An introduction to

enumerative
algebraic combinatorics
bijective

IMSc

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Chapter 4

The n! garden
(2)

IMSc

18 February 2016

Laguerre histories

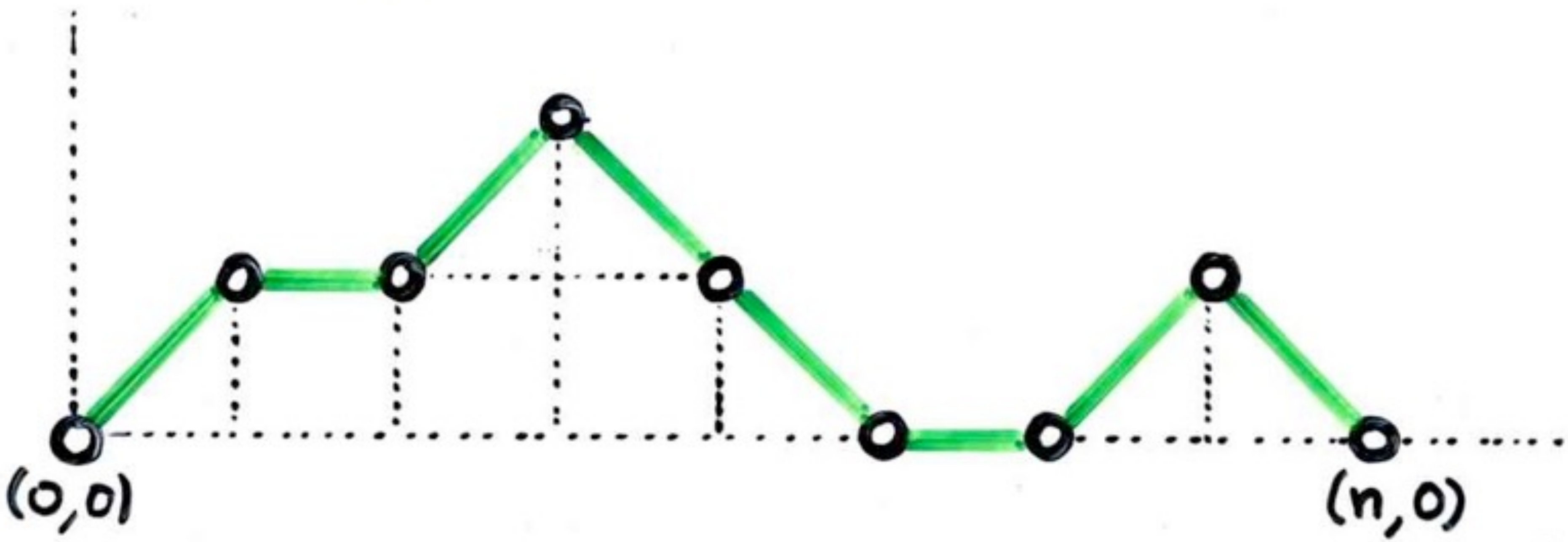
Laguerre histories

definition

Laguerre histories (γ_c, f) n

Laguerre histories (γ_c, f) n

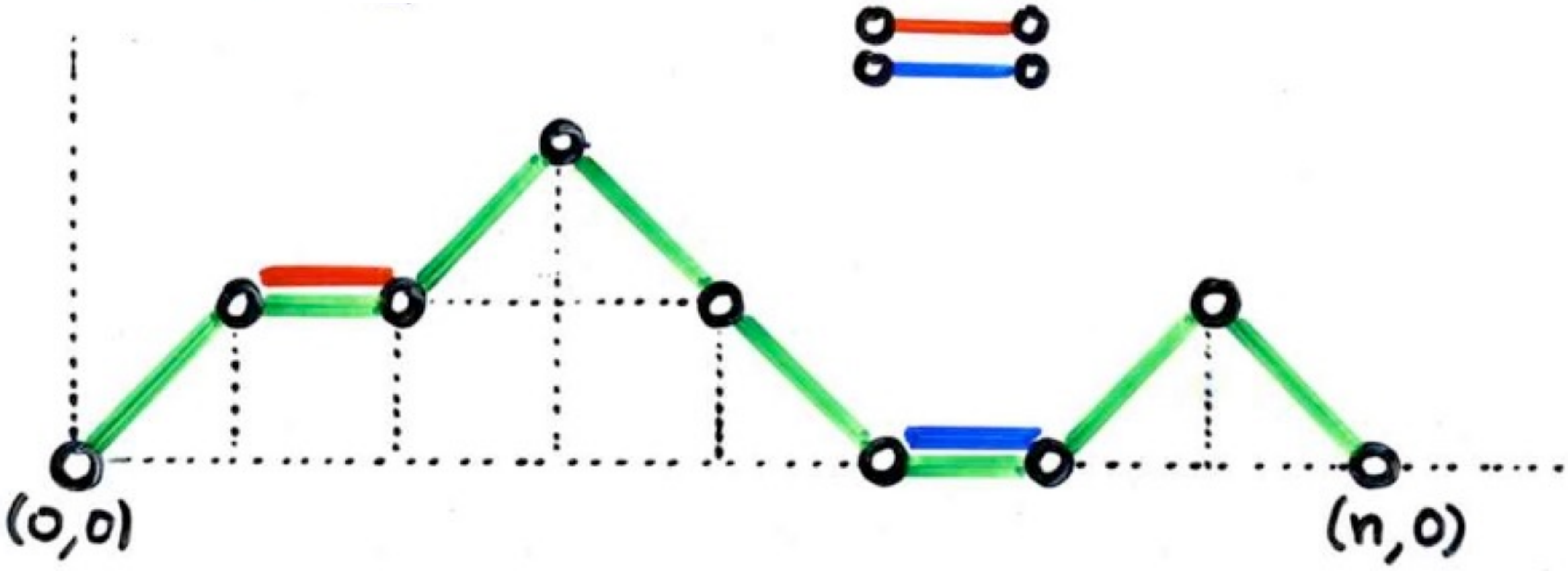
Motzkin path

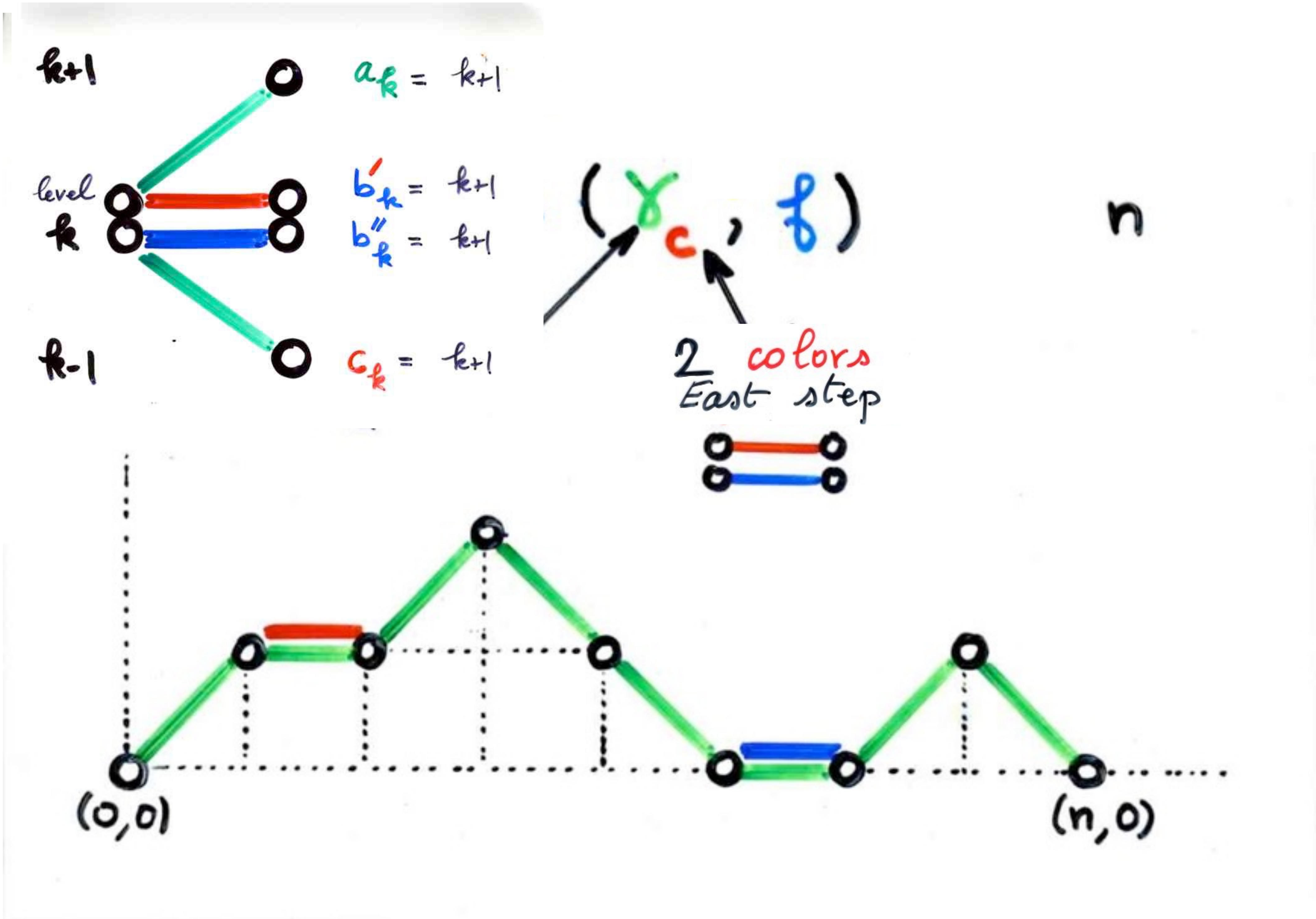


Laguerre histories (γ_c, f) n

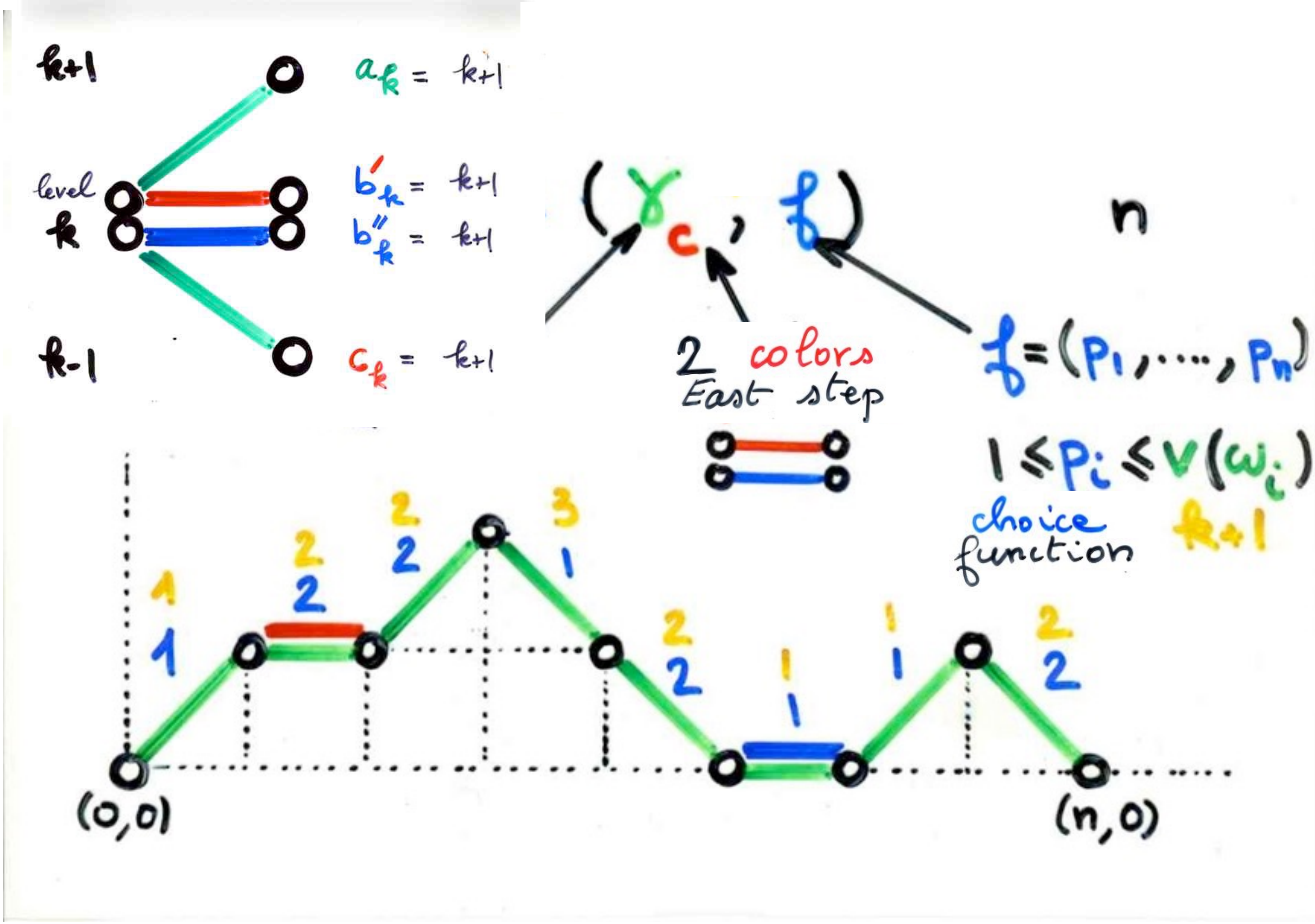
Motzkin path

2 colors East step





k+1
a_k = k+1
level
k
b'_k = k+1
b''_k = k+1
k-1
c_k = k+1
(γ_c , f)
n
2 colors
East step
(0,0)
(n,0)



k+1
level
k
k-1
a_k = k+1
b'_k = k+1
b''_k = k+1
c_k = k+1
(γ_c , f)
n
2 colors
East step
f = (p_1 , … , p_n)
1 ≤ p_i ≤ v(ω_i)
choice
function
k+1
1
1
2
2
2
2
3
1
2
2
1
1
1
1
2
2
(0,0)
(n,0)

Bijection

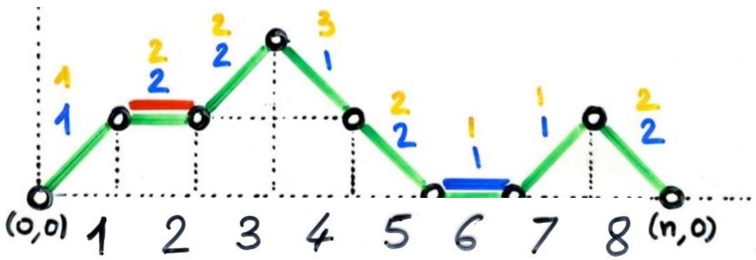
Laguerre histories $(\omega; (p_1, \dots, p_n))$ $\longleftrightarrow$ permutations

$(n+1)!$

bijection

Laguerre histories ⟶ permutations

description with binary trees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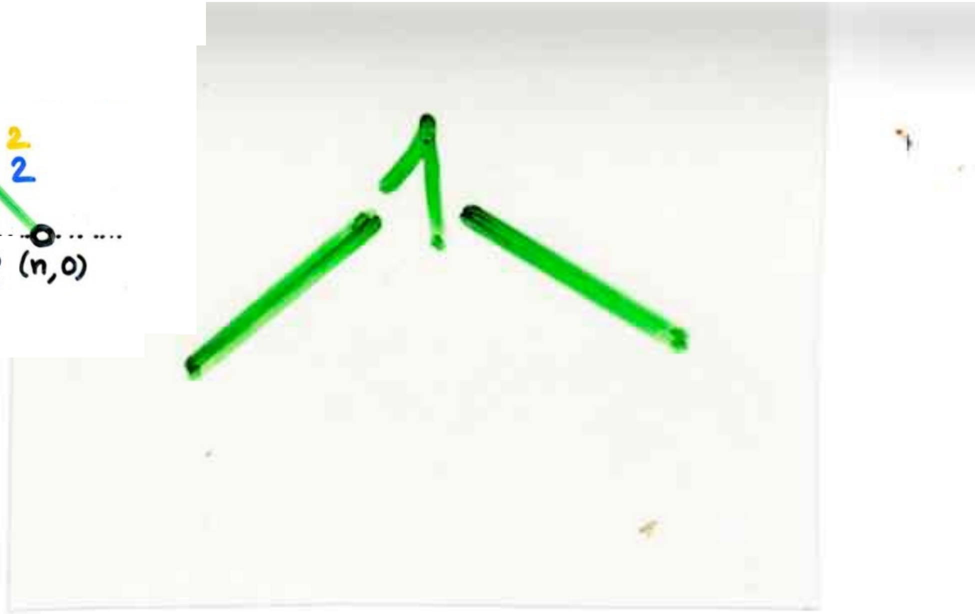

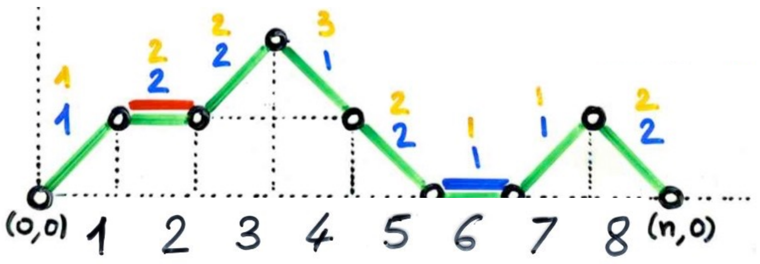
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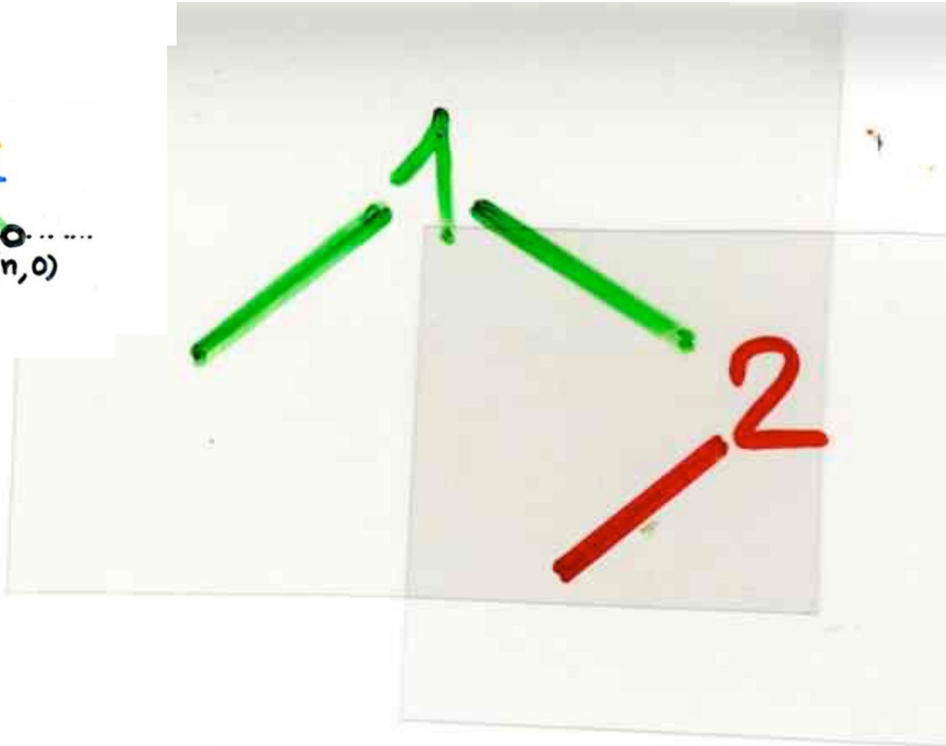
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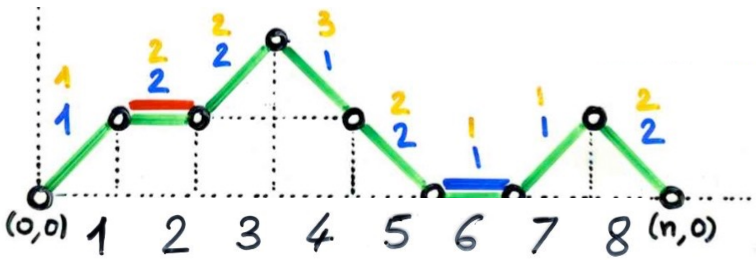
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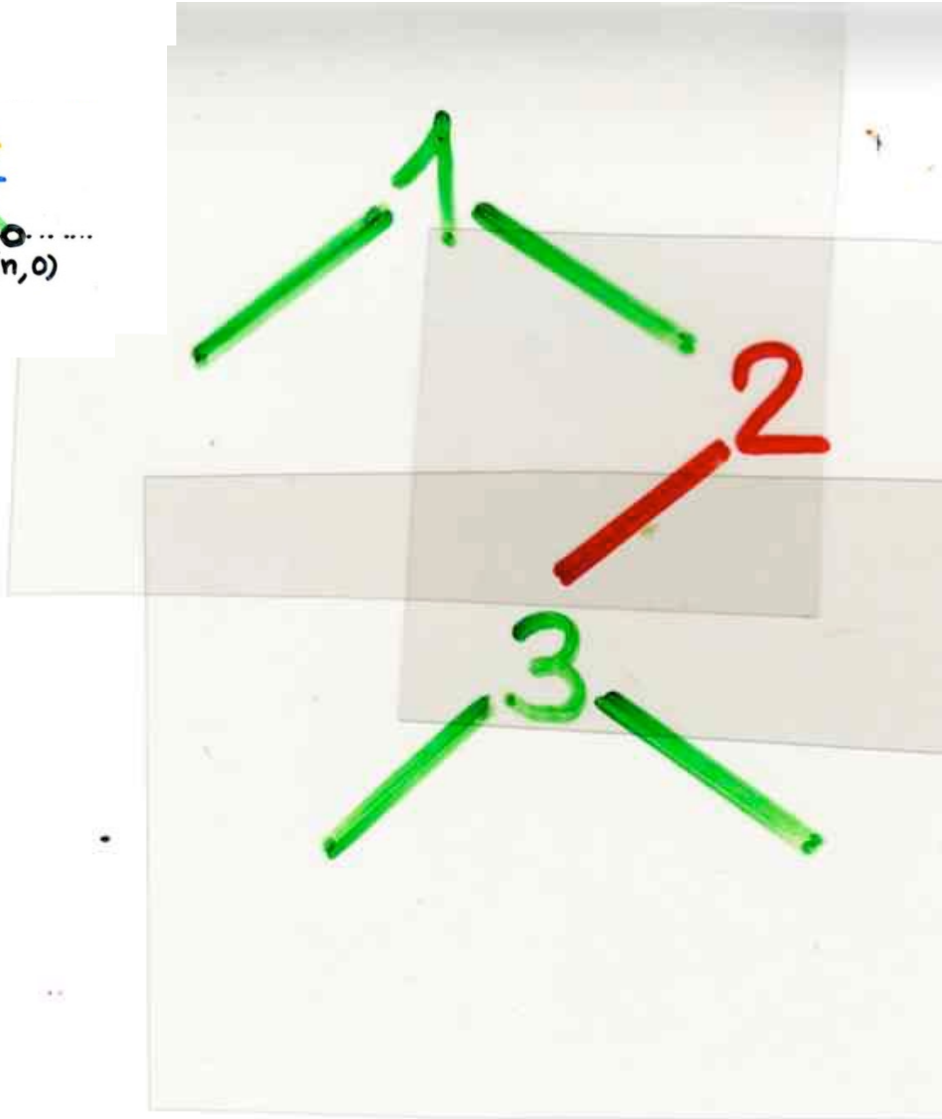
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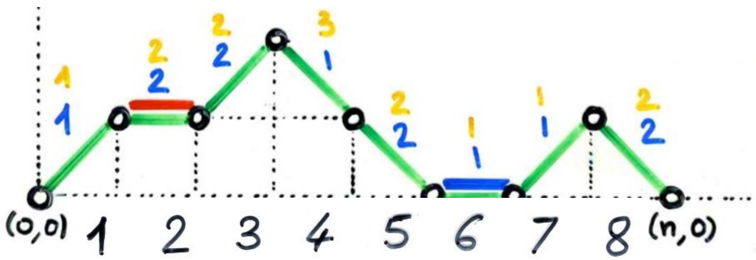

(0,0)
1 2 3 4 5 6 7 8
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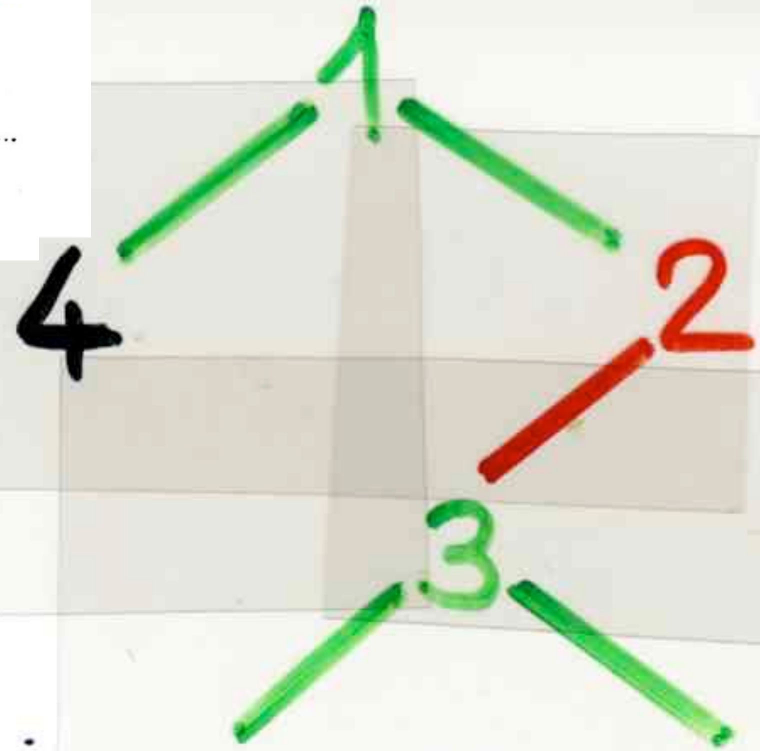

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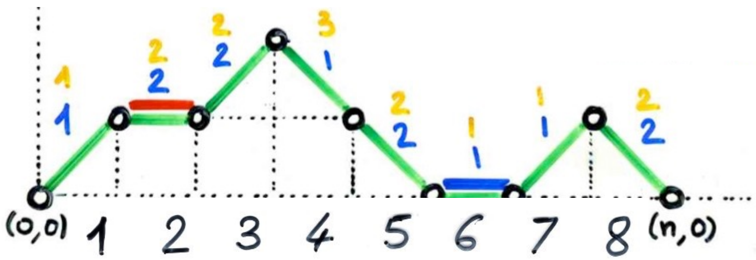

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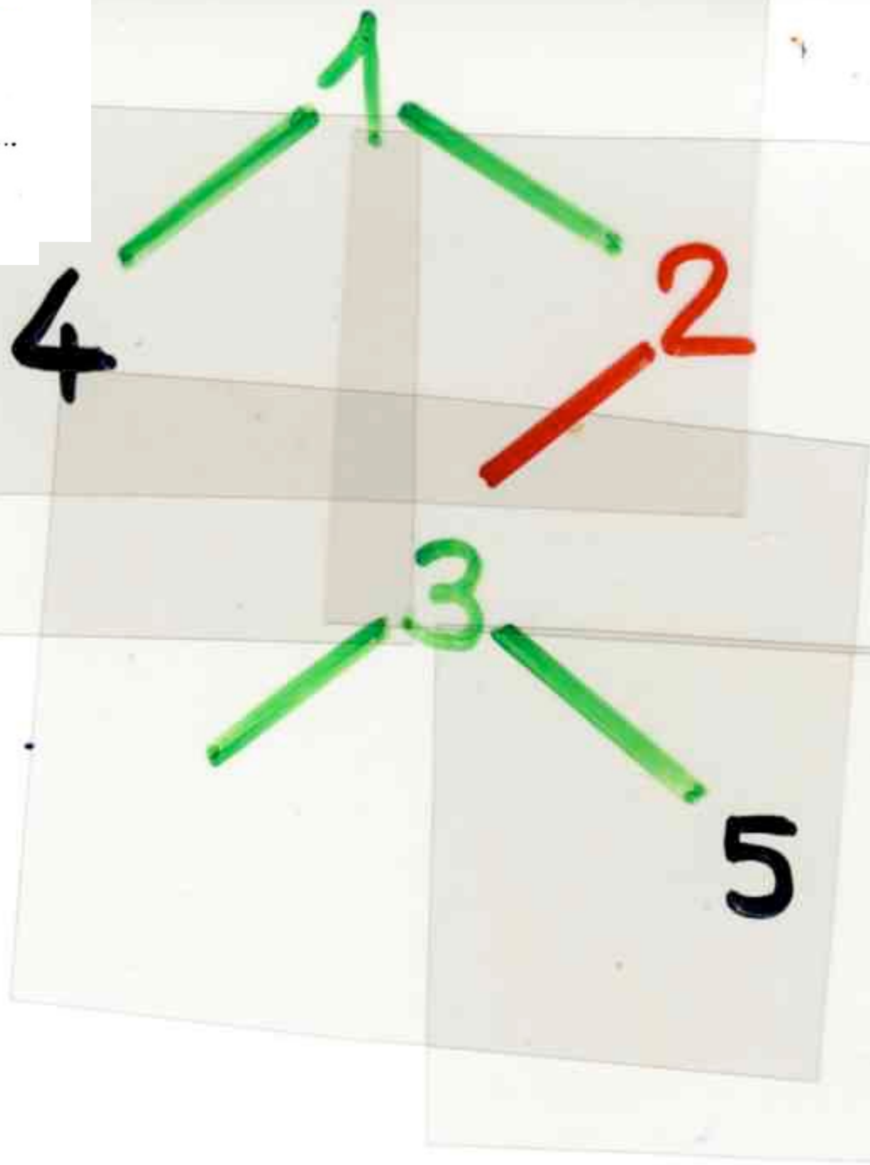

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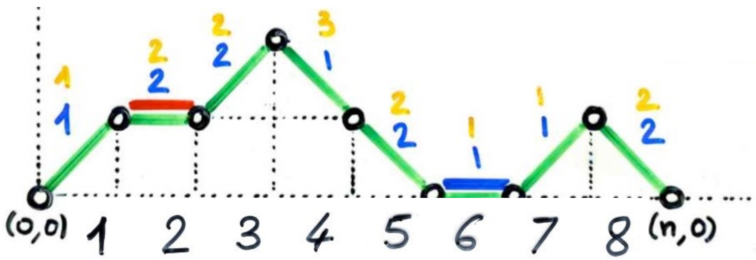
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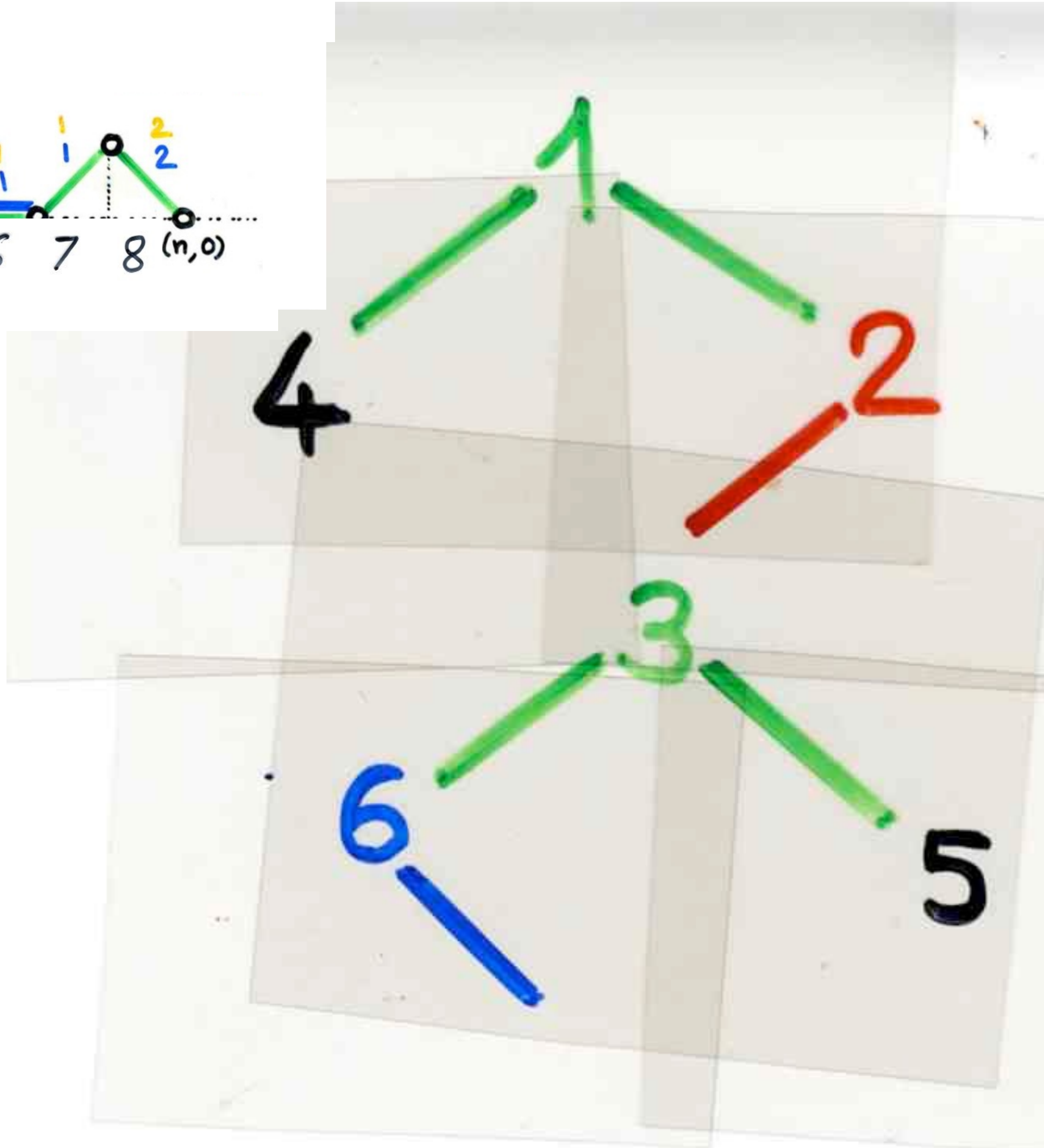
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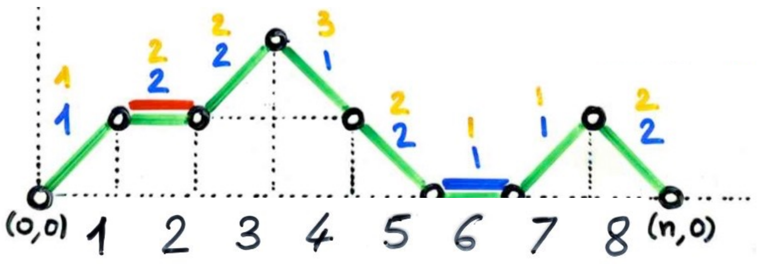

