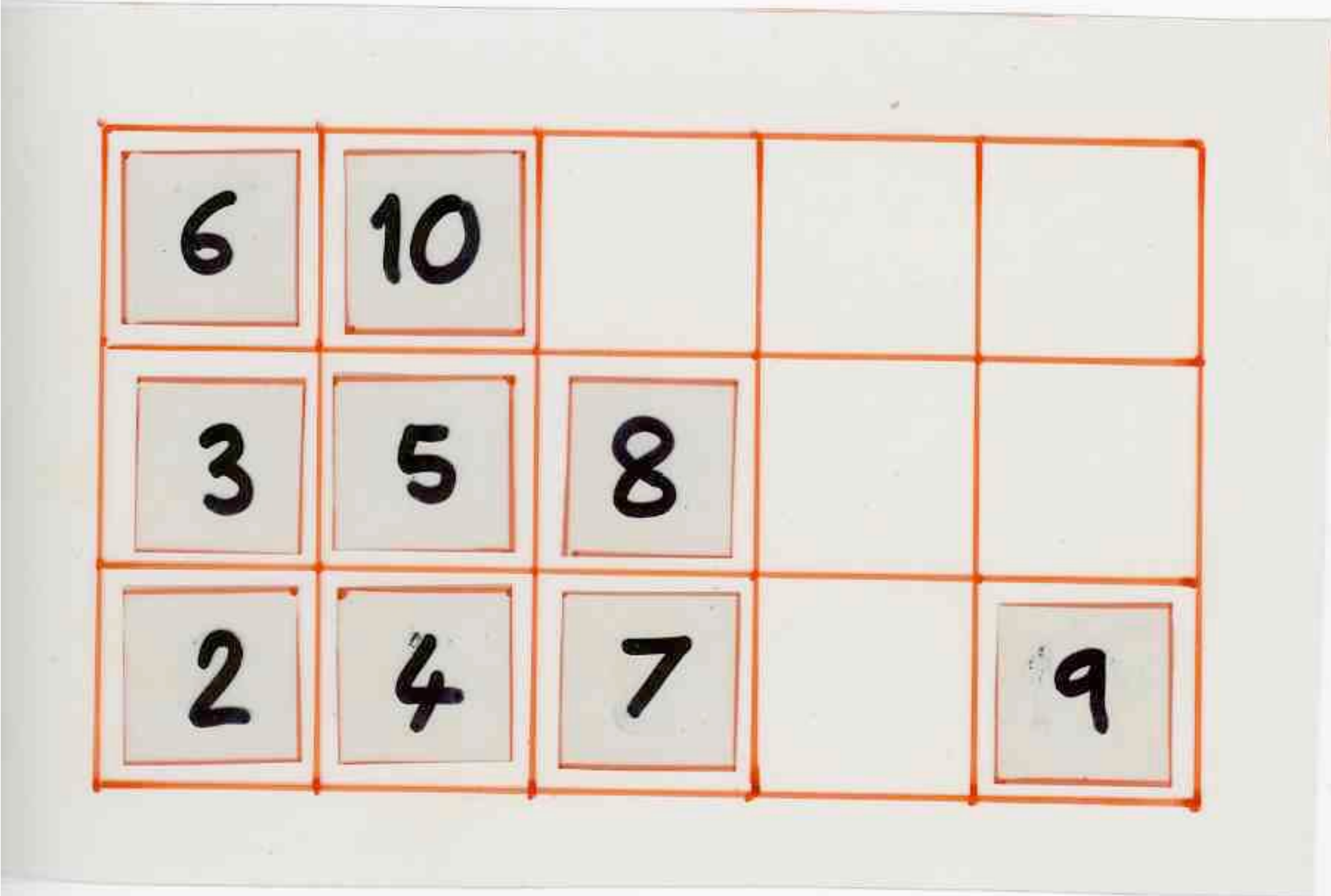
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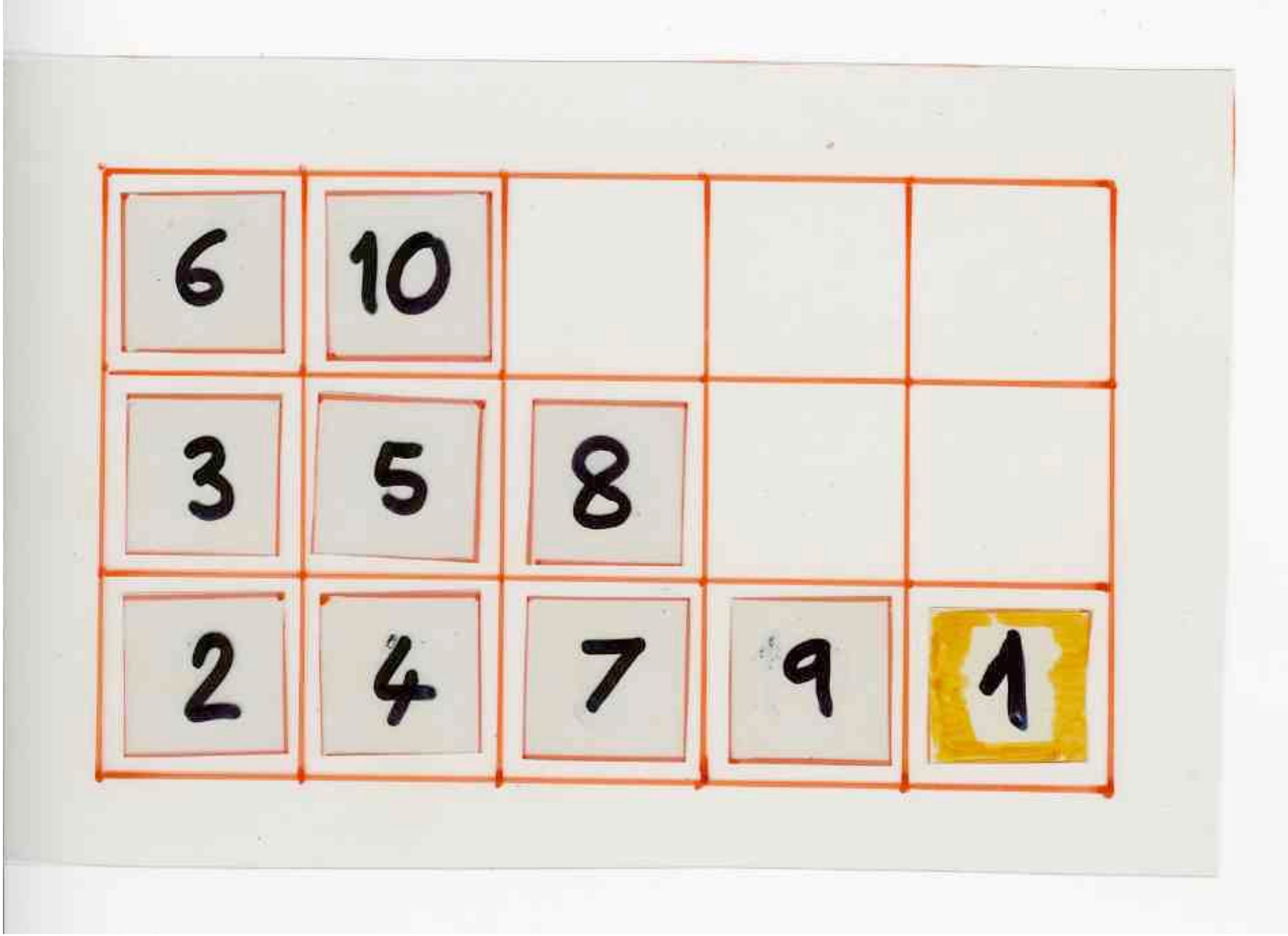
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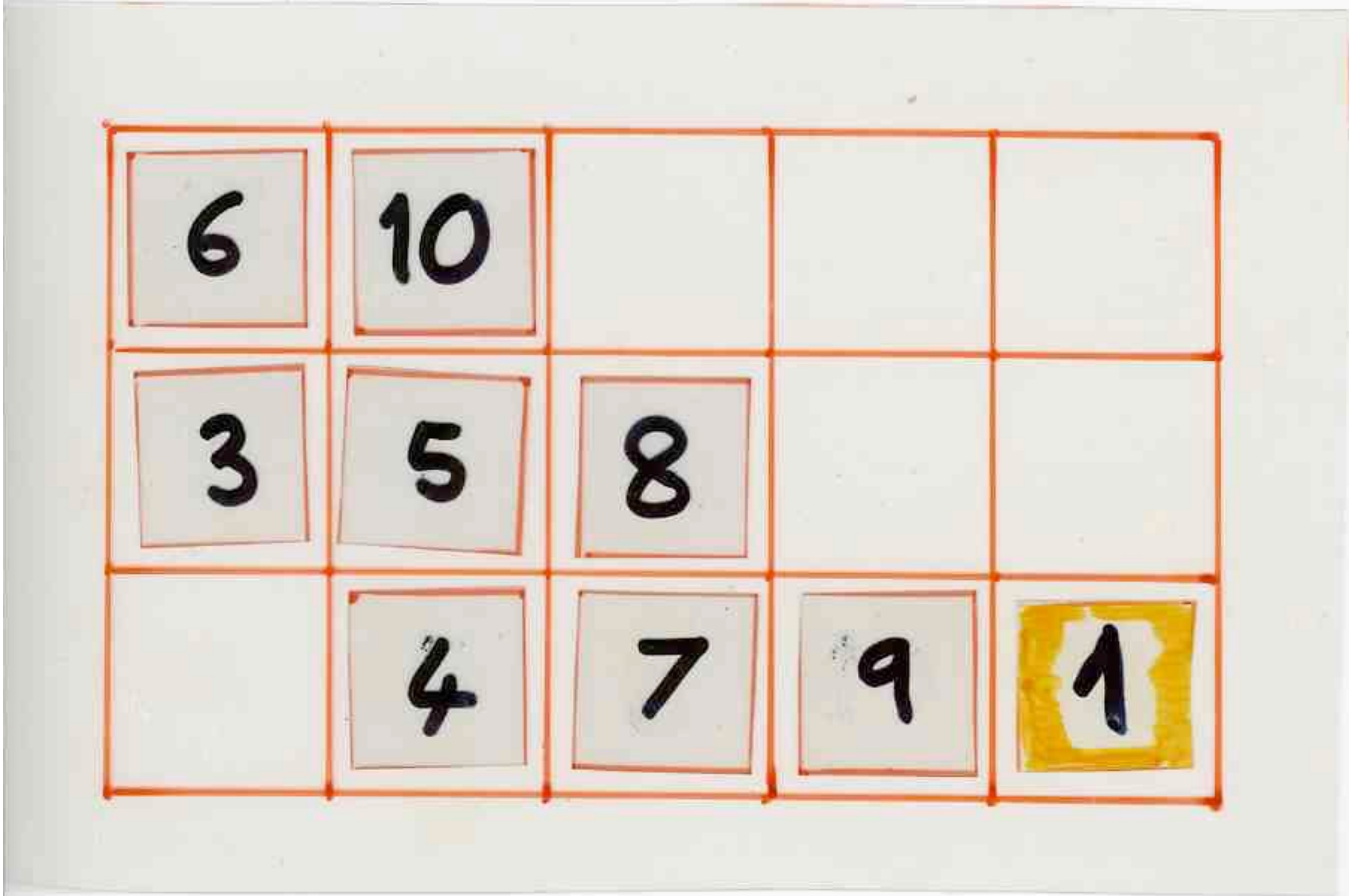
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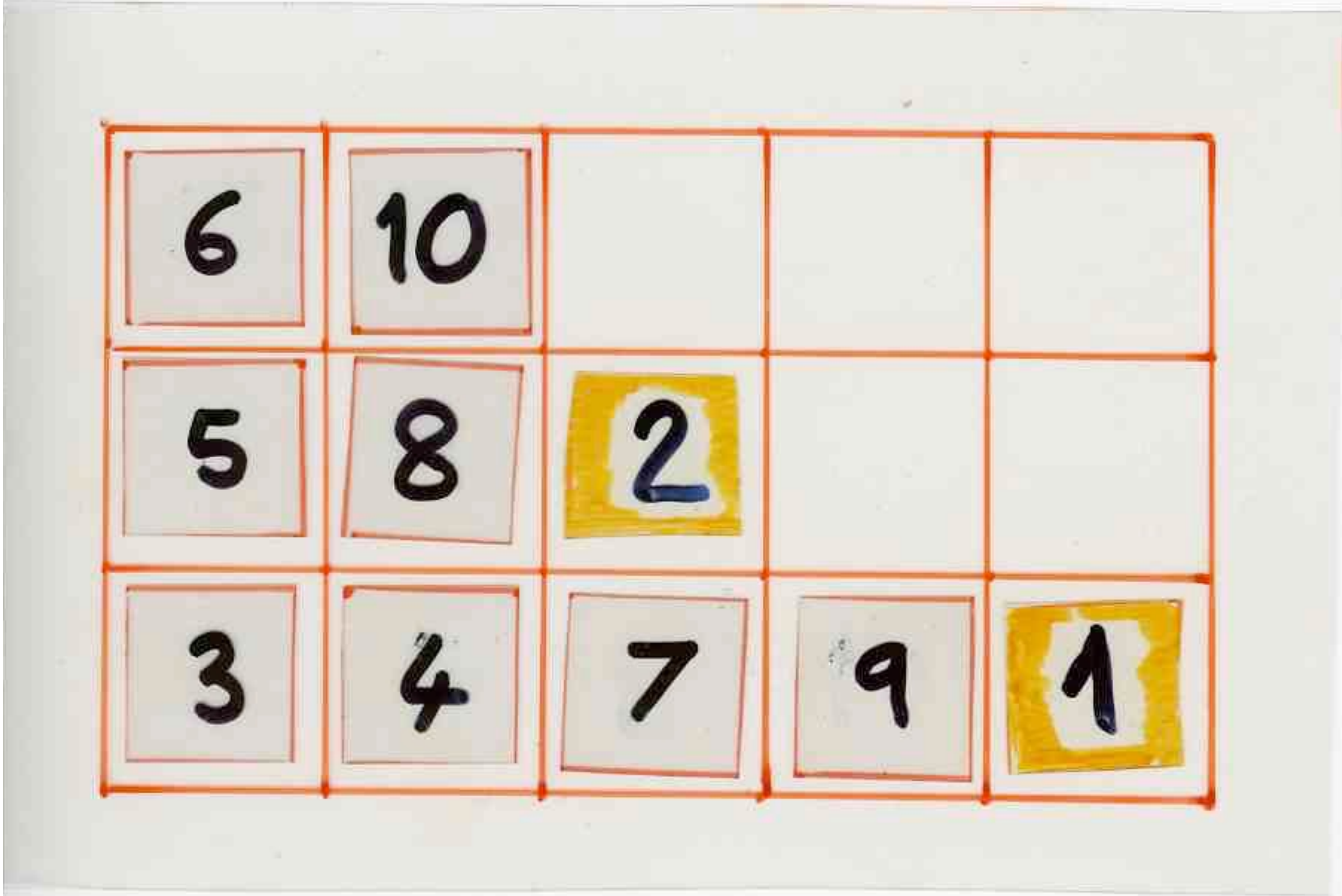
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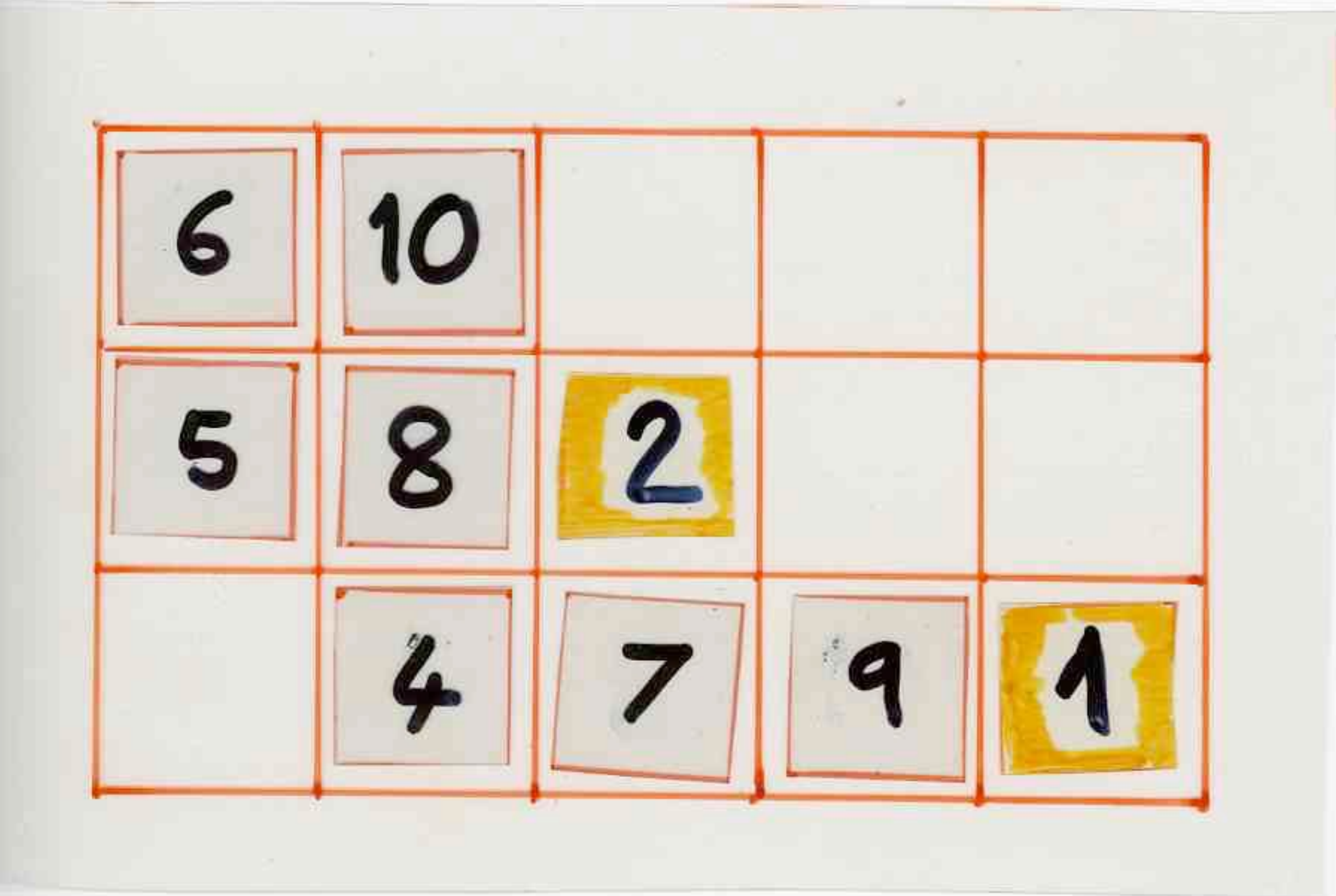
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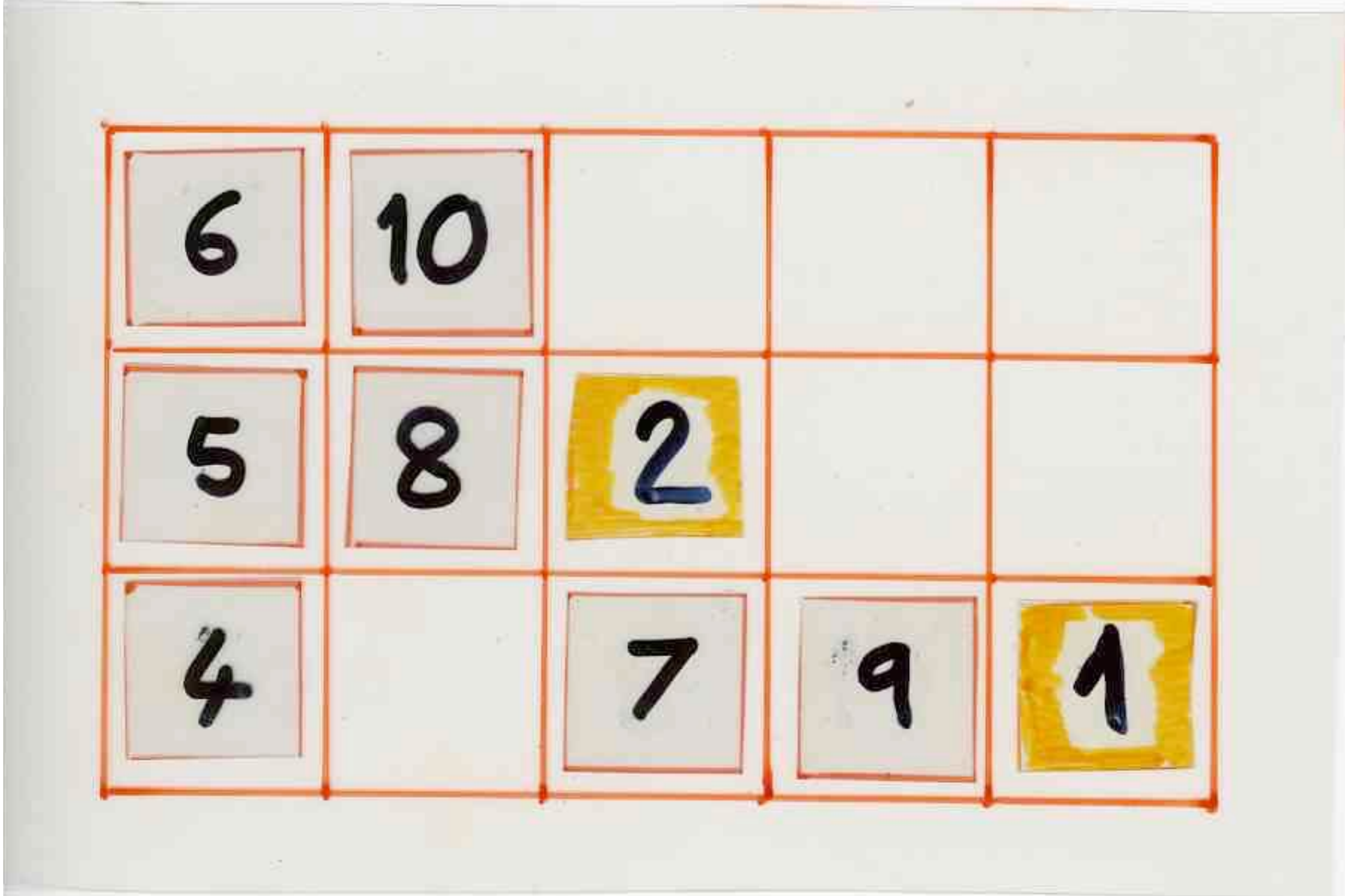
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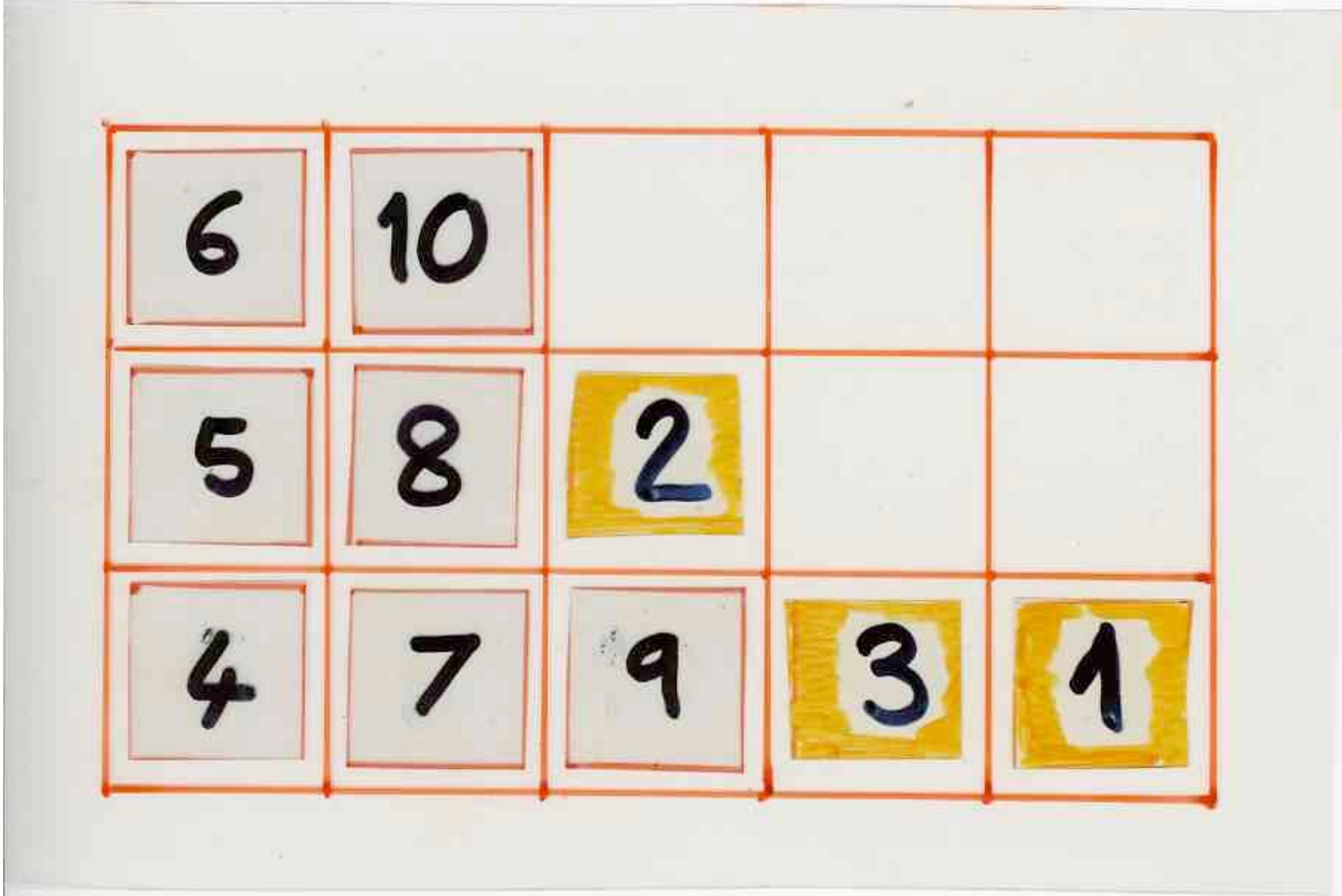
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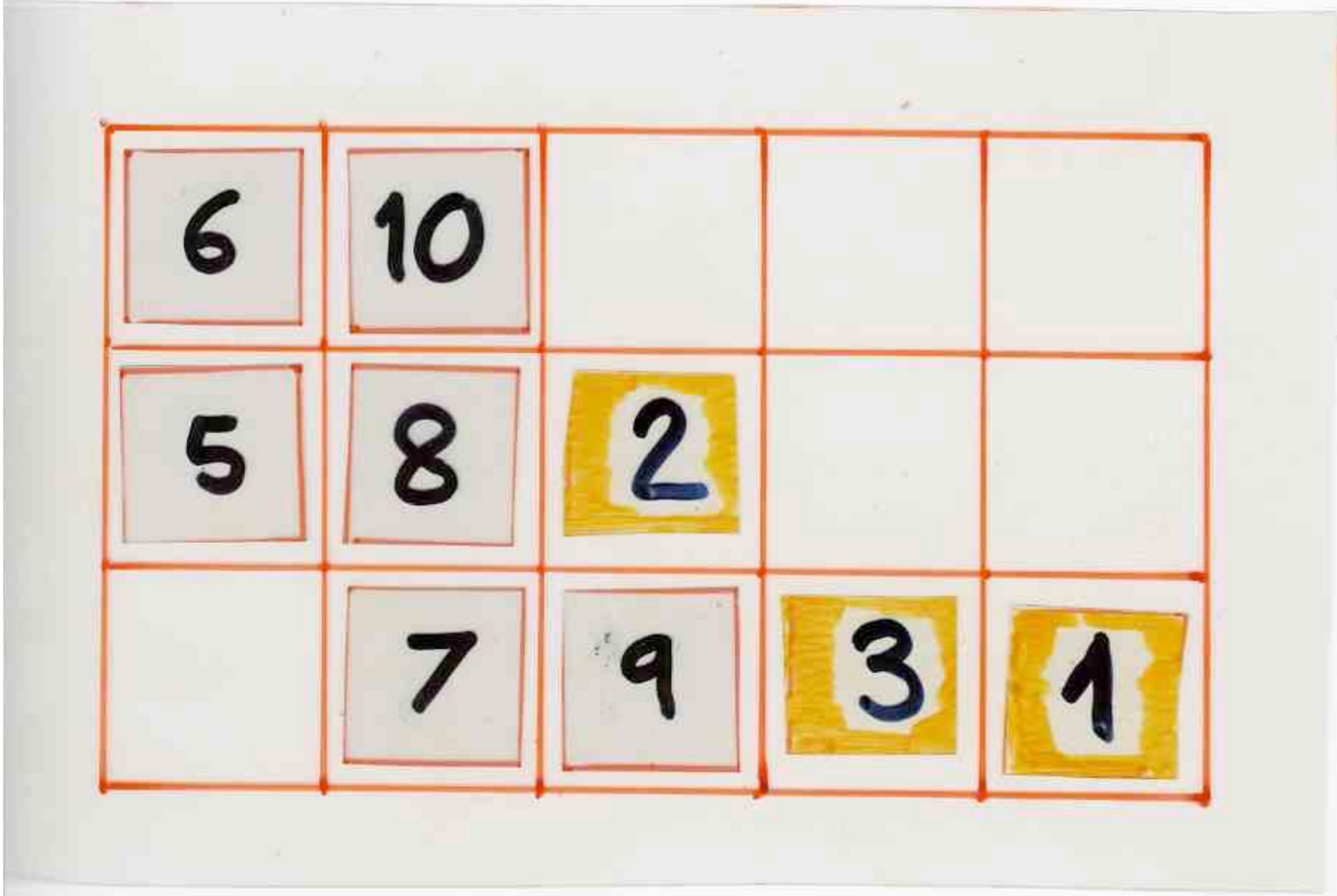
