An introduction to

enumerative
algebraic combinatorics
bijective

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Chapter 3

exponential structures
and
exponential generating functions
(2)

IMSc

11 February 2016

Weighted species

example assembleés of permutations

$$\left\{ [1\ 4\ 3\ 9\ 7]\ [2\ \ 5]\ [6\ 8] \right\}$$

$$E \circ S\,(t) = \exp\left(\frac{t}{1-t}\right)$$

example assemblées of permutations

$$\left\{ \overset{x}{[1\ 4\ 3\ 9\ 7]} \quad \overset{x}{[2\ \ 5]} \quad x[6\ \ 8] \right\}$$

$$(E \circ S)_v(t) = \exp\left(\frac{xt}{1-t}\right)$$

variable x is counting the number of components

weighted species F_v

exercise

Prove that the number of "assemblées" of permutations on n elements having k components is $\binom{n-1}{k-1} \frac{n!}{k!}$

(called Lah numbers)

$\mathbb{K}$ commutative ring

<u>Definition</u> weighted species F_v

$\alpha \in F[U] \longrightarrow v(\alpha) \in \mathbb{K}$

weight

(or valuation)

of the F-structure α

$f : U \longrightarrow V$

$\alpha \in F[U] \xrightarrow{F[f]} \beta \in F[V]$

$v(\alpha) = v(\beta)$

Definition generating power series $\mathbf{F}_v(t)$

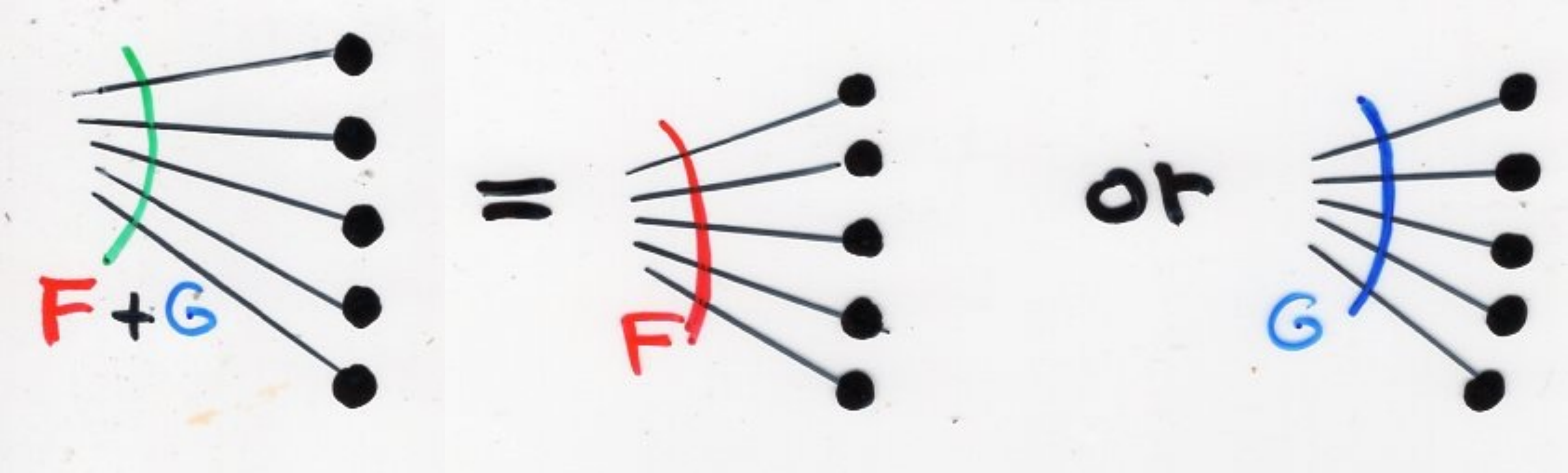
$$\mathbf{F}_v(t) = \sum_{n \geq 0} p_n \frac{t^n}{n!}$$

$$p_n \in \mathbb{K}$$

$$p_n = \sum_{\substack{\alpha \in \mathbf{F}[U] \\ \text{with } |U|=n}} v(\alpha)$$

operations on weighted species

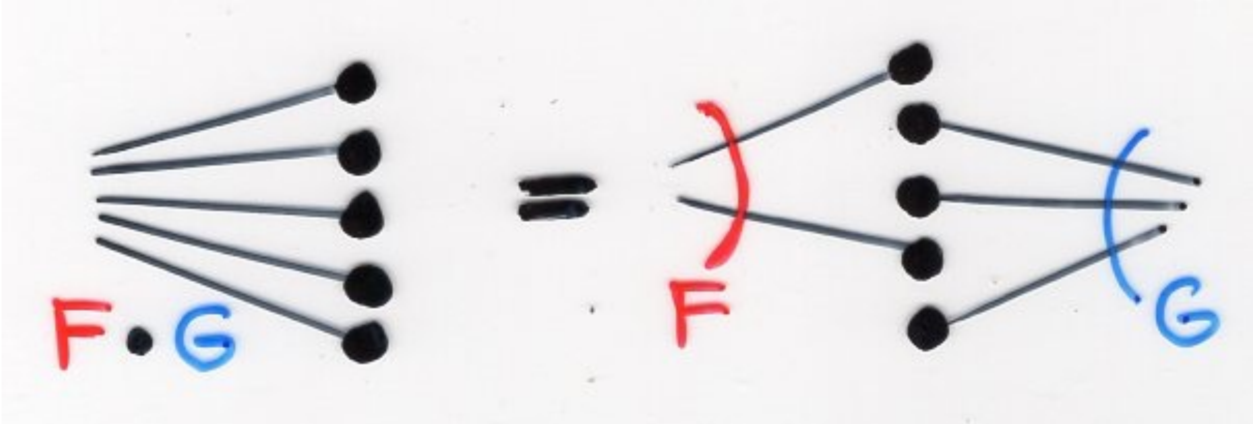
sum $\quad F_{v_1} + G_{v_2} \quad (F+G)_v$