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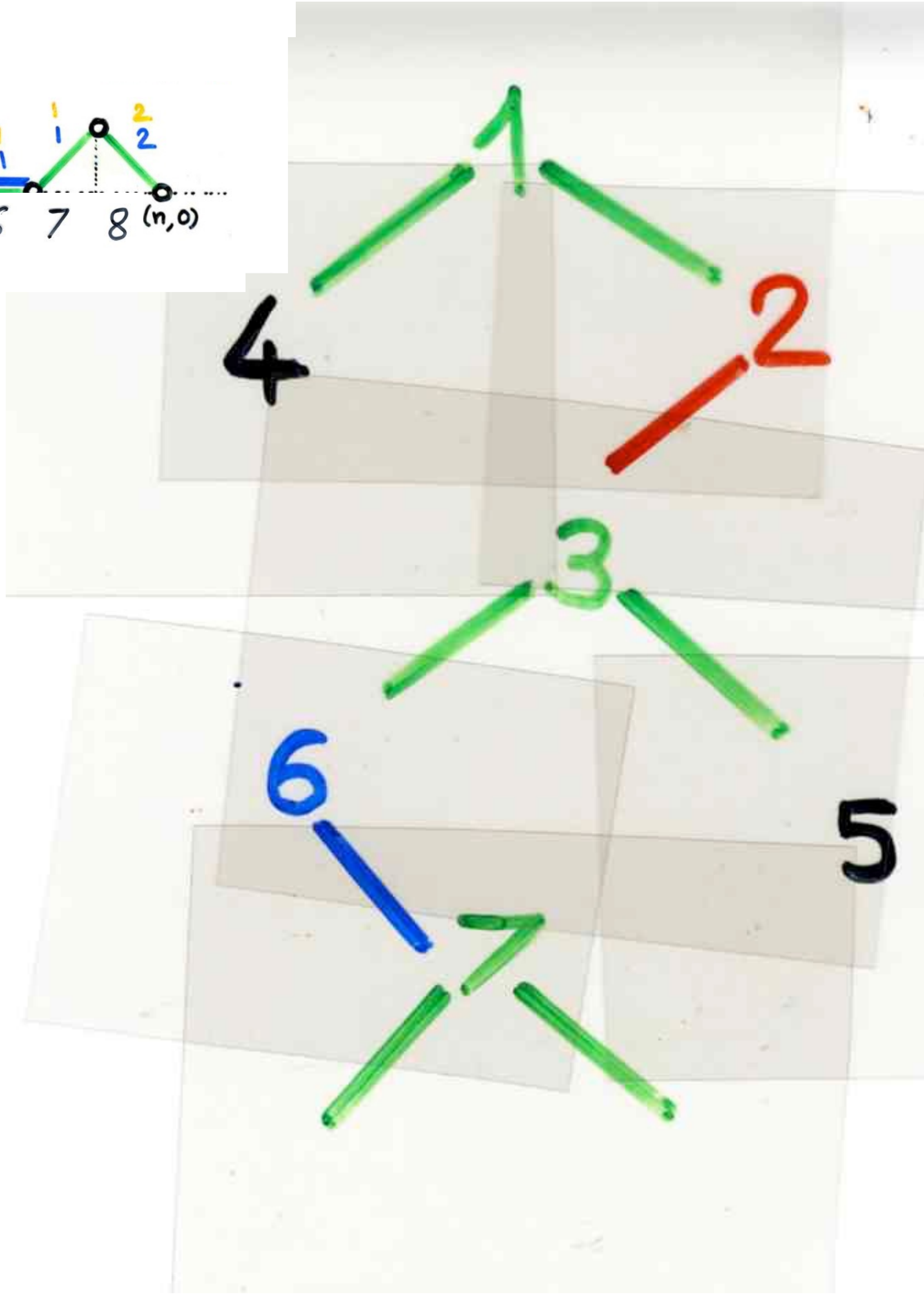

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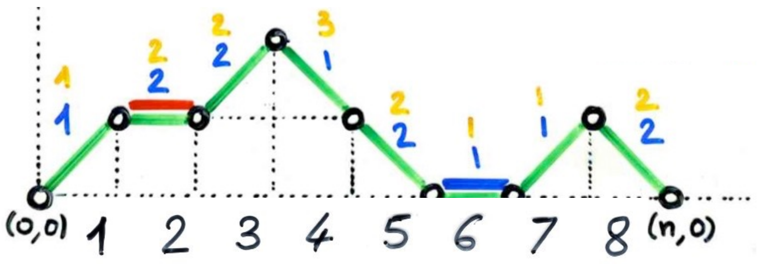
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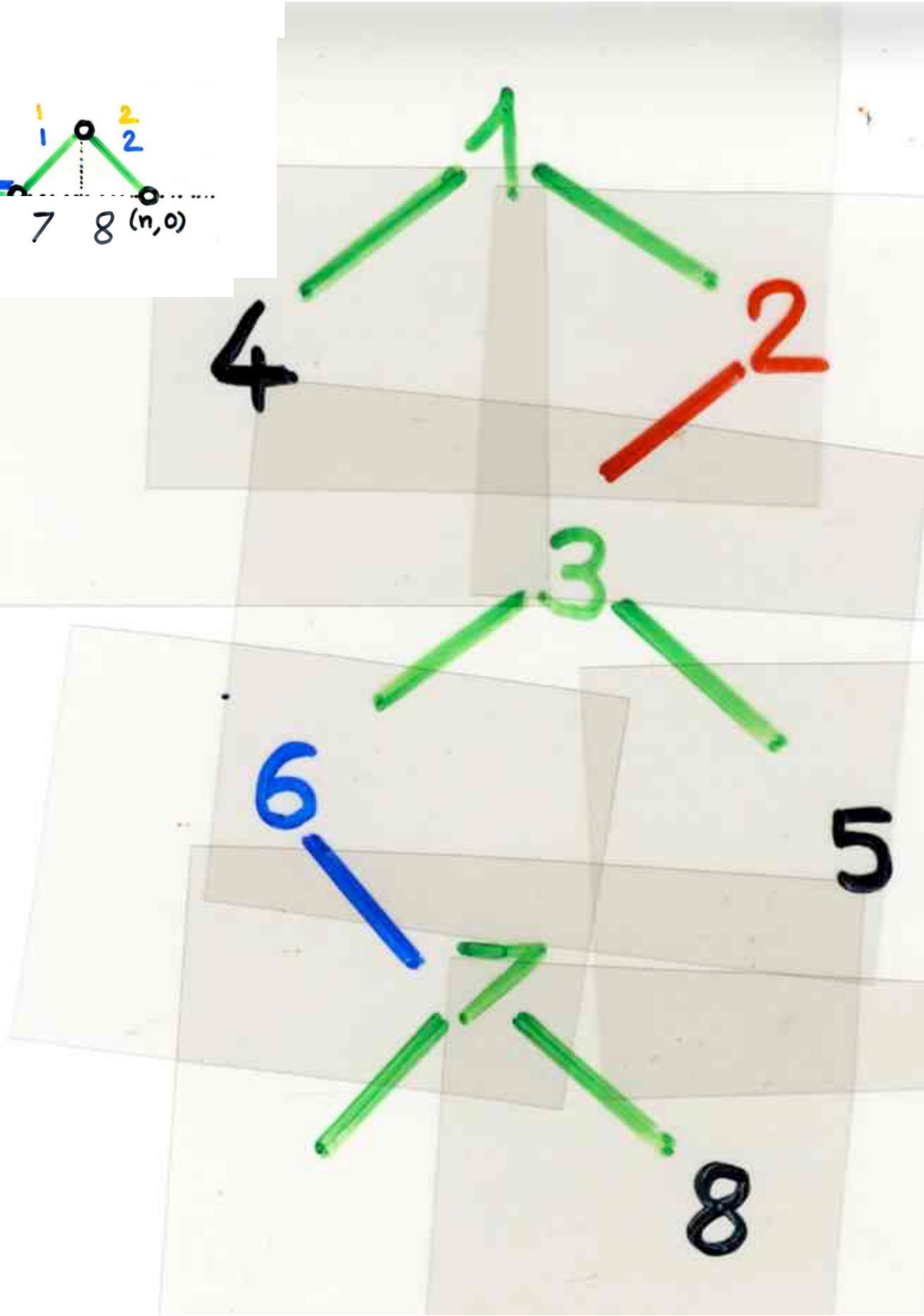
(n,0)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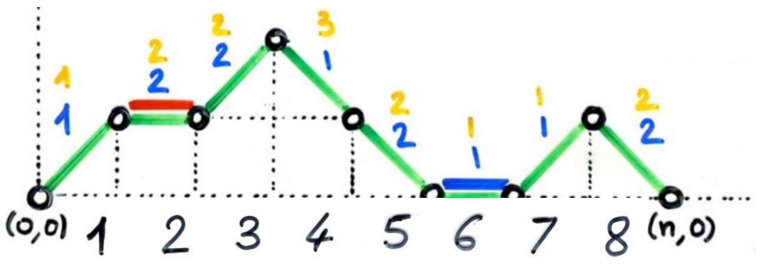

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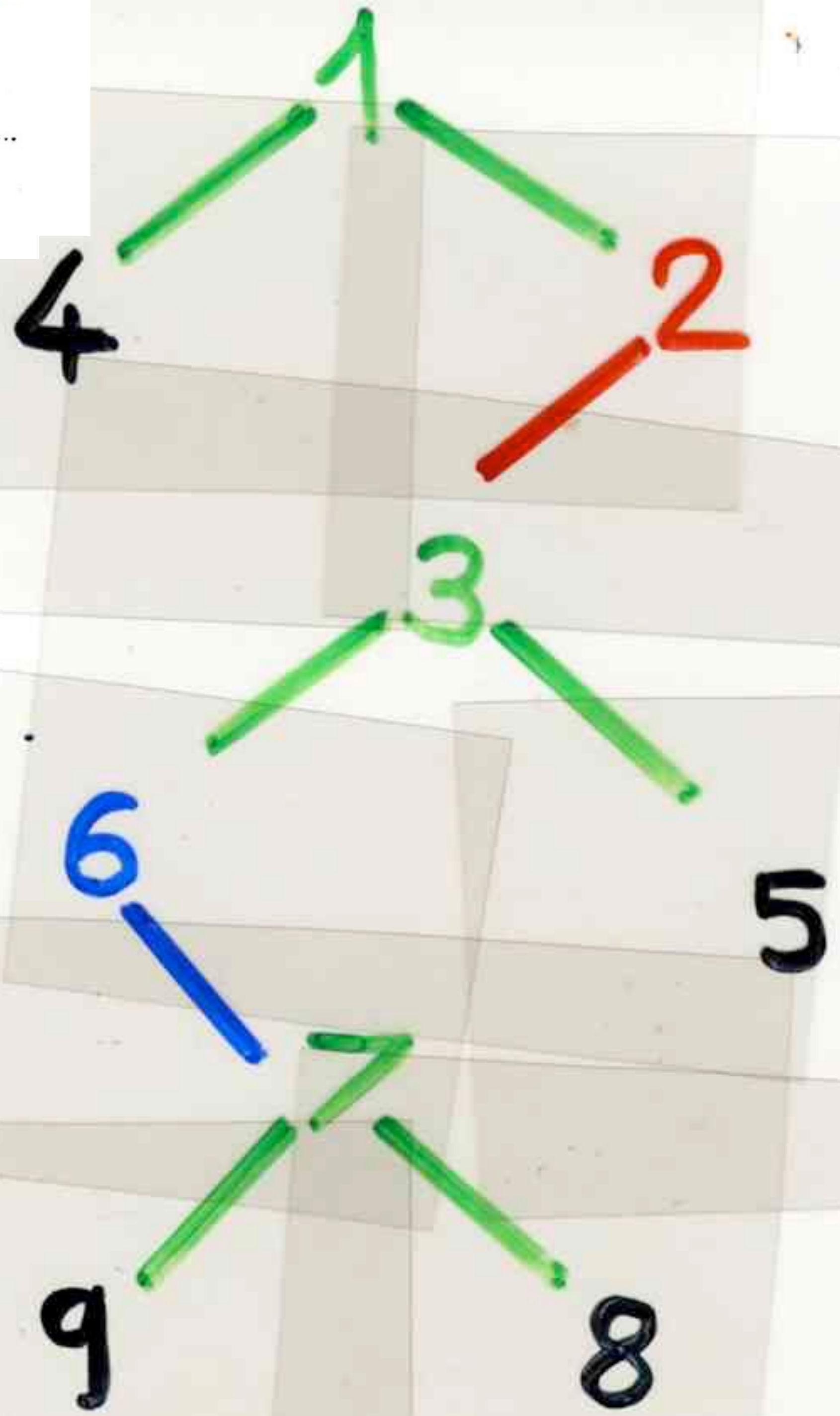

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9
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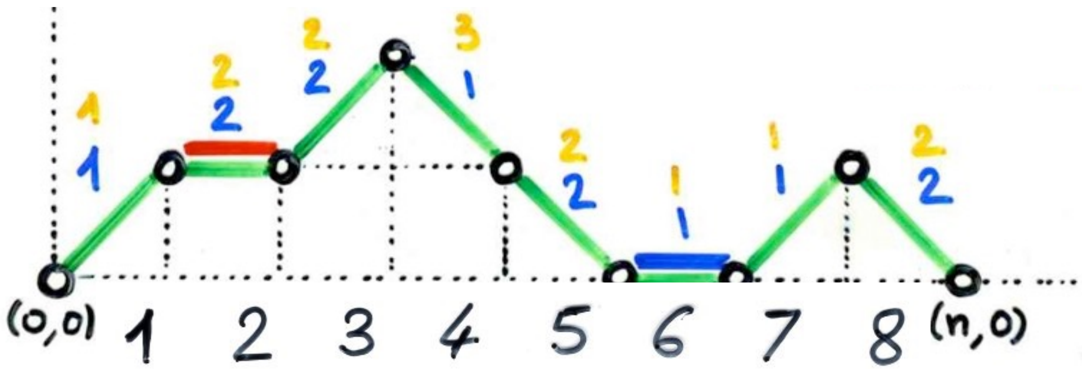
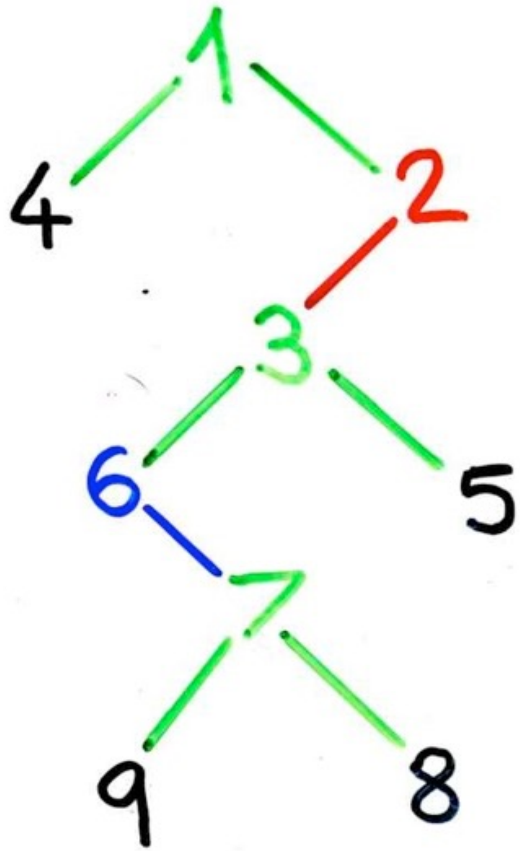




4 1 6 9 7 8 3 5 2

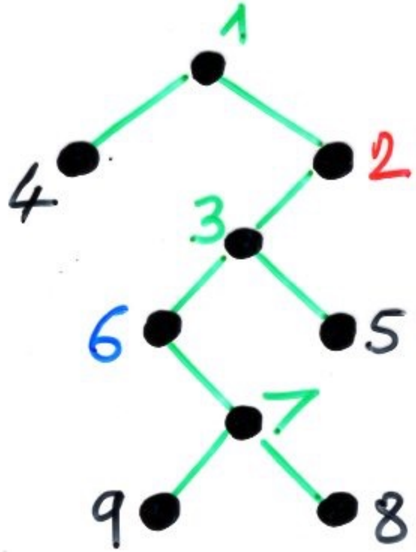
$$\mathcal{L}_n \xrightarrow{\ \varphi\ } \mathcal{T}_{n+1} \xrightarrow{\ \Pi\ } \mathfrak{S}_{n+1}$$

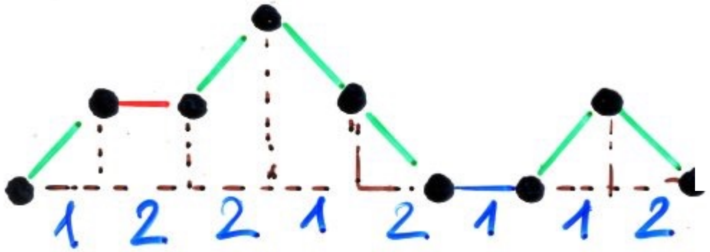
Laguerre histories — increasing binary trees — permutations

$$h = (\omega_c\ ; (p_1, \dots, p_n))$$

ω_c: 2-colored Motzkin path

$(p_1, \dots, p_n)$: choice function





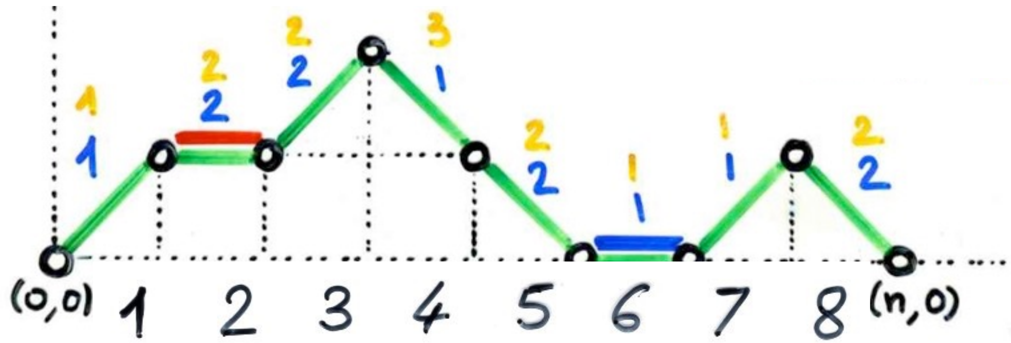
$$\sigma = 4\,1\,6\,9\,7\,8\,3\,5\,2$$

bijection

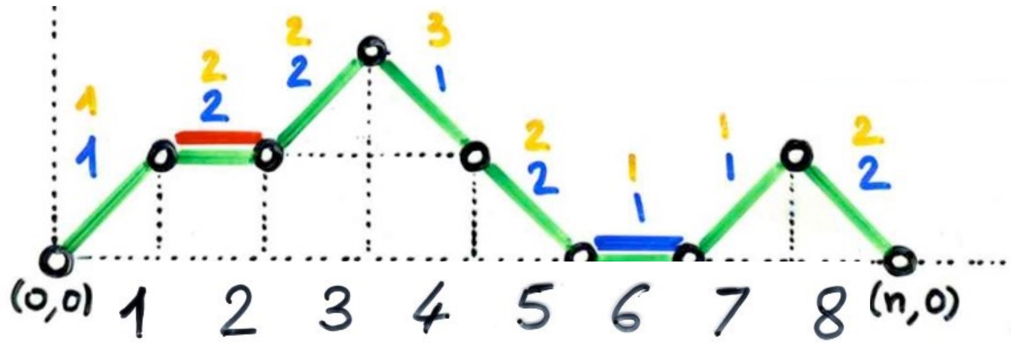
Laguerre histories ⟶ permutations

description with words

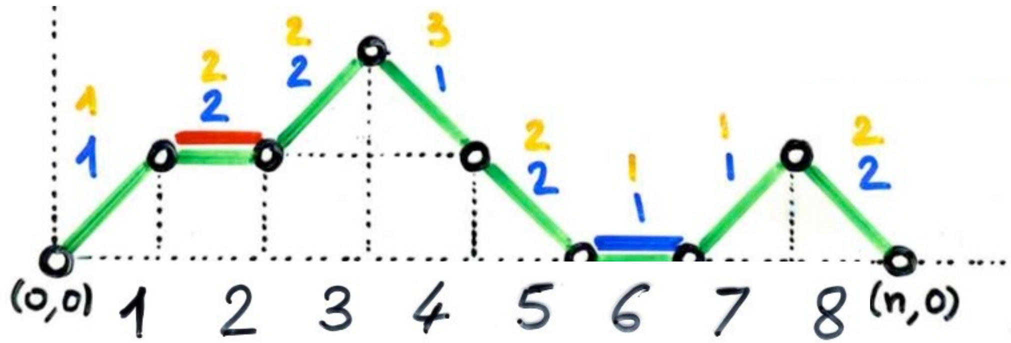
$h = (\omega_c ; (p_1, \ldots, p_n))$

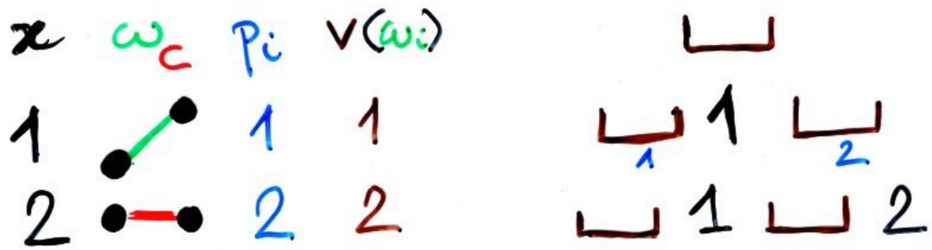


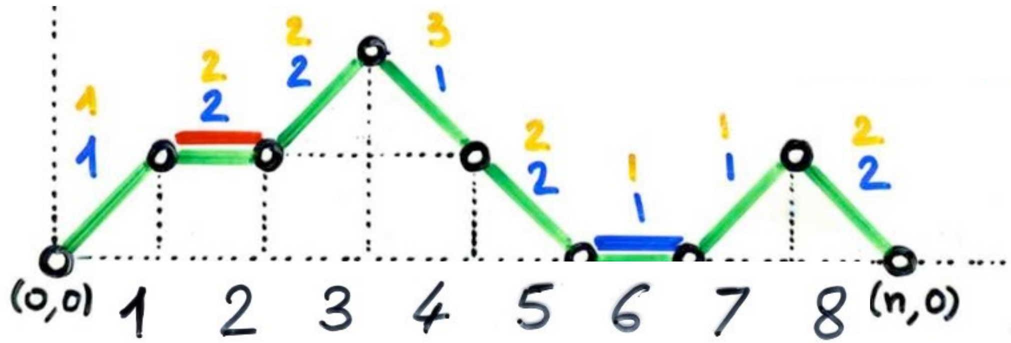
$x \quad \omega_c \quad p_i \quad v(\omega_i)$ ␣



x	ω_c	p_i	$V(\omega_i)$
1		1	1


(0,0)
1 2 3 4 5 6 7 8
(n,0)
1 1
2 2
2 2
3 1
2 2
1 1
1 1
2 2


x ω_c p_i v(ω_i)
1 1 1
2 2 2
1
1 2
1 2


(0,0)
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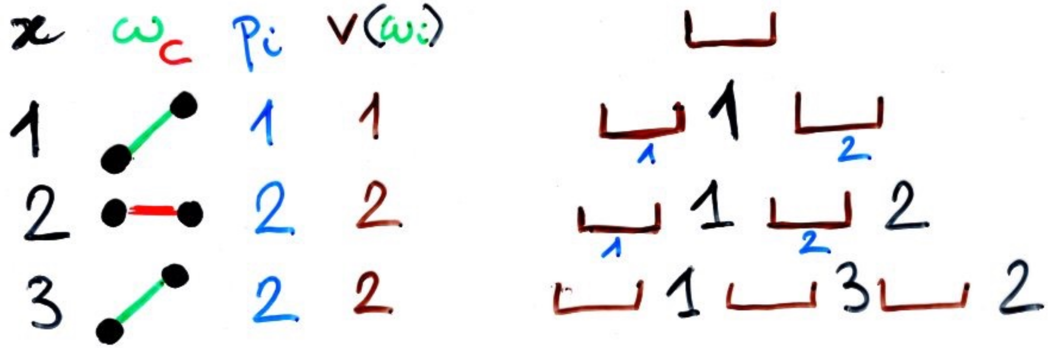
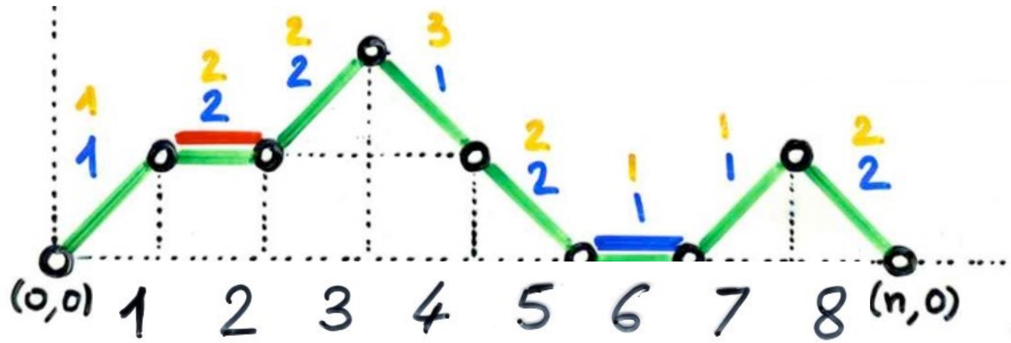
(n,0)


x
ω_c
p_i
v(ω_i)
1
2
3



(0,0)
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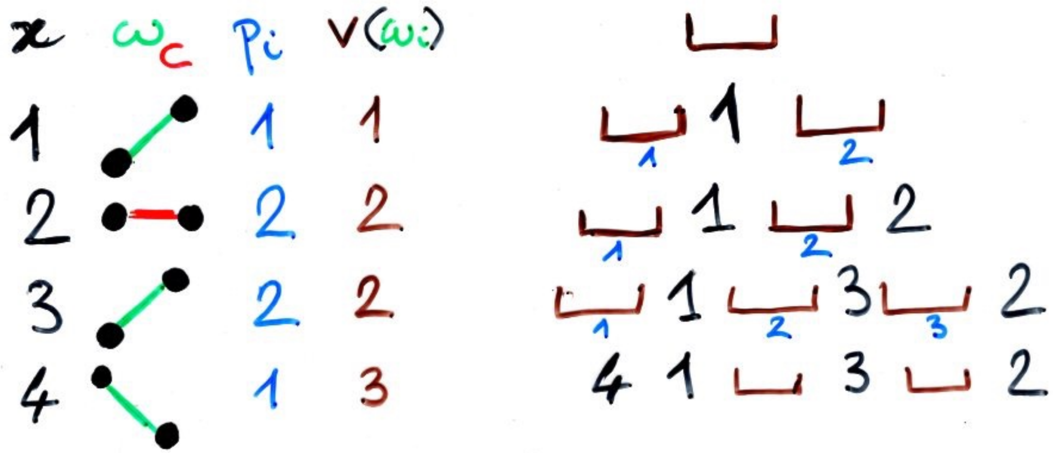
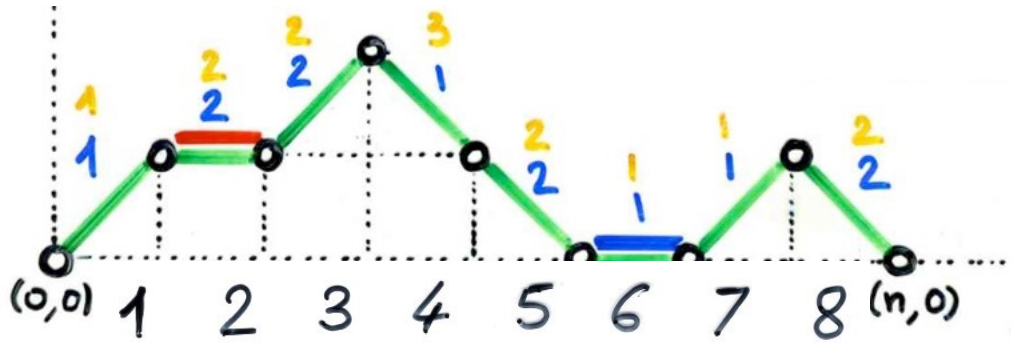
(n,0)



x
ω_c
p_i
$v(\omega_i)$
1 1 1
2 2 2
3 2 2
4 1 3
4 1 3 2


(0,0)
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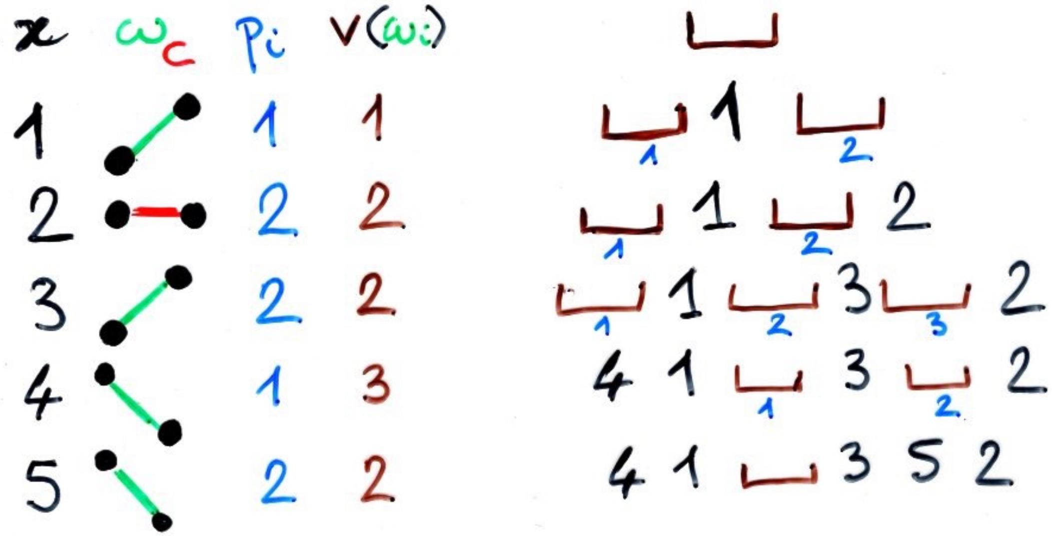
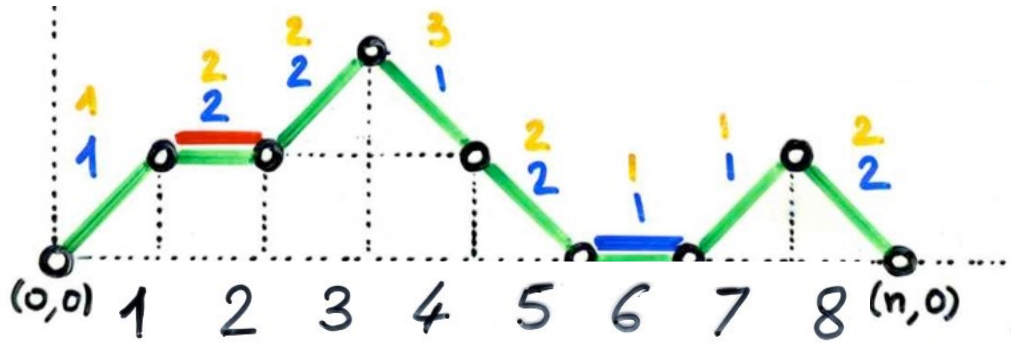
(n,0)


x
ω_c
p_i
$v(\omega_i)$
4 1 3 5 2



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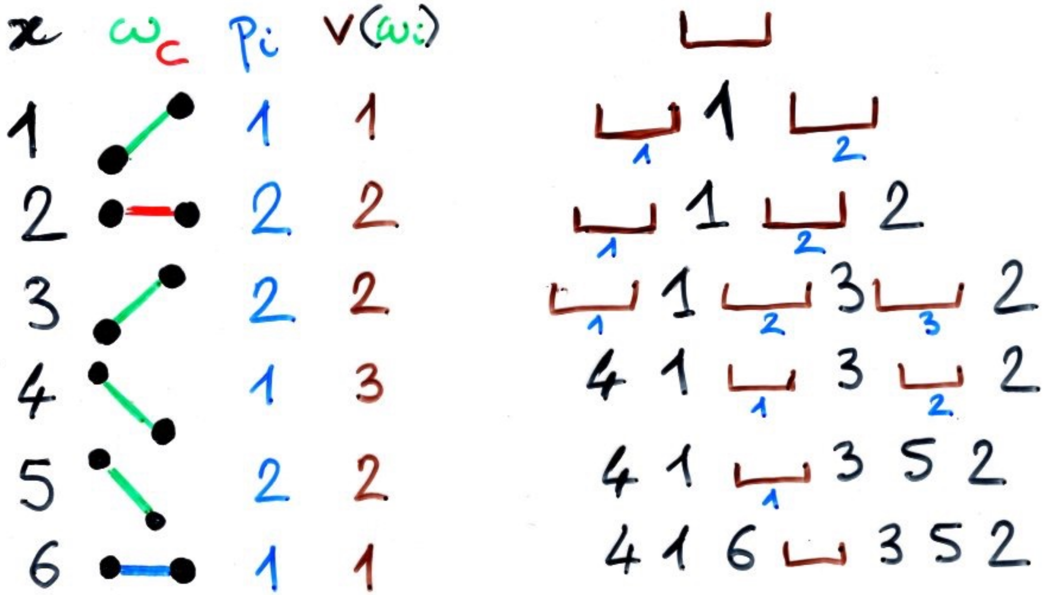
(n,0)



x
ω_c
p_i
v(ω_i)
1 1 1
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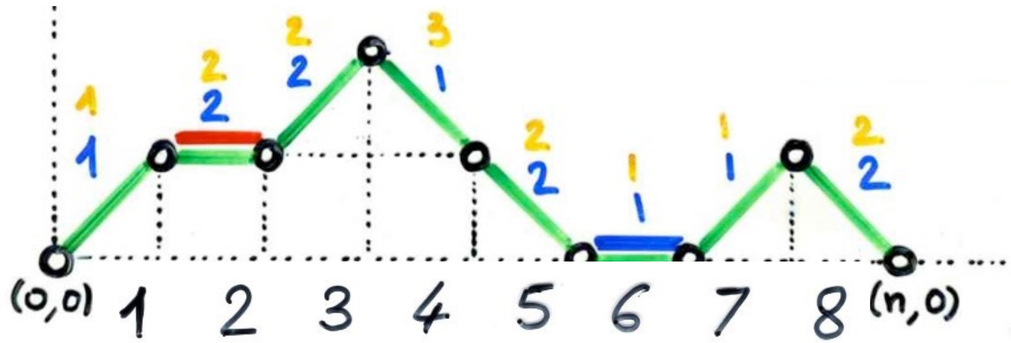
3 2 2
4 1 3
5 2 2
6 1 1
1 2
1 2
1 3 2
4 1 3 2
4 1 3 5 2
4 1 6 3 5 2