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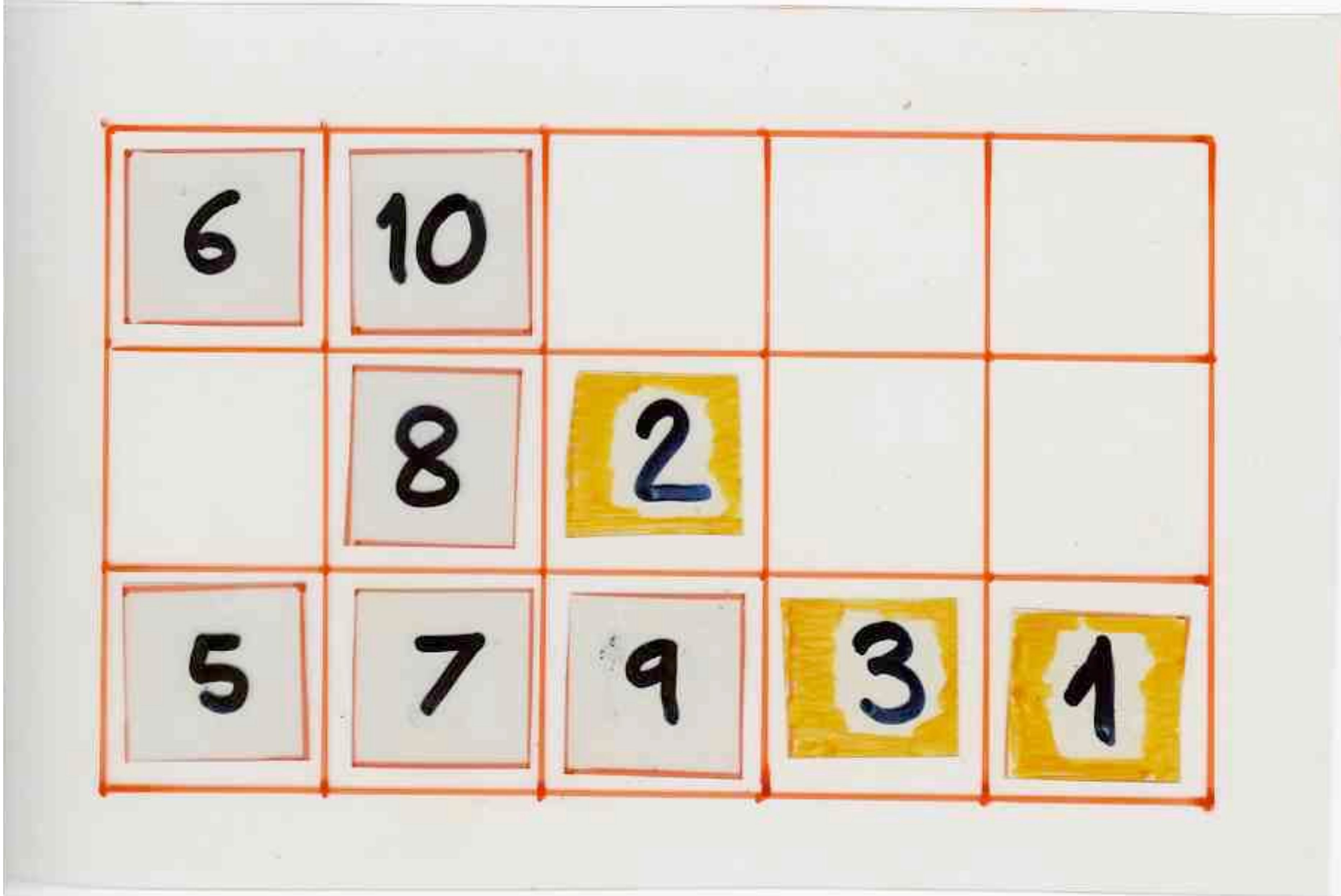
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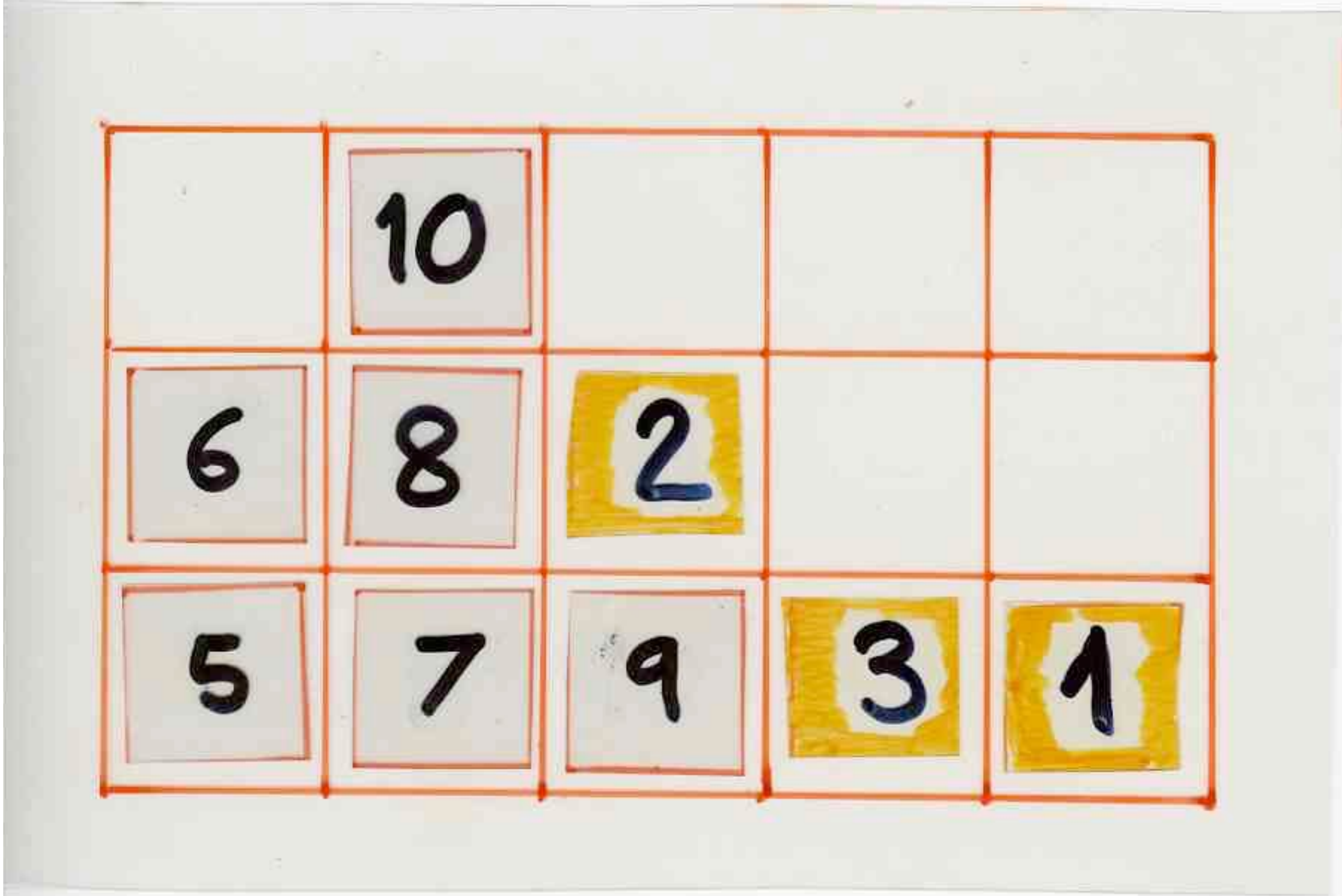
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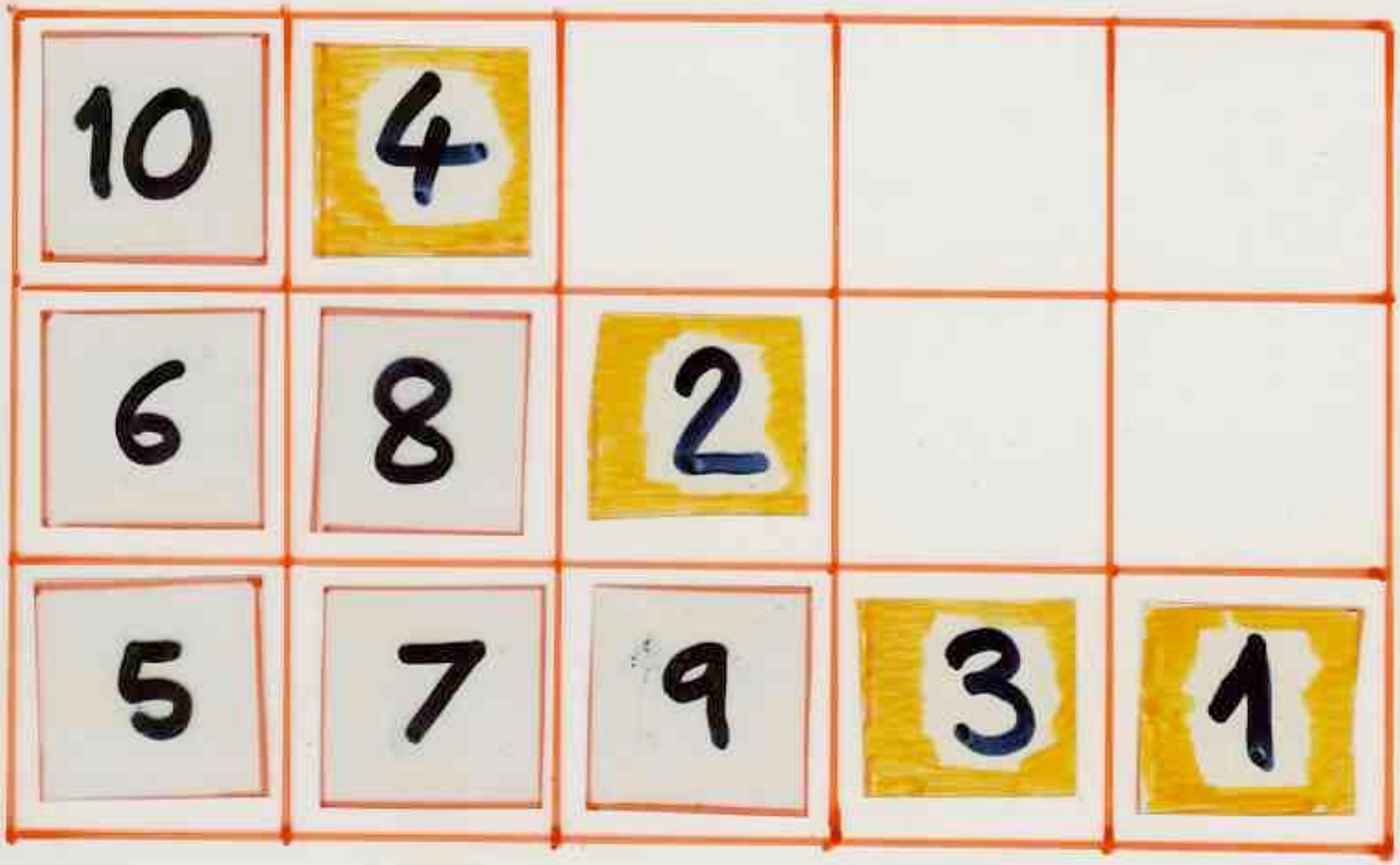
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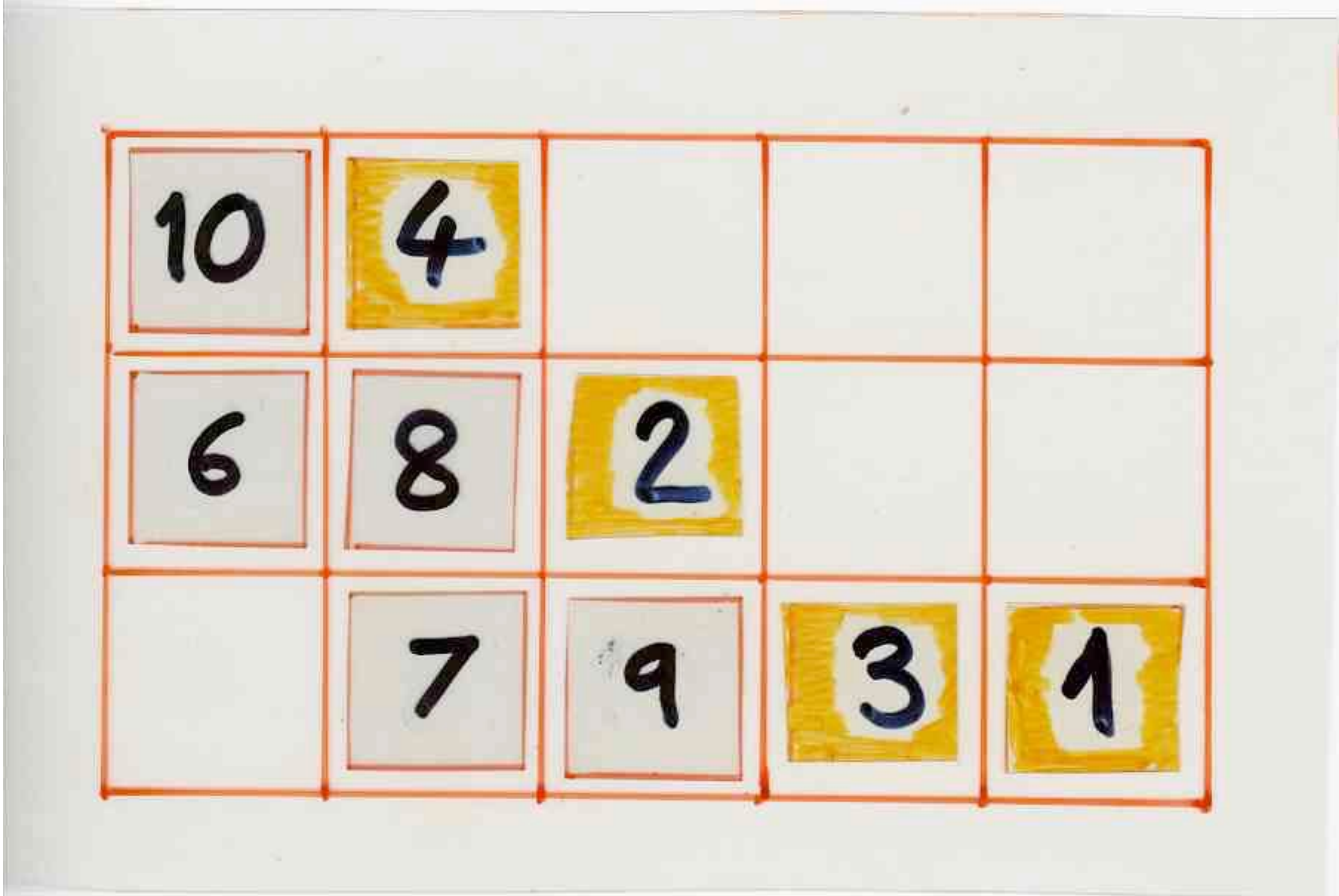
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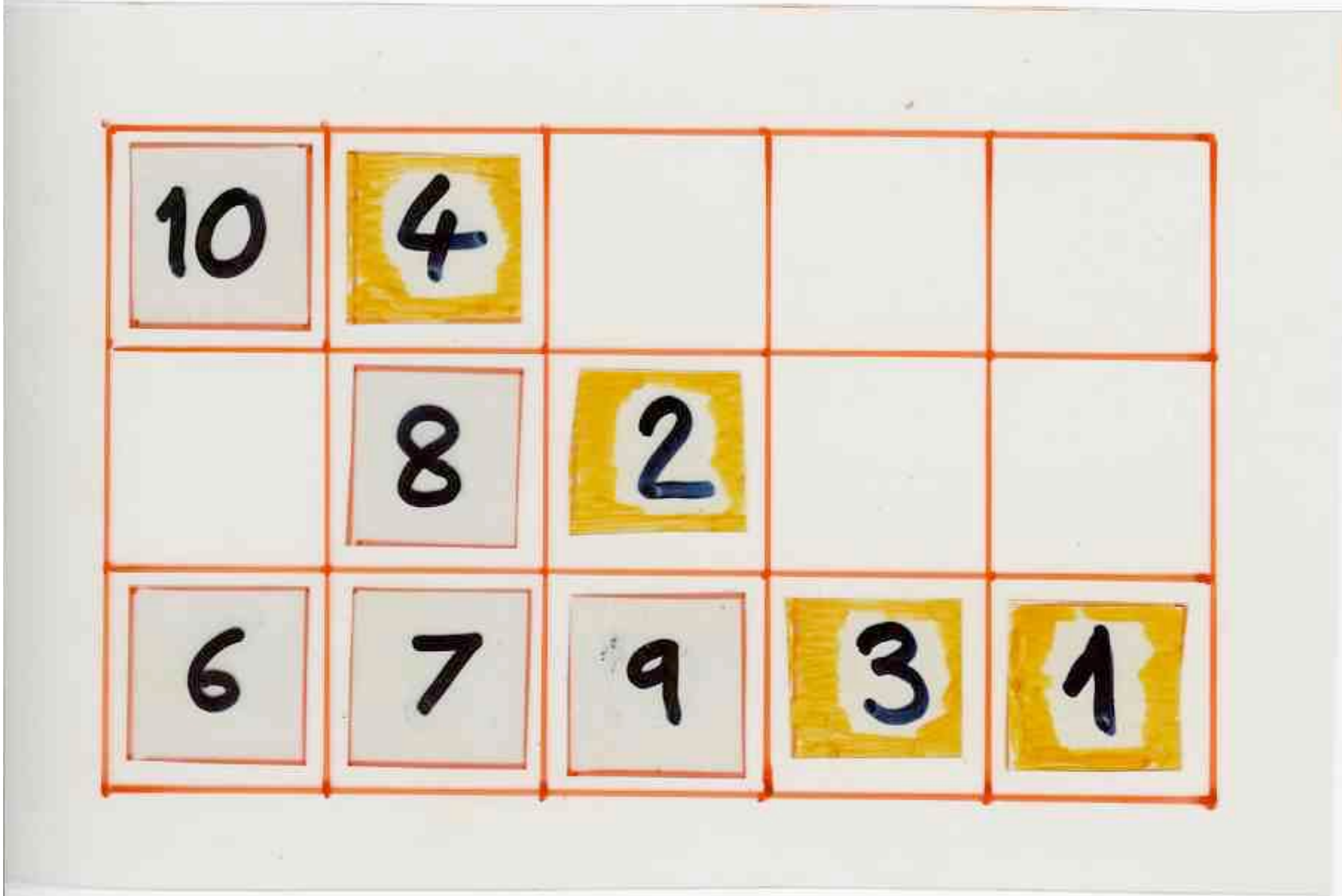
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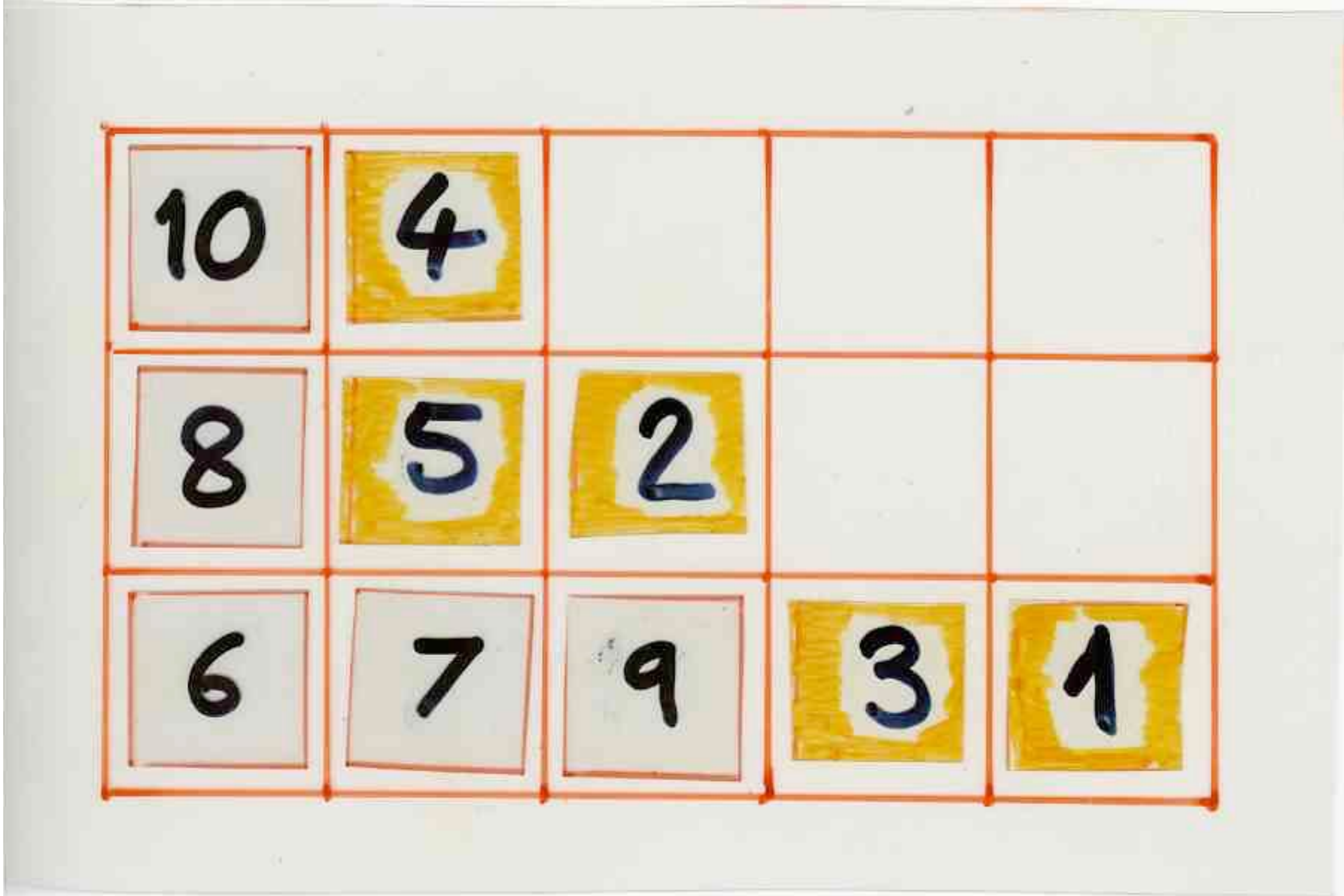
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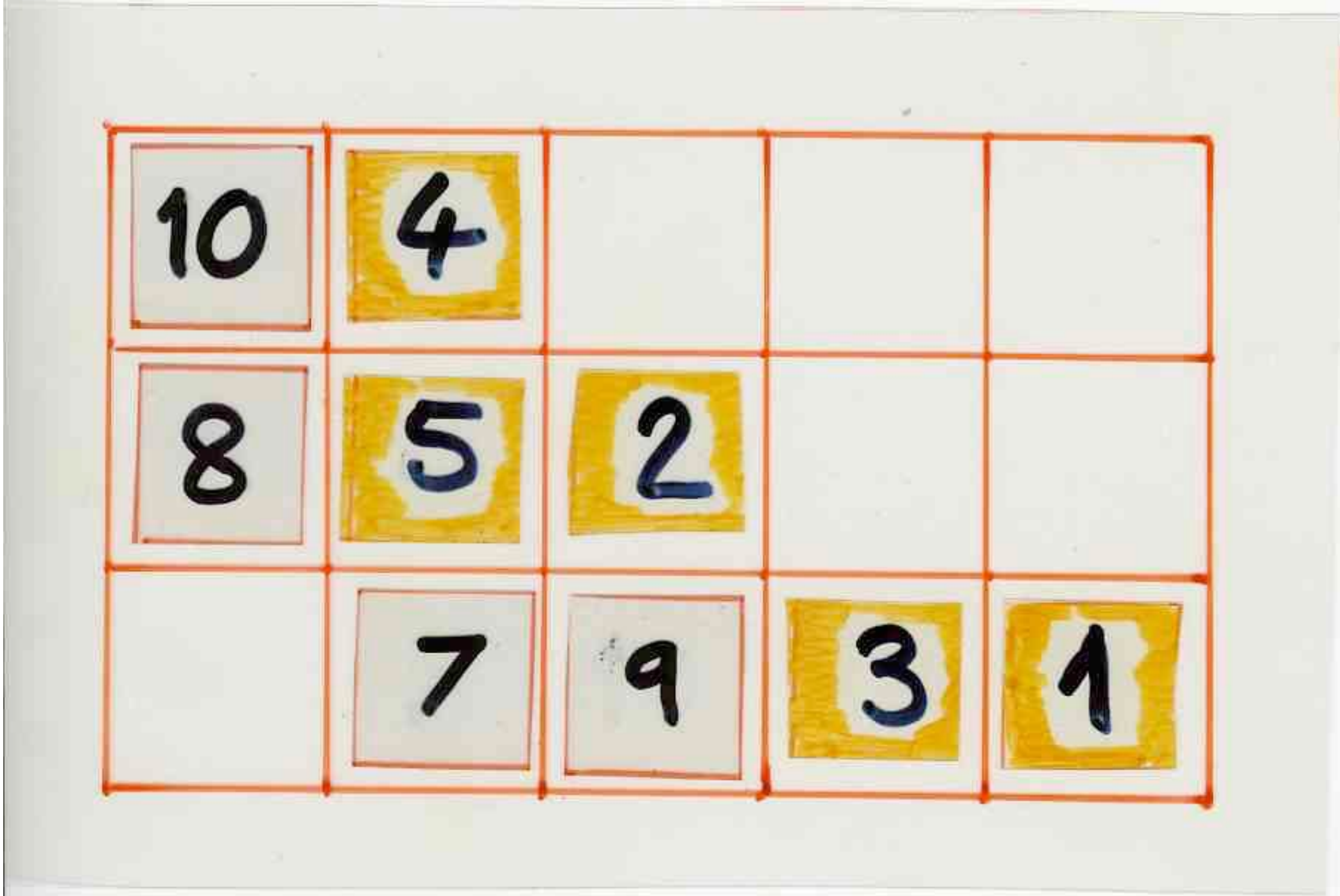