<u>Proposition</u> $(F+G)_v(t) = F_v(t) + G_v(t)$

operations on weighted species

product $\quad F_{v_1} \cdot G_{v_2} \quad (F \cdot G)_v$



$\gamma = (U_1, U_2, \alpha, \beta)$

$\alpha \in F[U_1] \quad \beta \in G[U_2]$

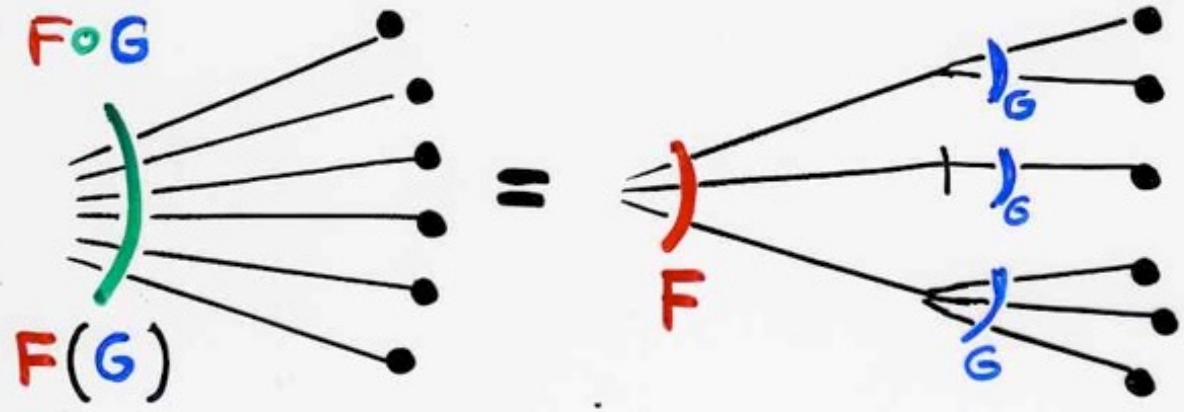
$\{U_1, U_2\}$ partition of U

$\gamma \in F \cdot G[U] \qquad v(\gamma) = v_1(\alpha)\, v_2(\beta)$

<u>Proposition</u> $\quad (F \cdot G)_v(t) = F_v(t) \cdot G_v(t)$

substitution $\quad F_{v_1} \circ G_{v_2} \quad (F \circ G)_v$

$$\gamma = (\{U_1, \dots, U_k\} \, ; \, \alpha \, ; \, \beta_1, \dots, \beta_k) \in F \circ G[U]$$

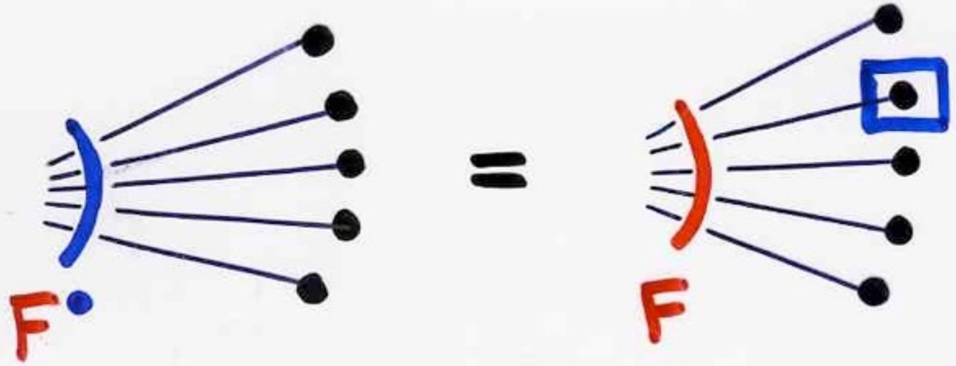


$$v(\gamma) = v(\alpha) \cdot v(\beta_1) \cdots v(\beta_k)$$

<u>Proposition</u> $\quad (F \circ G)_v(t) = F_v\big(G_v(t)\big)$

pointed

$(F_{v_1})^{\bullet}$ $(F^{\bullet})_{v}$



$\alpha \in F[U]$ $x \in U$ $(\alpha, x) \in F^{\bullet}[U]$

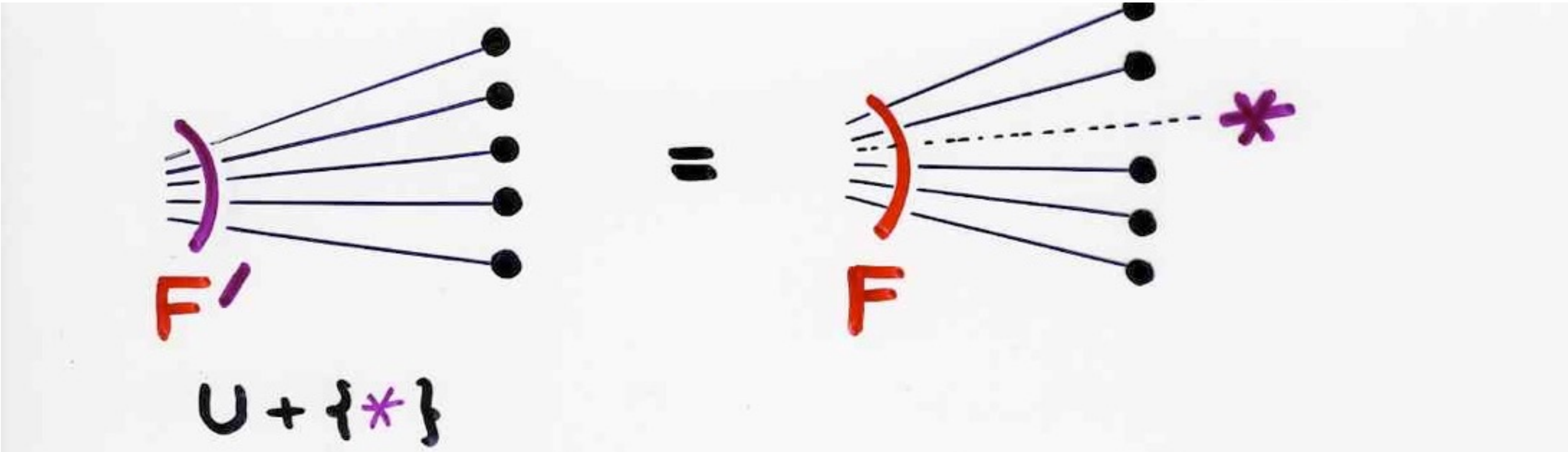
$\gamma = (\alpha, x) \in F^{\bullet}[U]$ $v(\gamma) = v_1(\alpha)$