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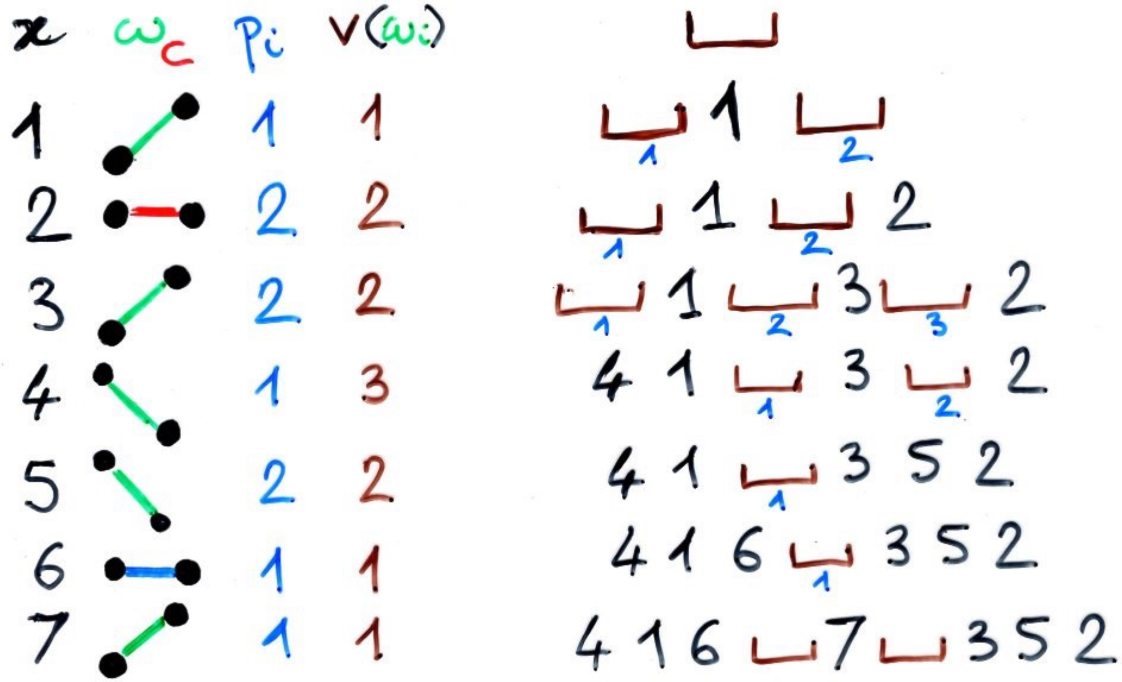
(0,0)
1 2 3 4 5 6 7 8
(n,0)



x ω_c p_i v(ω_i)
1 1 1
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4 1 3
5 2 2
6 1 1
7 1 1
1
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1 3 2
1 2 3
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1 2
4 1 3 5 2
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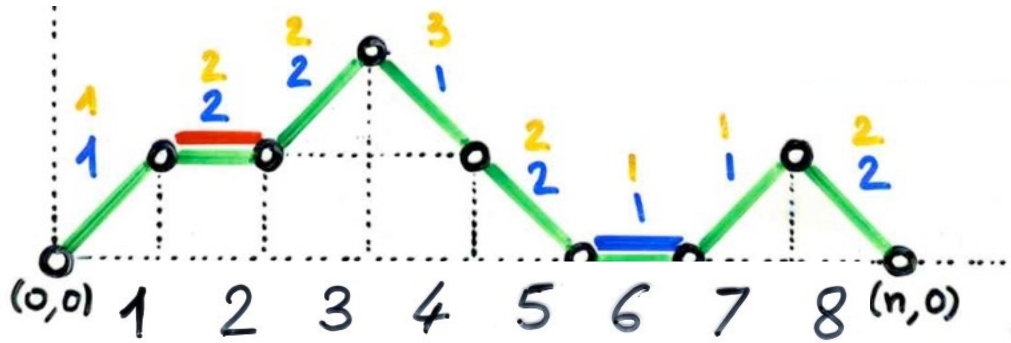
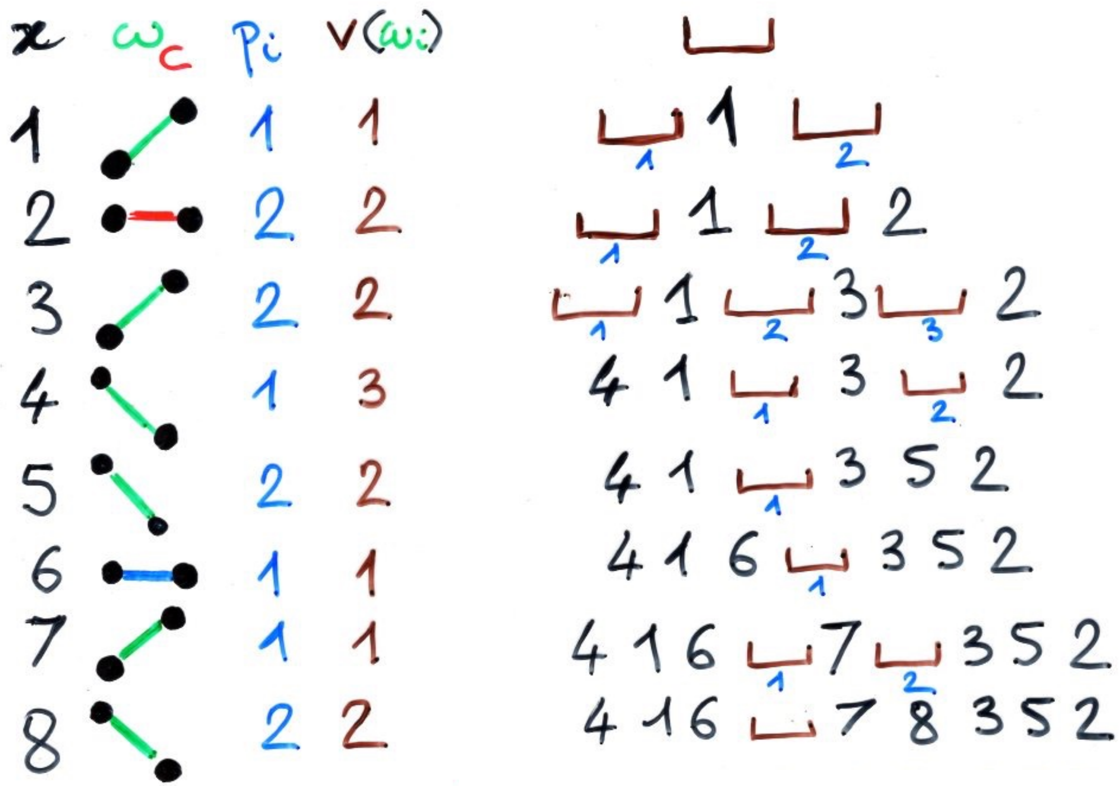
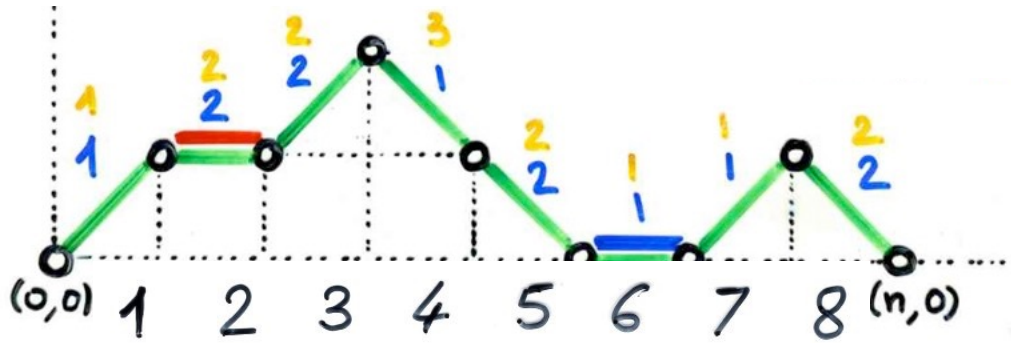
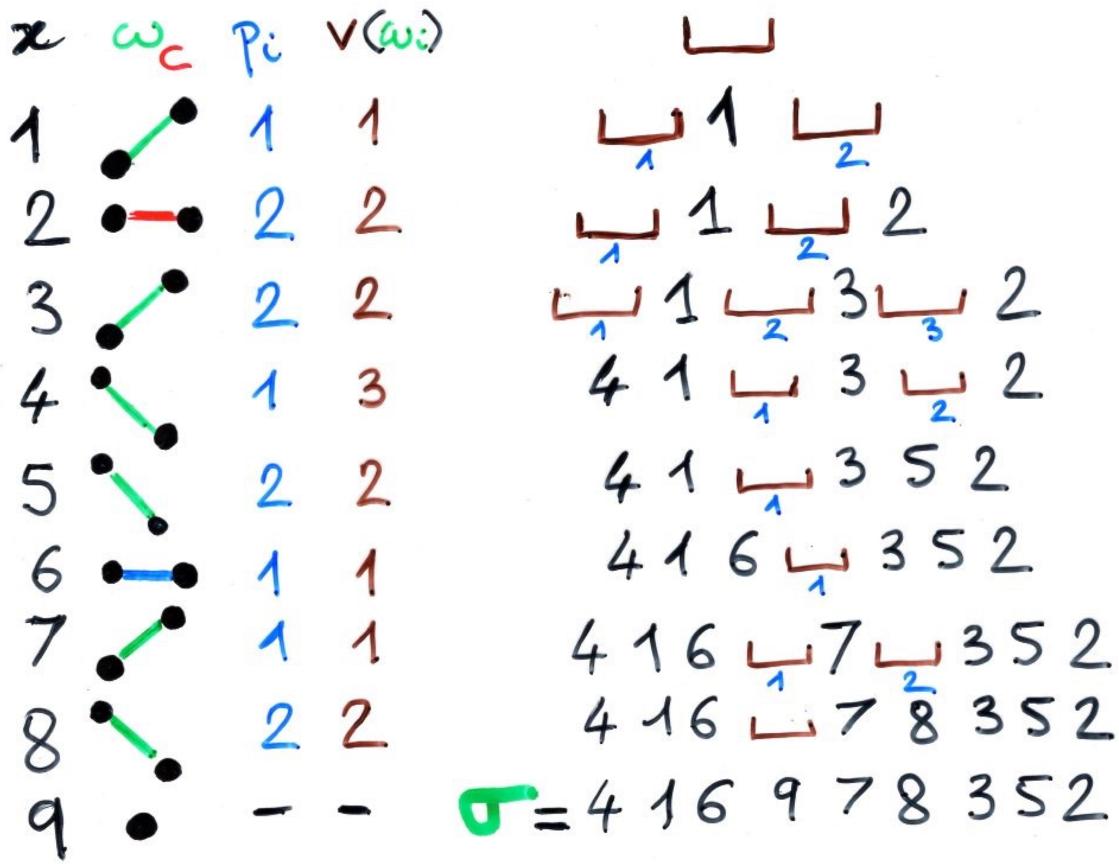
4 1 6 3 5 2
1
4 1 6 7 3 5 2


(0,0)
1 2 3 4 5 6 7 8
(n,0)

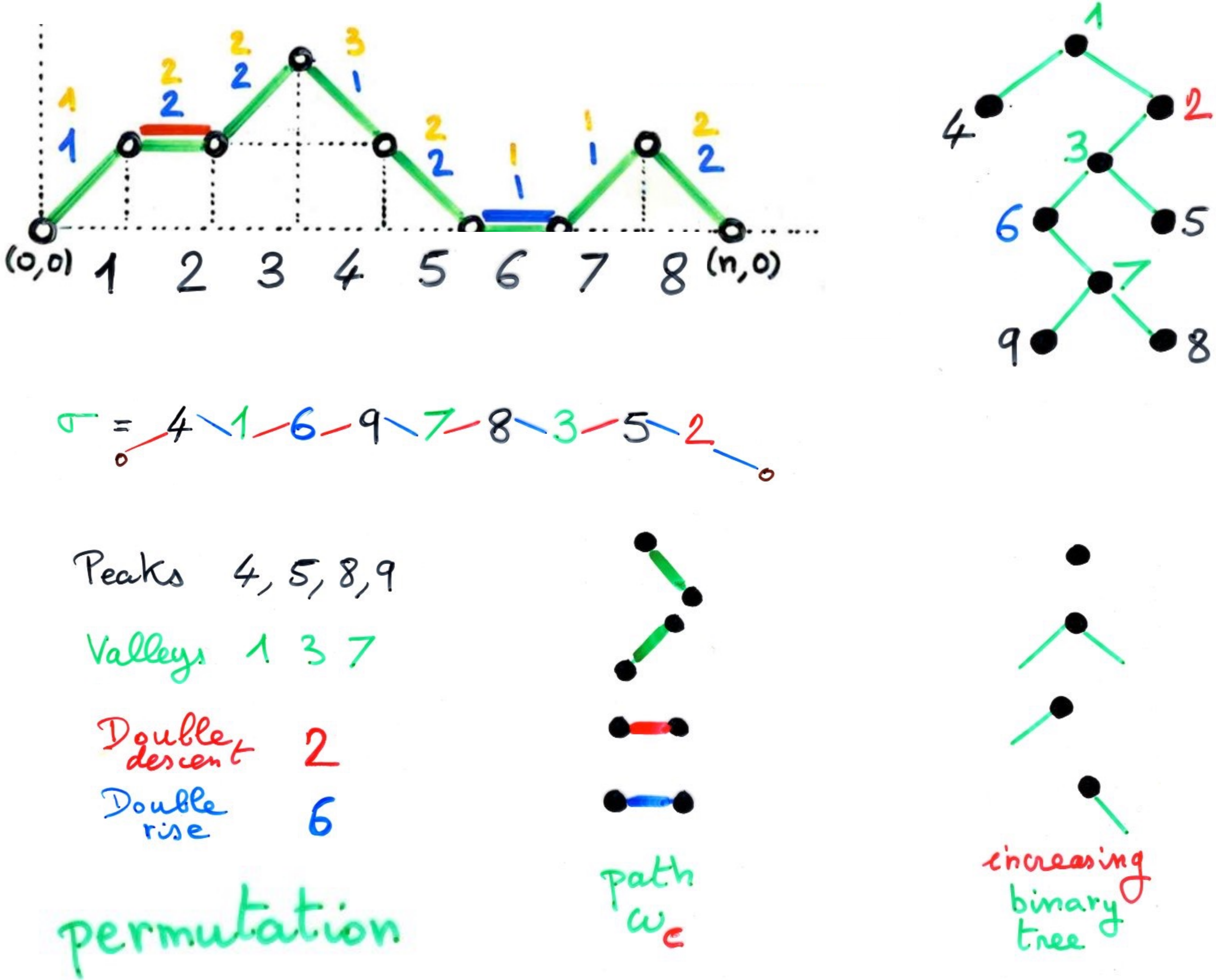

x ω_c p_i v(ω_i)
1 1 1
2 2 2
3 2 2
4 1 3
5 2 2
6 1 1
7 1 1
8 2 2
4 1 6 7 8 3 5 2


(0,0)
1 2 3 4 5 6 7 8
(n,0)


x
ω_c
p_i
v(ω_i)
σ = 4 1 6 9 7 8 3 5 2

reciprocal bijection

permutations $\longrightarrow$ Laguerre histories



(0,0) 1 2 3 4 5 6 7 8 (n,0)
σ = 4 1 6 9 7 8 3 5 2
Peaks 4, 5, 8, 9
Valleys 1 3 7
Double descent 2
Double rise 6
permutation
path ω_c
increasing binary tree

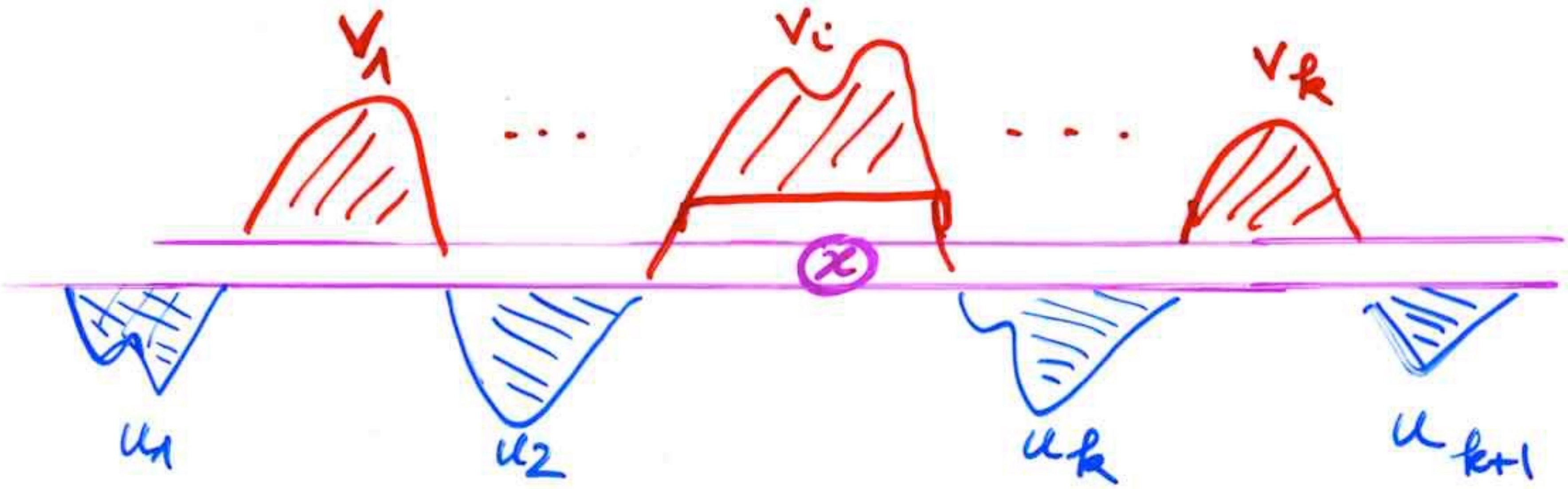
Def - $\sigma \in \mathfrak{S}_n$, $x \in [1, n]$

x- decomposition

$$\begin{cases} \bullet \ \sigma = u_1 v_1 \ldots u_k v_k u_{k+1} \\ \bullet \ \text{letters } (u_i) < x \\ \bullet \ \text{letters } (v_j) \geq x \\ \bullet \ \text{words } v_1, u_2, \ldots, u_k, v_k \ \text{ non empty} \end{cases}$$

ex. $\sigma = 4\,1\,6\,9\,7\,8\,3\,5\,2$, $x = 3$

$$\underbrace{\ }_{u_1}\overbrace{4}^{v_1}\underbrace{1}_{u_2}\overbrace{6\,9\,7\,8\,3\,5}^{v_2}\underbrace{2}_{u_3}$$



v_1
v_i
v_k
x
u_1
u_2
u_k
u_{k+1}

reciprocal bijection

$$\sigma \in \mathfrak{S}_{n+1} \longrightarrow \left(\omega_c \,;\, (p_1, \dots, p_n)\right)$$

$$\omega_c = \omega_1 \cdots \omega_n$$

- ω_i is
  i^{th} step

iff

i valley

i peak

i double rise

i double descent

- $p_i = j$ iff i is a letter of v_j in the i-decomposition of σ

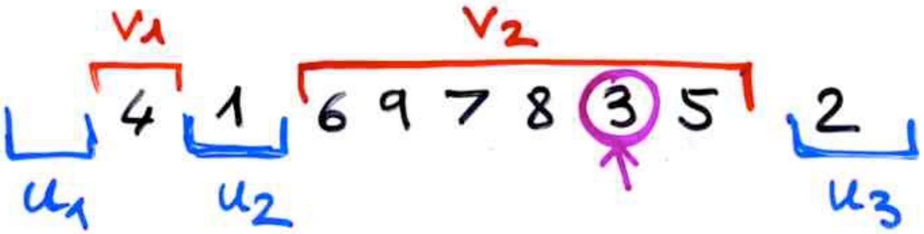
$$\sigma = u_1 v_1 \dots v_j \dots u_k v_k u_{k+1}$$

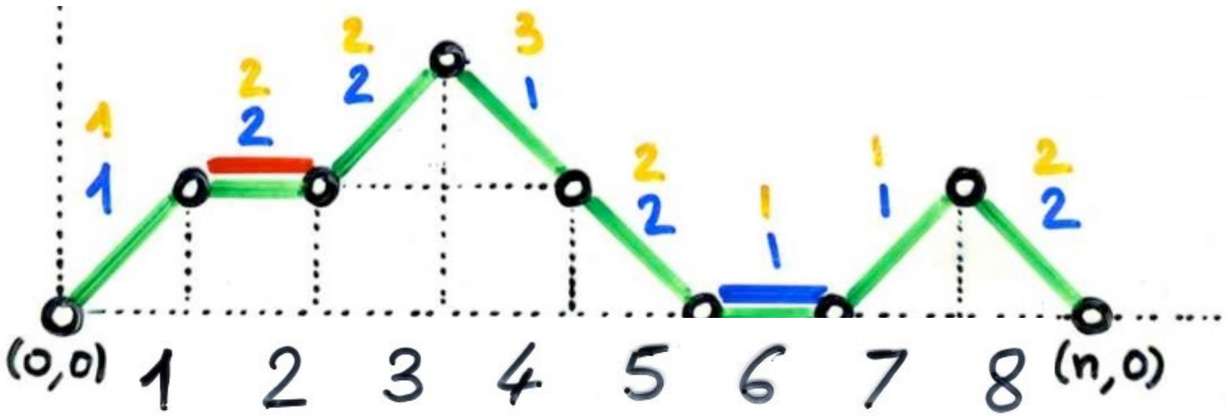
example

- $p_i = j$ iff i is a letter of v_j in the i-decomposition of σ

$$\sigma = u_1 v_1 \dots v_j \dots u_k v_k u_{k+1}$$

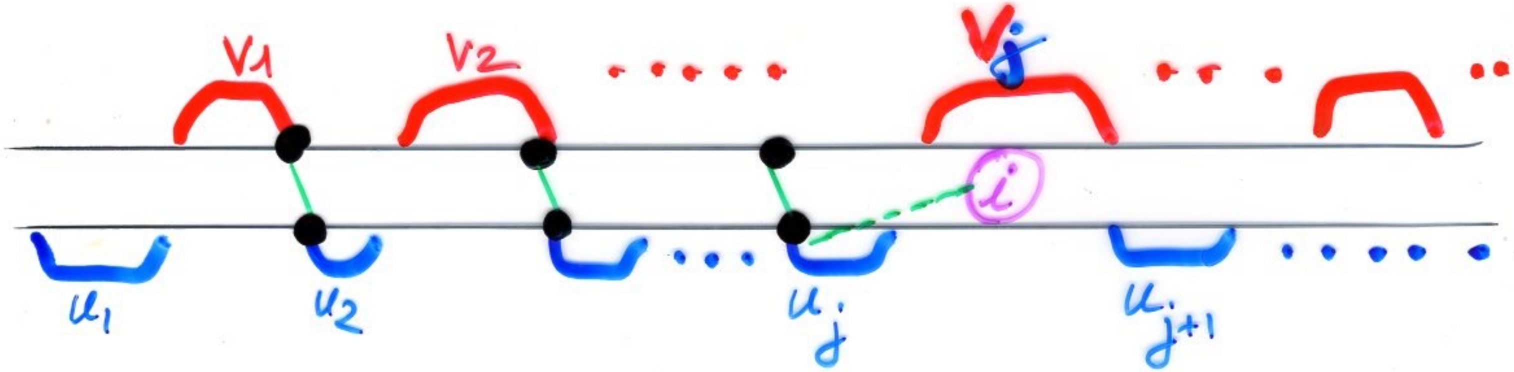
ex. $\sigma = 4\,1\,6\,9\,7\,8\,3\,5\,2$, $x = 3$





<u>Lemma</u> $P_i = j$ is also defined by:

$j = 1 +$ number of triples (a, b, i) having the pattern $(31-2)$, that is $a = \sigma(k)$, $b = \sigma(k+1)$, $i = \sigma(\ell)$ with $k < k+1 < \ell$ and $b < i < a$



relation with

(formal) orthogonal polynomials

→ course on combinatorics of orthogonal polynomials

Orthogonal polynomials

Def. $\{P_n(x)\}_{n\geq 0}$ $\quad P_n(x) \in \mathbb{K}[x]$

orthogonal iff $\exists\ f : \mathbb{K}[x] \longrightarrow \mathbb{K}$

linear functional

$$\begin{cases} \text{(i)} & \deg(P_n(x)) = n \quad (\forall n \geq 0) \\ \text{(ii)} & f(P_k P_l) = 0 \quad \text{for } k \neq l \geq 0 \\ \text{(iii)} & f(P_k^2) \neq 0 \quad \text{for } k \geq 0 \end{cases}$$

$$\mathcal{L}(x^n) = \mu_n \quad (n \geqslant 0)$$

moments

$$\mathcal{L}(PQ) = \int_a^b P(x)\,Q(x)\,d\mu$$

mesure

Thm. (Favard)

- $\{P_n(x)\}_{n\geqslant 0}$ sequence of monic polynomials, $\deg(P_n)=n$
- $\{b_k\}_{k\geqslant 0}$, $\{\lambda_k\}_{k\geqslant 1}$ coeff. in $\mathbb{K}$

orthogonality $\Longleftrightarrow$

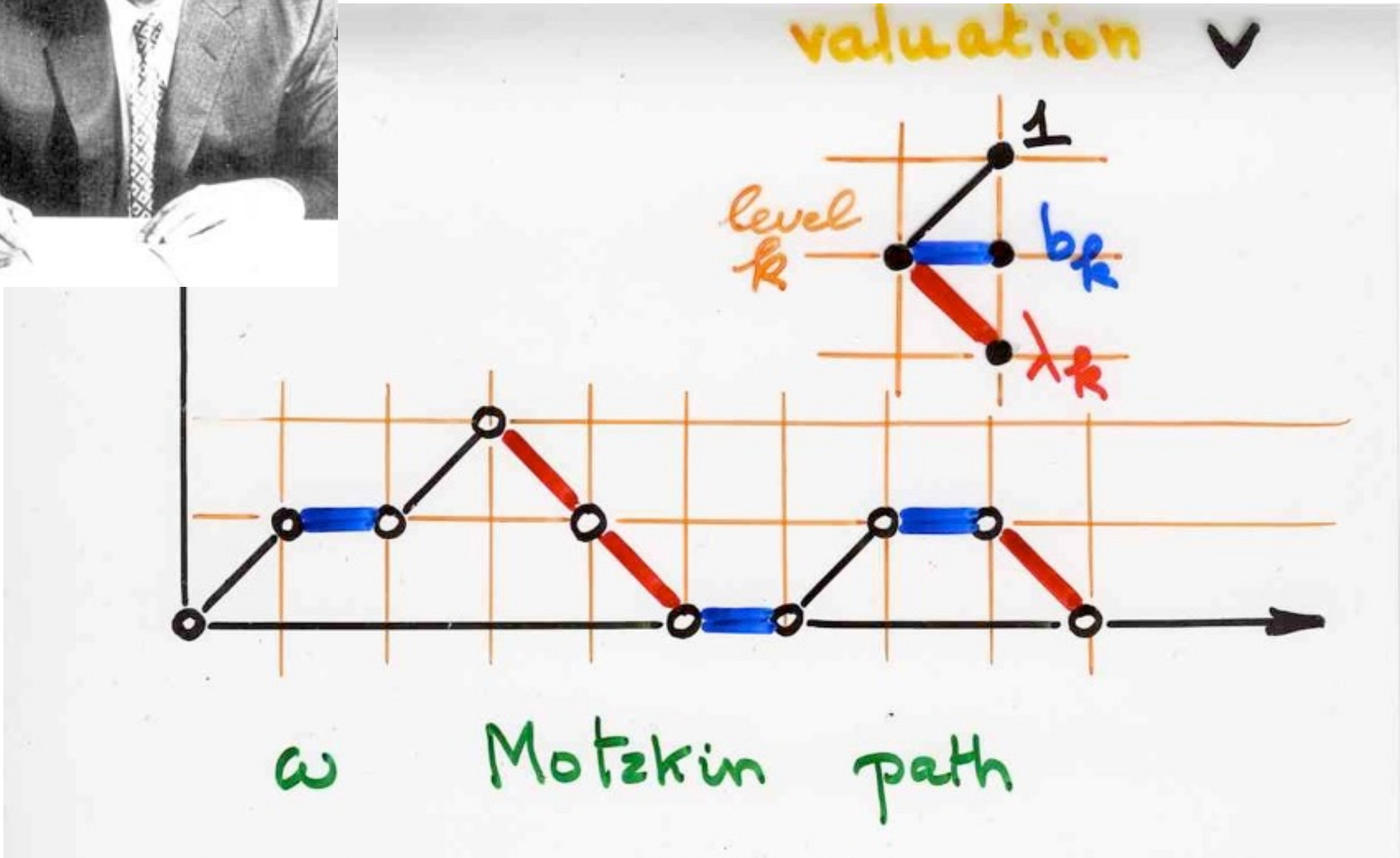
$$P_{k+1}(x) = (x-b_k)P_k(x) - \lambda_k P_{k-1}(x)$$