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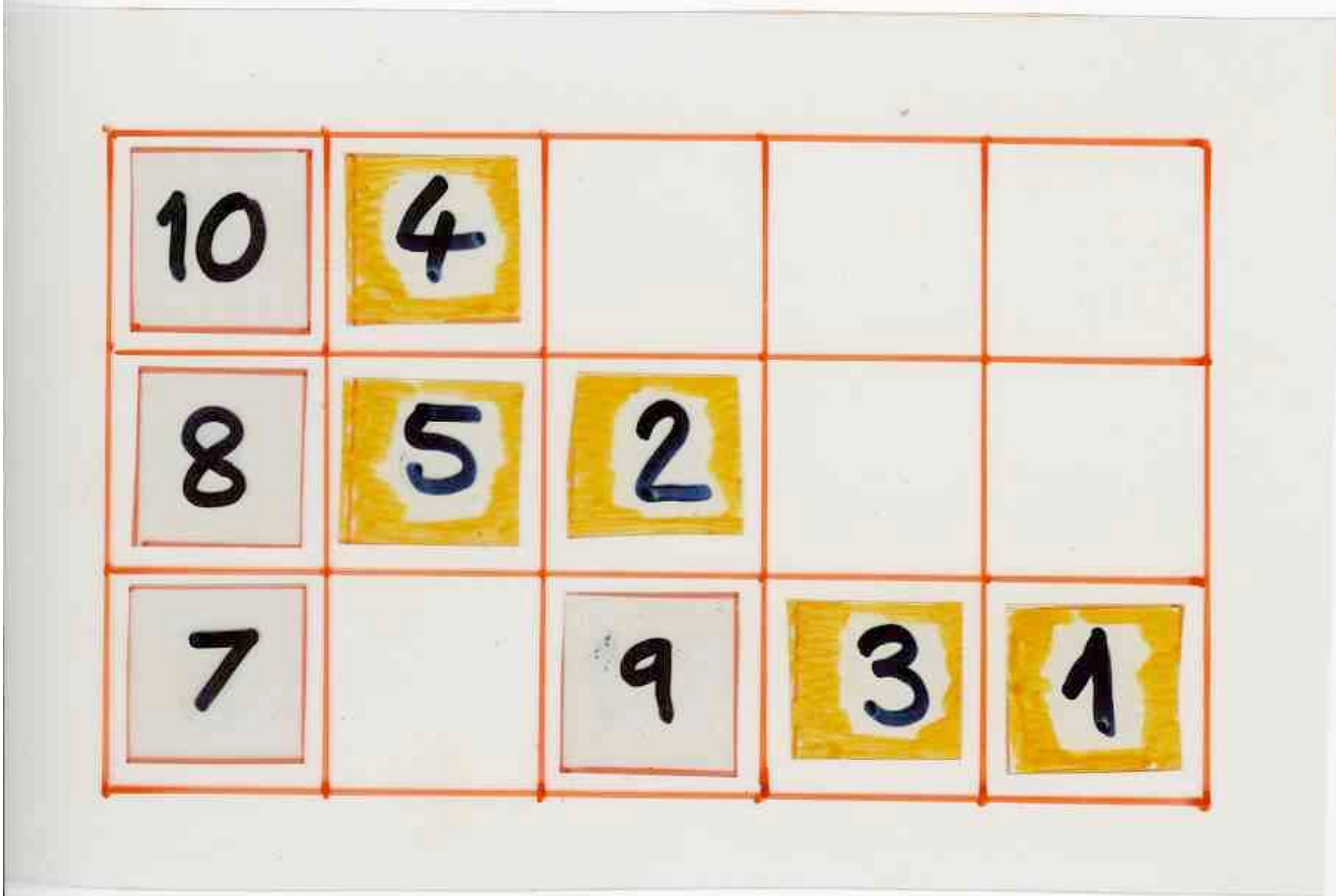
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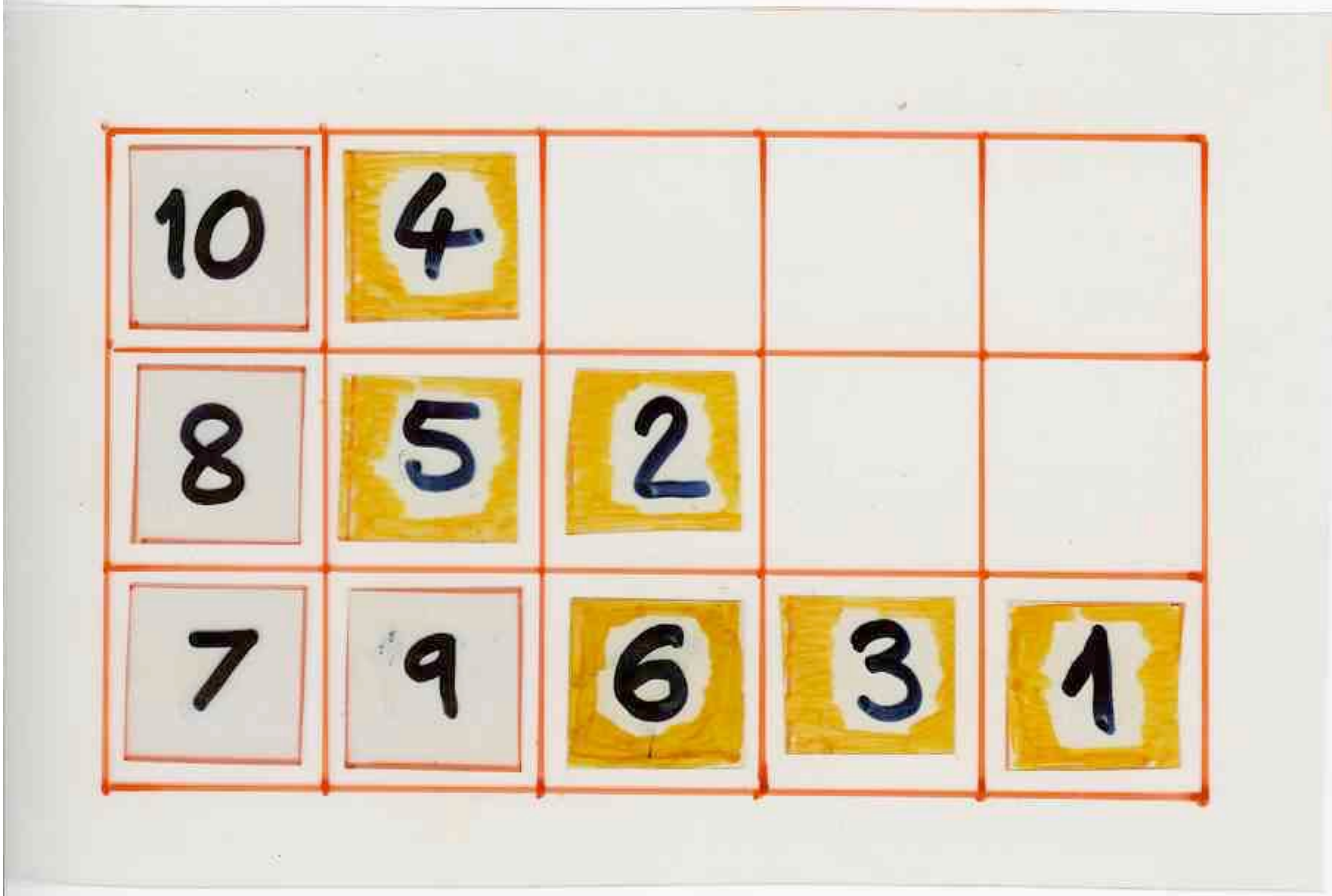
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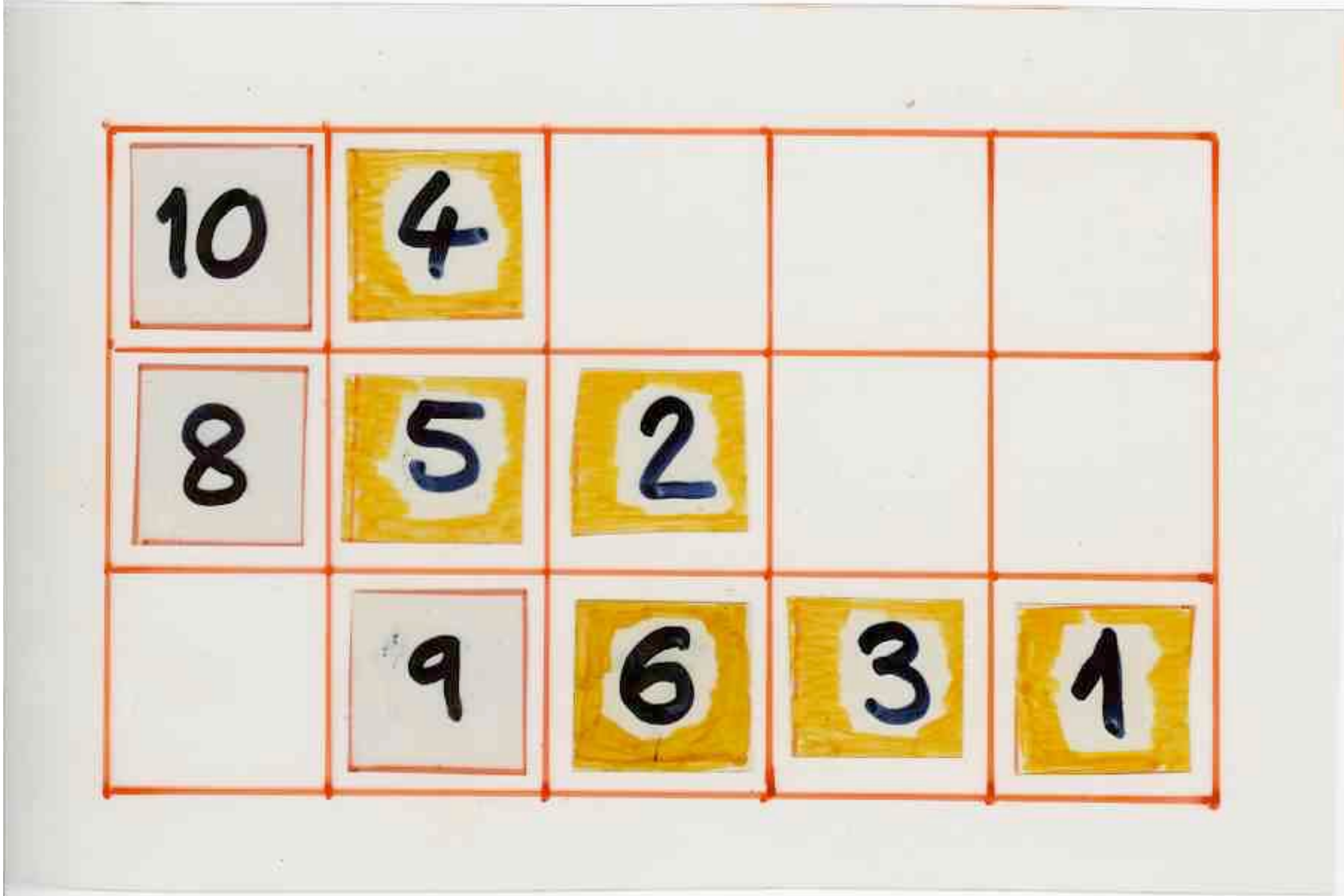
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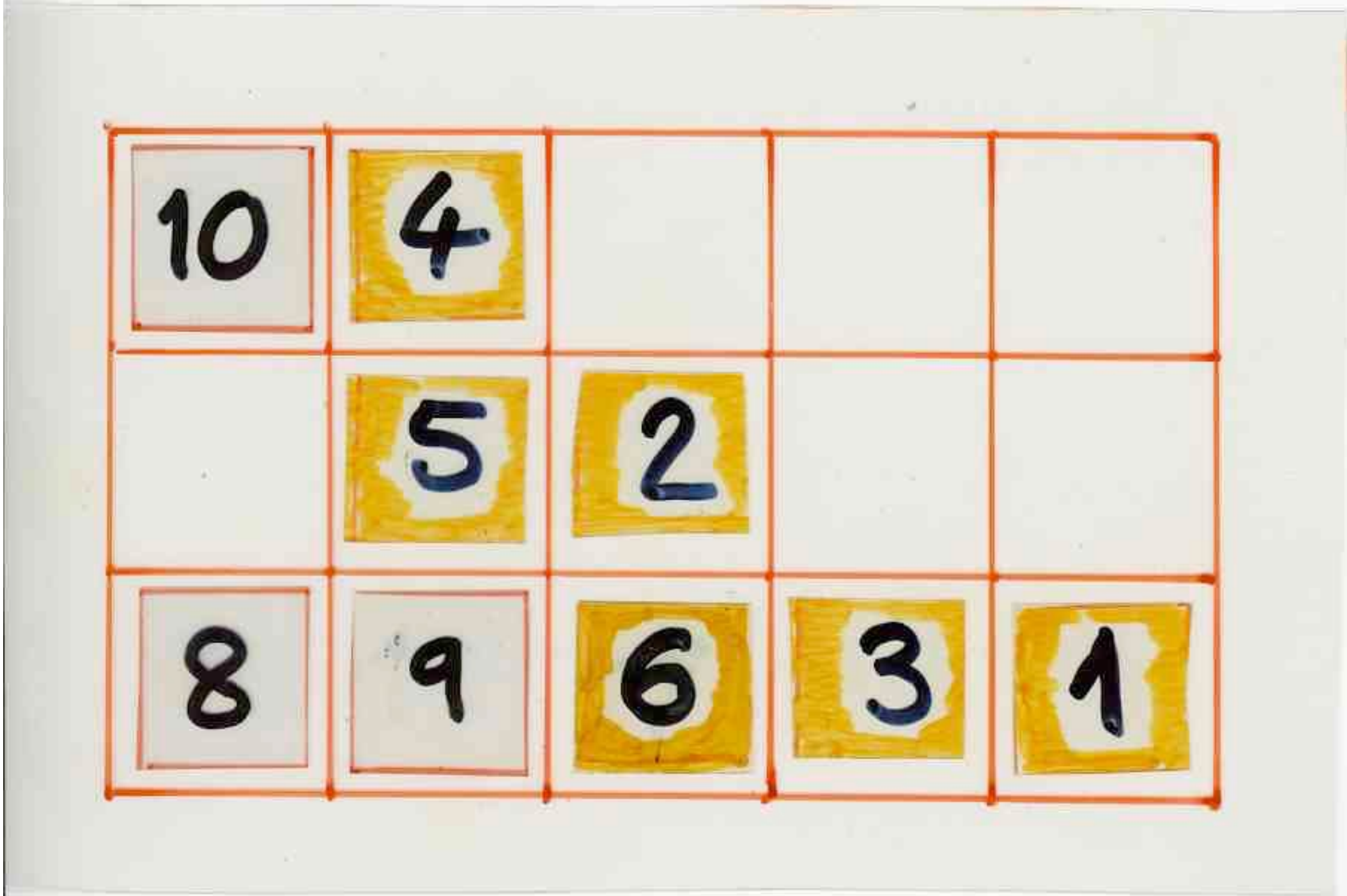
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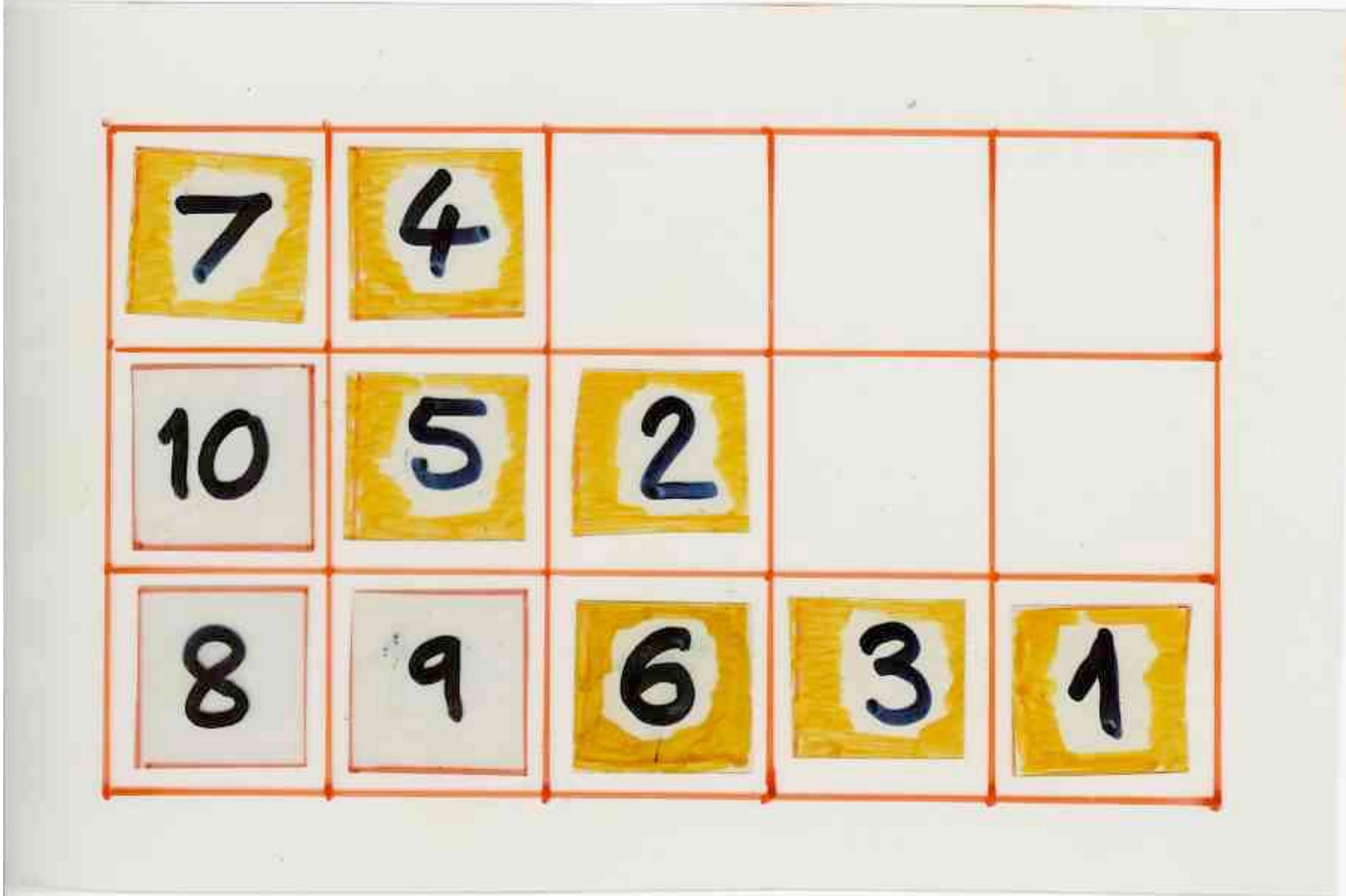
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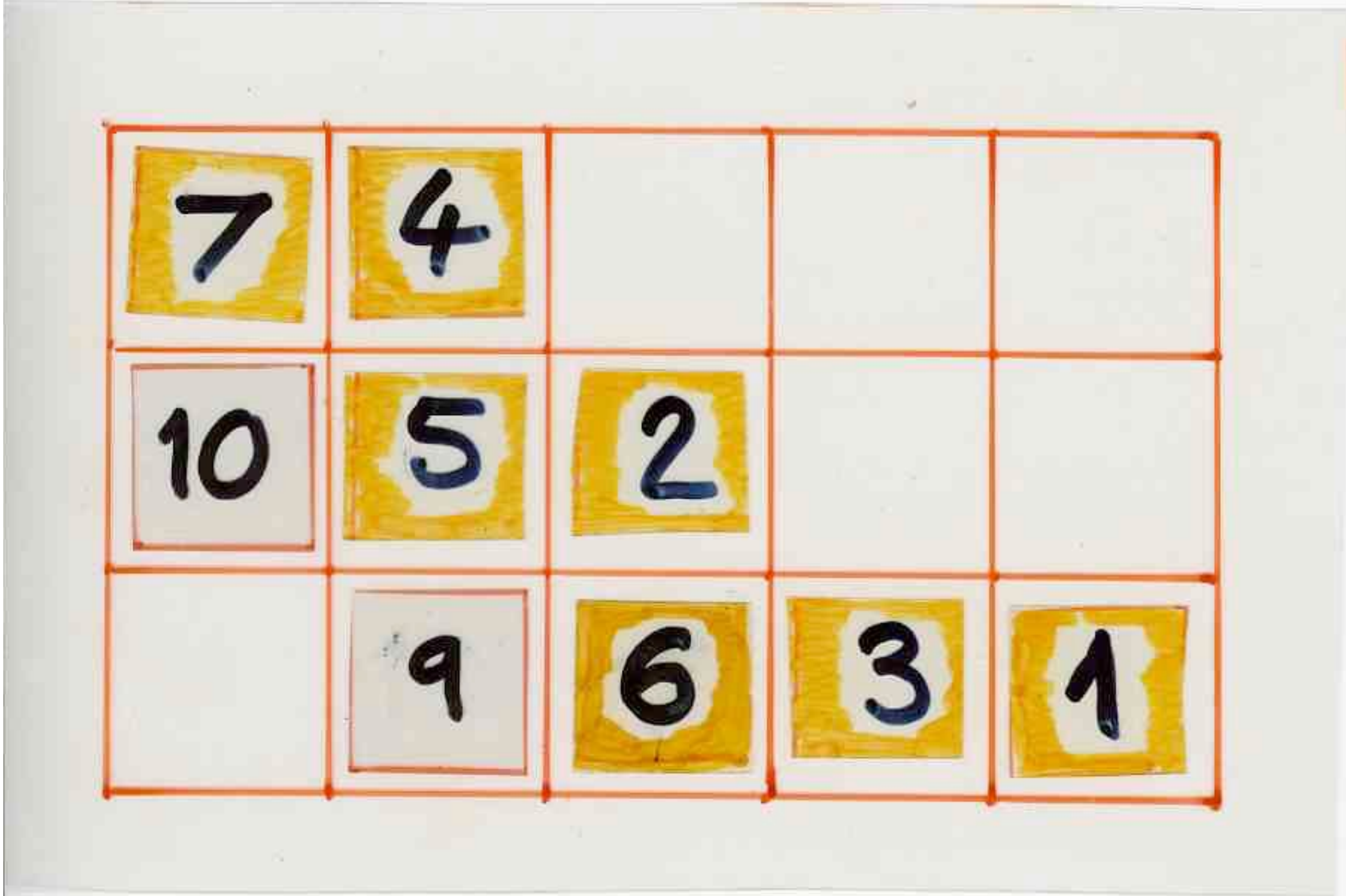
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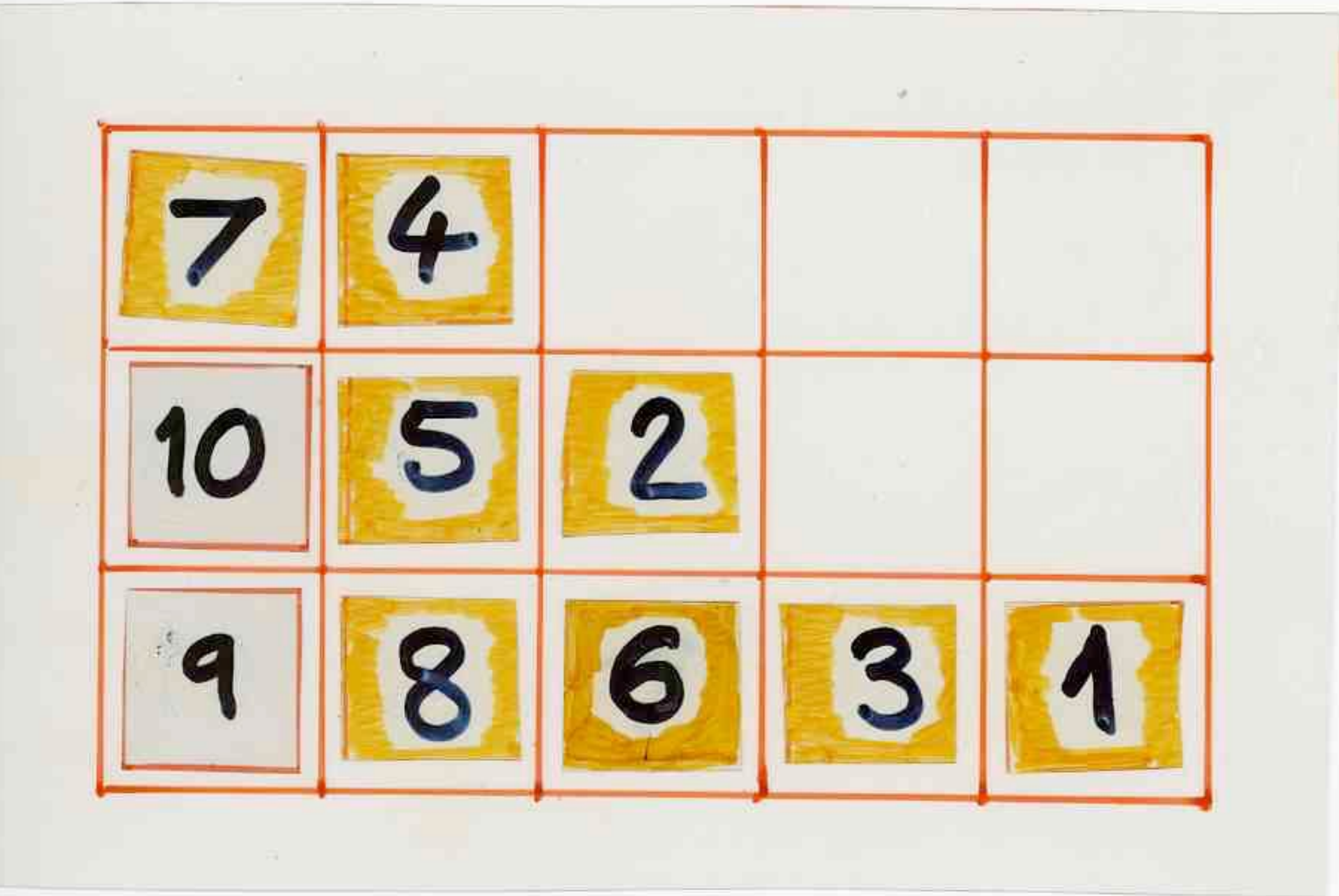