Proposition

$$F^{\bullet}_{v}(t) = t \frac{d}{dt} F_{v}(t)$$

derivative

$(F_{v_1})'$ $\quad$ $(F')_v$

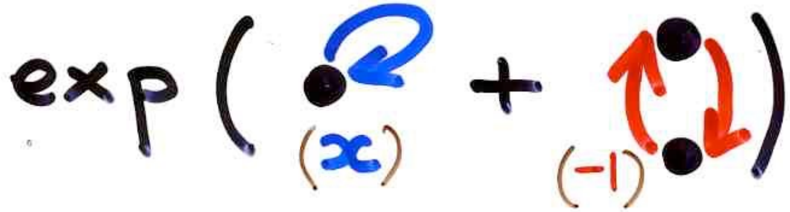


$\gamma \in F'[U]$ $\quad$ $\gamma \in F[U + \{*\}]$ $\quad$ $v(\gamma) = v_1(\gamma)$

<u>Proposition</u>

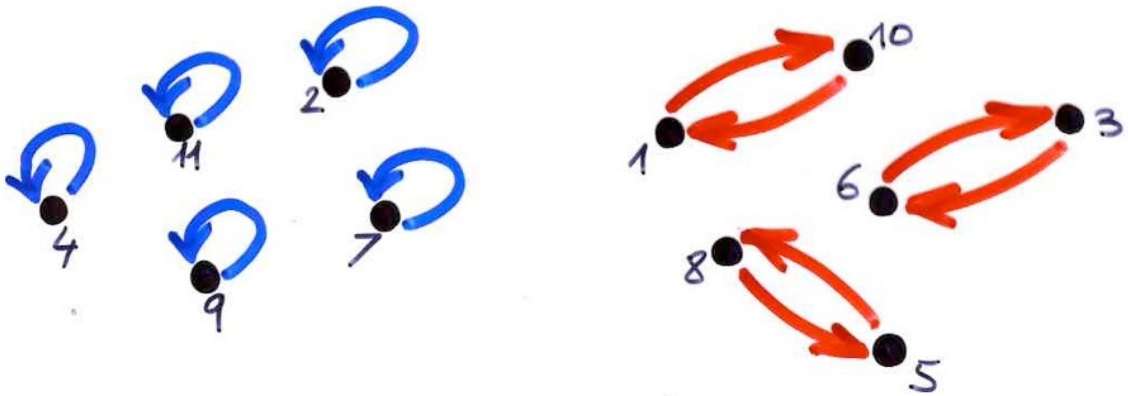
$$F'_v(t) = \frac{d}{dt} F_v(t)$$

Examples:
some orthogonal polynomials

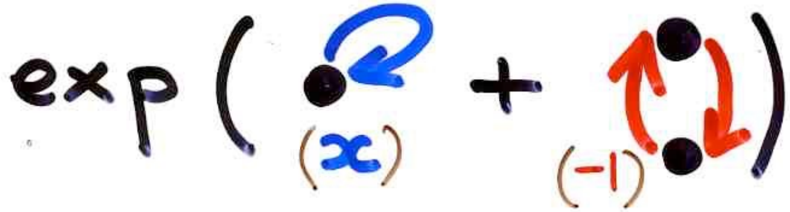


$$\sum_{n\geqslant 0} H_n(x)\frac{t^n}{n!} = \exp\left(xt - \frac{t^2}{2}\right)$$

Hermite configurations

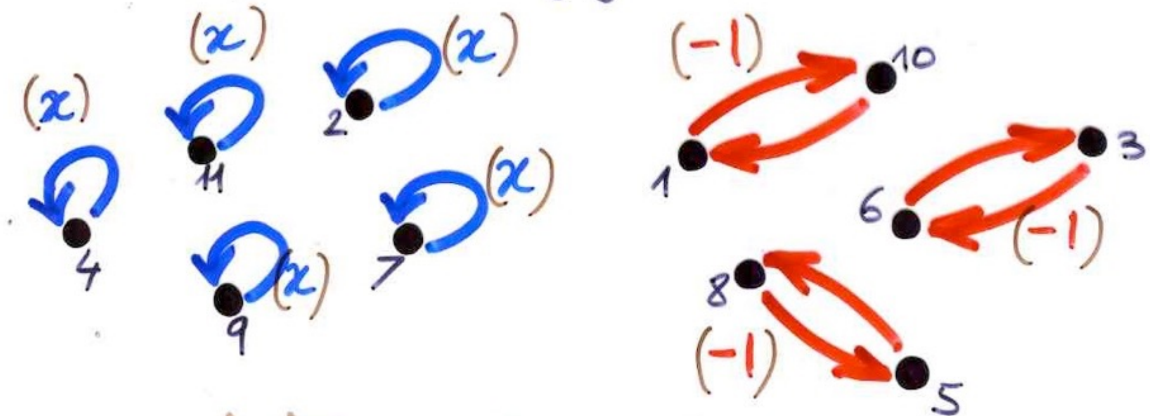


Charles Hermite
1822 - 1901



$$\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = \exp\left(xt - \frac{t^2}{2}\right)$$

Hermite configurations

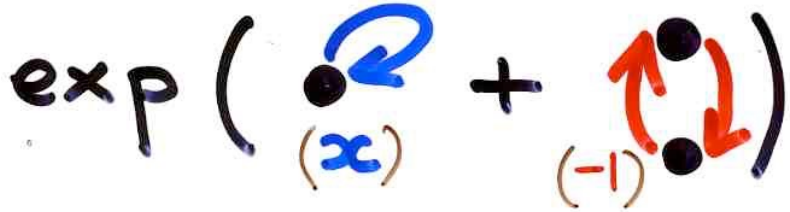


weight (x)
(−1)