$(\forall k\geqslant 1)$

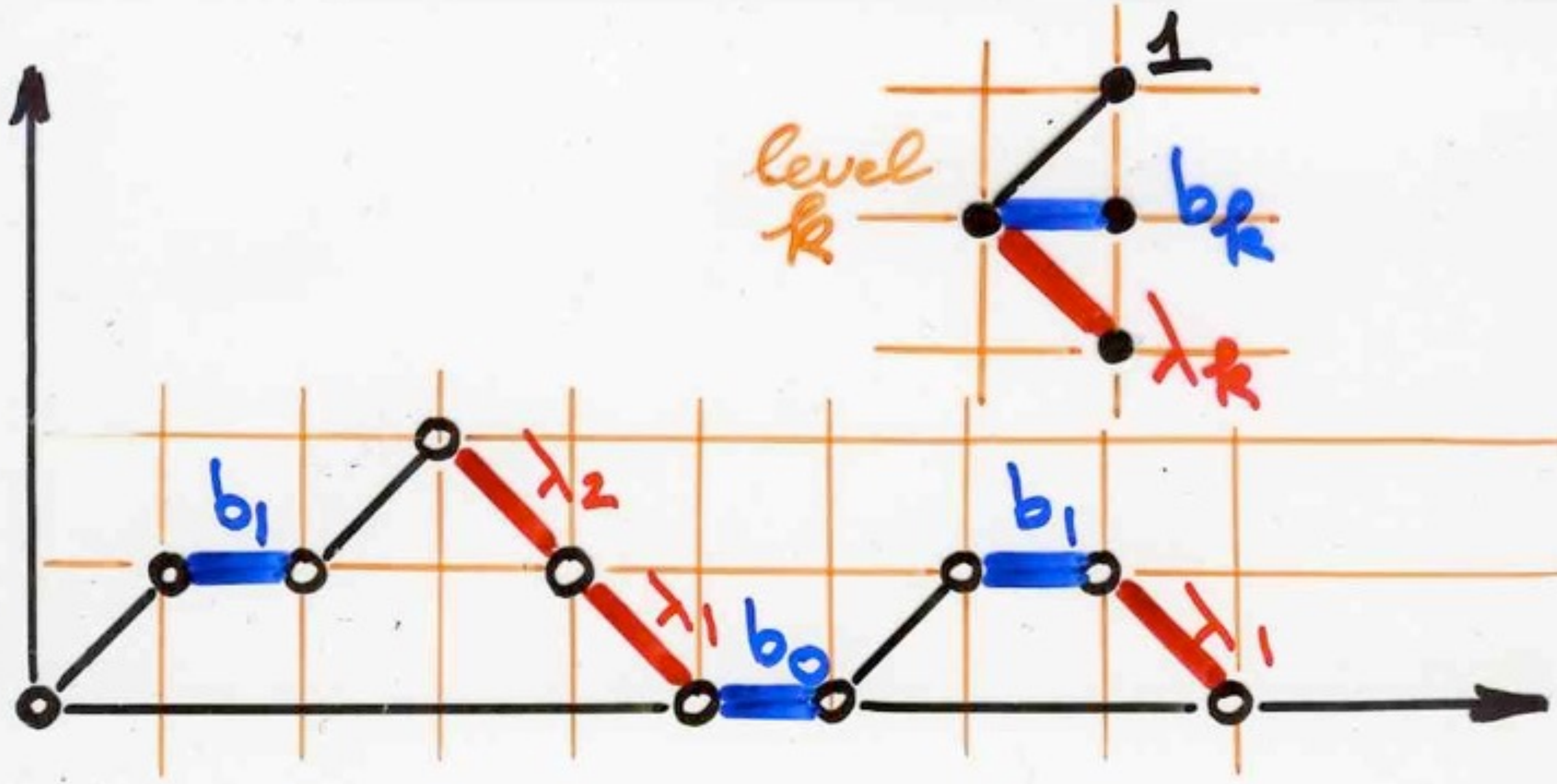
3 terms linear recurrence relation

$\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$

$b_k , \lambda_k \in \mathbb{K}$ ring



valuation v



ω Motzkin path

$$v(\omega) = b_0 \, b_1^2 \, \lambda_1^2 \, \lambda_2$$

$$f(x^n) = \mu_n \quad (n \geq 0)$$

moments

$$\mu_n = \sum_{\substack{\omega \\ \text{Motzkin path} \\ |\omega| = n}} v(\omega)$$

Laguerre histories
and
moment of Laguerre polynomials

Laguerre polynomial

$$\mu_n = (n+1)!$$

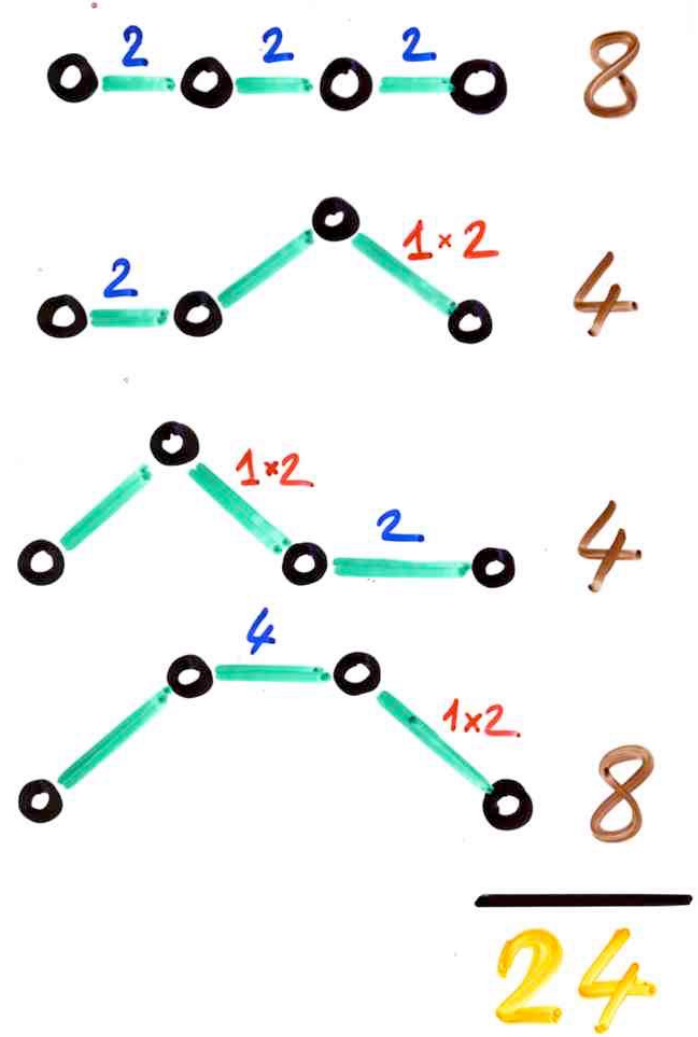
$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

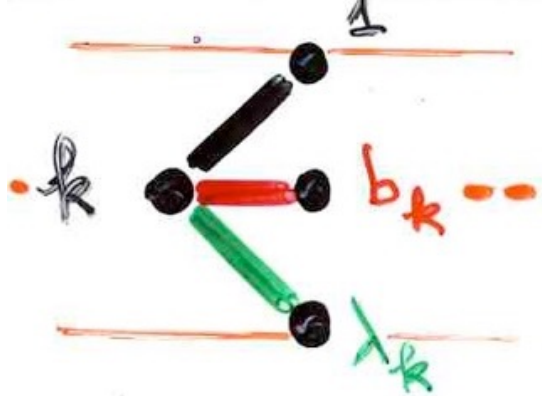
Laguerre $L_n^{(1)}(x)$

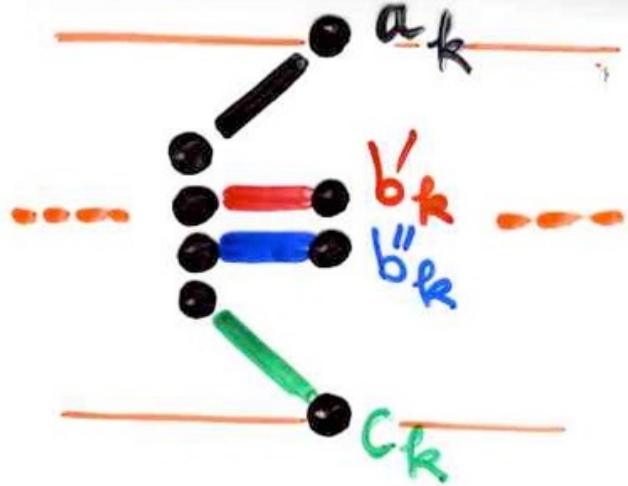
moment $\mu_n = (n+1)!$

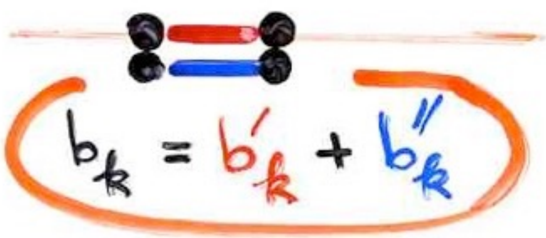
$b_k = 2k+2$

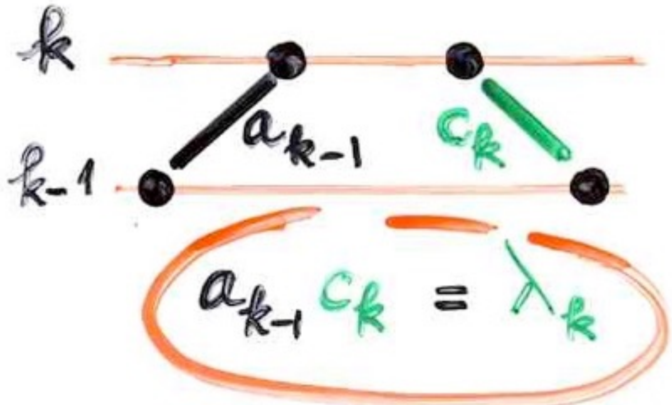
$\lambda_k = k(k+1)$

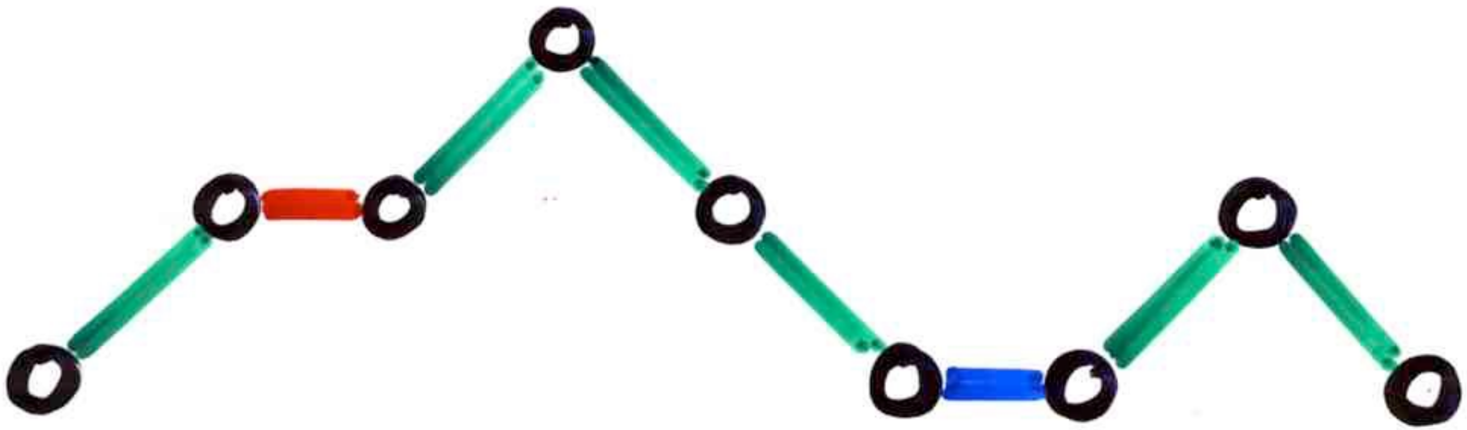


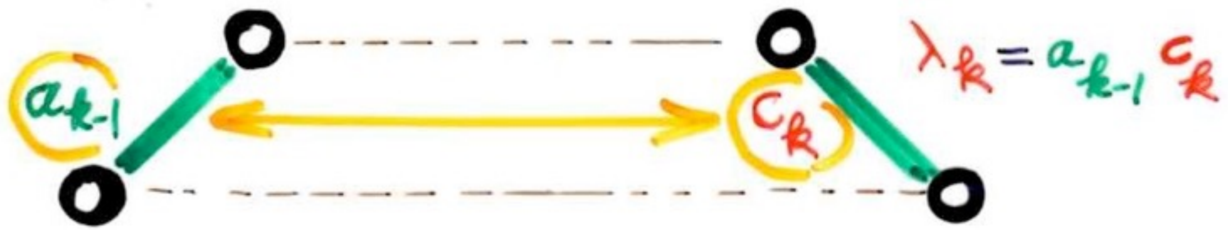

1
k
b_k
λ_k

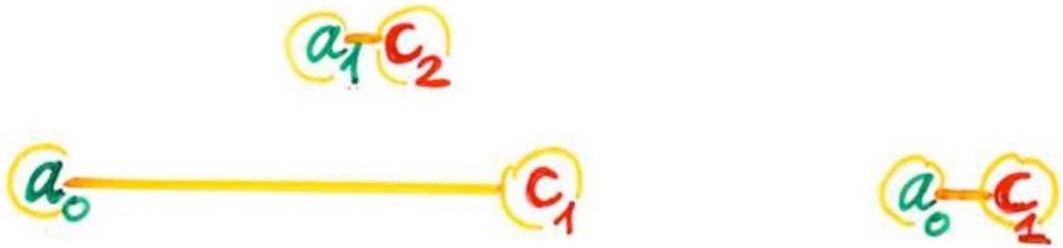

a_k
b'_k
b''_k
c_k


$b_k = b'_k + b''_k$


k
k-1
a_{k-1}
c_k
$a_{k-1} c_k = \lambda_k$




a_{k-1}
c_k
$\lambda_k = a_{k-1} c_k$


a_1
c_2
a_0
c_1
a_0
c_1

$$(n+1)! = \sum_{\substack{|\omega|=n \\ \text{Motzkin}}} v(\omega)$$

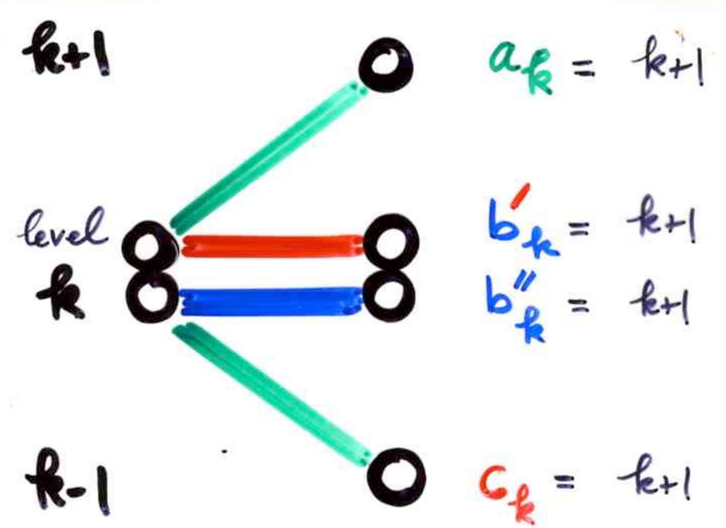
$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

$$= \sum_{\substack{|\omega|=n \\ \text{2-colored} \\ \text{Motzkin}}} v^*(\omega)$$

$$\begin{cases} b'_k = k+1 \\ b''_k = k+1 \\ a_k = k+1 \\ c_k = k+1 \end{cases}$$

$$\lambda_k = a_{k-1}\, c_k$$

$$b_k = b'_k + b''_k$$



weigthed Laguerre histories

Laguerre $L_n^{(\alpha)}$

$$b_k = 2k + \alpha + 1 \; ; \quad \lambda_k = k(k + \alpha)$$

$$(n+1)! = \sum_{\substack{|\omega|=n \\ \text{Motzkin}}} v(\omega) = \sum_{\substack{|\omega|=n \\ \text{2-colored} \\ \text{Motzkin}}} v^*(\omega)$$

$$\begin{cases} b_k = 2k+2 \\ \lambda_k = k(k+1) \end{cases}$$

$$\begin{cases} b'_k = k+1 \\ b''_k = k+1 \\ a_k = k+1 \\ c_k = k+1 \end{cases}$$

$$\lambda_k = a_{k-1} c_k$$

$$b_k = b'_k + b''_k$$

Laguerre $L_n^{(\alpha)}$

$$b_k = 2k + \alpha + 1 \quad ; \quad \lambda_k = k(k+\alpha)$$

 $a_k = k+1$  $\begin{cases} b'_k = k+\alpha \\ b''_k = k+1 \end{cases}$; $c_k = k+\alpha$ 

$(k \geq 0)$ $\qquad (k \geq 0) \qquad (k \geq 1)$

$$\lambda_k = a_{k-1}\, c_k$$

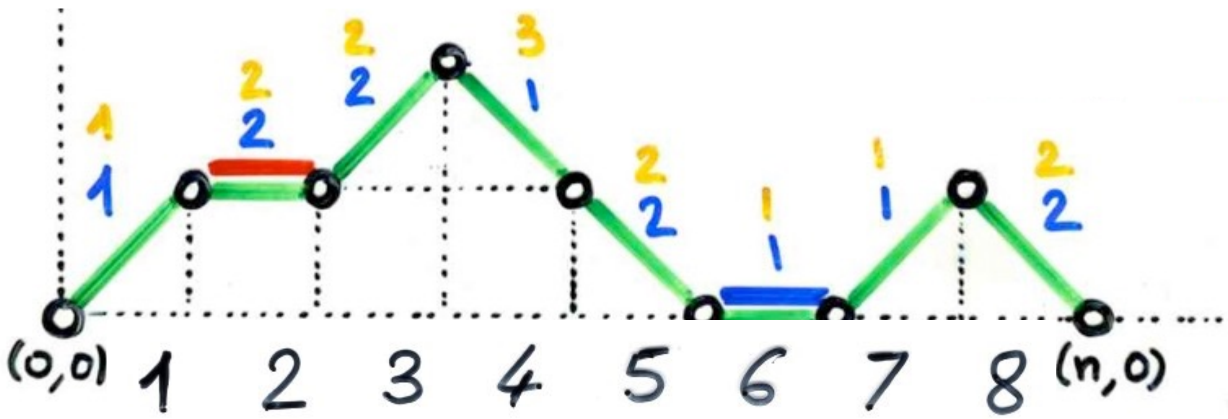
$$b_k = b'_k + b''_k$$

Laguerre polynomial $L_n^{(\alpha)}(x)$

$$h = (\omega_c\,;(p_1,\dots,p_n))$$

$$\omega_c = \omega_1 \cdots \omega_n$$

weighted Laguerre histories



put a weight α for each choice $p_i = 1$

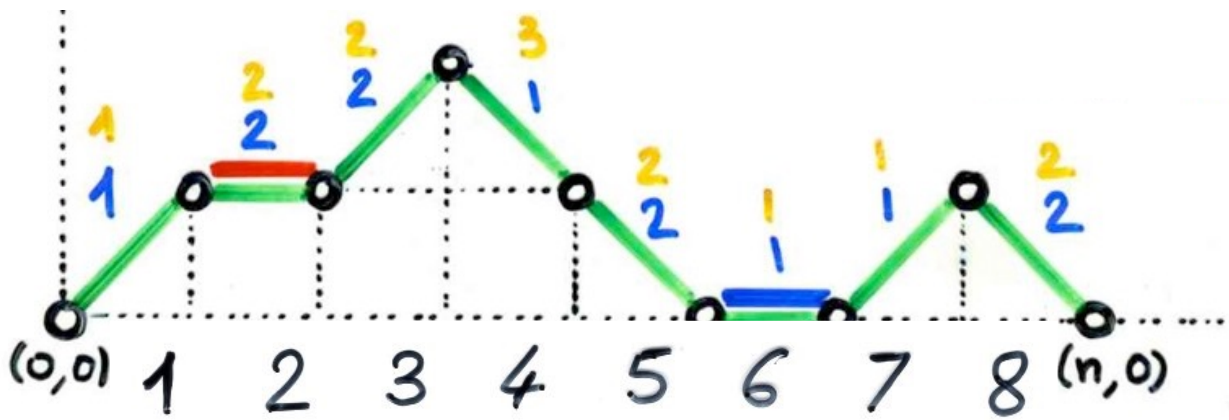
with ω_i = { blue East step 
or South-East step 

this is equivalent to say that the element i is a lr-max element of the permutation σ (except $i = n+1$)

<u>example</u> σ = ④ 1 ⑥ ⑨ 7 8 3 5 2

↑ lr-max ↗

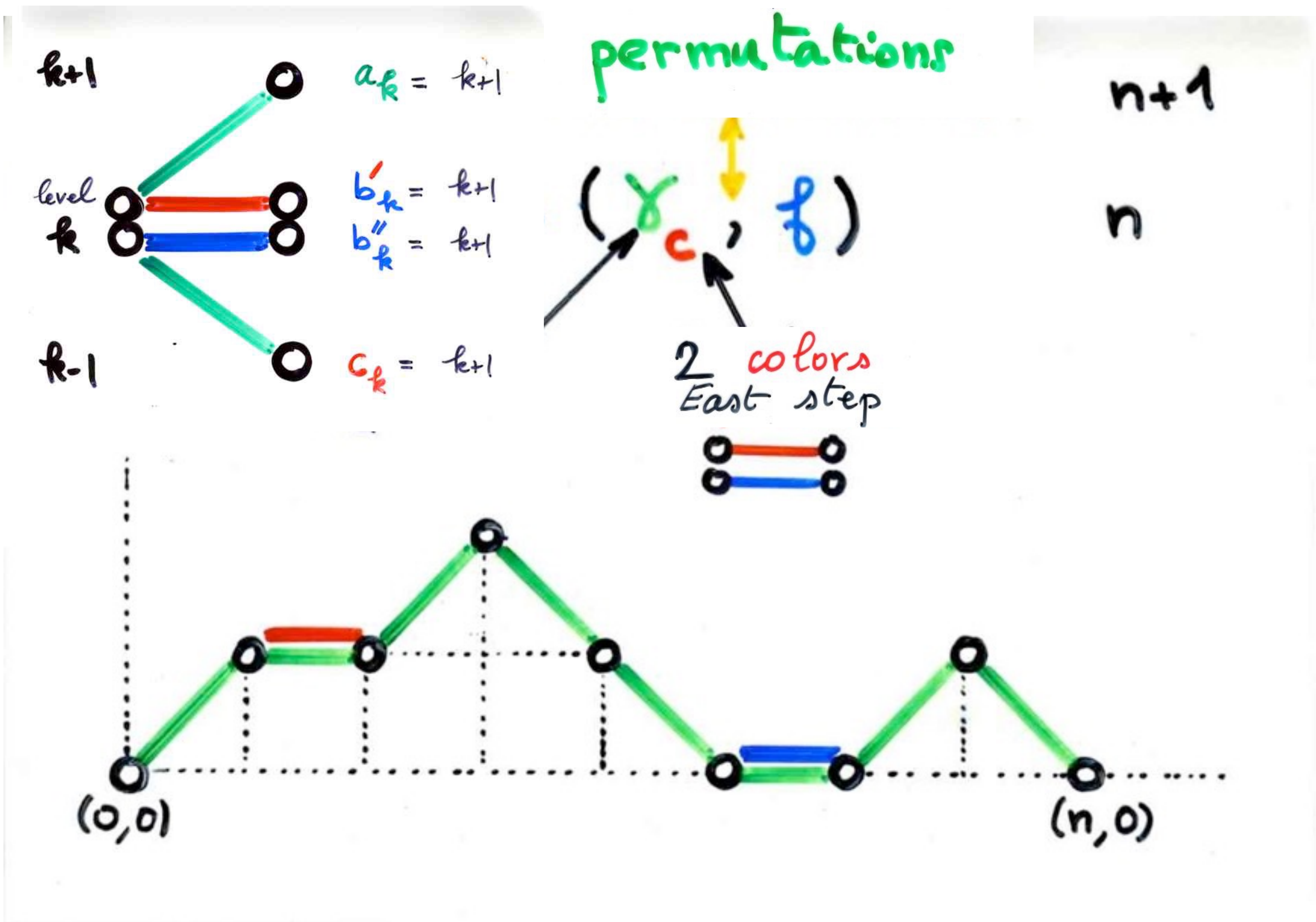
{ ω_4 =  , $p_4 = 1$
{ ω_6 =  , $p_6 = 1$



Corollary The moments of the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ are:

$$\mu_n = (\alpha+1)(\alpha+2)\ldots(\alpha+n)$$

restricted Laguerre histories



permutations
k+1
level k
k-1
a_k = k+1
b'_k = k+1
b''_k = k+1
c_k = k+1
(γ_c , f)
2 colors
East step
n+1
n
(0,0)
(n,0)

valuation corresponding to the $(n+1)!$

(enlarged) Laguerre histories
(of length n)

Laguerre $L_n^{(1)}(x)$

$\mu_n = (n+1)!$

$a_k = k+1$
$b'_k = k+1$
$b''_k = k+1$
$c_k = k+1$

valuation corresponding to the $(n+1)!$

(enlarged) Laguerre histories (of length n)

Laguerre $L_n^{(1)}(x)$

$\mu_n = (n+1)!$

$\mu_n = n!$ $L_n^{(0)}(x)$

a_k	=	$k+1$	$k+1$
b'_k	=	$k+1$	k
b''_k	=	$k+1$	$k+1$
c_k	=	$k+1$	k

valuation corresponding to the $n!$

restricted Laguerre histories (of length n)

$\alpha = 0 \quad L_n^{(0)}(x)$ $\qquad \sigma(1) = n+1$

$$\begin{cases} b_k = 2k+1 \\ \lambda_k = k^2 \end{cases}$$

3-terms linear recurrence relation

$$\mu_n = n!$$

$\beta = \alpha + 1 \qquad \alpha = 0$

$$a_k = k + \beta \qquad \begin{cases} b'_k = k \\ b''_k = k + \beta \end{cases} \qquad c_k = k$$

$(k \geqslant 0)$ $\qquad$ $(k \geqslant 1)$

For restricted Laguerre histories
put a weight β for each choice
$p_i = 1$ with ω_i = { or

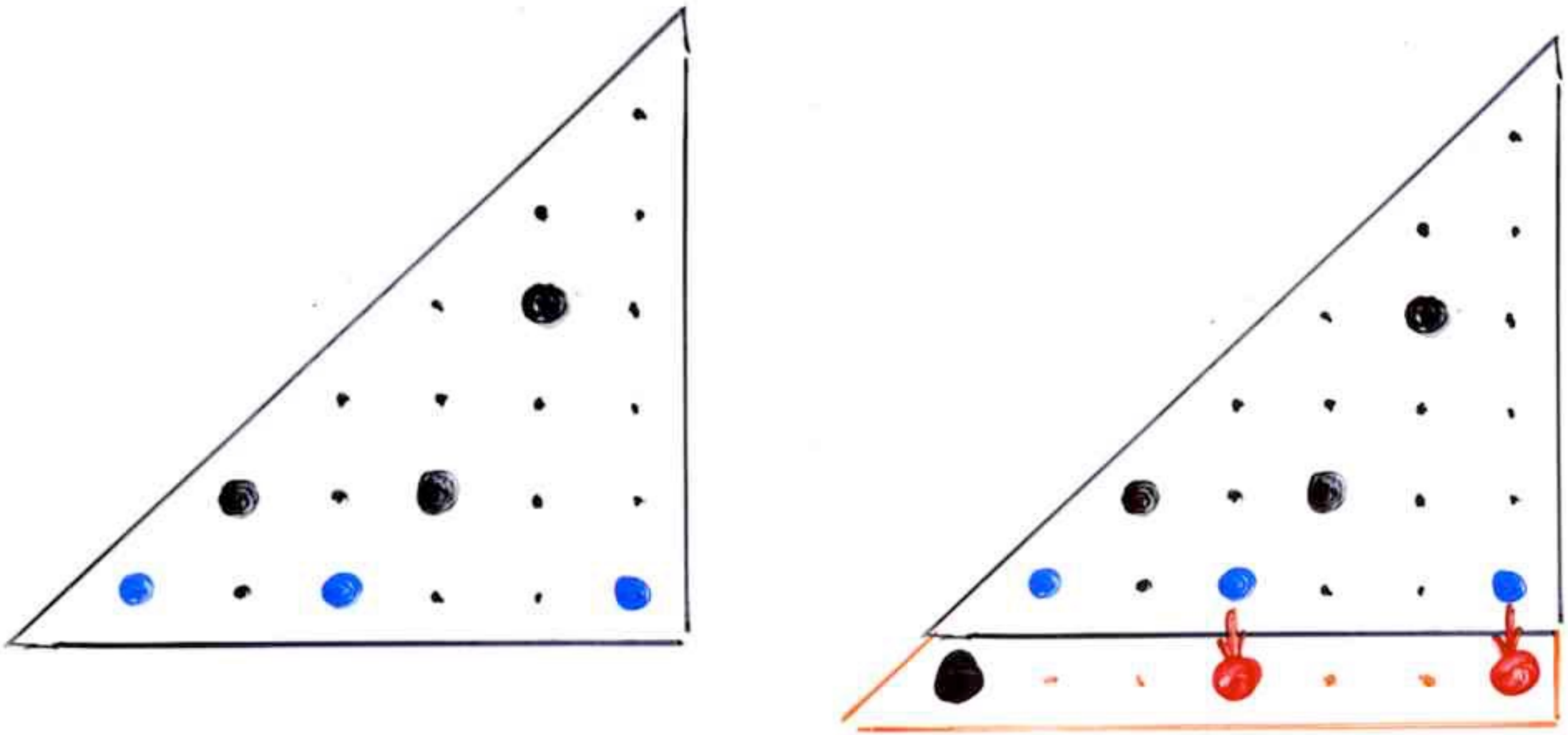
this is equivalent to say that
the element i is a lr-min element
of the permutation σ

Corollary $\mu_n = \beta(\beta+1)\cdots(\beta+n-1)$

remark

$$\beta(\beta+1)\ldots(\beta+n-1) = \sum_{k=1}^{n} s_{n,k}\,\beta^k$$

$$(n+1)! = \sum_{k=1}^{n} s_{n,k}\,2^k$$



$$(n+1)! = \sum_{k=1}^{n} s_{n,k} 2^k$$

Hermite histories

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

moments
Hermite
polynomials

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of
involutions
no fixed point
on $\{1, 2, \ldots, 2n\}$