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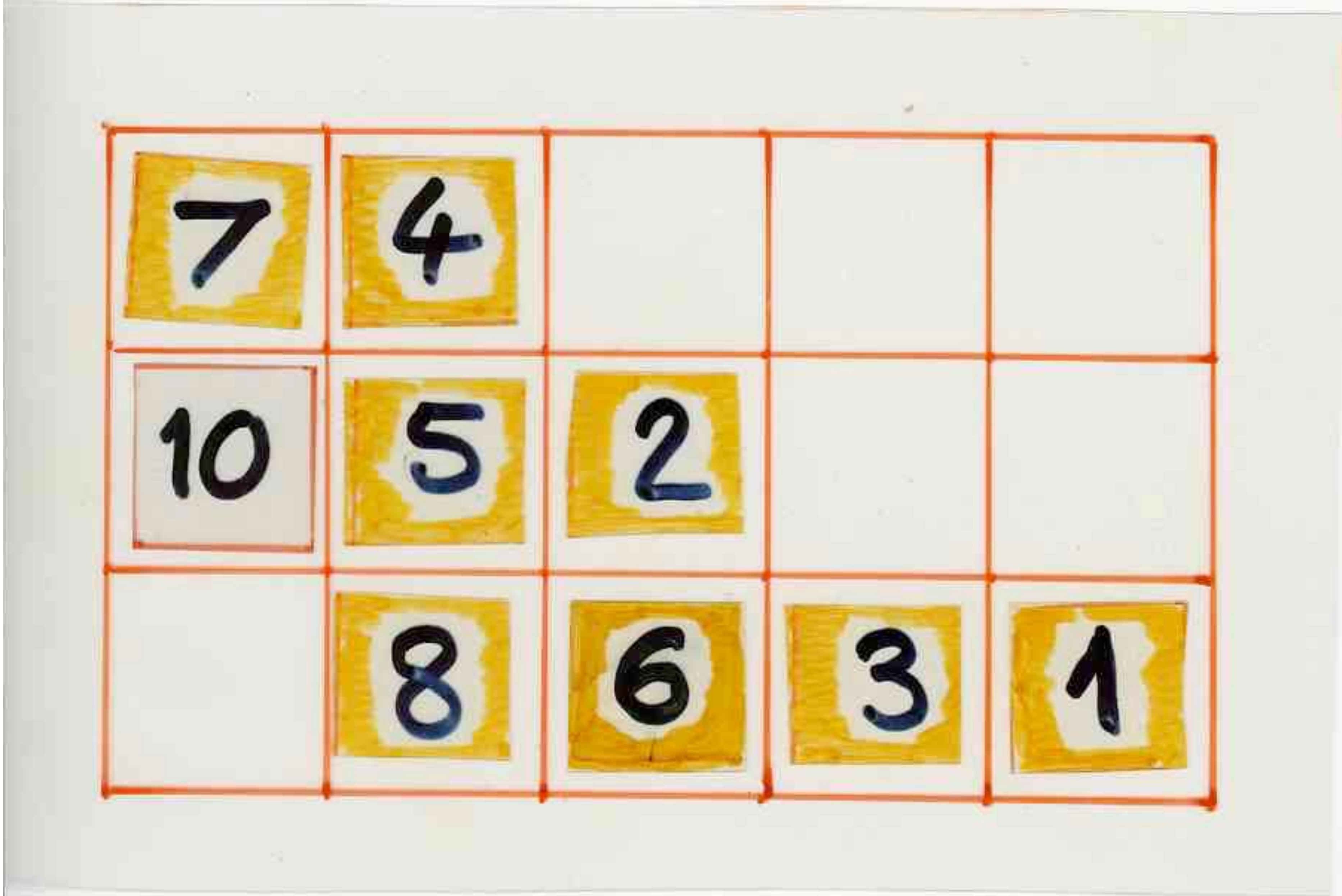