$$H_n(x) = \sum v(\alpha)$$

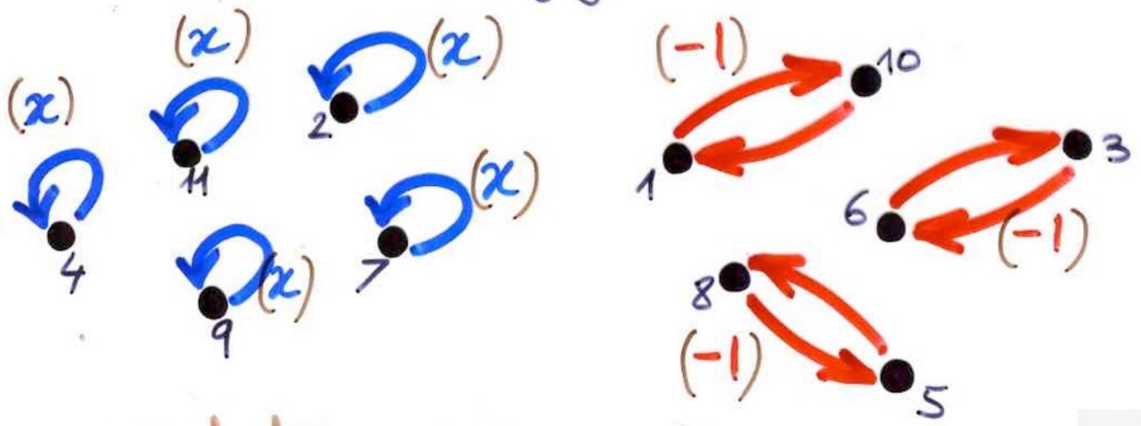
α involutions on $[1, n]$

Charles Hermite
1822 - 1901



$$\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = \exp\left(xt - \frac{t^2}{2}\right)$$

Hermite configurations



weight (x)

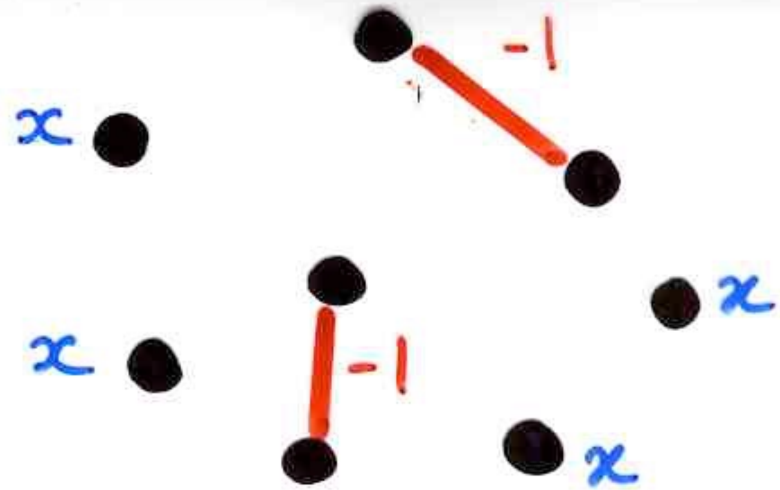
(-1)

$$H_n(x) = \sum v(\alpha)$$

α involutions on $[1, n]$

ex: Hermite

$$H_n(x) = \sum_{\substack{\text{matching } \gamma \\ \text{of } K_n}} (-1)^{|\gamma|} x^{\text{fix}(\gamma)}$$



matching polynomials
of K_n the complete graph
(→ see Ch 1)

exercise

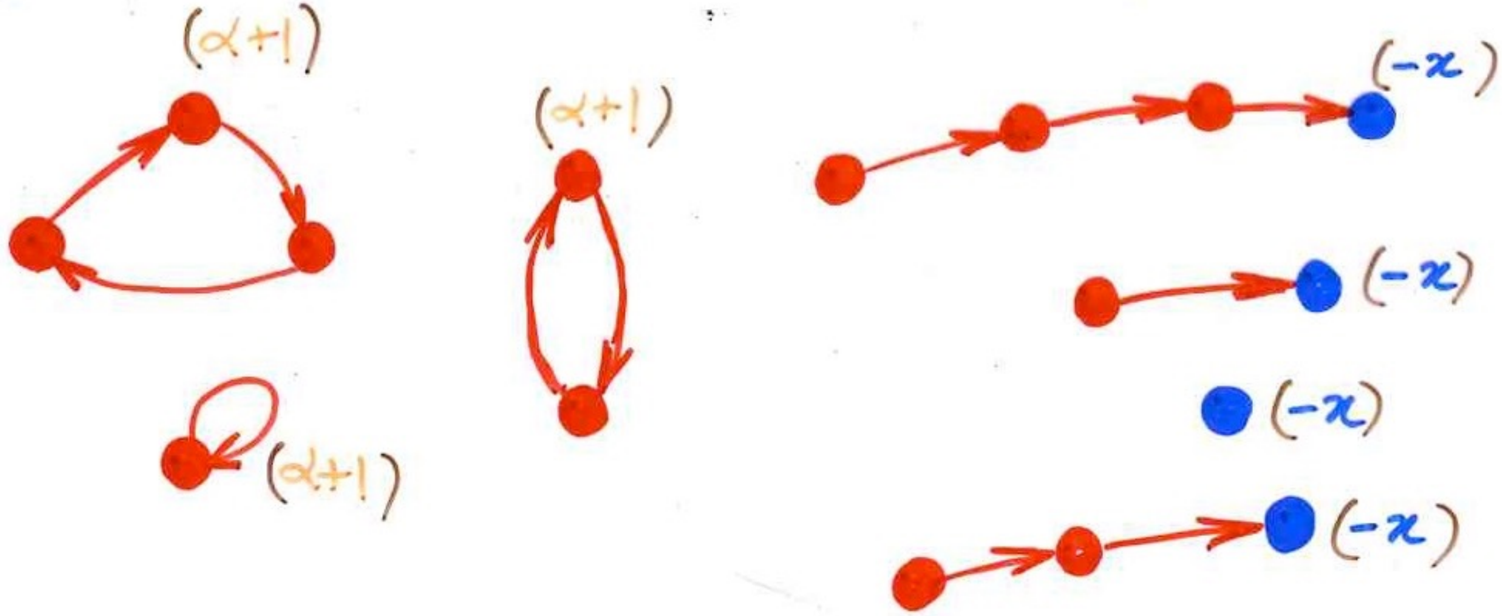
$$H_n(x) = \sum_{0 \leq 2k \leq n} (-1)^k \frac{n!}{2^k \, k! \, (n-2k)!} x^{n-2k}$$

Laguerre polynomial

Laguerre $L_n^{(\alpha)}(x)$

$$\sum_{n\geq 0} L_n^{(\alpha)}(x)\,\frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}}\,\exp\left(\frac{-xt}{1-t}\right)$$