Hermite history

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

$$h = (\omega \; ; \; f)$$

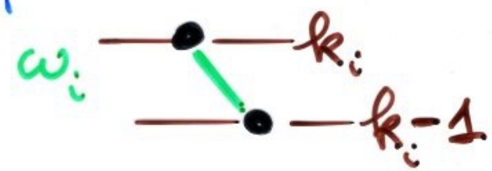
ω: Dyck path; f: choice function

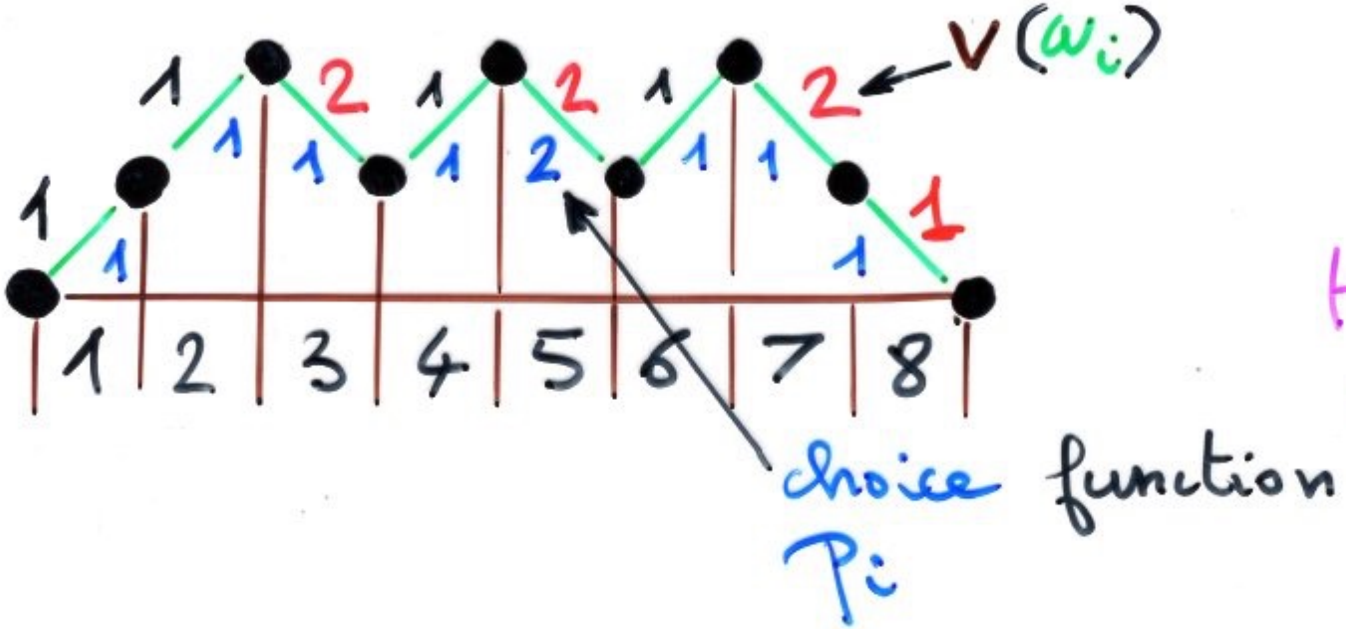
$\omega = \omega_1 \dots \omega_{2n}$

$f = (p_1, \dots, p_{2n})$

$p_i = 1$

$1 \leq p_i \leq v(\omega_i) = k_i$





Hermite histories

$\mu_{2n+1} = 0$

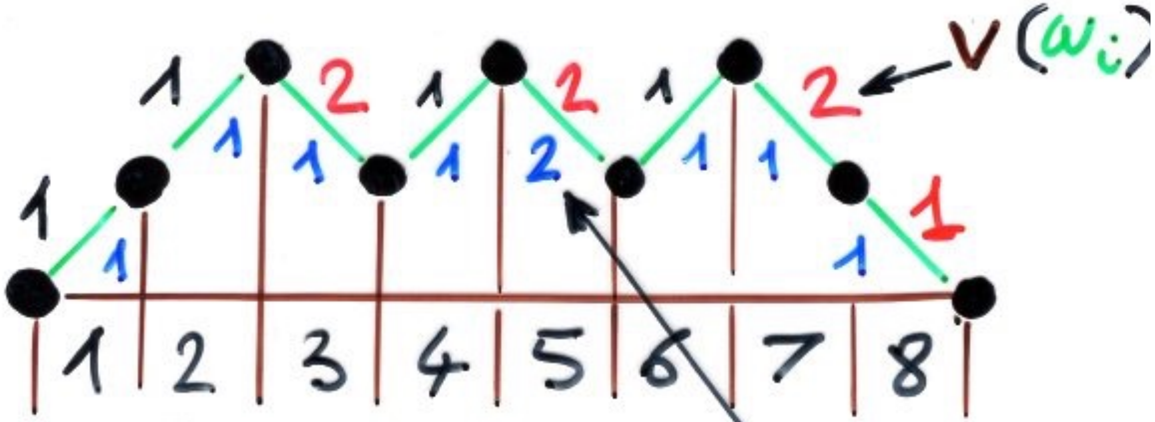
$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of involutions no fixed point on $\{1, 2, \ldots, 2n\}$

chord diagrams

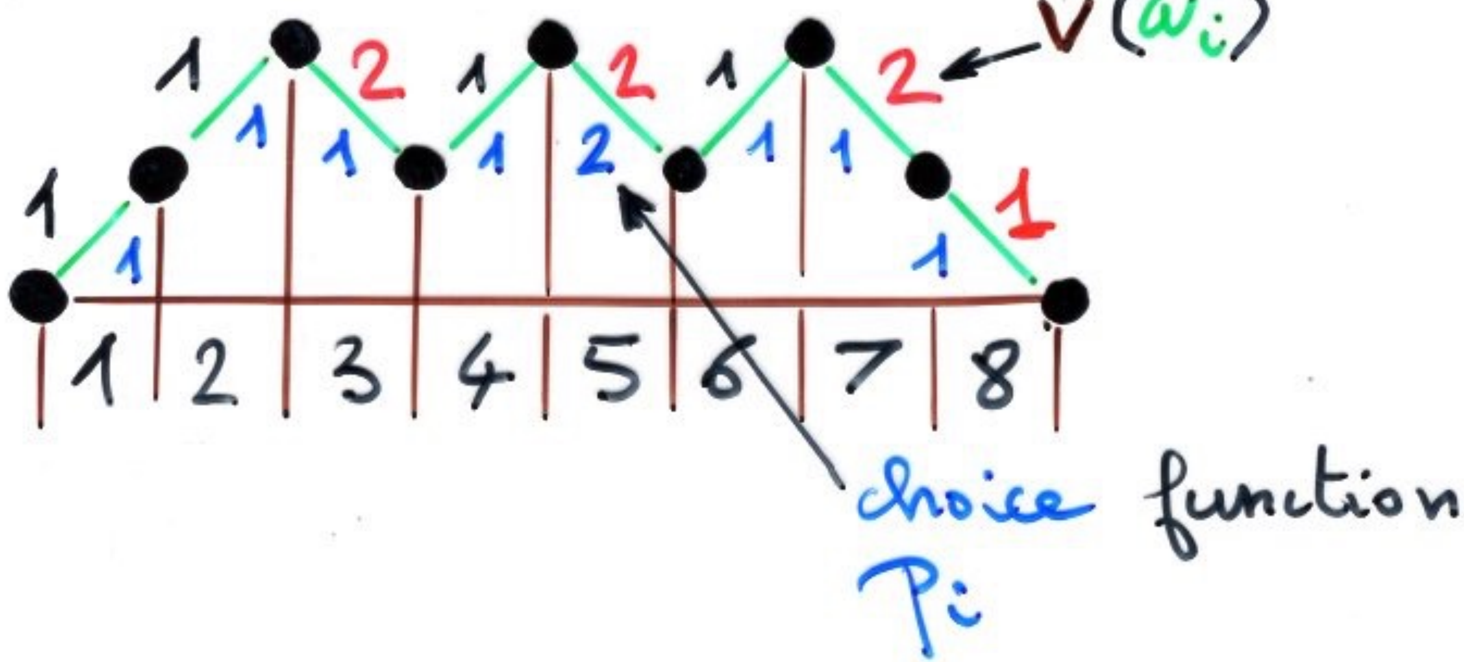
perfect matching

bijection

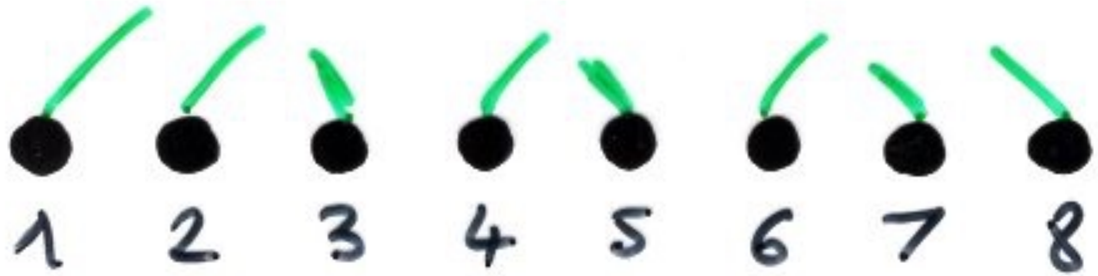


Hermite histories

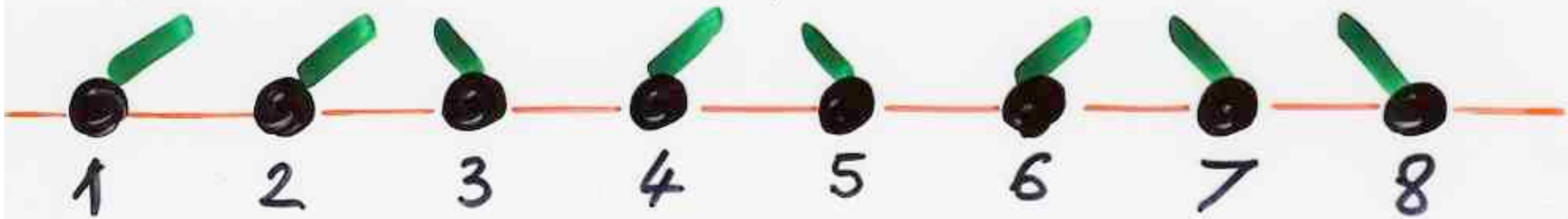
choice function p_i

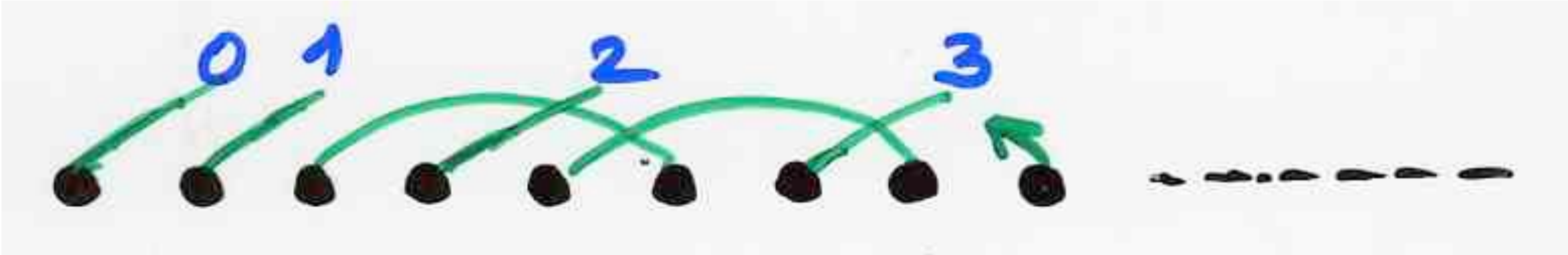


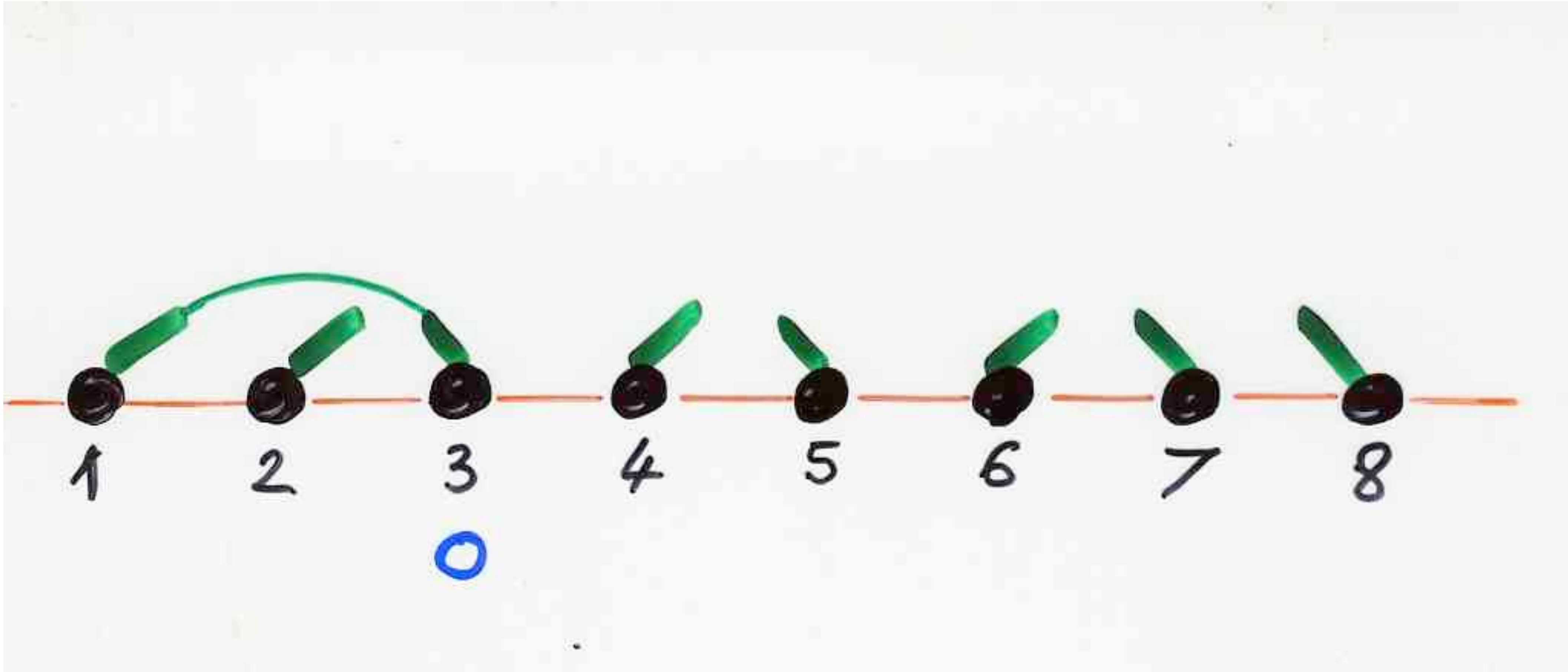
V(w_i)
choice function
P_i
1
2
3
4
5
6
7
8