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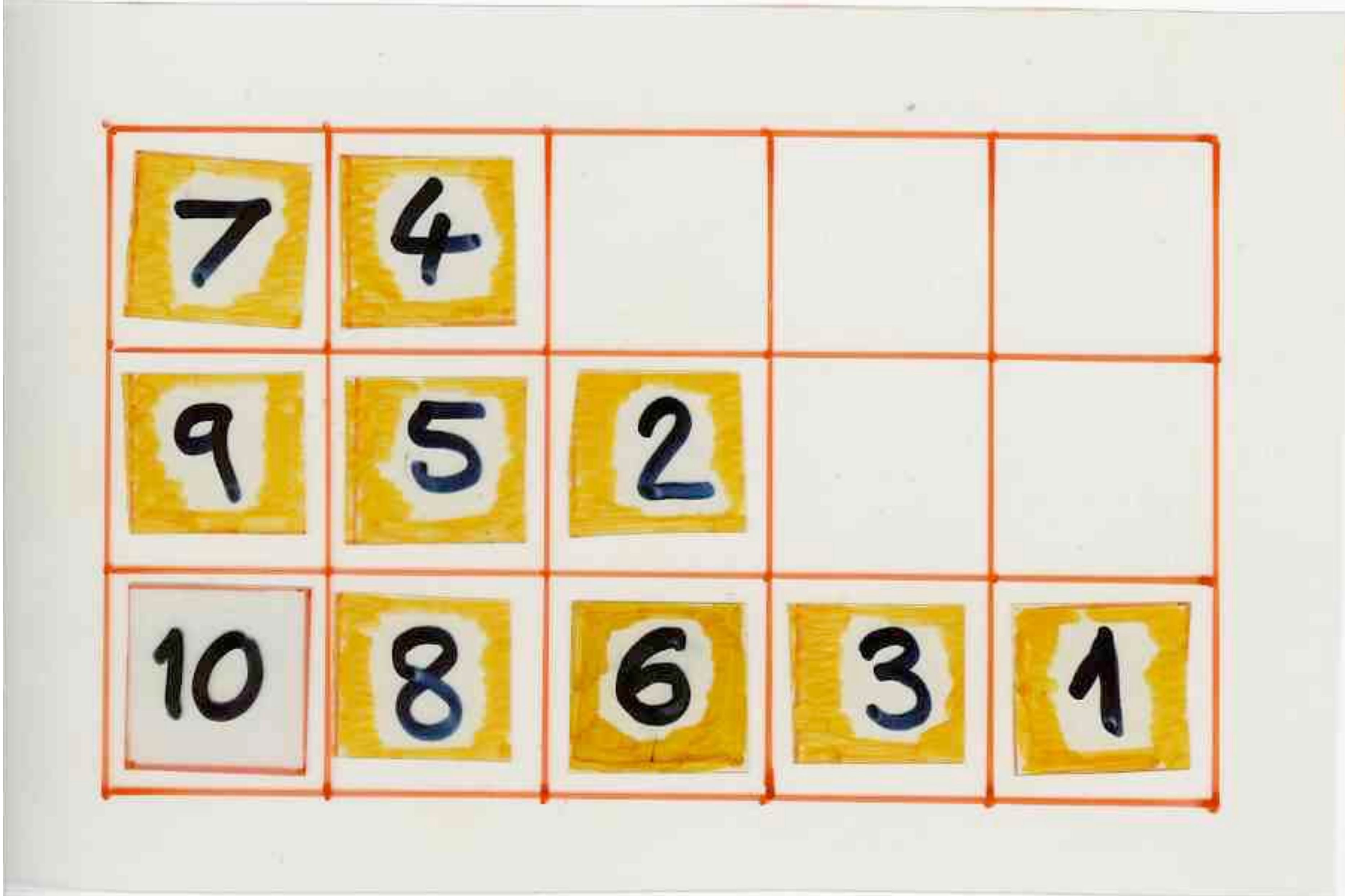
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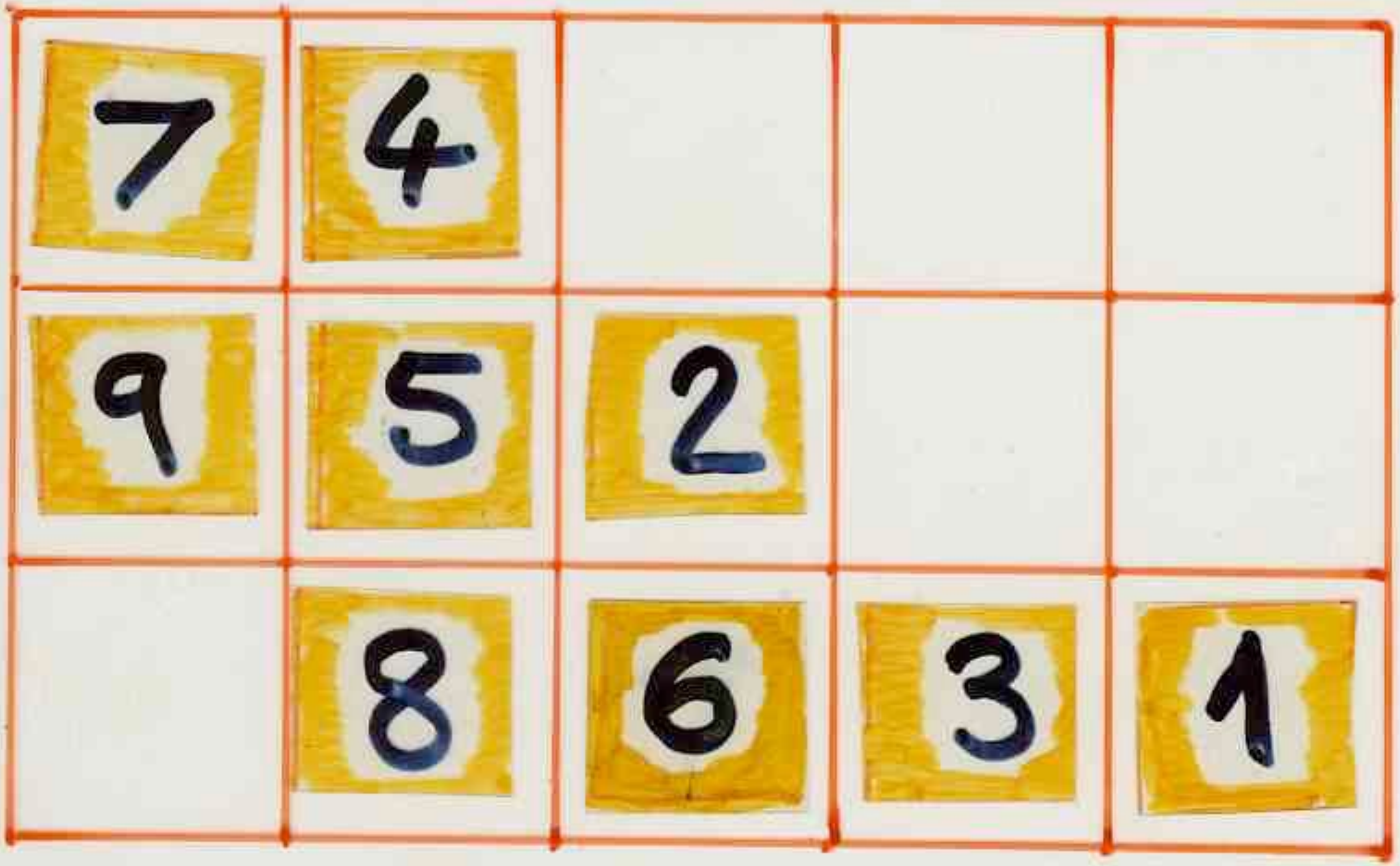
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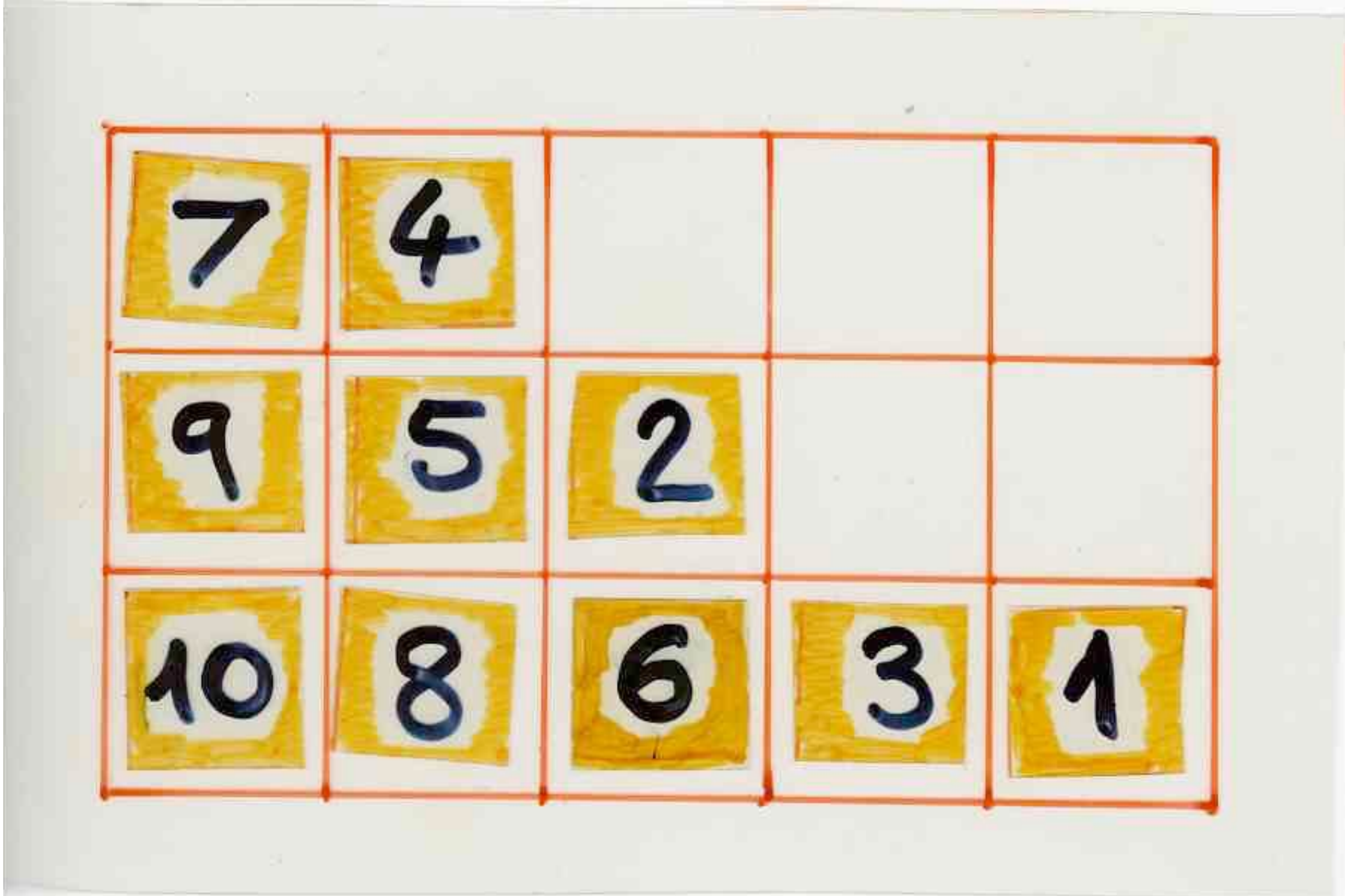
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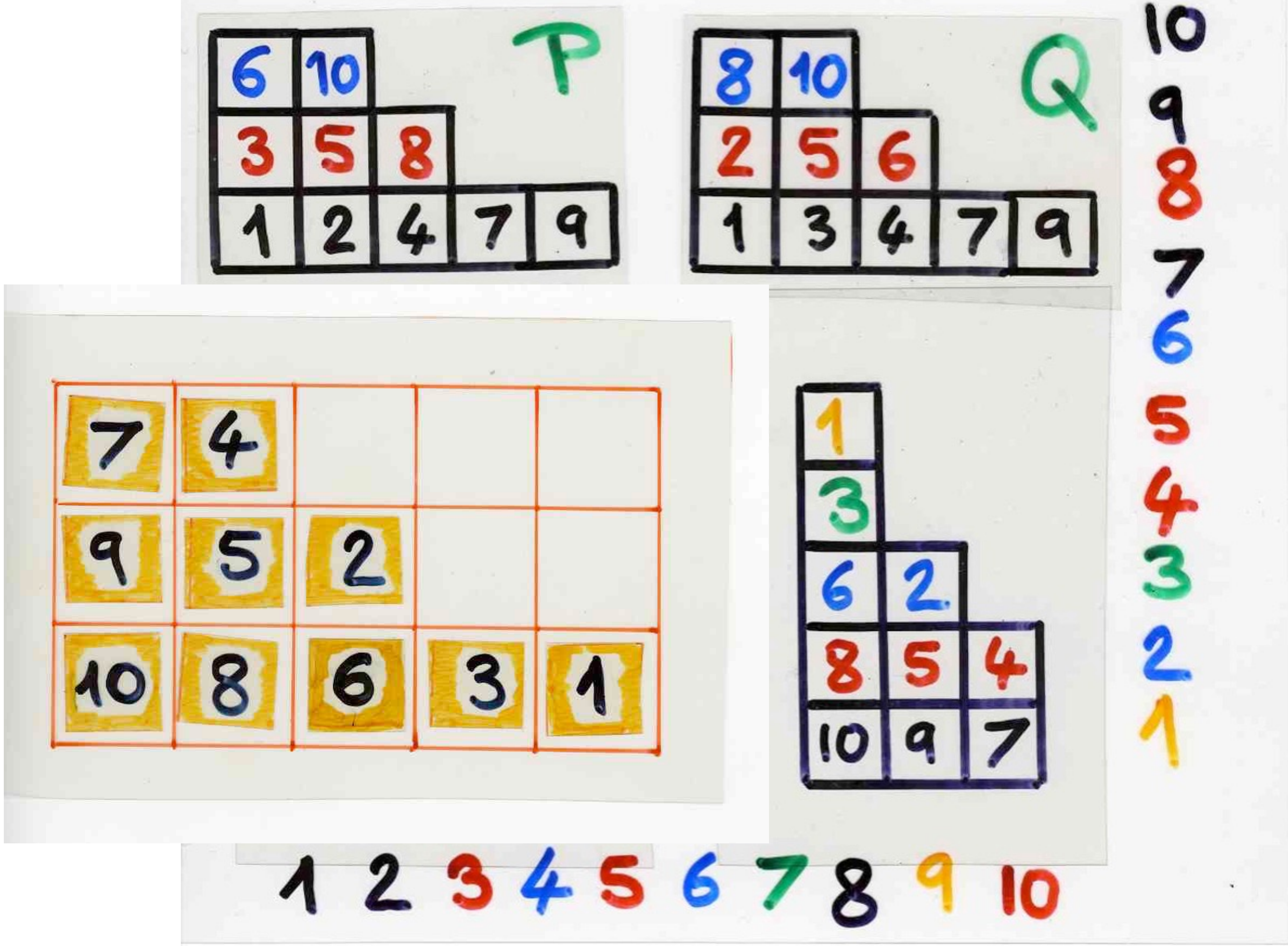
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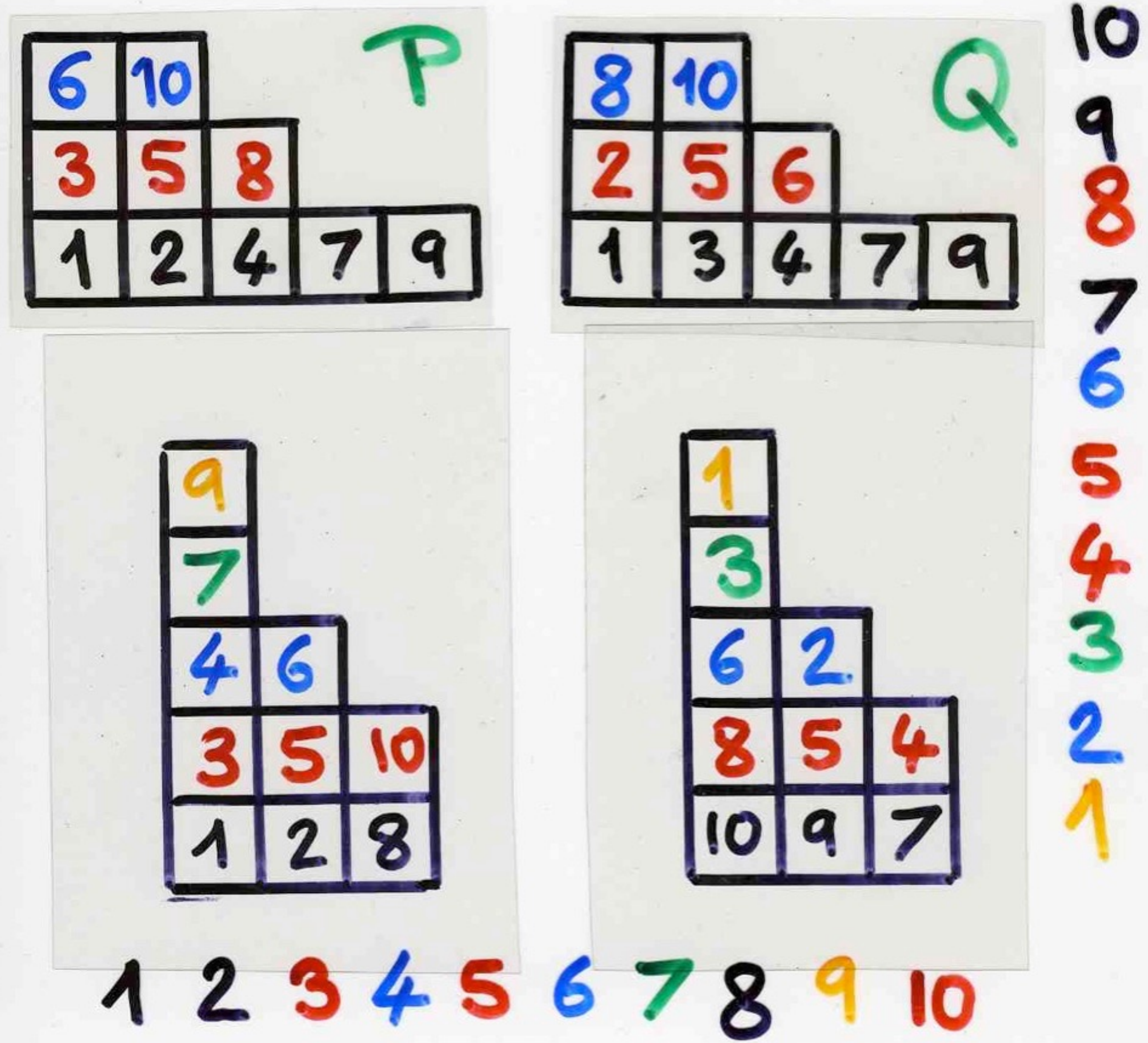
P
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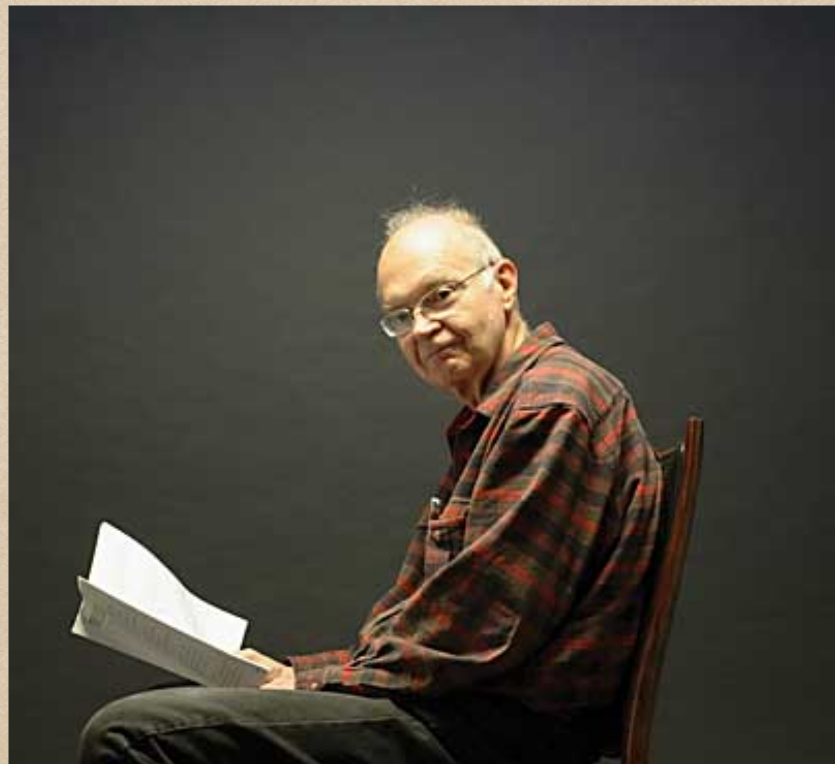
complements