Laguerre configuration



$$L_n^{\alpha}(x) = \sum_{LC} v(LC)$$

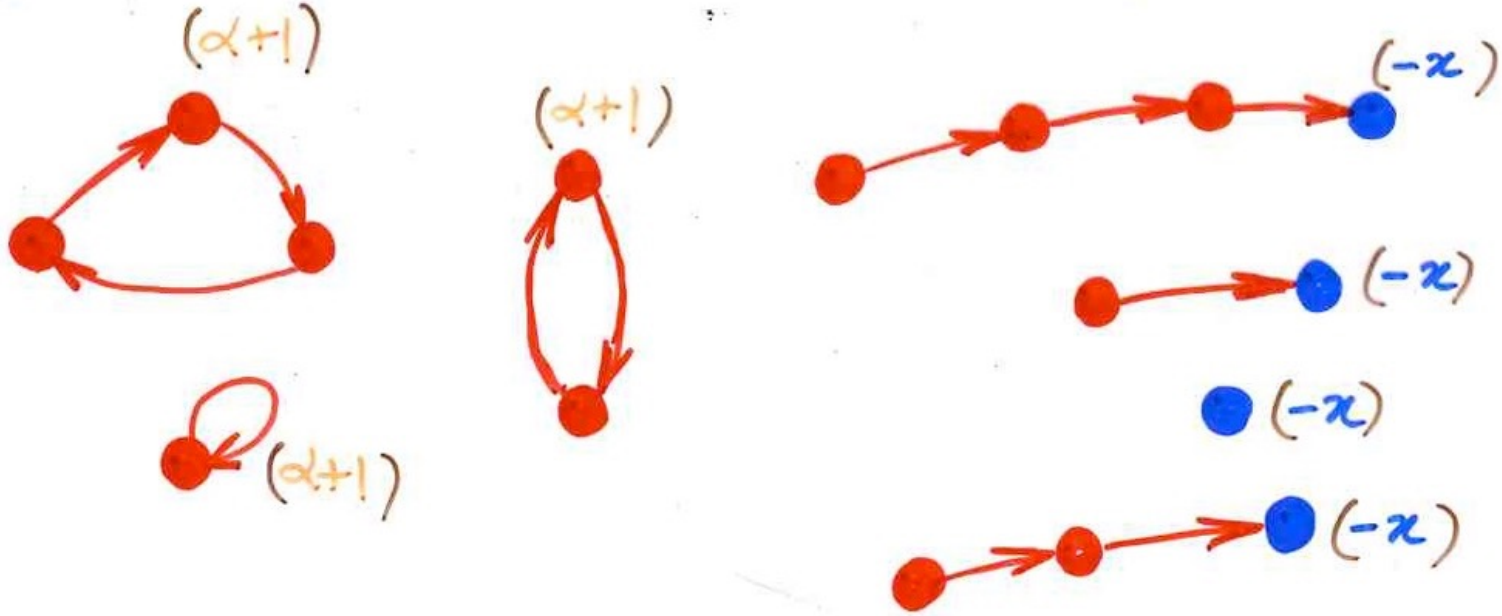
LC
Laguerre configurations on $[1,n]$

$$v(LC) = (\alpha+1)^{i}(-x)^{j}$$

i = number cycles 

j = number of chains 

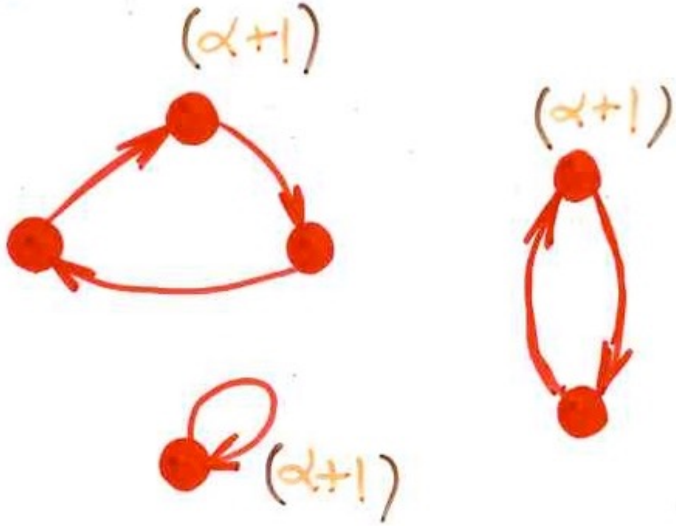
Laguerre configuration



Laguerre $\quad L_n^{(\alpha)}(x)$

$$\sum_{n\geq 0} L_n^{(\alpha)}(x)\,\frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}}\,\exp\left(\frac{-xt}{1-t}\right)$$

Laguerre configuration



$$\exp\left((\alpha+1)\log\frac{1}{(1-t)}\right)$$

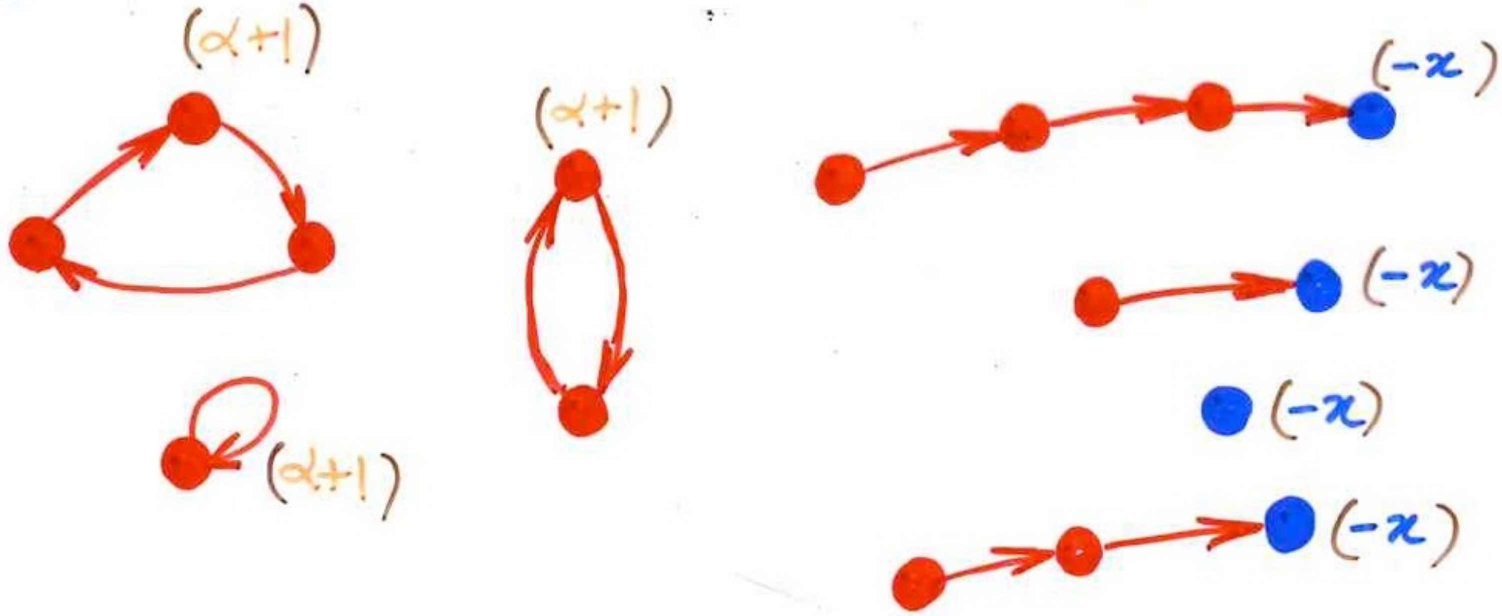
$$= \exp\left(\log\frac{1}{(1-t)^{(\alpha+1)}}\right)$$