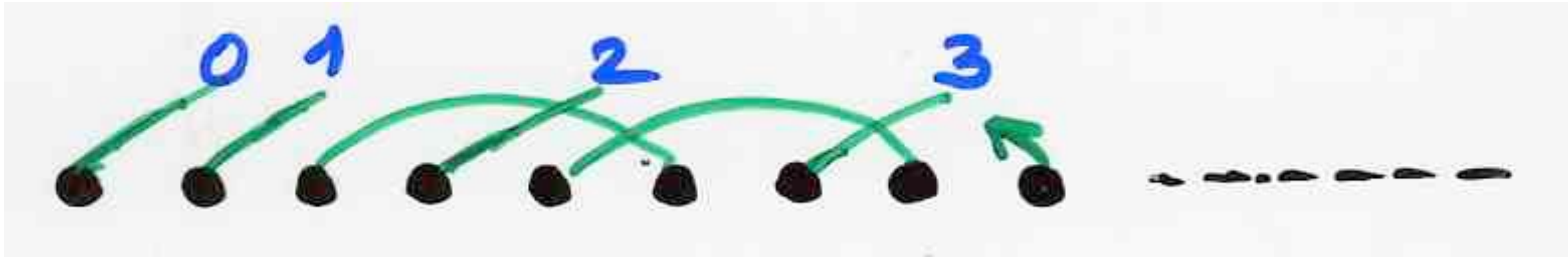


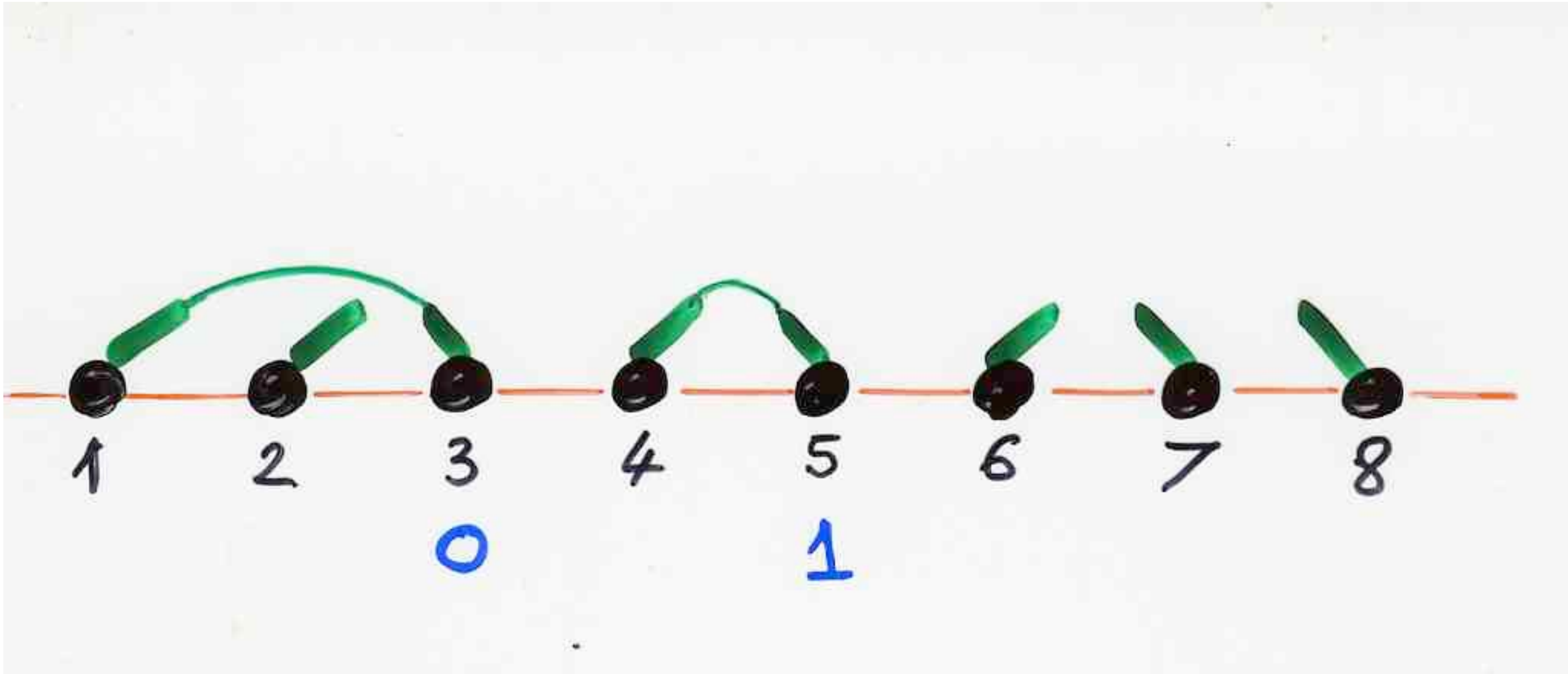
1
2
3
4
5
6
7
8

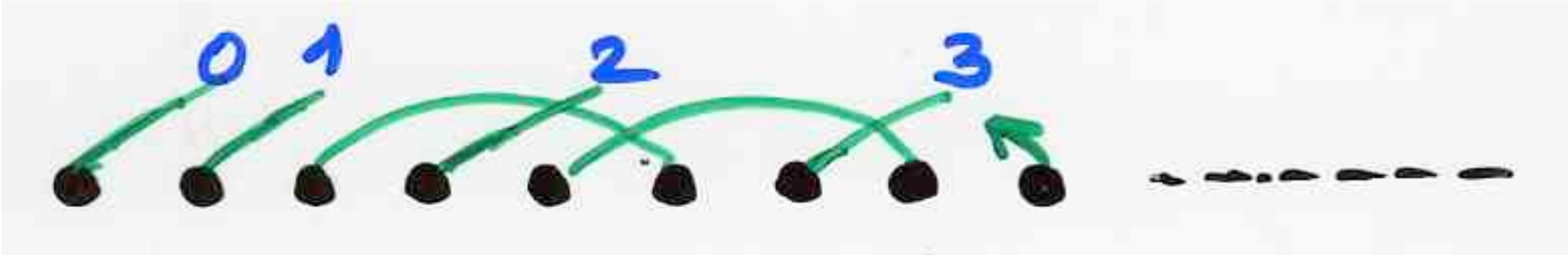

1
2
3
4
5
6
7
8

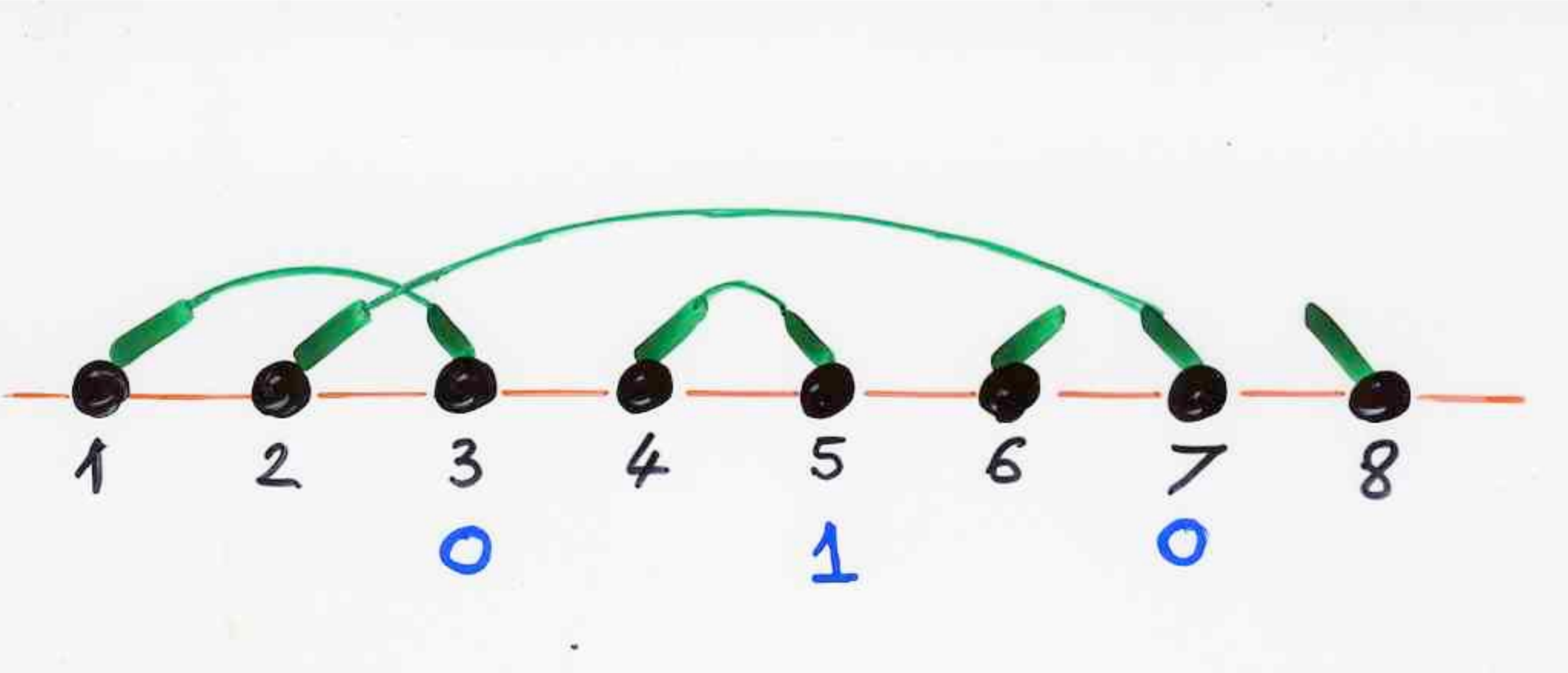

0
1
2
3

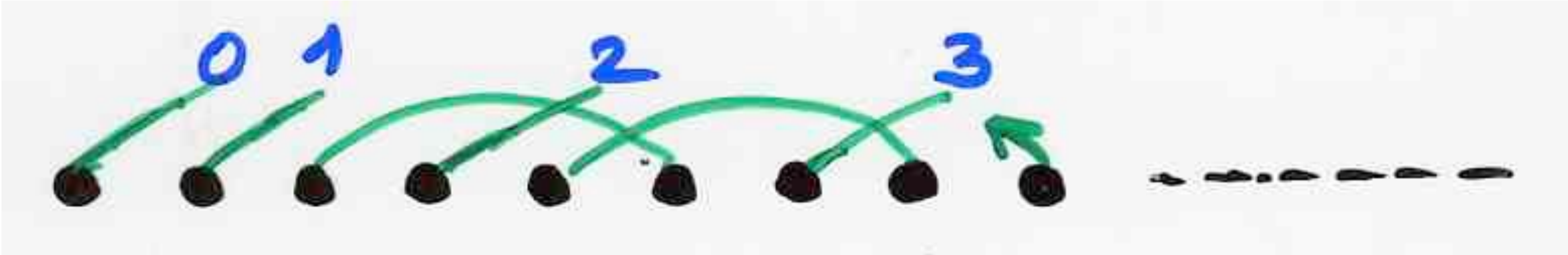

1
2
3
4
5
6
7
8
0

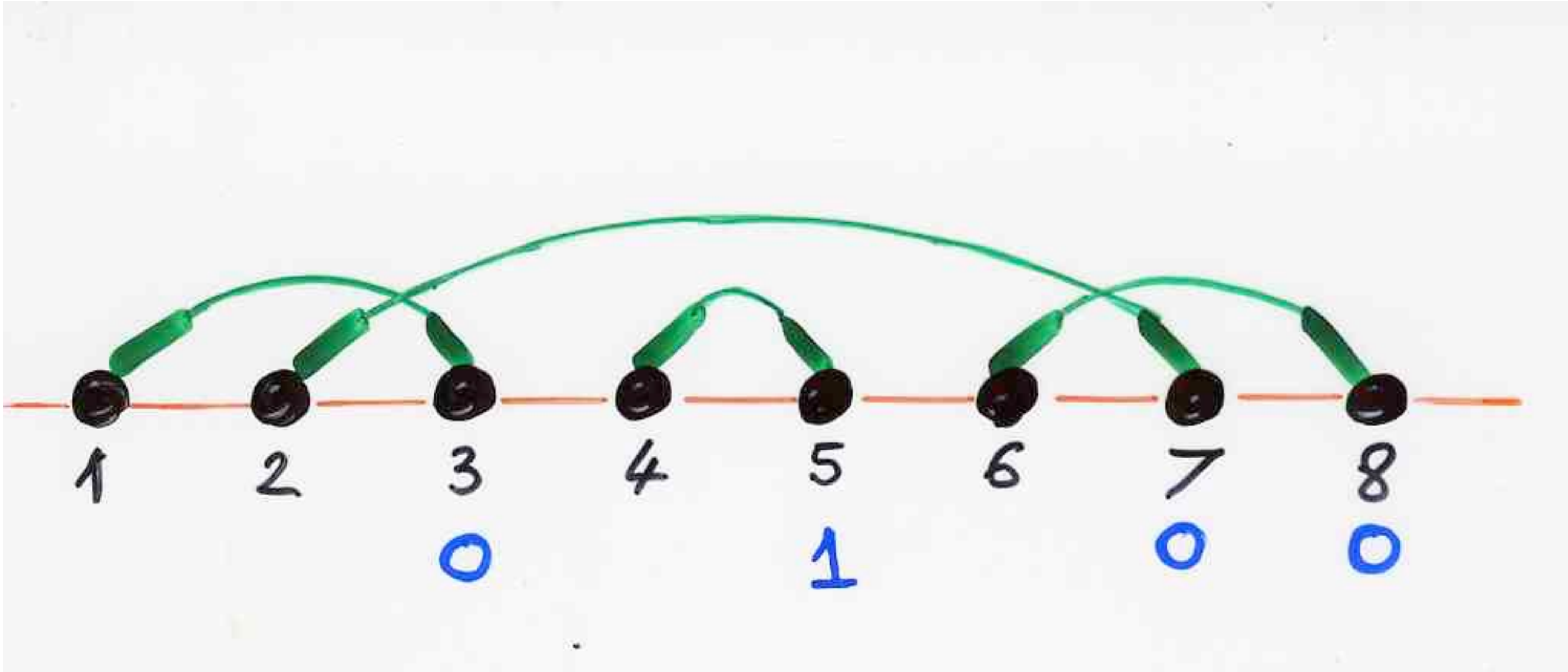

0
1
2
3


1
2
3
4
5
6
7
8
0
1


0
1
2
3


1
2
3
4
5
6
7
8
0
1
0


0
1
2
3


1
2
3
4
5
6
7
8
0
1
0
0

q-analog of Hermite histories

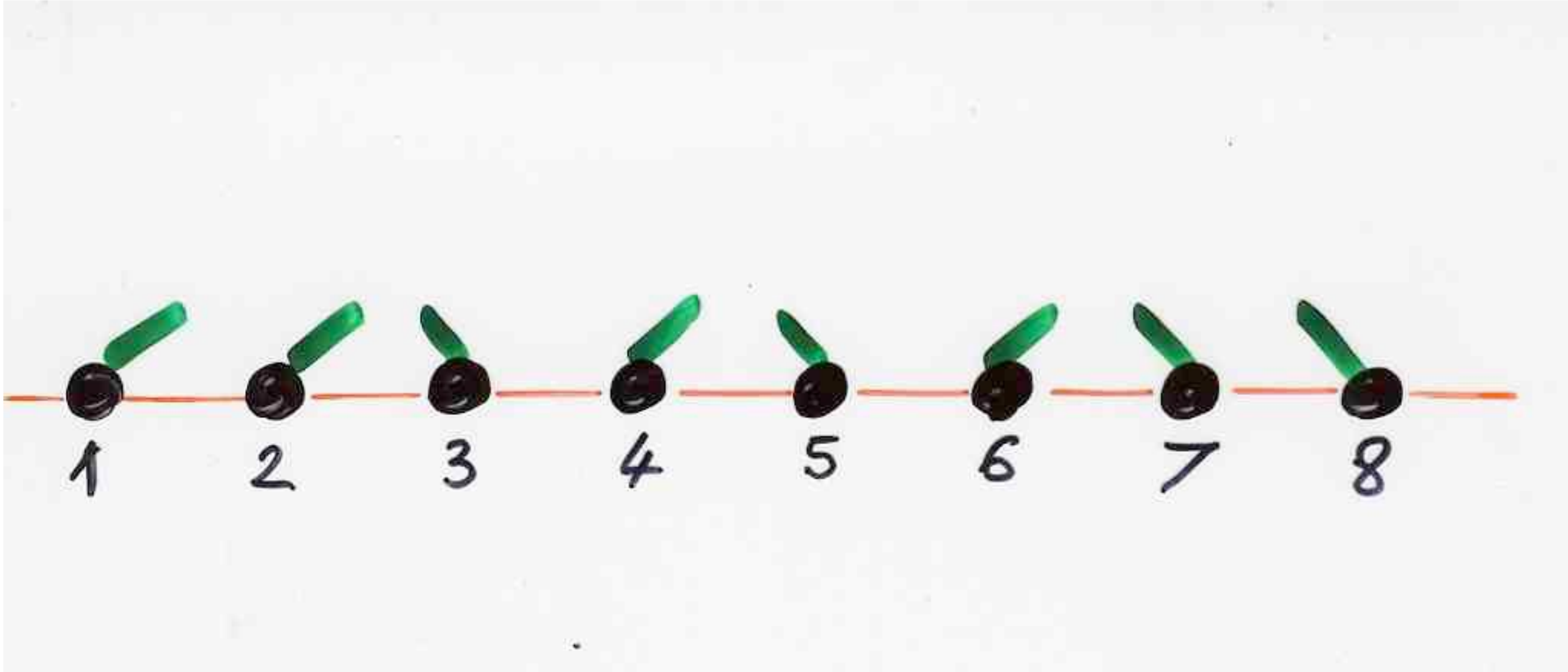
q-Hermite

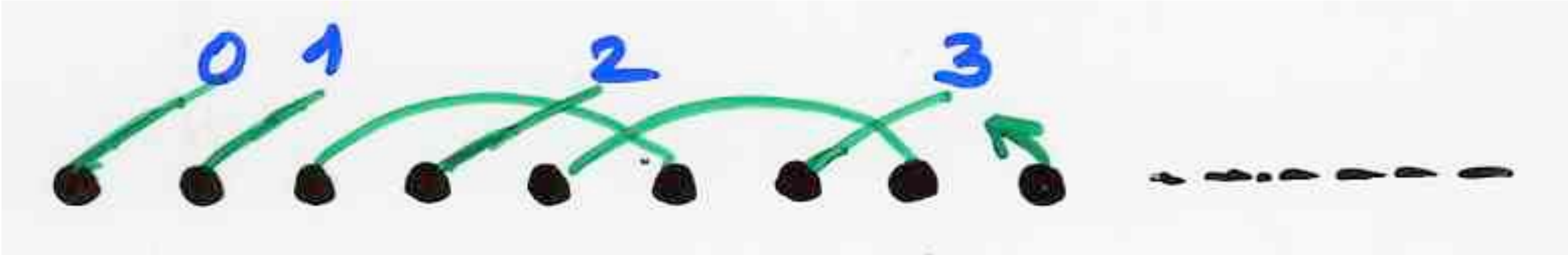
$H_n^{I}(x;q)$

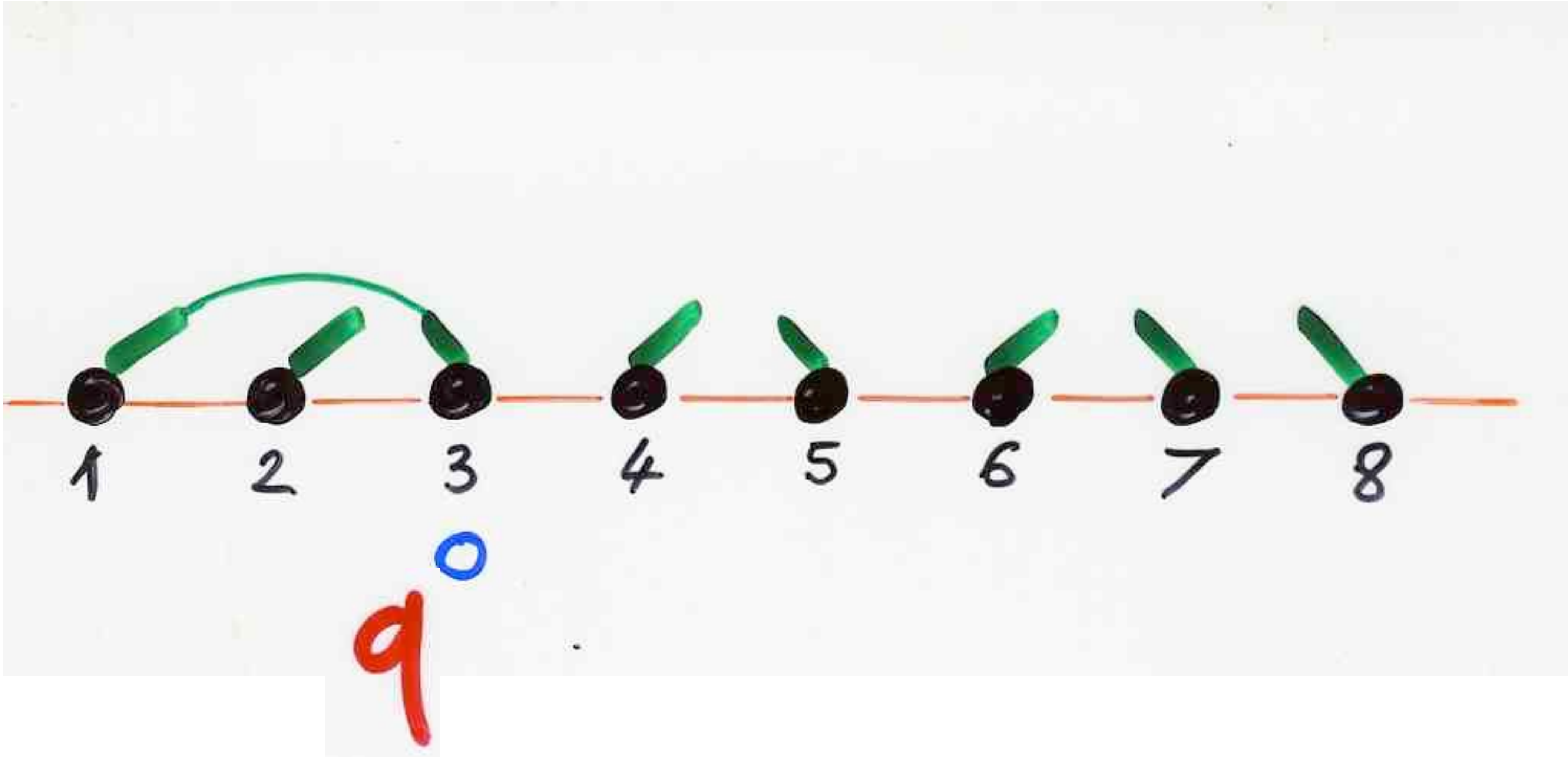
$b_k = 0$

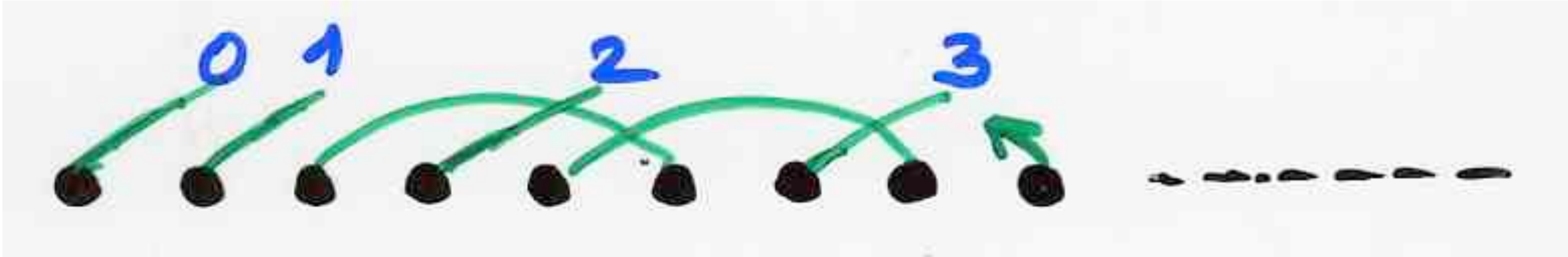
$\lambda_k = [k]_q$

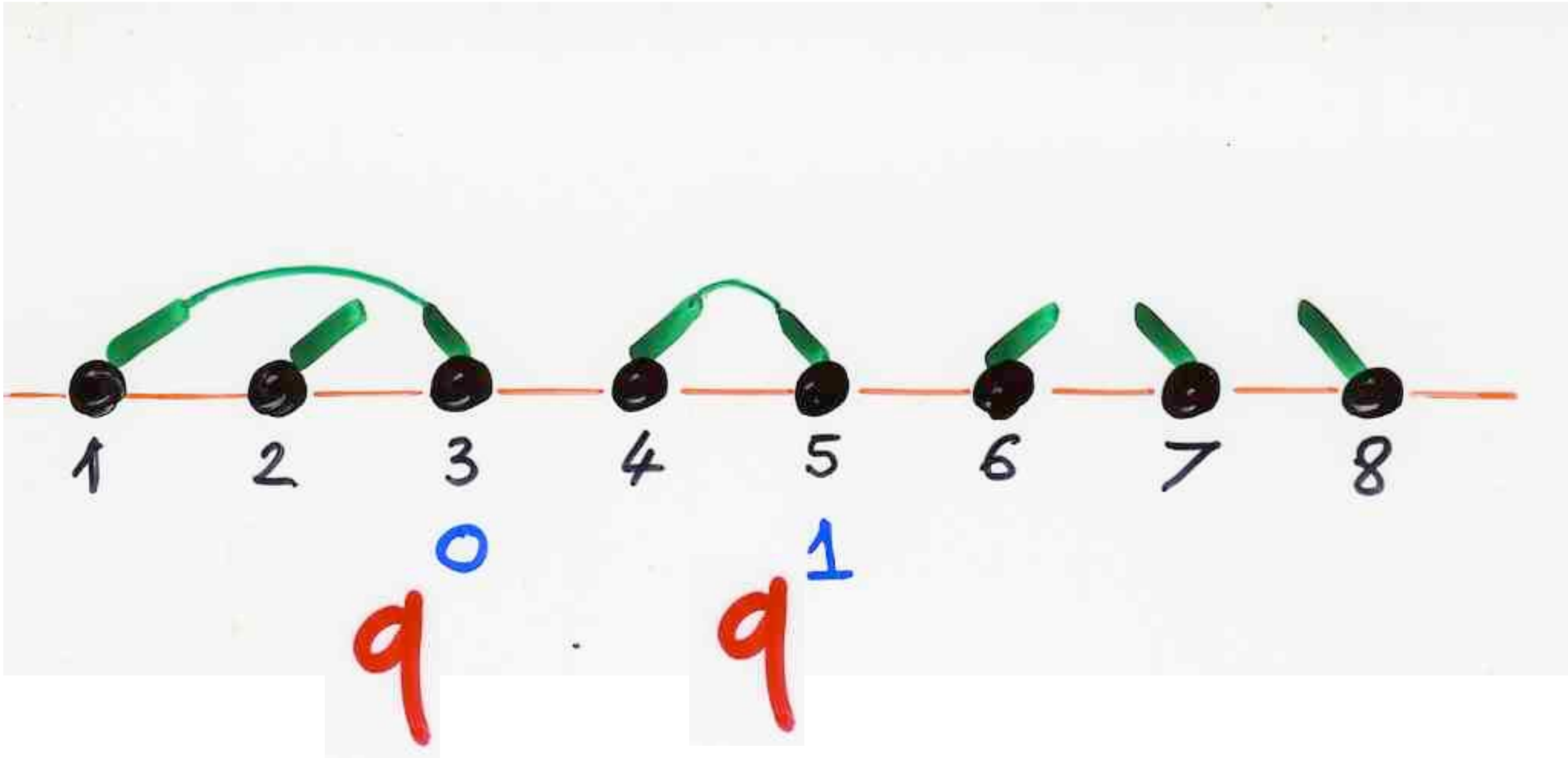
$= 1 + q + \ldots + q^{k-1}$

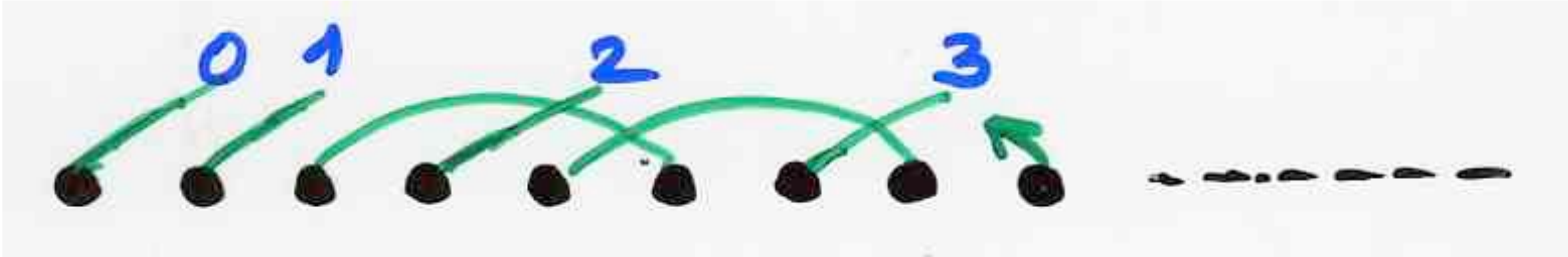

1 2 3 4 5 6 7 8

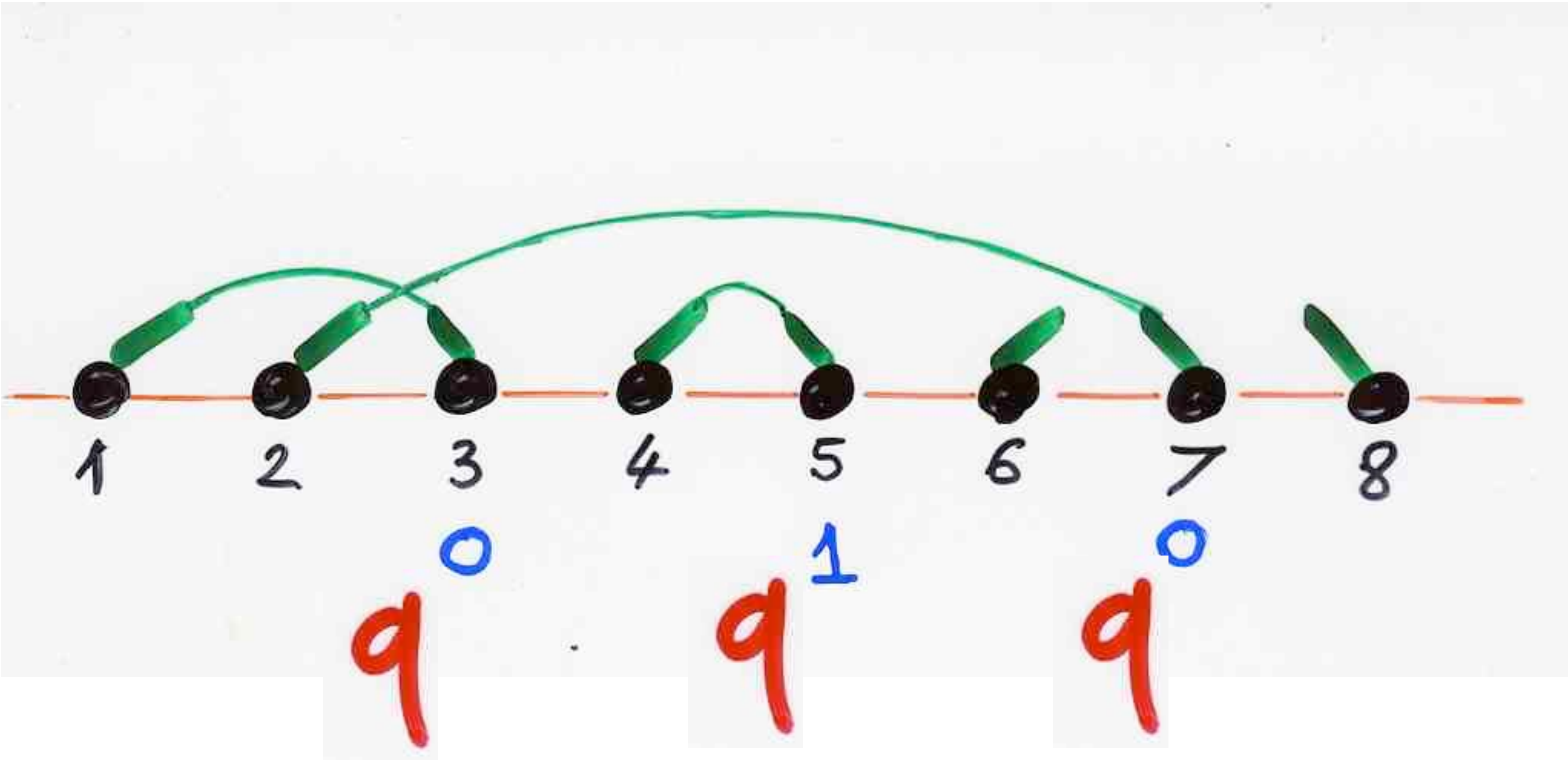

0
1
2
3


1
2
3
4
5
6
7
8
9
0

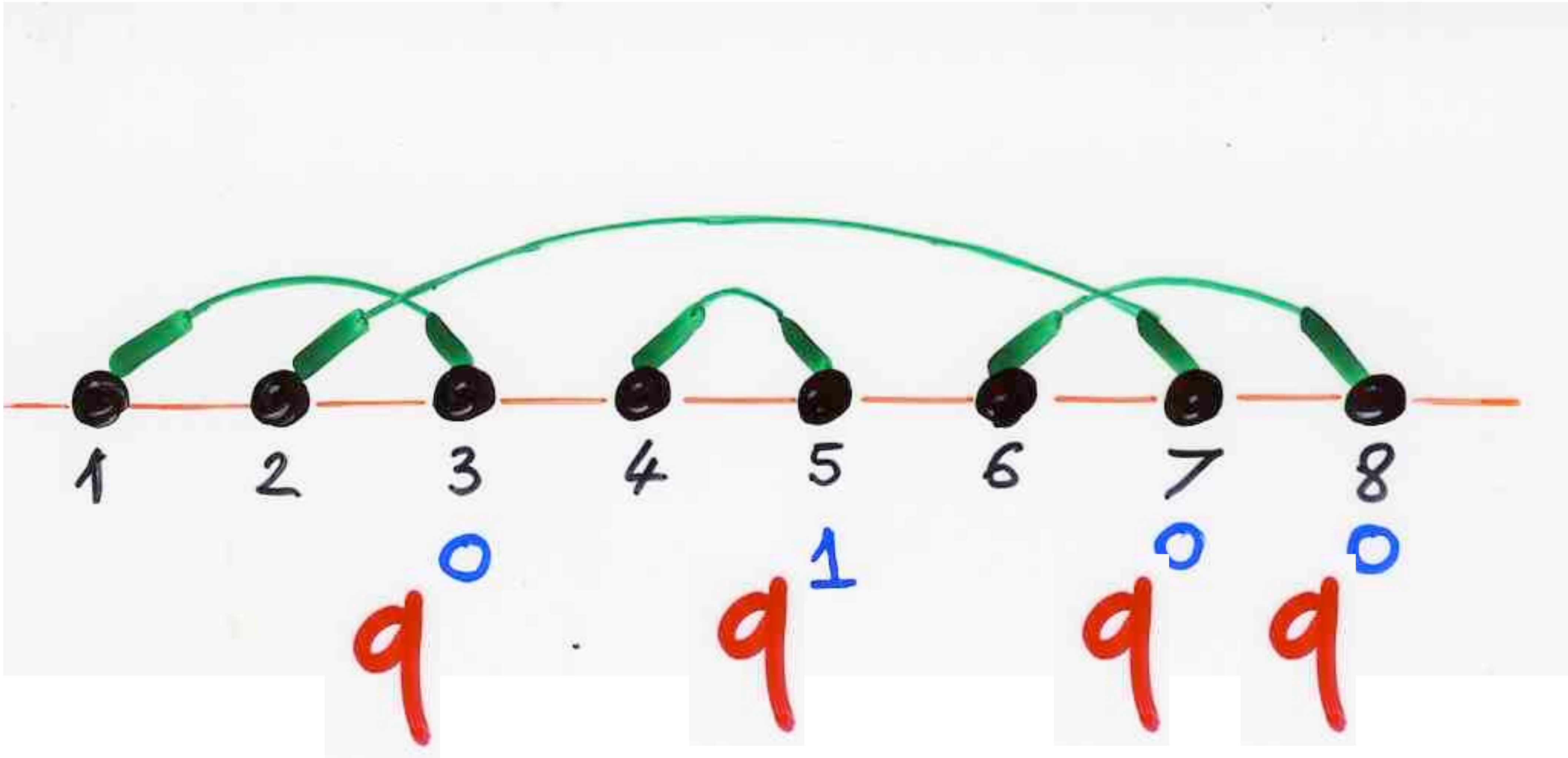

0 1 2 3

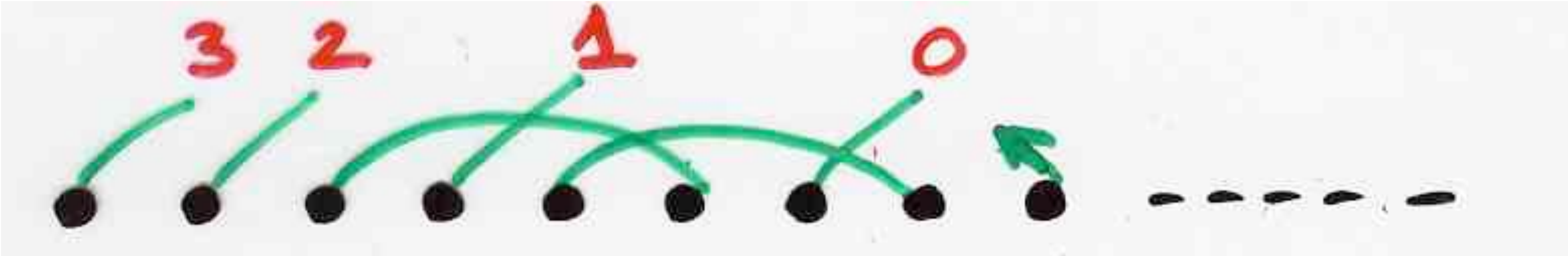

1 2 3 4 5 6 7 8
q 0
q 1

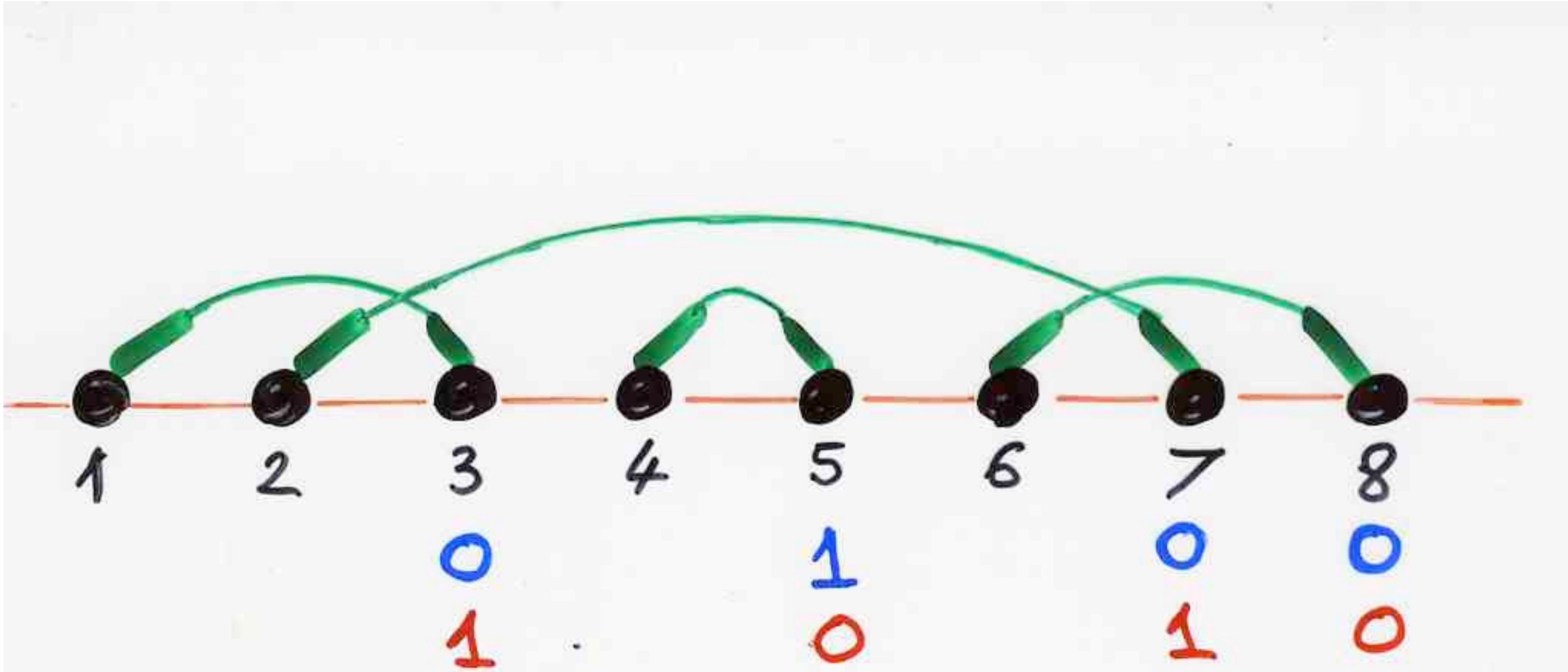

0 1 2 3

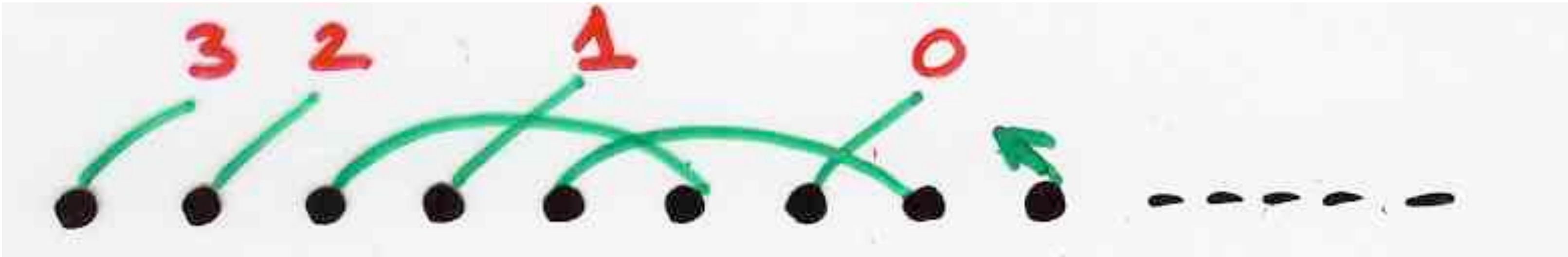

1 2 3 4 5 6 7 8
0 1 0
q q q

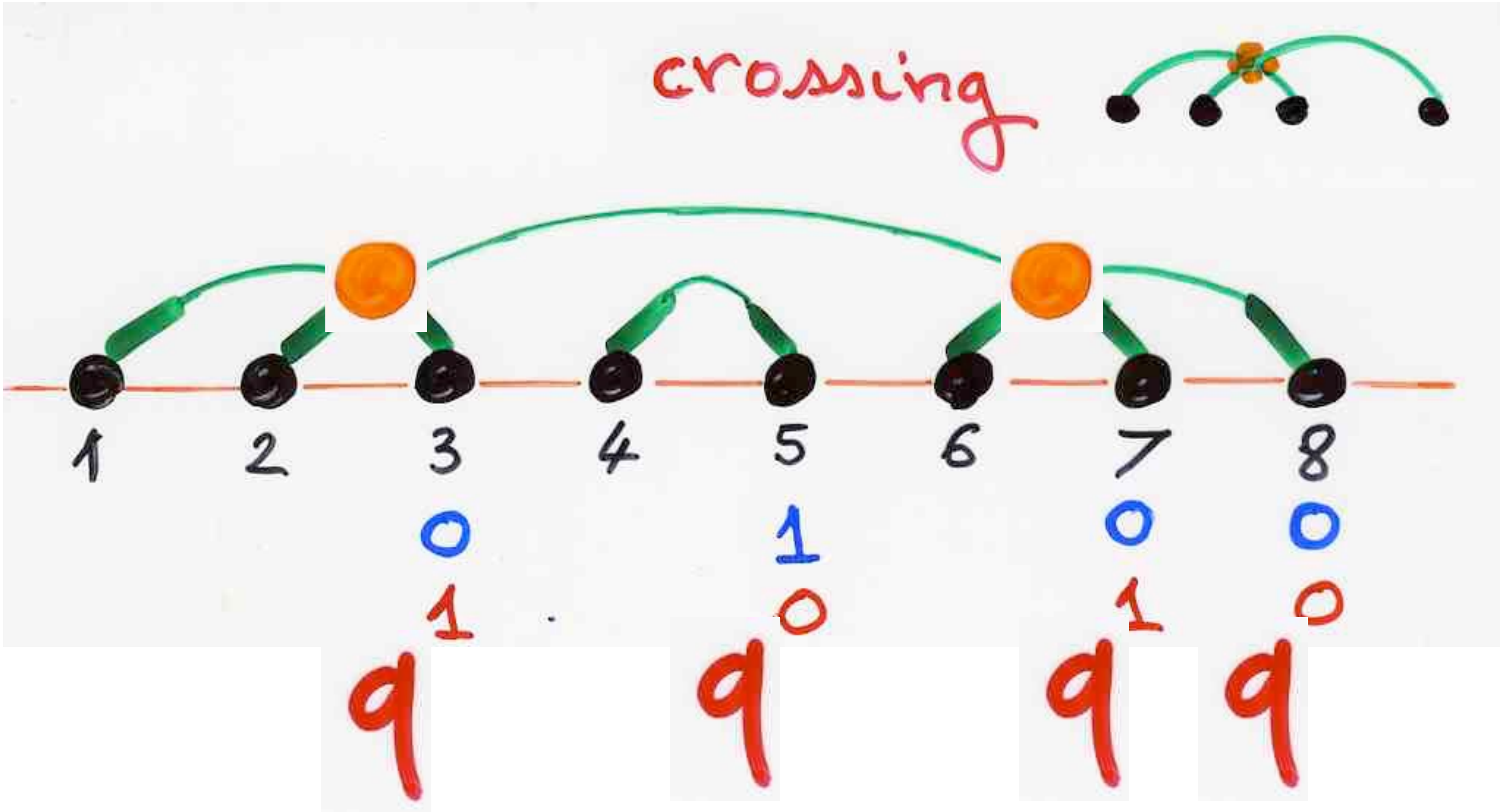

nesting


1 2 3 4 5 6 7 8
0 1 0 0
q q q q


3 2 1 0

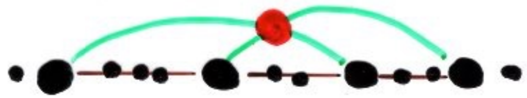

1 2 3 4 5 6 7 8
0 1 0 0
1 0 1 0


3
2
1
0

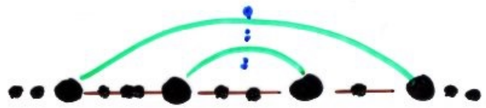

crossing
1
2
3
4
5
6
7
8
0
1
0
0
1
0
1
0
9
9
9
9

$$\sum_{\substack{\text{chord diagrams } I\\ [1,2n]}} q^{cr(I)} = \sum_{\substack{I\\ \text{chord diagrams}\\ [1,2n]}} q^{nest(I)} = \sum_{\substack{h\\ \text{Hermite histories}\\ |h|=2n}} q^{sum(h)}$$

cr (I) = number of crossings



nest(I) = number of nestings



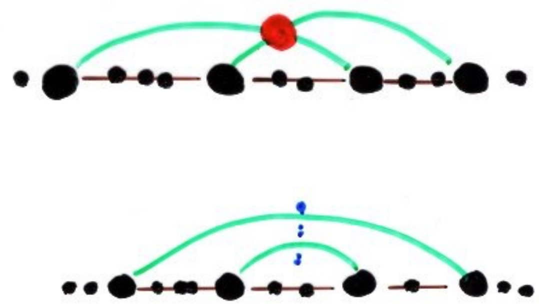
$$sum(h) = \sum_i (p_i - 1)$$

the philosophy of « histories »

and its q-analogues

exercise

$$\sum_{\substack{I \\ \text{chord diagrams} \\ [1,2n]}} q^{cr(I)} t^{nest(I)}$$



(q,t)-polynomial

symmetric in q and t

weighted histories

q-Laguerre

continuous q-Laguerre polynomials

discrete

$$\begin{cases} b_k = [k+1]_q + [k+1]_q \\ \lambda_k = [k]_q \times [k+1]_q \end{cases}$$