Knuth's transpositions

D. Knuth, 1970



Knuth's transpositions (1970)

$$\sigma = \sigma(1) \cdots \underbrace{\sigma(i)}_{x} \underbrace{\sigma(i+1)}_{y} \cdots \sigma(n)$$

$x \longleftrightarrow y$

$z = \sigma(i-1)$ or $\sigma(i+1)$ is between x and y

$$x < z < y \quad \text{or} \quad y < z < x$$

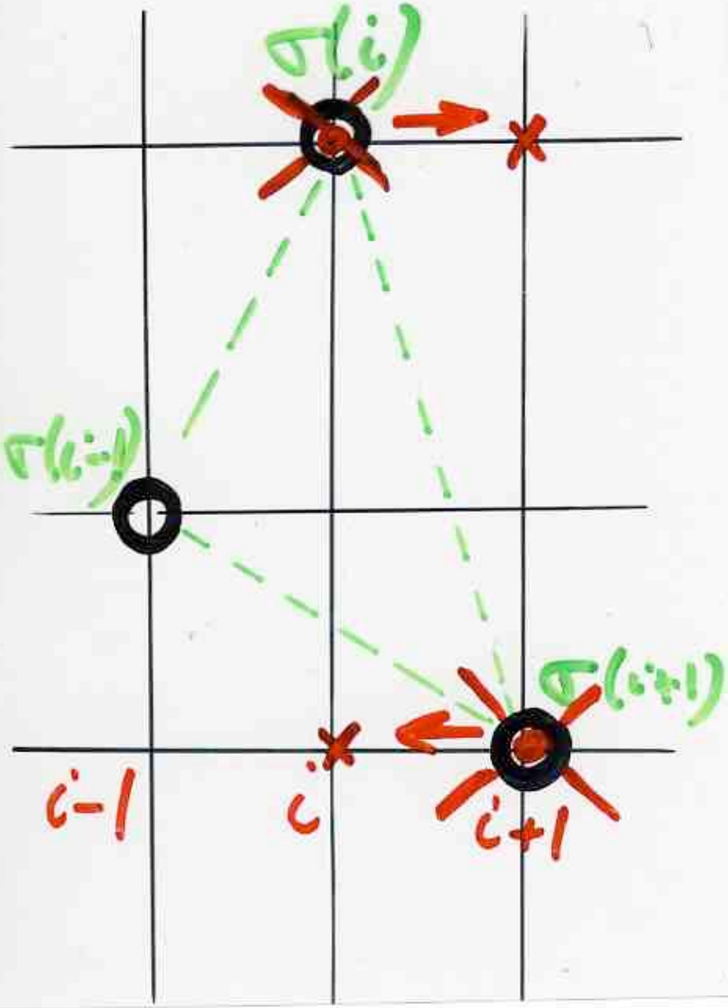
$$P(\sigma) = P(\tau)$$

$$\tau = \sigma(1) \cdots y\, x \cdots \sigma(n)$$

Knuth

Transpositions

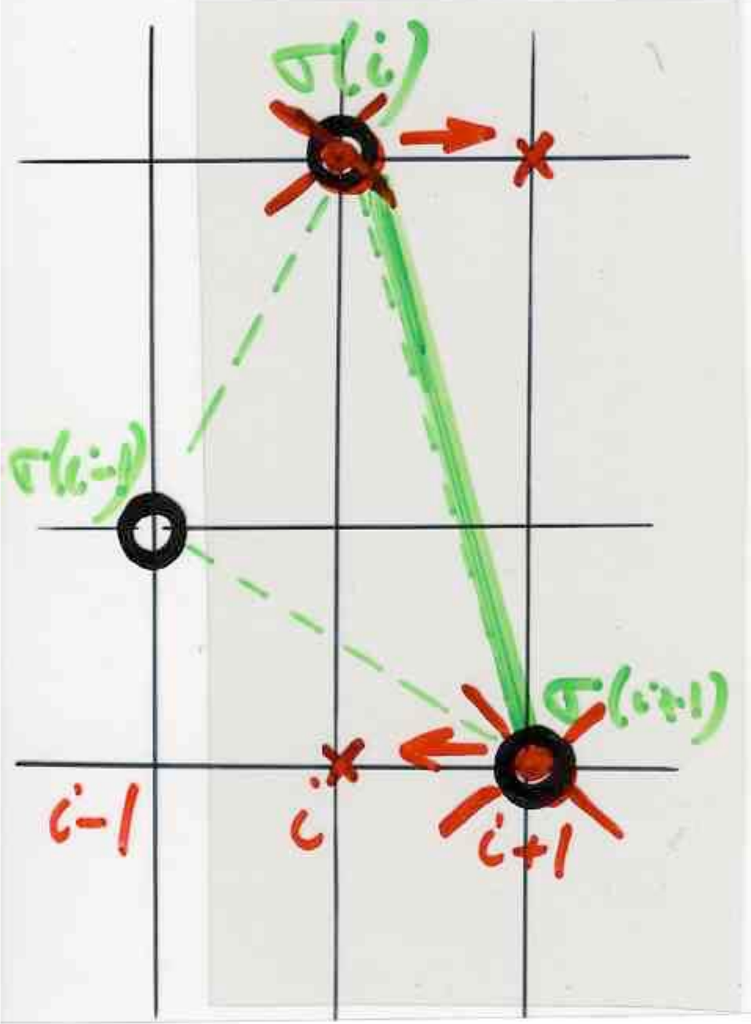
(1970)