$$= \frac{1}{(1-t)^{(\alpha+1)}}$$

Laguerre $L_n^{(\alpha)}(x)$

$$\sum_{n\geq 0} L_n^{(\alpha)}(x)\,\frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}}\,\exp\left(\frac{-xt}{1-t}\right)$$

Laguerre configuration



exercise

$$L_n^{(\alpha)}(x) = (\alpha+1)_n \; {}_1F_1\left[\begin{matrix}-n \\ \alpha+1\end{matrix} ; x\right]$$

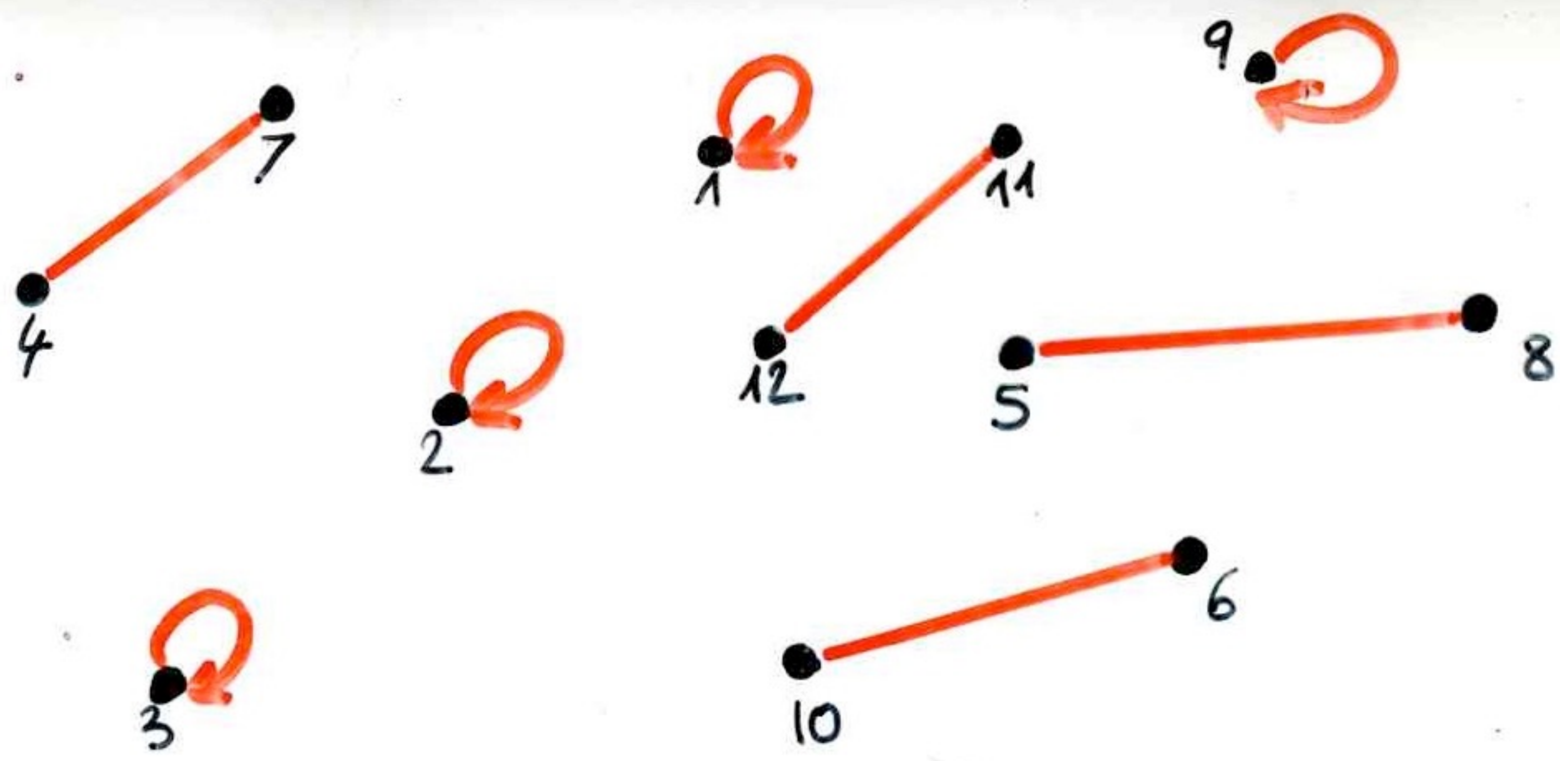
$$= \sum_{i+j=n} \binom{n}{i} (\alpha+1+j)_i \, (-x)_j$$