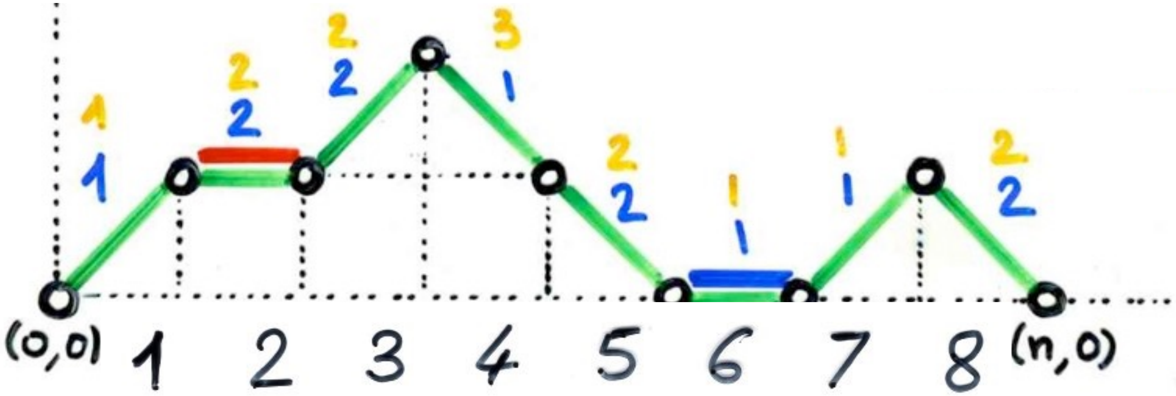
$$\begin{cases} b'_k = [k+1]_q \\ b''_k = [k+1]_q \\ a_k = [k+1]_q \\ c_k = [k+1]_q \end{cases}$$

weighted q-Laguerre histories

$$q^{\left[\sum_{i=1}^{n}(p_i-1)\right]}$$

choice function

$h = (\omega_c ; (p_1, \dots, p_n))$



x	ω_c	pos	v
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n =$ 8		2	2
9	•		

␣

␣ 1 ␣

␣ 1 ␣ 2

␣ 1 ␣ 3 ␣ 2

4 1 ␣ 3 ␣ 2

4 1 ␣ 3 5 2

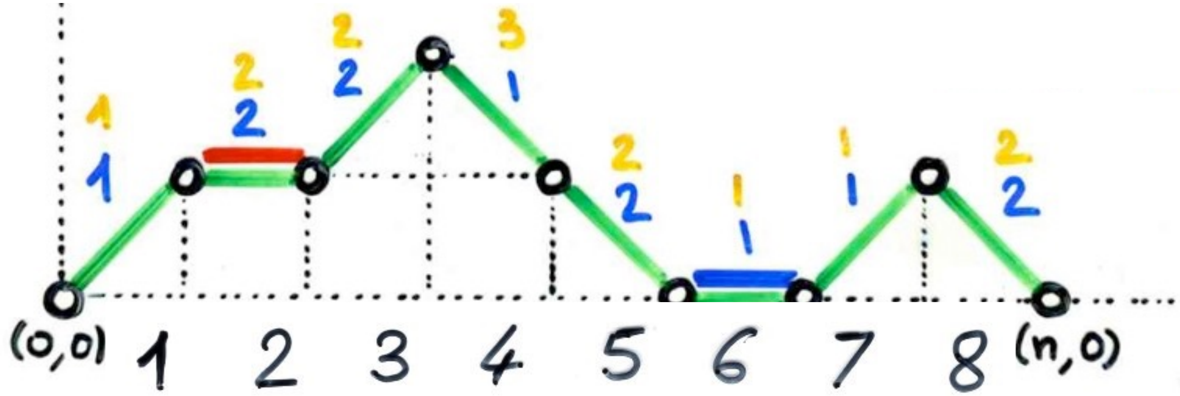
4 1 6 ␣ 3 5 2

4 1 6 ␣ 7 ␣ 3 5 2

4 1 6 ␣ 7 8 3 5 2

4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$

"q-analogue" of Laguerre histories



choice function

$i=$	1	2	3	4	5	6	7	8
$p_i=$	1	2	2	1	2	1	1	2
$p_i-1=$	0	1	1	0	1	0	0	1

␣
␣ 1 ␣
␣ 1 ␣ 2
␣ 1 ␣ 3 ␣ 2
4 1 ␣ 3 ␣ 2
4 1 ␣ 3 5 2
4 1 6 ␣ 3 5 2
4 1 6 ␣ 7 ␣ 3 5 2
4 1 6 ␣ 7 8 3 5 2
4 1 6 9 7 8 3 5 2 $= \sigma$ $\in \mathfrak{S}_{n+1}$

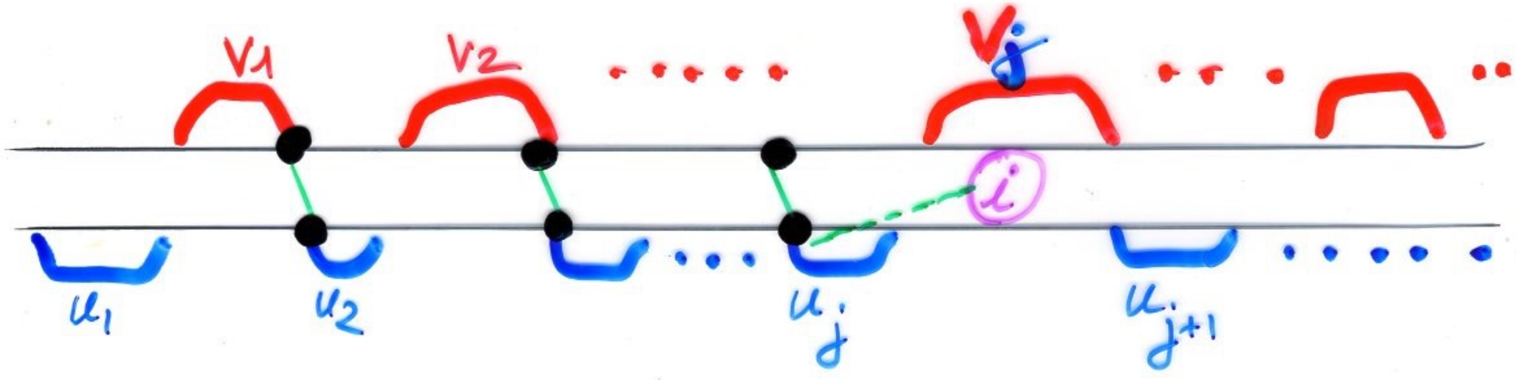
weighted q-Laguerre histories

q^4

weighted
q- Laguerre
histories

$$q^{\left[\sum_{i=1}^{n} (p_i - 1)\right]}$$

this is also q^m where m is the number of subsequences (a, b, c) of σ having the pattern $(31-2)$



Complements

q-Hermite and q-Laguerre second kind

discrete
continuous

q - Laguerre
polynomials

q - Laguerre II

if $\mu_n = [n!]_q$

then
$$\begin{cases} b_k = q^k \left([k]_q + [k+1]_q \right) \\ \lambda_k = q^{2k-1} [k]_q \times [k]_q \end{cases}$$

→ subdivided Laguerre histories
A. de Médicis, X.V. (1994)

$$L^{II\,\beta}_n(x;q) \qquad \begin{cases} b^{(\beta)}_{k,q} = q^k\left([k]_q + [k+1;\beta]_q\right) \\ \lambda^{(\beta)}_{k,q} = q^{2k-1}[k]_q \cdot [k;\beta]_q \end{cases}$$

$$\mu^{(\beta)}_{n,q} = [n;\beta]_q\,!$$

$$= [1;\beta]_q\,[2;\beta]_q \cdots [n;\beta]_q$$

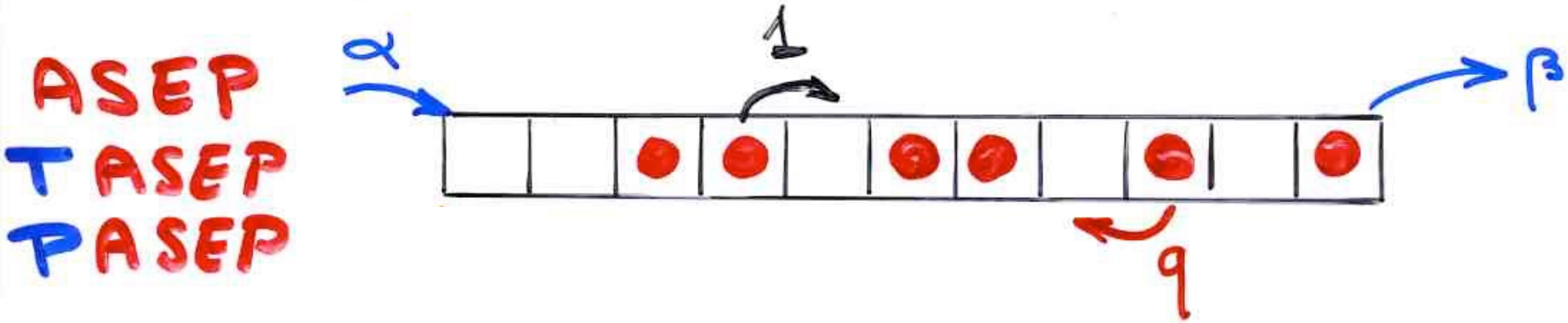
q- Laguerre polynomials

q- Laguerre I

then
$$\begin{cases} b_k = ([k]_q + [k+1]_q) \\ \lambda_k = [k]_q \times [k]_q \end{cases}$$

$$\mu_n = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \left(\sum_{i=0}^{k} q^{i(k+1-i)} \right)$$

Corteel, Josuat-Vergès, Prellberg, Rubey (2008) y


ASEP
TASEP
PASEP
α
1
β
q

discrete q-Hermite polynomials

q-Hermite II

$$\begin{cases} \mu_{2n+1} = 0 \\ \mu_{2n} = [1]_q \times [3]_q \times \cdots \times [2n-1]_q \end{cases}$$

then
$$\begin{cases} b_k = 0 \\ \lambda_k = q^{k-1} [k]_q \end{cases}$$

continuous q-Hermite polynomials

q-Hermite I

$$\begin{cases} b_k = 0 \\ \lambda_k = [k]_q \end{cases}$$

$$\begin{cases} \mu^{I}_{2n+1,q} = 0 \\ \mu^{I}_{2n,q} = \dfrac{1}{(1-q)^n} \displaystyle\sum_{j=0}^{n} (-1)^j \, t_{n,j} \, q^{j(j+1)/2} \end{cases}$$

$$t_{n,j} = \binom{2n}{n-j} - \binom{2n}{n+j+1}$$

Riordan (1975) Touchard (1952)

Penaud (1995)

the philosophy of « histories »

and its q-analogues

Complements

Laguerre histories

and

orthogonal Scheffer polynomials

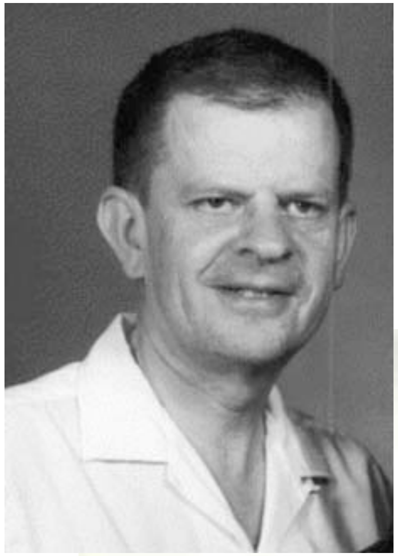
orthogonal polynomials

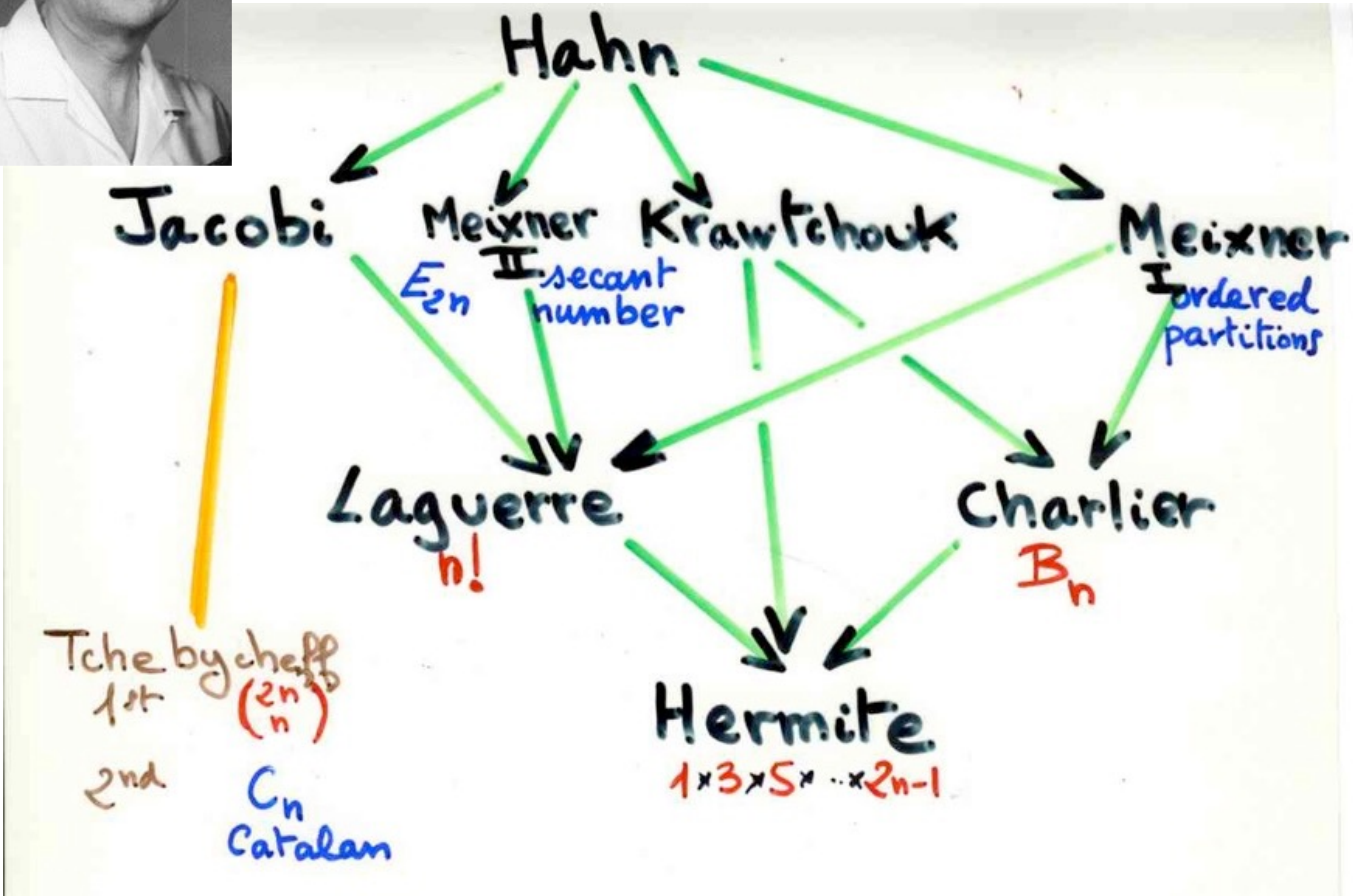
(binomial type)
Sheffer type

$$\sum P_n(x) \frac{t^n}{n!} = g(t) e^{x f(t)}$$

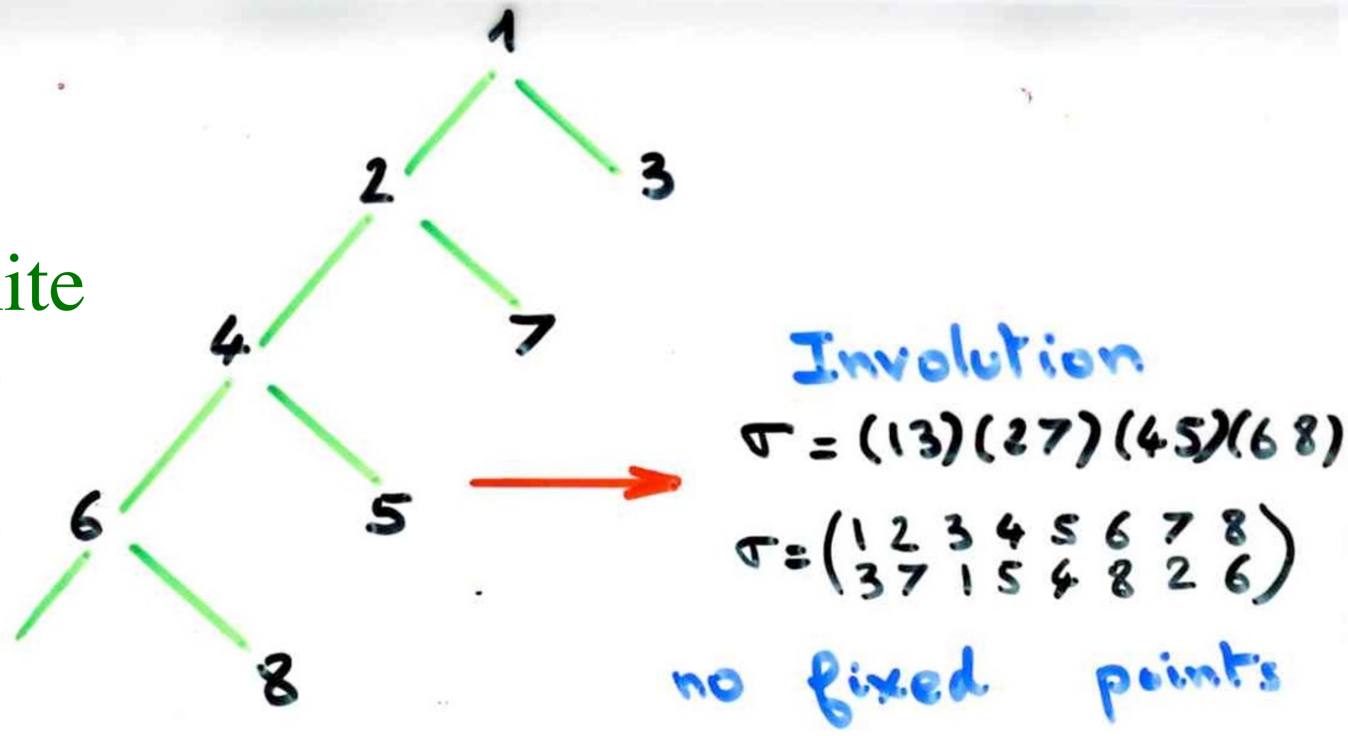
- Hermite H_n
- Laguerre $L_n^{(\alpha)}$
- Charlier $C_n^{(a)}$
- Meixner I $M_n^{I\,(\alpha,\beta)}$
- Meixner II $M_n^{II\,(\delta,\eta)}$

Askey-Wilson

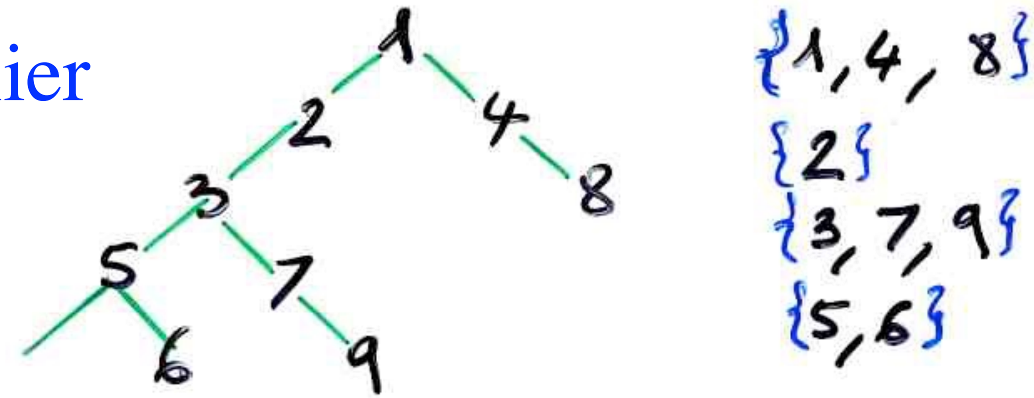




Hermite

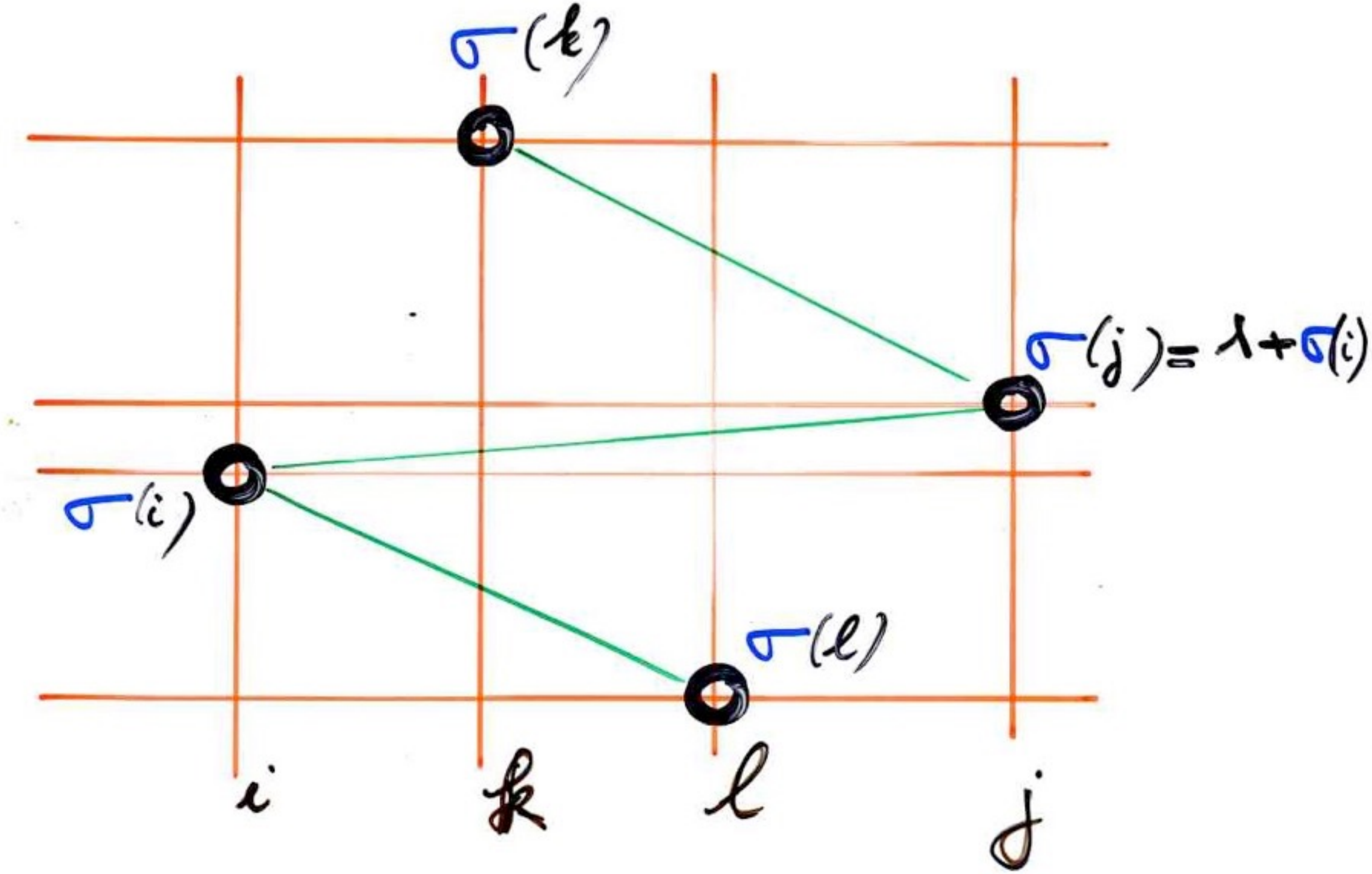


Charlier

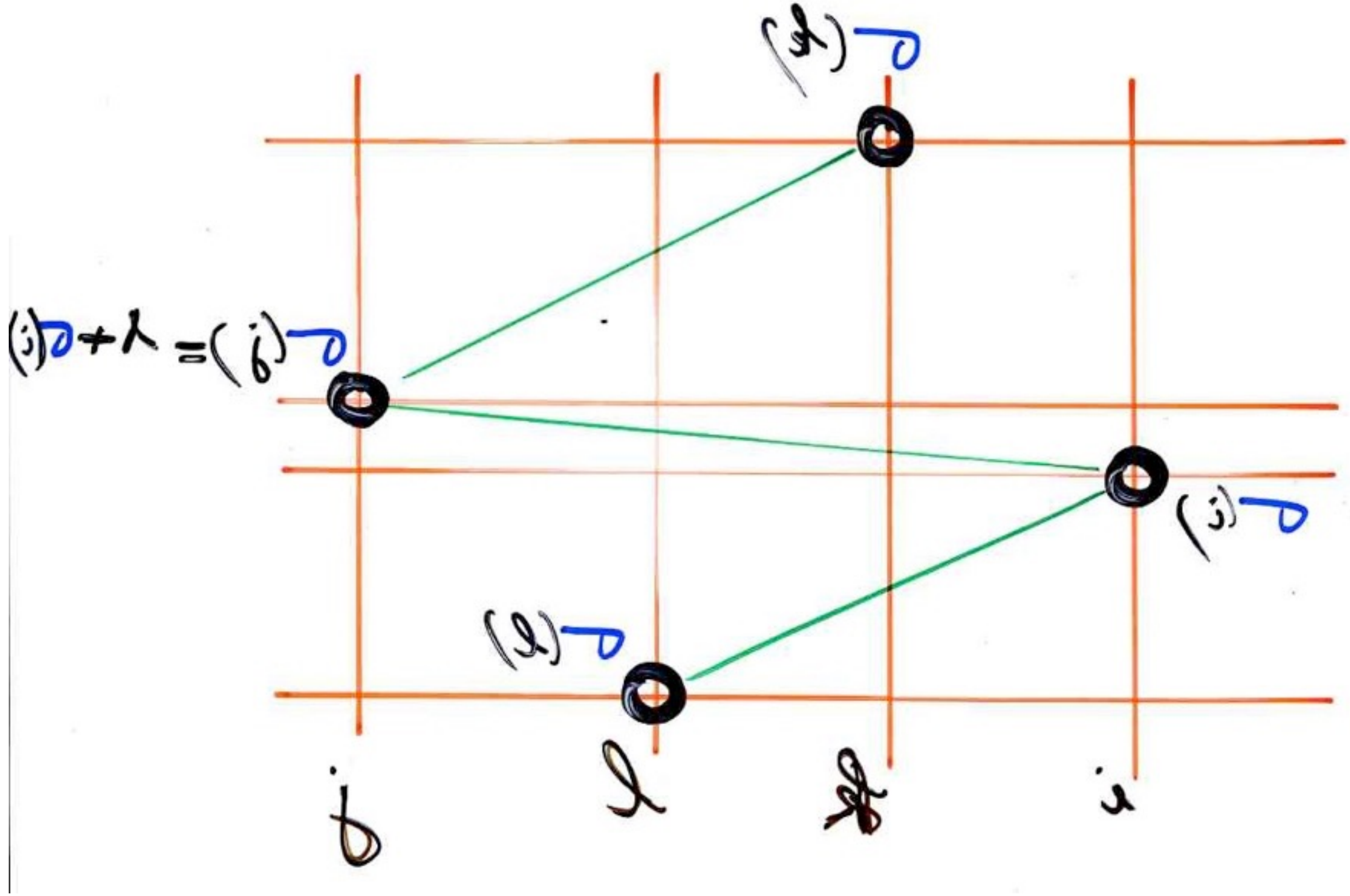


Complements

Baxter permutations



σ(k)
σ(j) = 1 + σ(i)
σ(i)
σ(l)
i
k
l
j


d(k)
d(j) = λ + d(i)
d(i)
d(l)
j
l
k
i

$\sigma = 5\ 1\ 2\ 4\ 3\ 9\ 7\ 8\ 6$

Chung, Graham, Hoggatt, Kleiman (1978)

$$B(n) = \frac{1}{\binom{n+1}{1}\binom{n+1}{2}} \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}$$

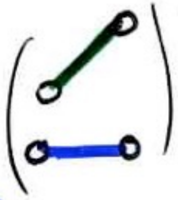
Mallows (1979)

nb of Baxter permutations having $(k-1)$ rises

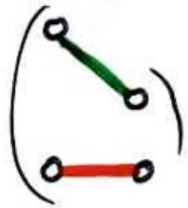
$\sigma(i) < \sigma(i+1)$

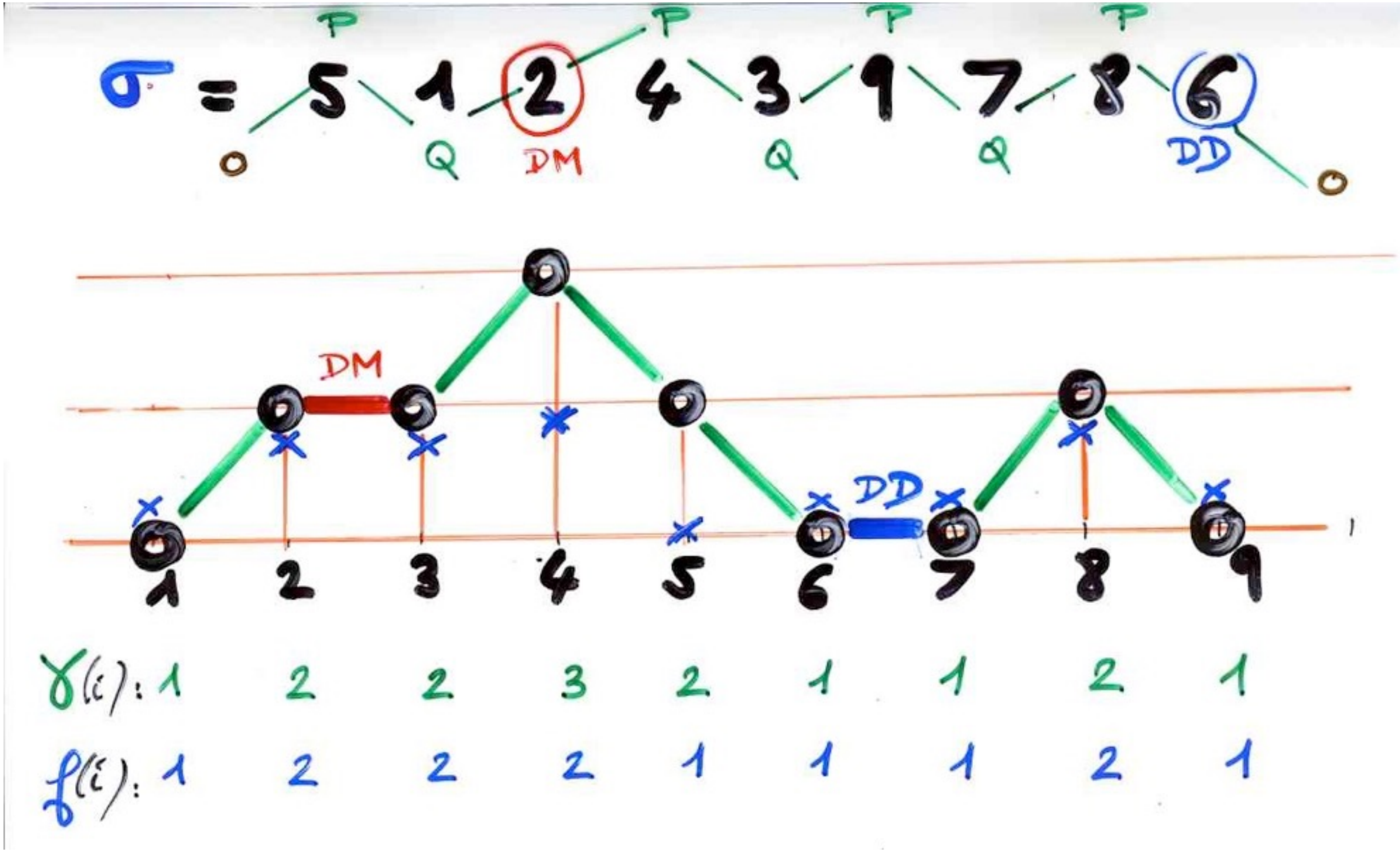
Prop $\sigma \longrightarrow (\gamma_c, f)$