Knuth

Transpositions

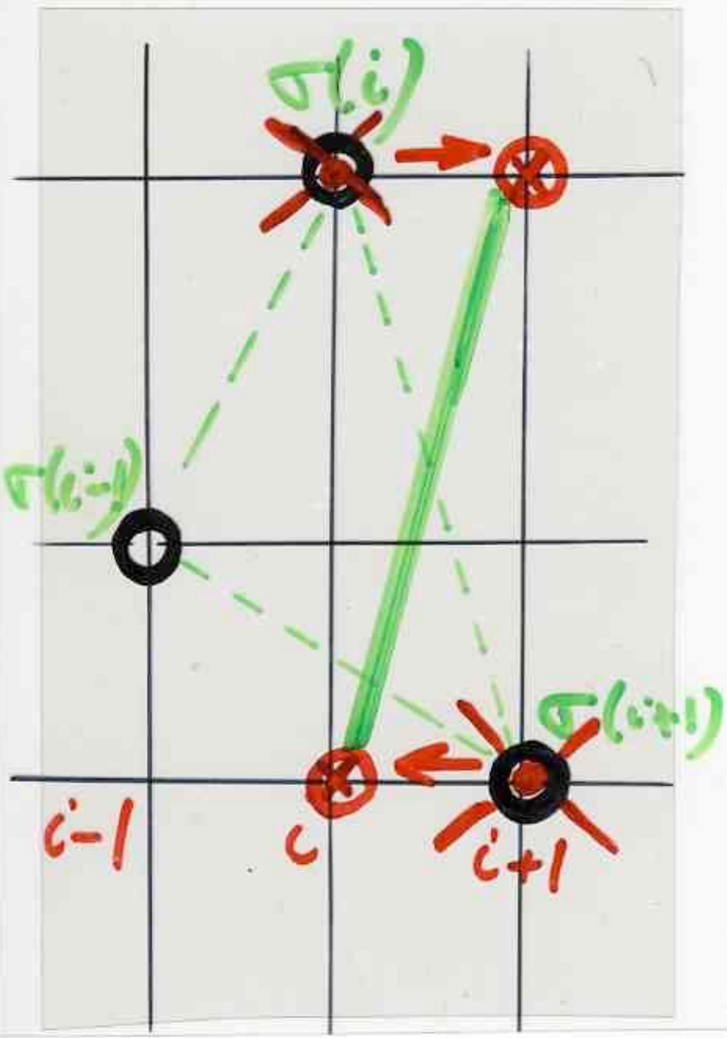
(1970)



Knuth

Transpositions

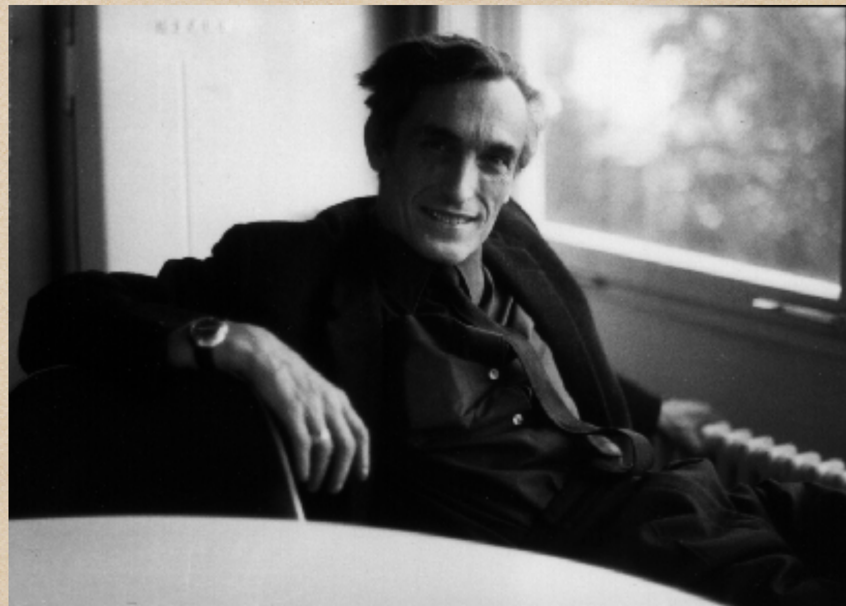
(1970)



complements

plactic monoid

A. Lascoux



M.P. Schützenberger

Plactic monoid

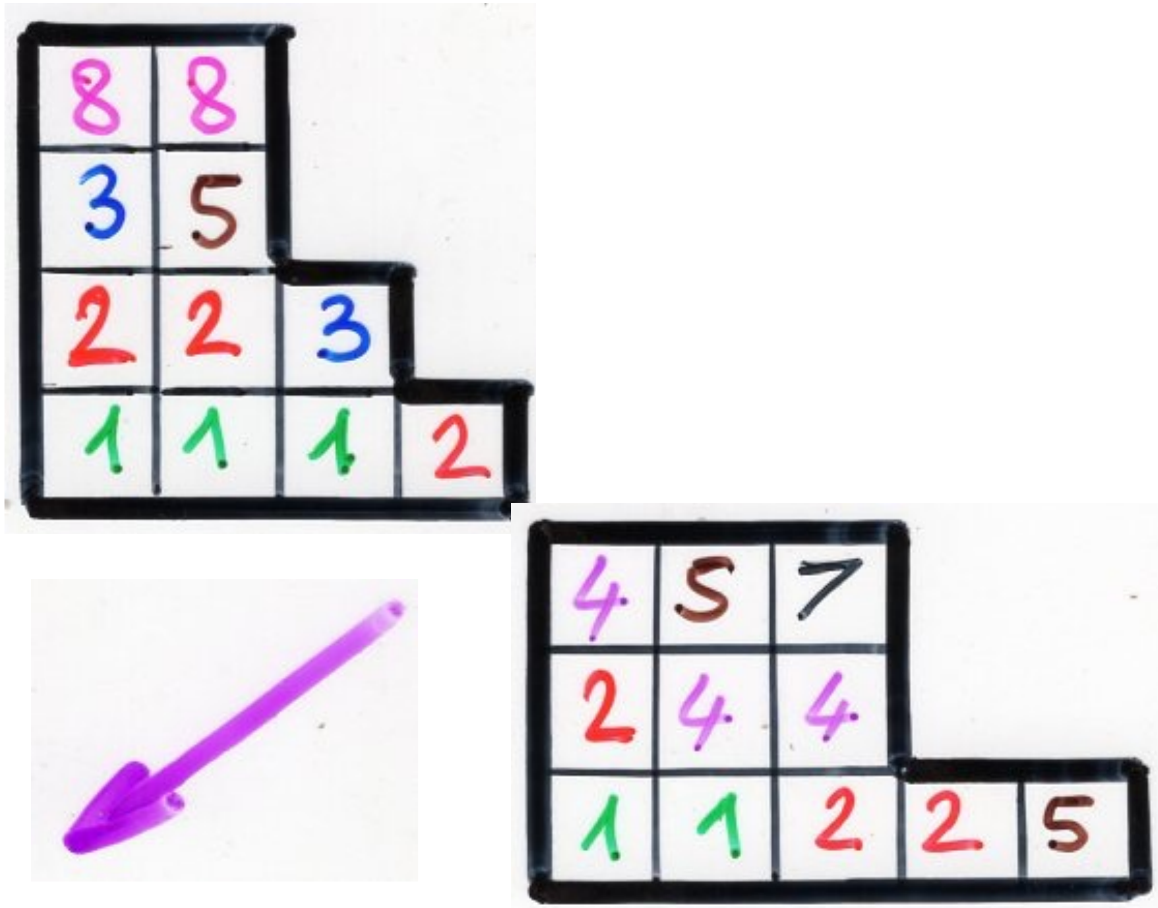
X alphabet

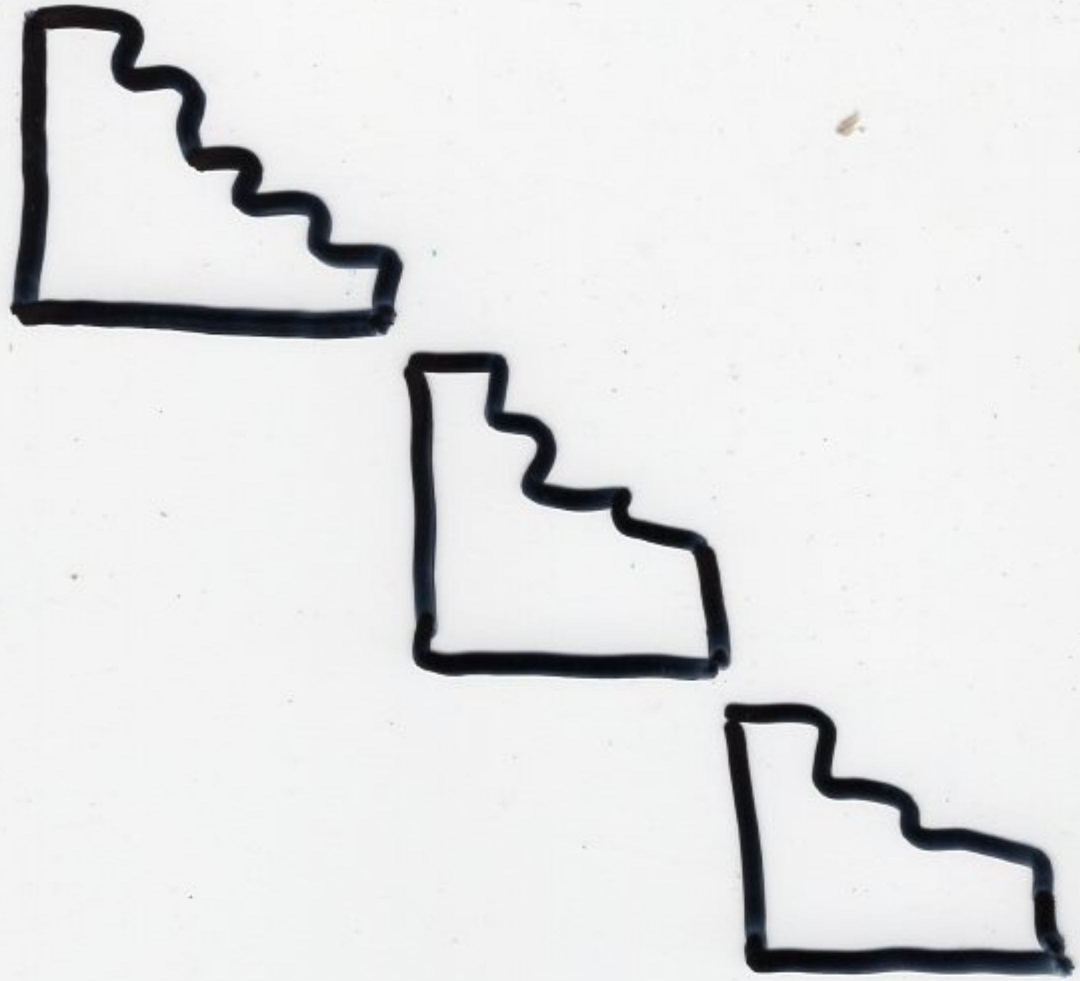
$X^* / \equiv$

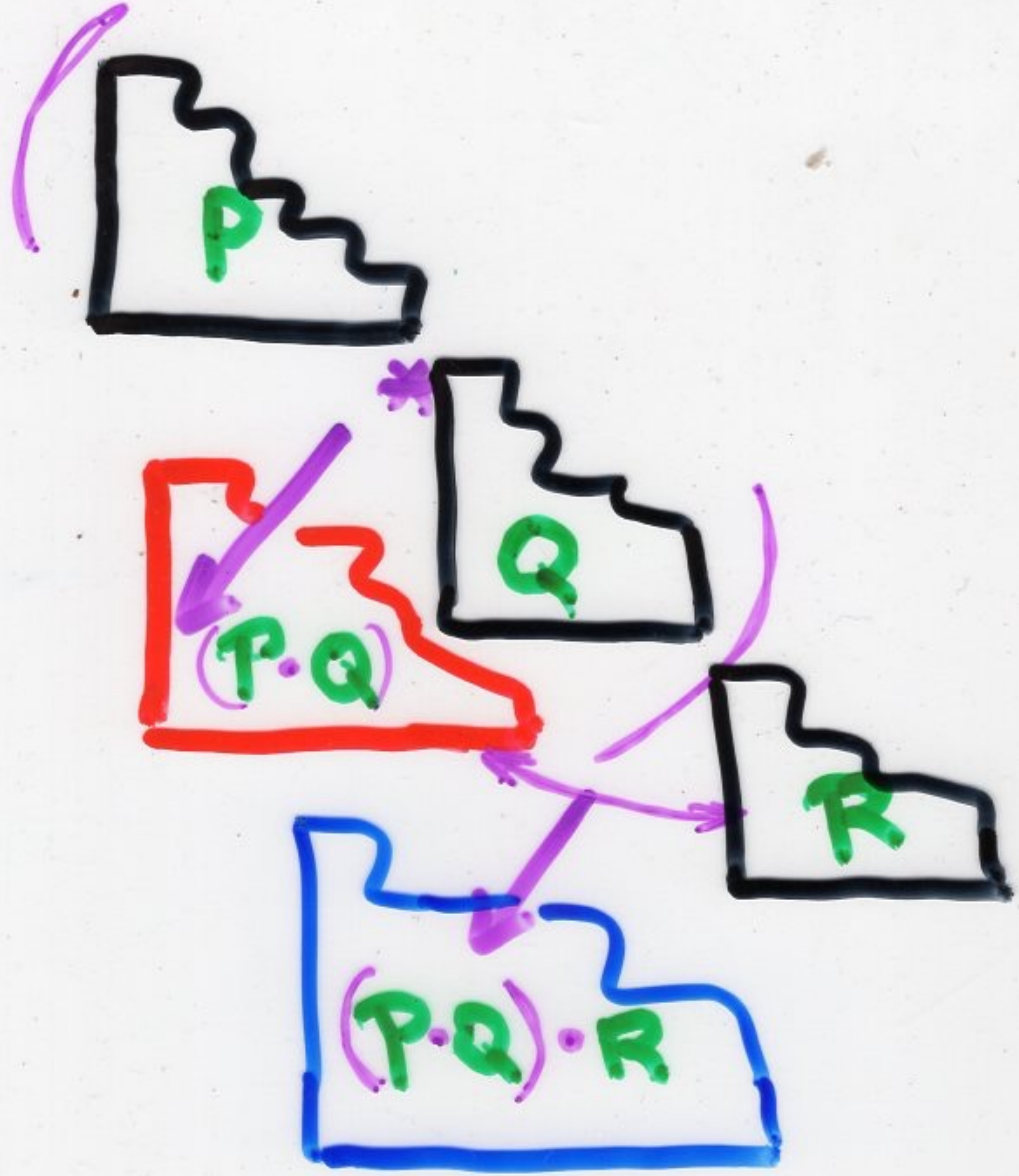
$$\begin{cases} yzx \equiv yxz & x < y \leq z \\ xzy \equiv zxy & x \leq y < z \end{cases}$$