Mehler identity for Hermite polynomials

$$\sum_{n\geq 0} H_n(x) H_n(y) \frac{t^n}{n!} = (1-4t^2)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\sum_{n\geq 0} H_n(x) \quad \frac{t^n}{n!} =$$

$$H_n(y)$$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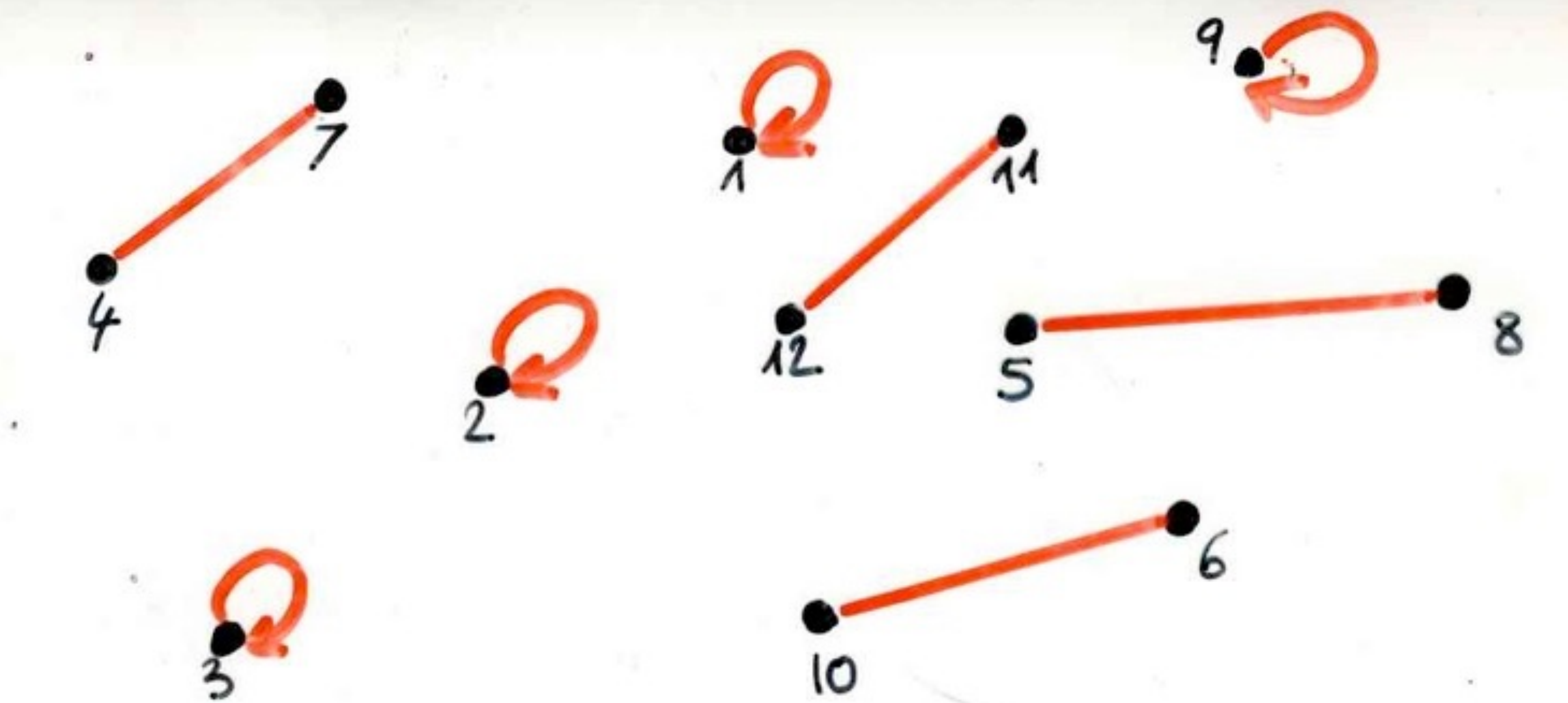
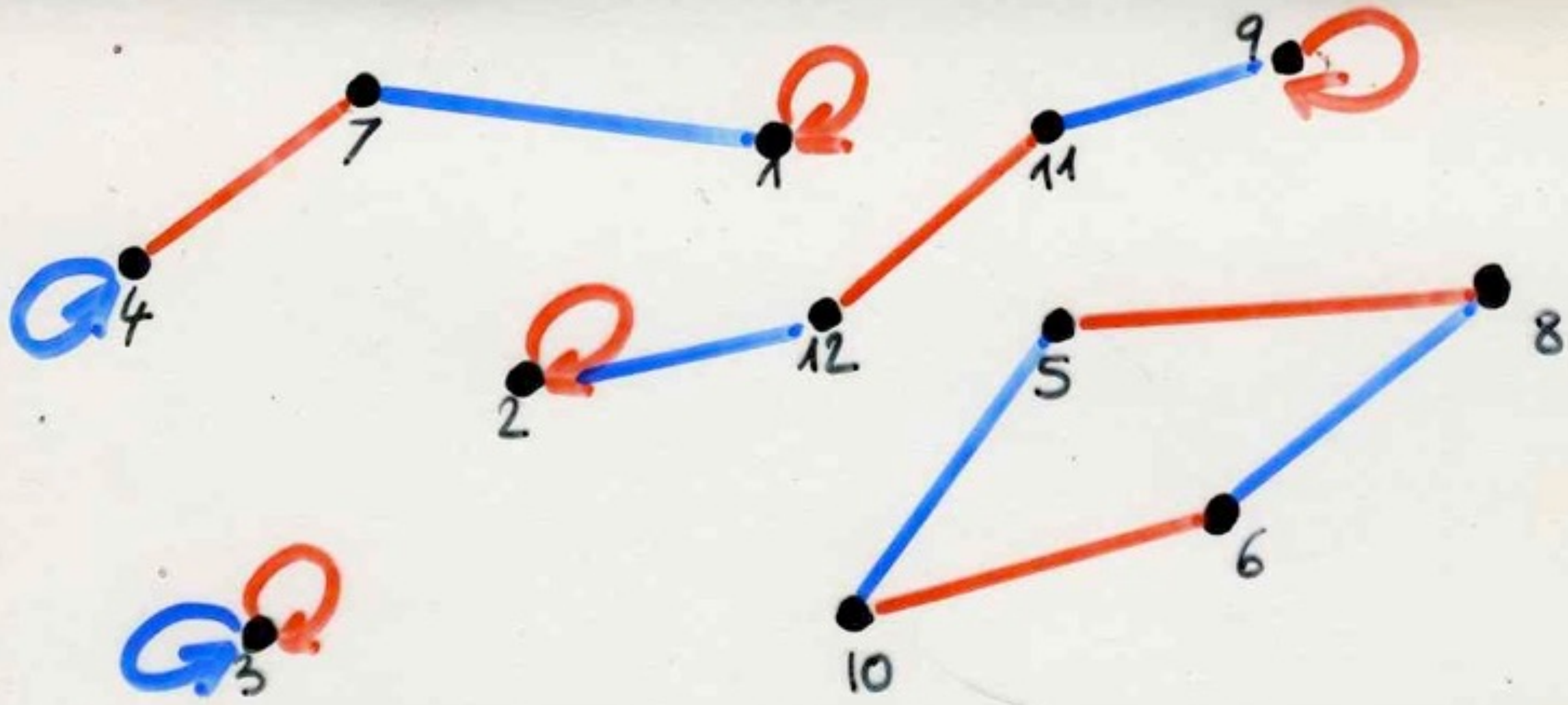