σ Baxter permutation iff

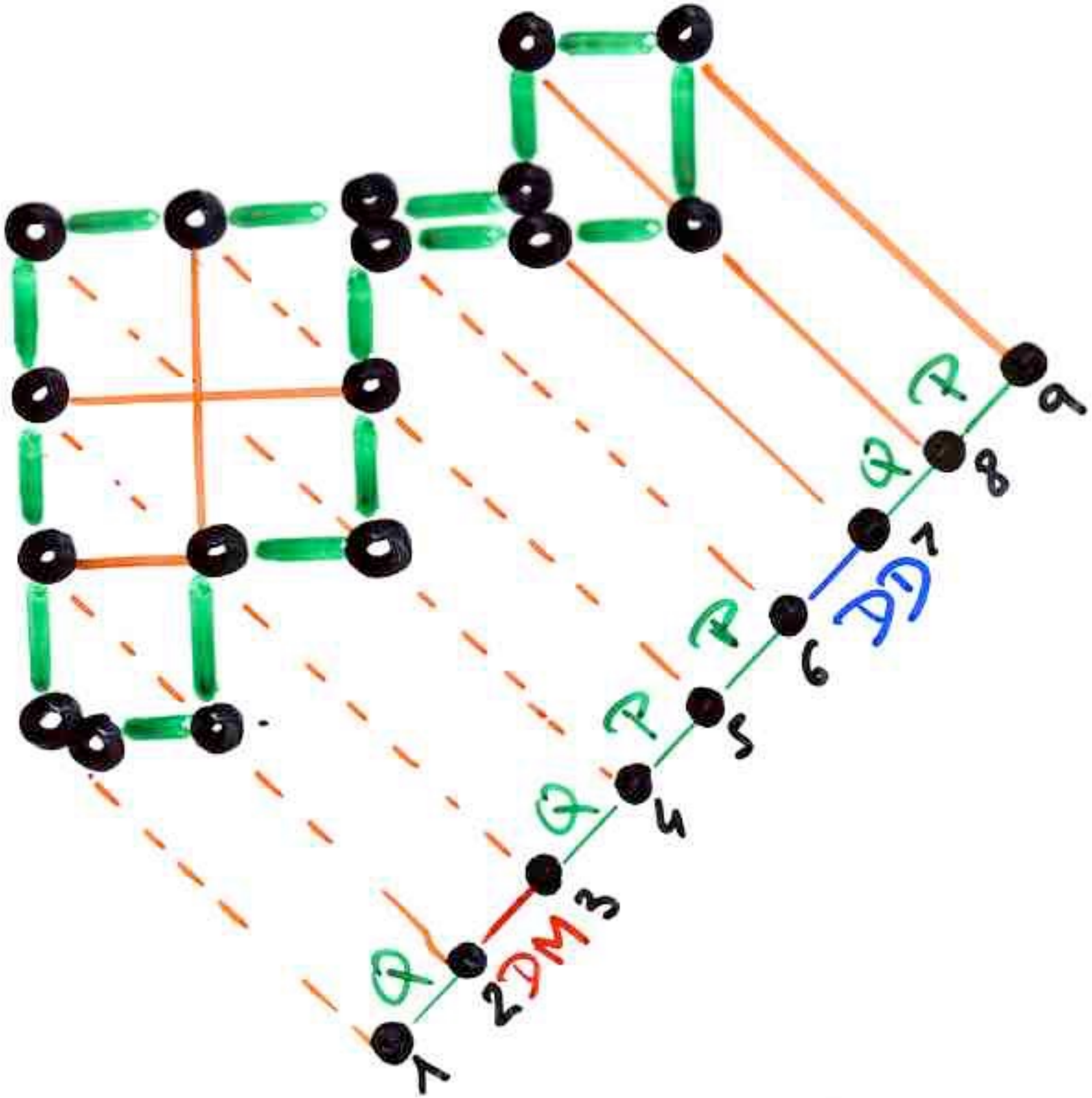
- for $i \in Q$ trough or DD double descent  $\Rightarrow f(i+1) = \begin{cases} f(i) \text{ or} \\ f(i)+1 \end{cases}$

γ

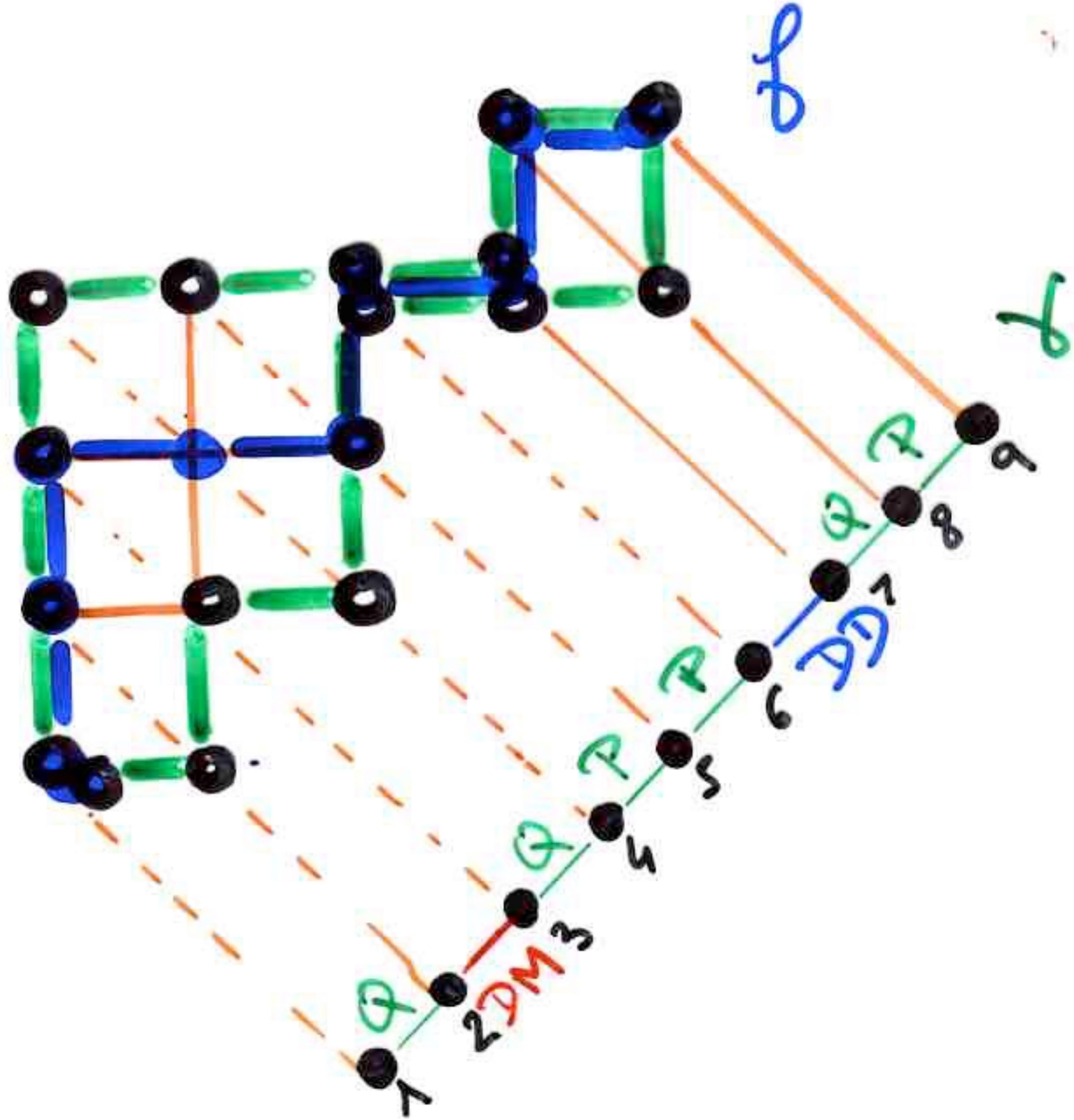
- for $i \in P$. peak DM double rise  $\Rightarrow f(i+1) = \begin{cases} f(i) \text{ or} \\ f(i)-1 \end{cases}$



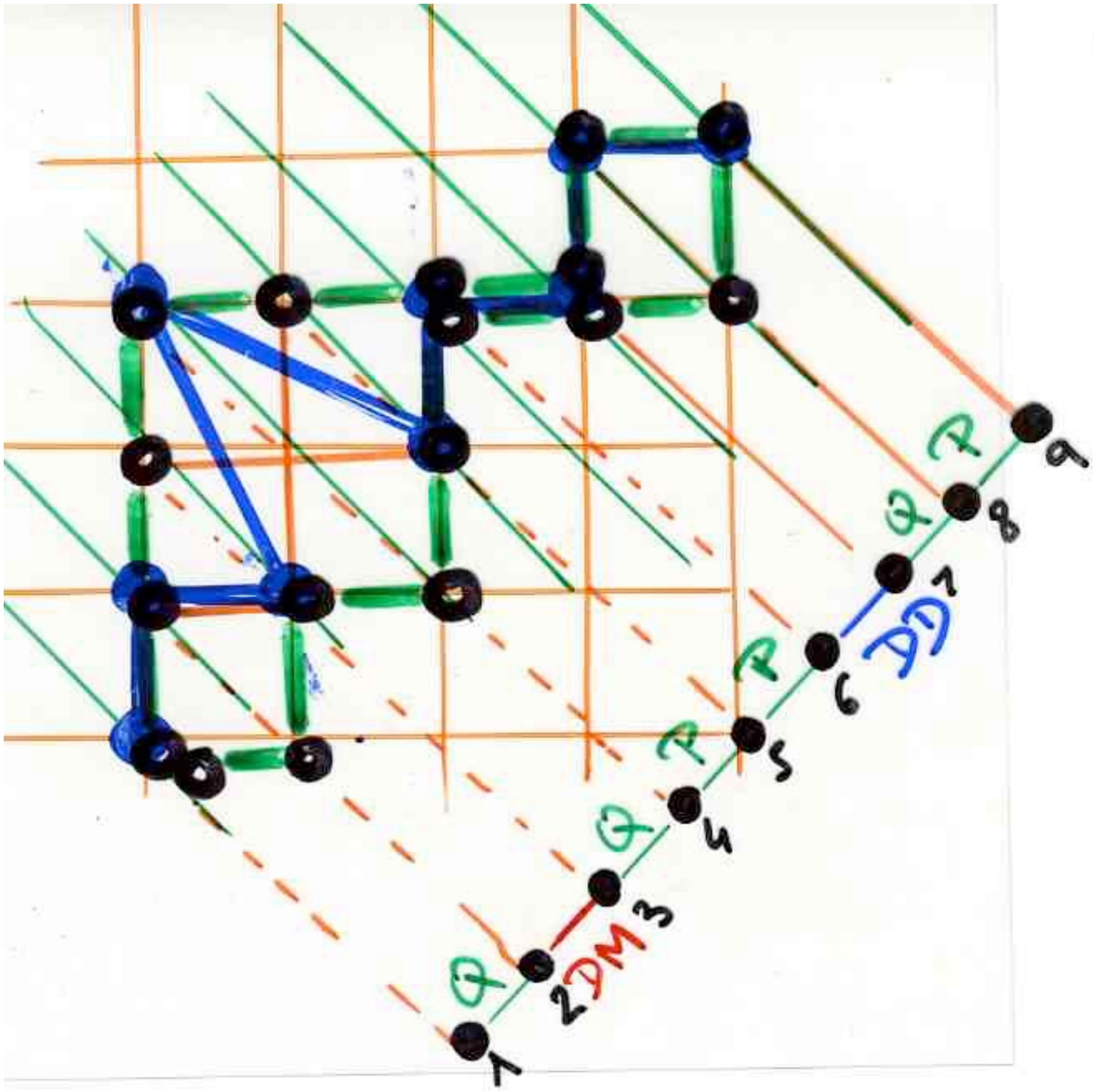
σ = 5 1 2 4 3 9 7 8 6
P P P P
o Q DM Q Q DD o
DM
DD
1 2 3 4 5 6 7 8 9
γ(i): 1 2 2 3 2 1 1 2 1
f(i): 1 2 2 2 1 1 1 2 1



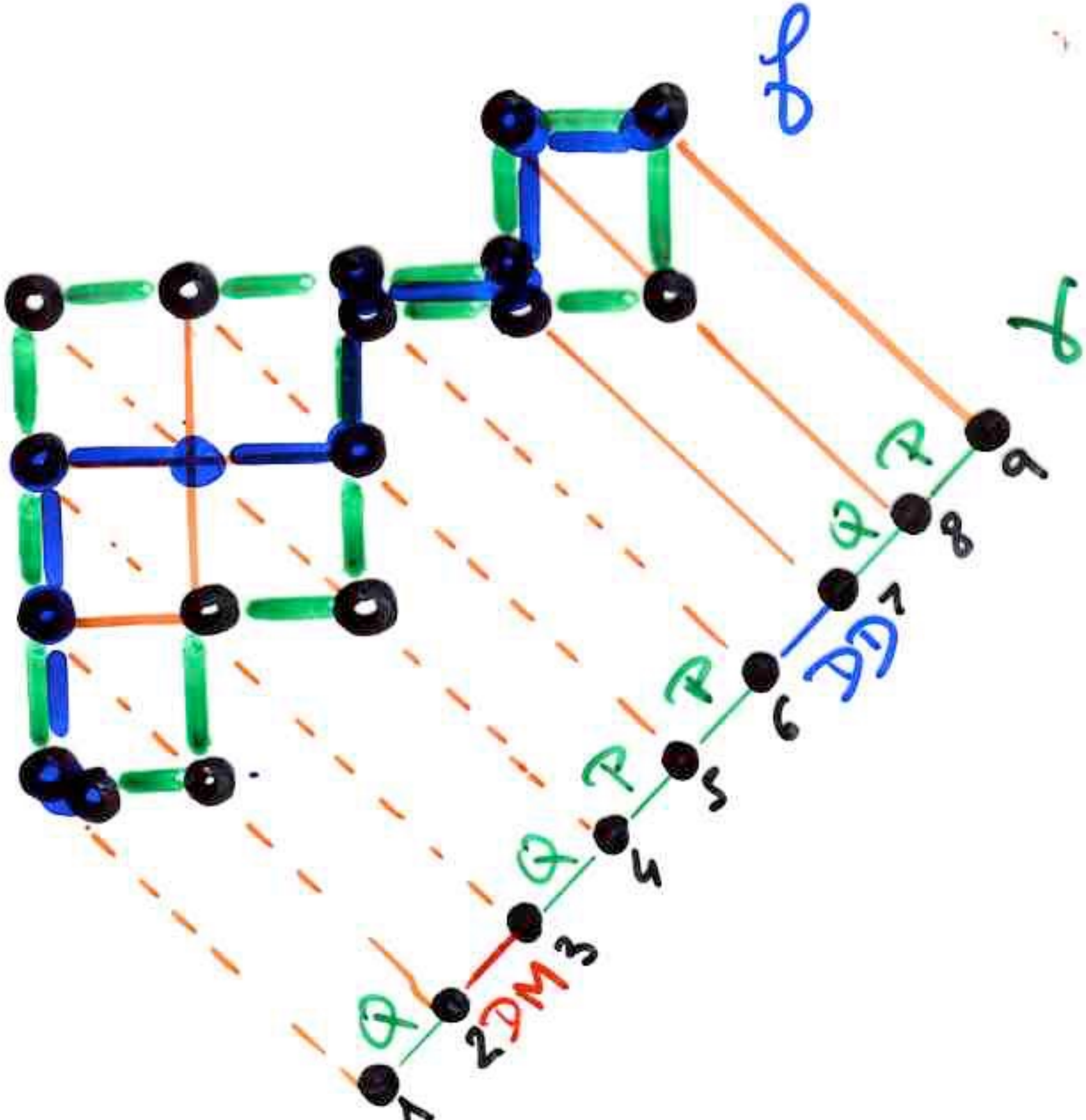
1
2
3
4
5
6
7
8
9

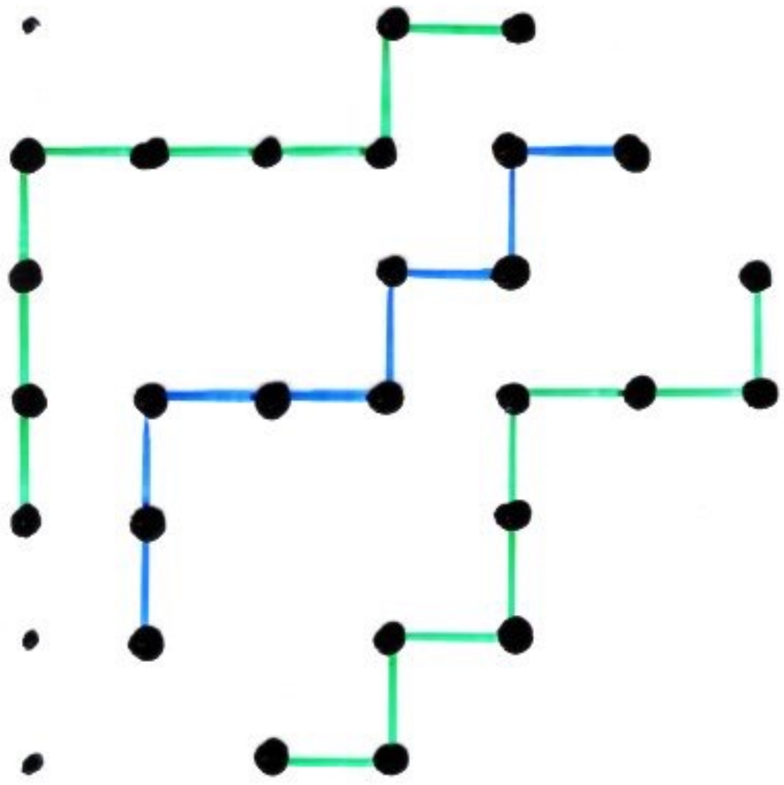


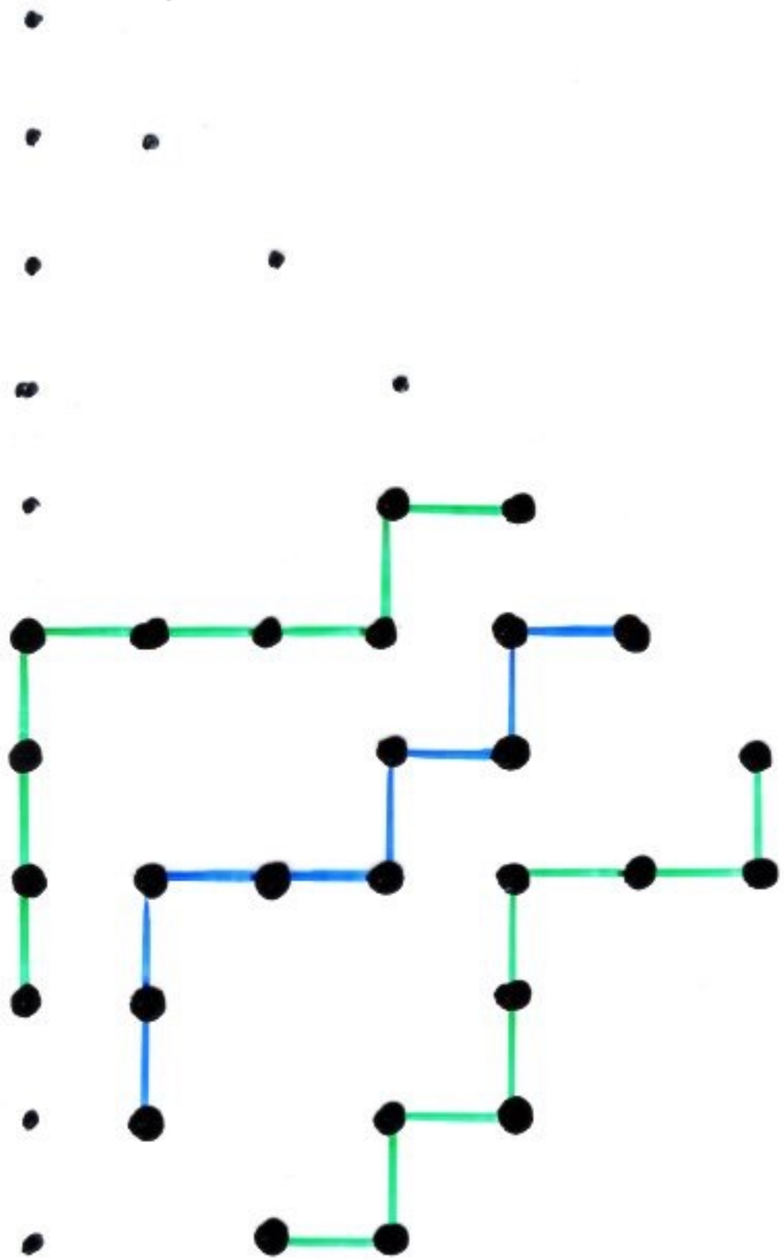
Q
DM
Q
P
P
AD
Q
P
1
2
3
4
5
6
7
8
9

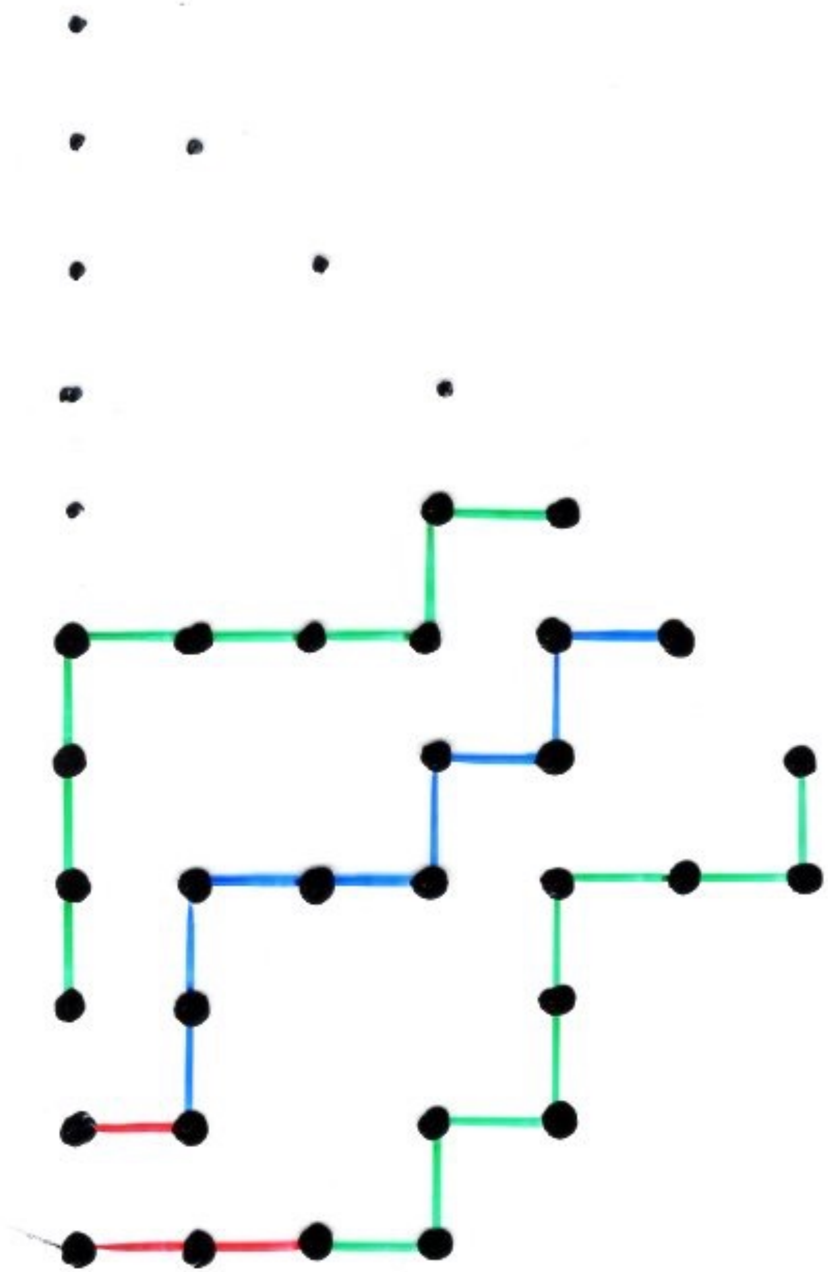


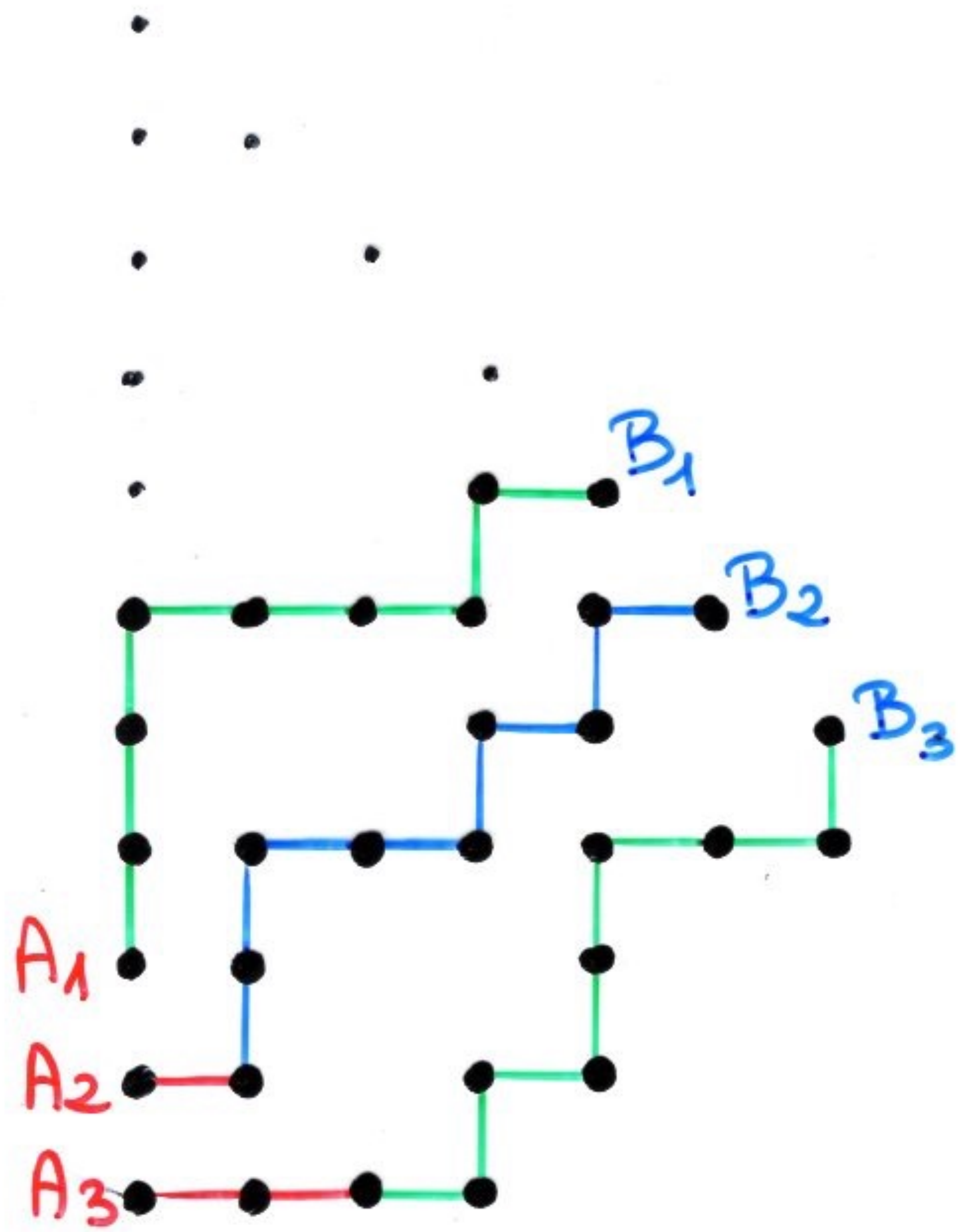
1
Q
2
DM
3
Q
4
P
5
P
6
AD
7
Q
8
P
9





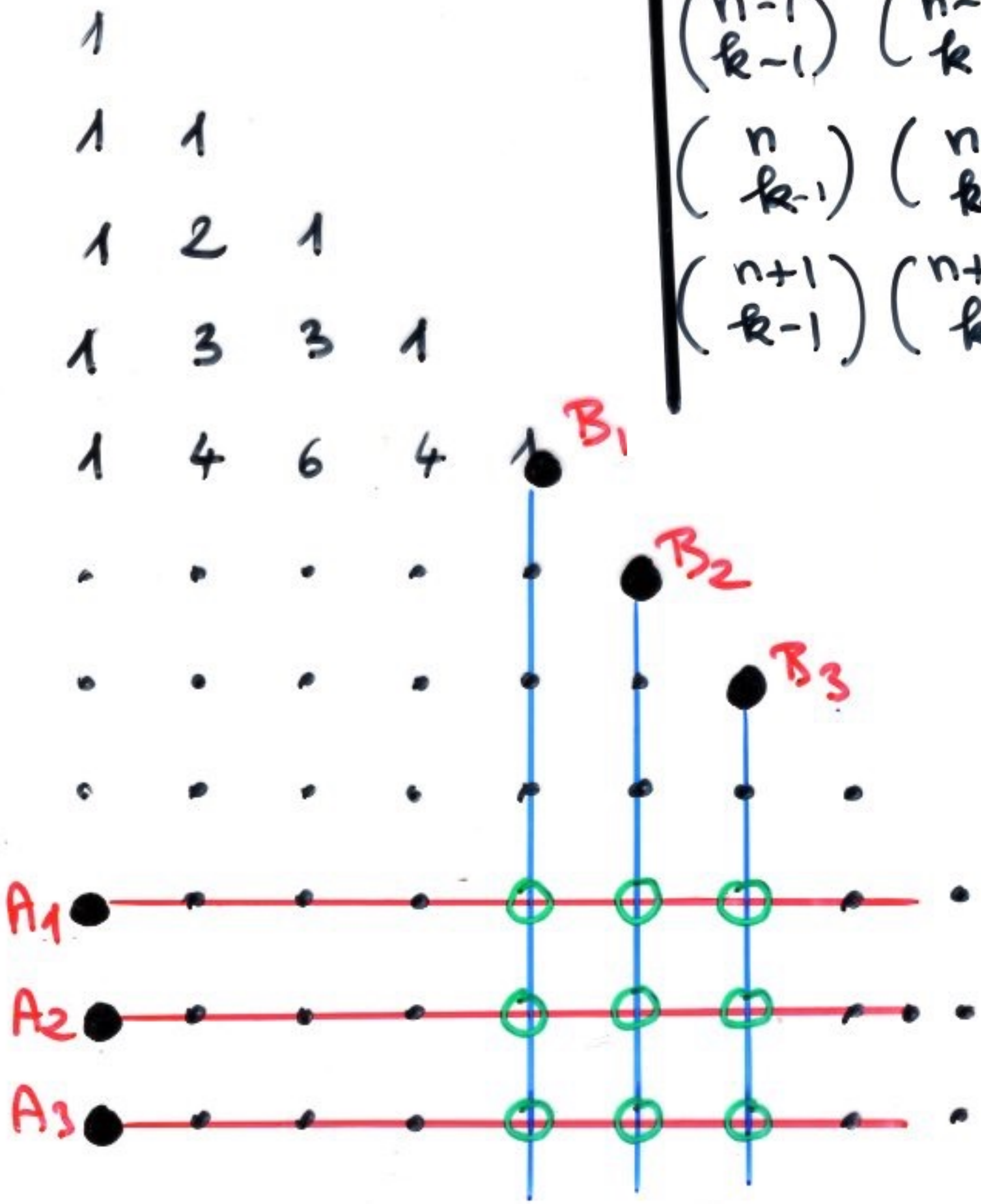






B1
B2
B3
A1
A2
A3

$$\begin{vmatrix} \binom{n-1}{k-1} & \binom{n-1}{k} & \binom{n-1}{k+1} \\ \binom{n}{k-1} & \binom{n}{k} & \binom{n}{k+1} \\ \binom{n+1}{k-1} & \binom{n+1}{k} & \binom{n+1}{k+1} \end{vmatrix}$$



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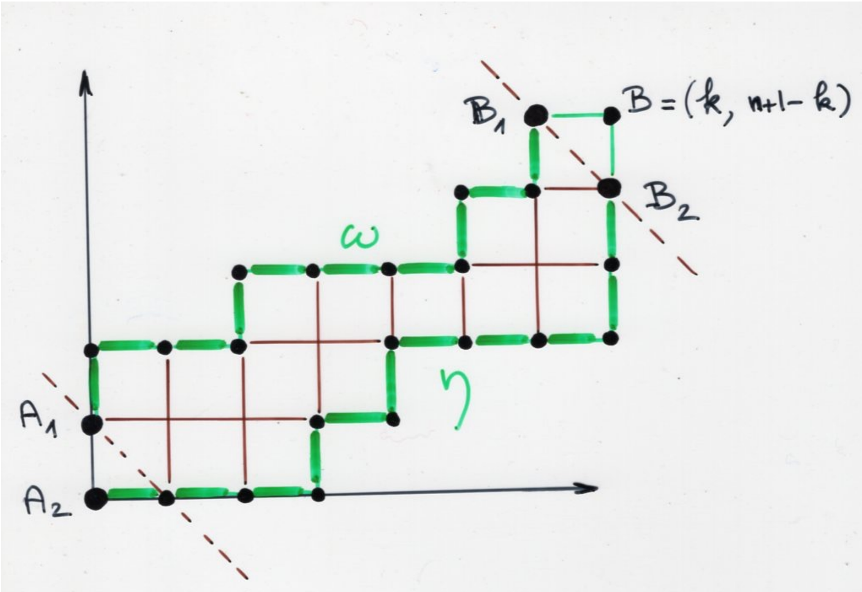
$$\begin{vmatrix} \binom{n-1}{k-1} & \binom{n-1}{k} & \binom{n-1}{k+1} \\ \binom{n}{k-1} & \binom{n}{k} & \binom{n}{k+1} \\ \binom{n+1}{k-1} & \binom{n+1}{k} & \binom{n+1}{k+1} \end{vmatrix}$$



$$\begin{vmatrix} \binom{n-1}{k-1} & \binom{n-1}{k} \\ \binom{n}{k-1} & \binom{n}{k} \end{vmatrix} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

Narayana numbers

(β)-distribution

Catalan numbers