A. Lascoux, M.P. Schützenberger (1978)

Plactic monoid



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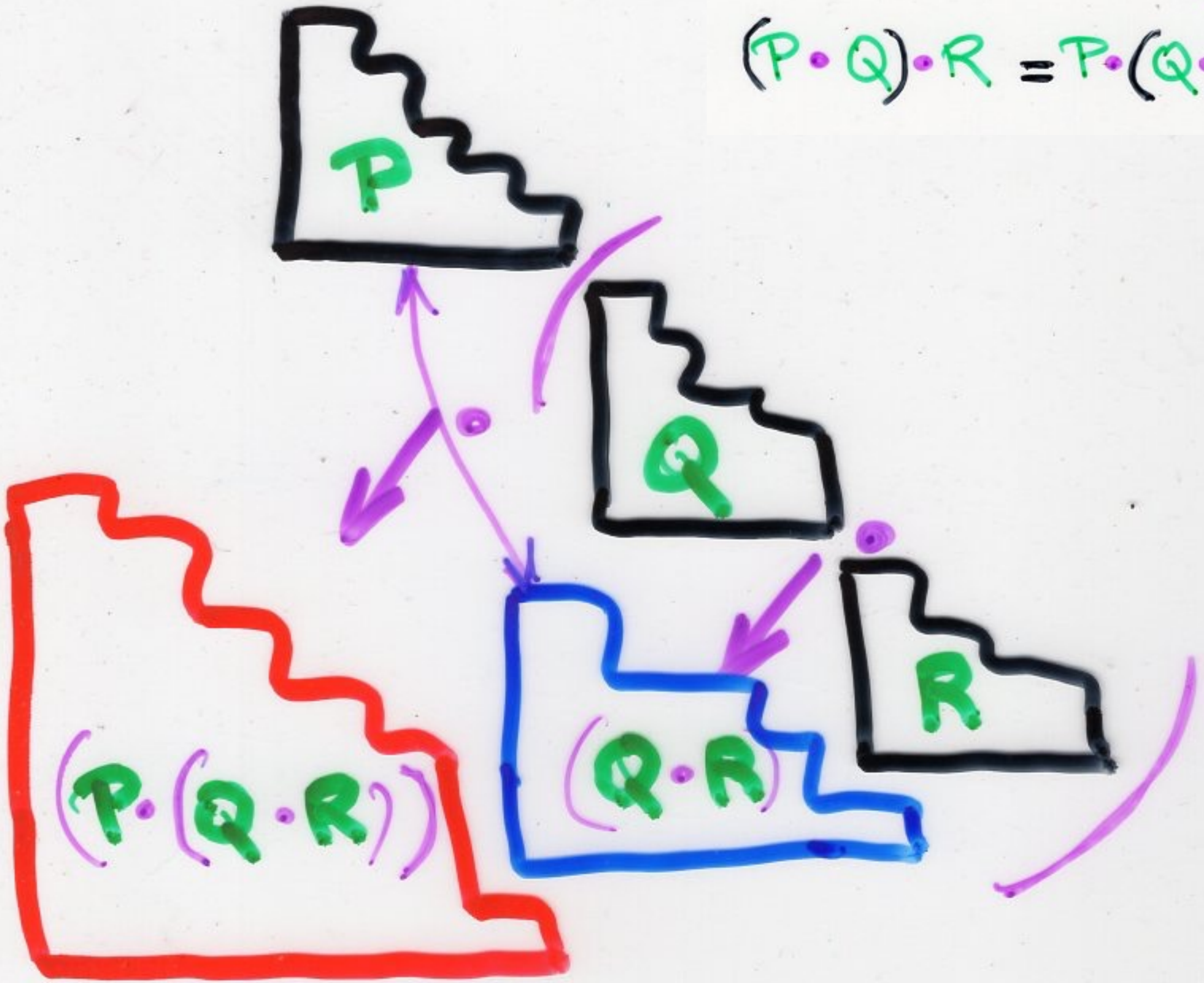
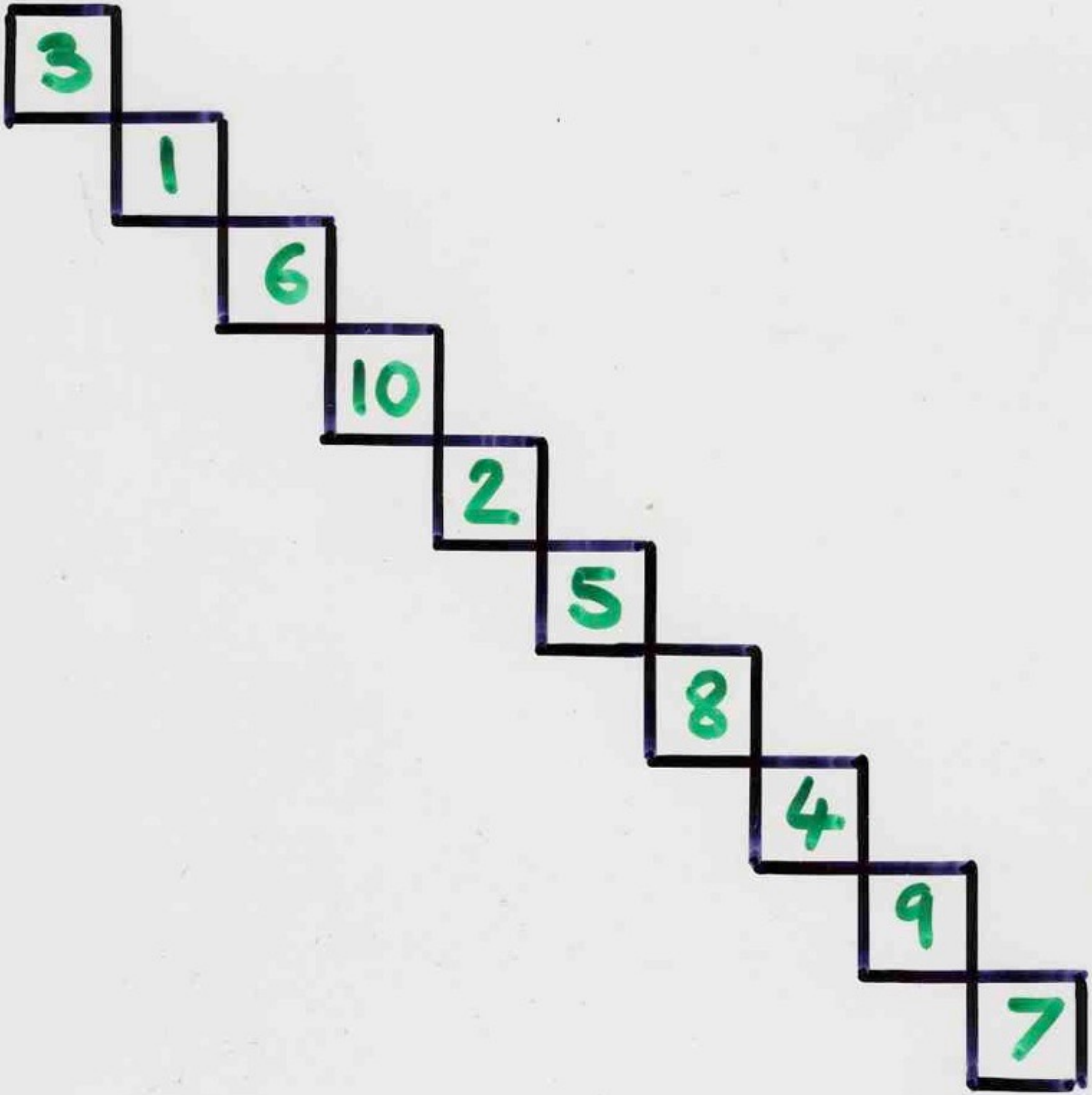
associativity

$$(P \cdot Q) \cdot R = P \cdot (Q \cdot R)$$

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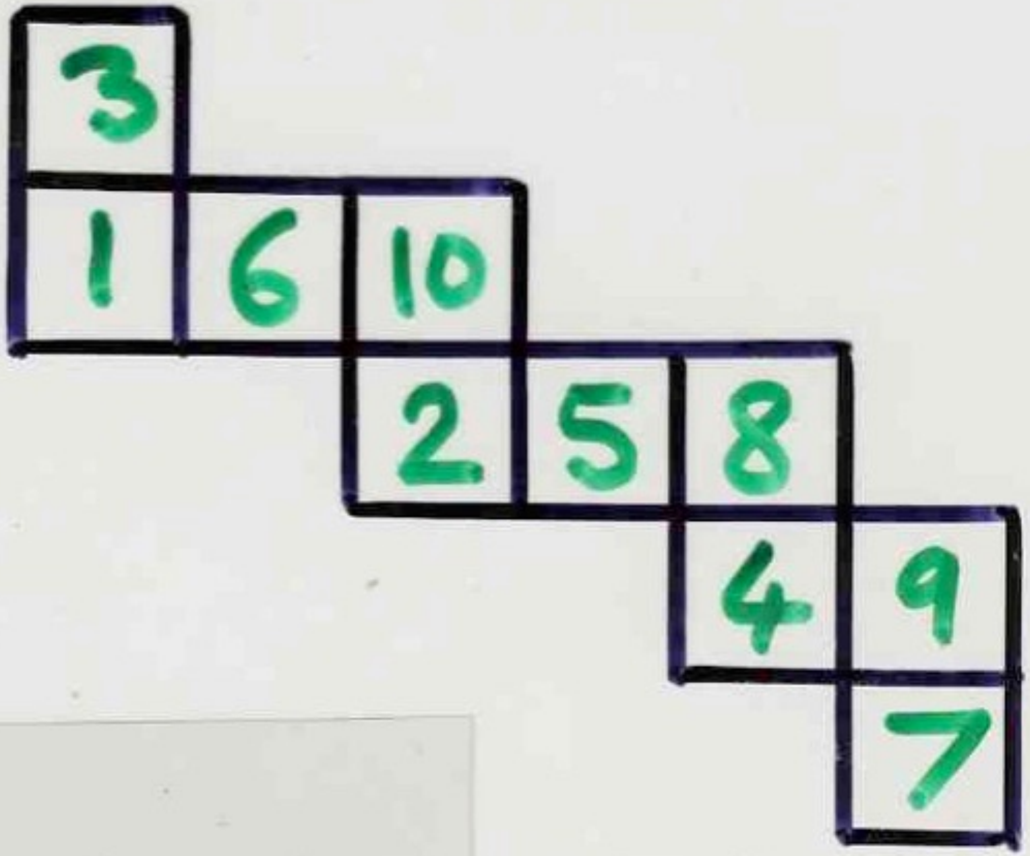
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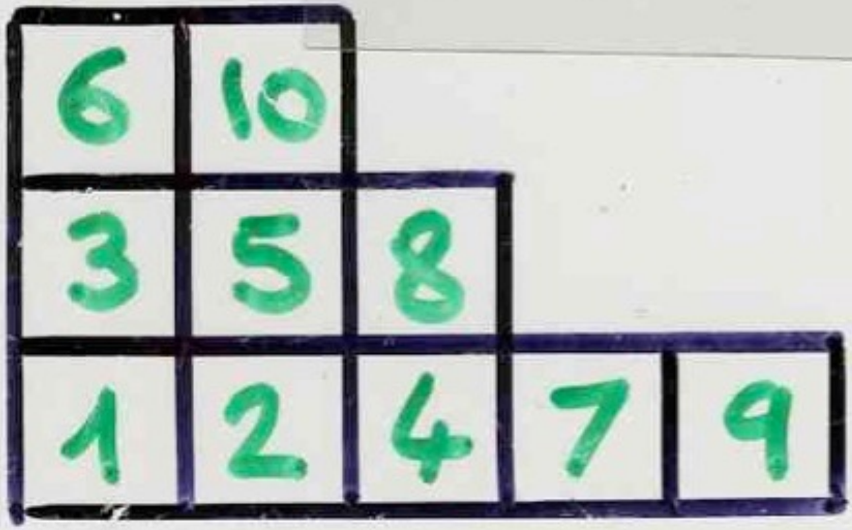
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complements

Schur functions

fonction de Schur

$$s_\lambda(x_1, x_2, \dots, x_n) = \frac{\det\left(x_j^{\,n-i+\lambda_i}\right)_{1\le i,j\le n}}{\det\left(x_j^{\,n-i}\right)_{1\le i,j\le n}}$$

$\lambda = (\lambda_1, \dots, \lambda_n)$

Issai Schur 1875–1941

théorie des invariants

Cauchy 1812

Jacobi 1841 det (homogène)

N. Trudi 1864

Schur Functions

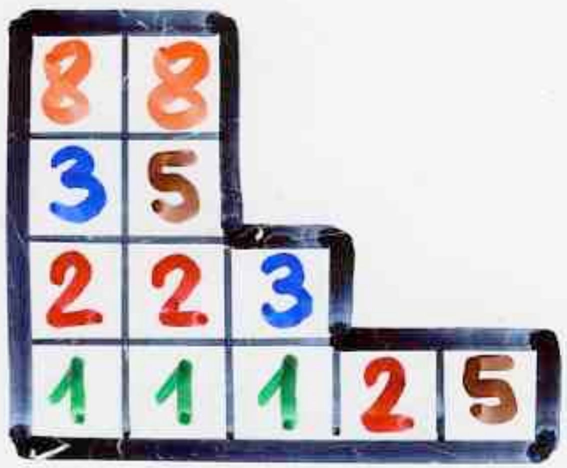
$$s_\lambda(x_1, x_2, \ldots, x_m) = \sum_{T} v(T)$$

T: Young tableau
shape λ
entries $1, 2, \ldots, m$

Jacobi (1841)
Schur (1901)

Littlewood-Richardson (1934)

basis of **symmetric** functions



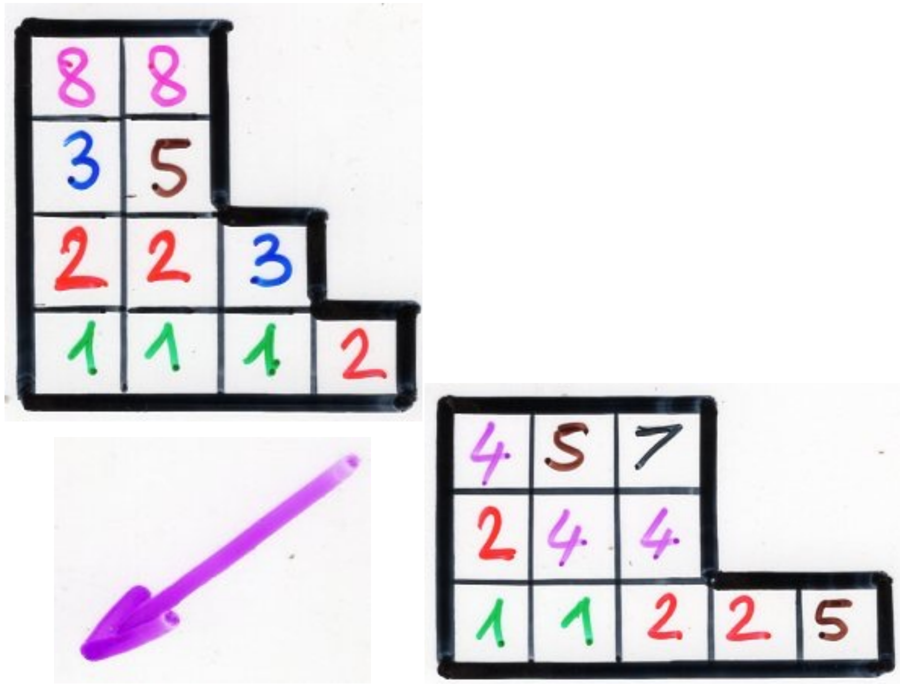
Schur functions

$$s_\lambda \, s_\mu = \sum_\nu g_{\lambda,\mu,\nu} \, s_\nu$$

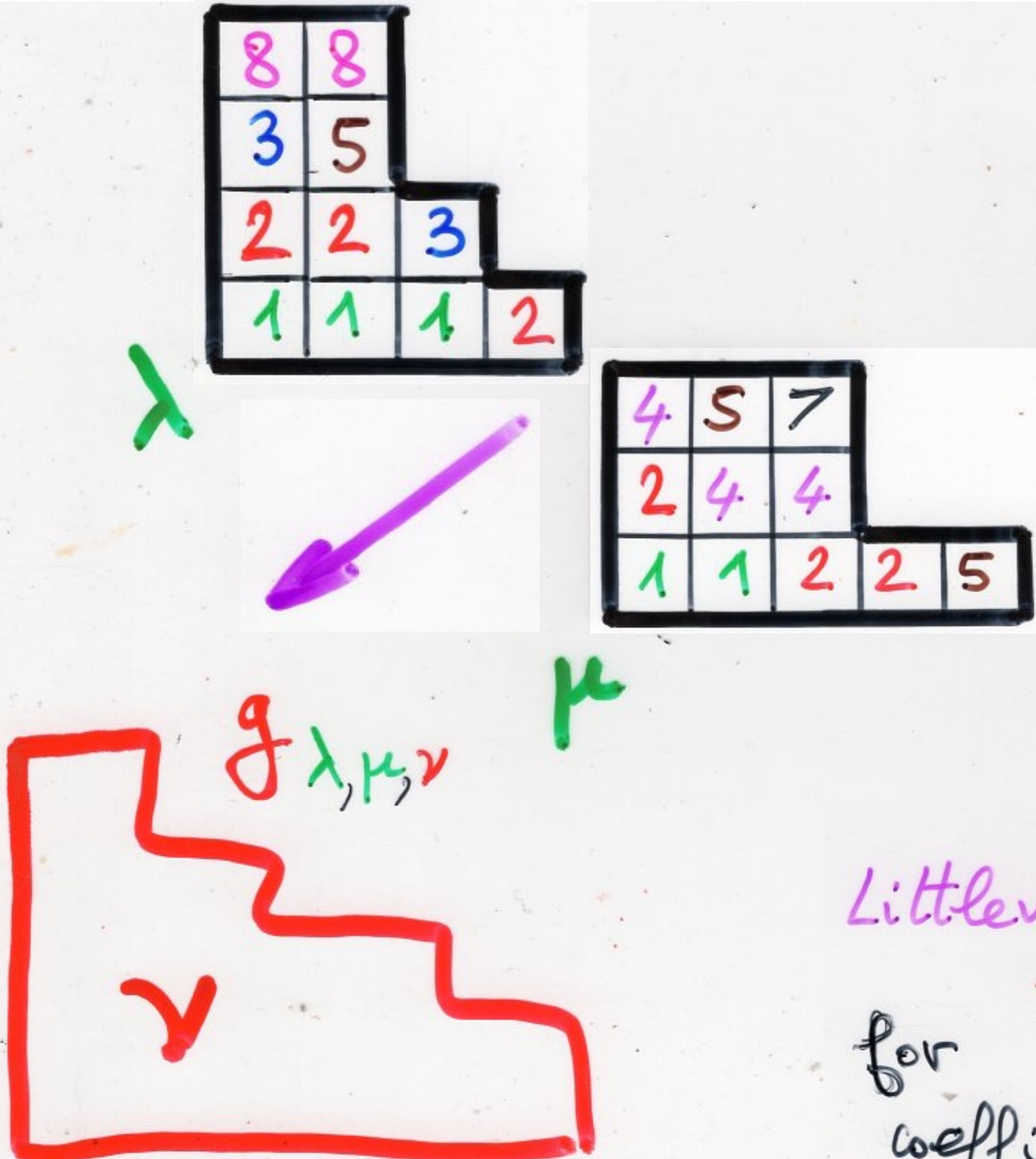
$s_\lambda(x_1, \ldots, x_m)$

Littlewood-
Richardson

Plactic monoid



Plactic monoid



Littlewood-Richarson rule (1934) for computing the coefficients $g_{\lambda,\mu,\nu}$