$$\sum_{n\geq 0} H_n(x) \quad \frac{t^n}{n!} =$$



$$\sum_{n\geq 0} H_n(x)H_n(y)\frac{t^n}{n!} =$$

$$(1-4t^2)^{-1/2} \exp\left[\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

here $H_n(x)$ is Hermite polynomial $H_n(t)$ with $t = 2x$

$H_n(x)$

$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)} + \frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)}+\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\frac{-4\,y^2\,t^2}{1-4t^2}$$

$$\frac{-4\,x^2\,t^2}{1-4t^2}$$

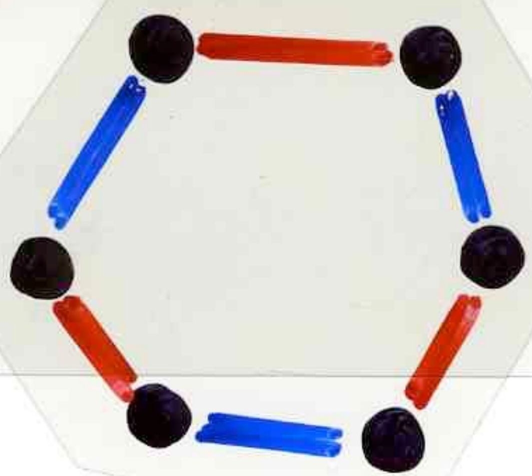
$$\frac{1}{2}\log\frac{1}{(1-4t^2)}$$

$$\frac{4xyt}{1-4t^2}$$

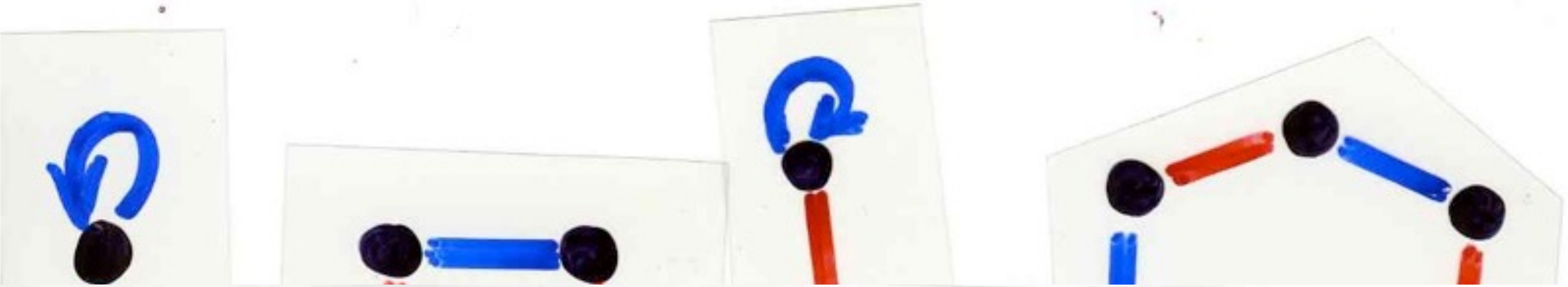
$$\frac{-4x^2t^2}{1-4t^2}$$

$$\frac{-4y^2t^2}{1-4t^2}$$

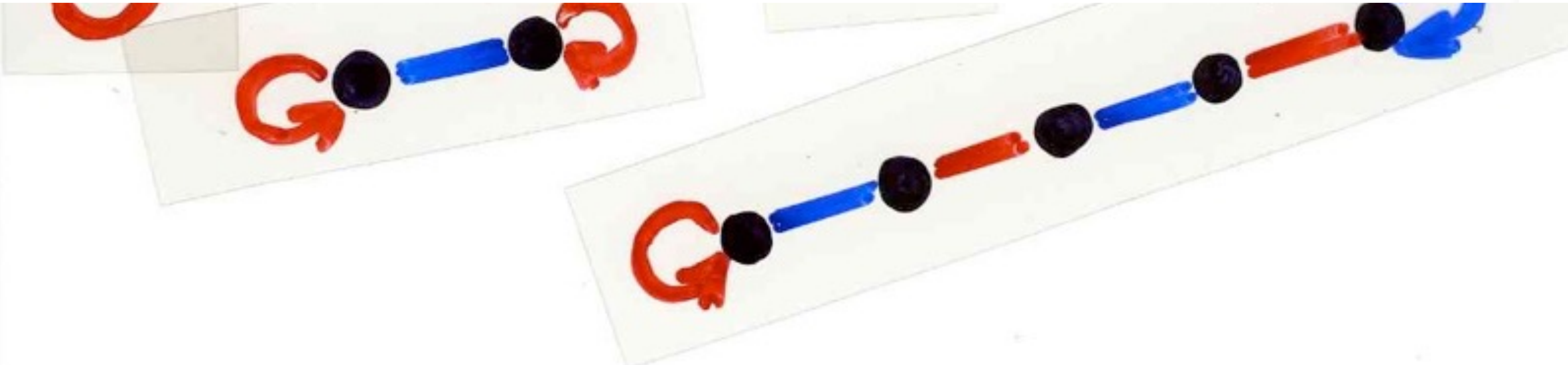
$$\frac{4xyt}{1-4t^2}$$



$$\frac{1}{2}\log\frac{1}{(1-4t^2)}$$



$$\sum_{n\geqslant 0} H_n(x)\, H(y)\, \frac{t^n}{n!} = (1-4t^2)^{-1/2} \exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$



Sheffer polynomials

Def- $\{P_n(x)\}_{n\geq 0}$ $\quad P_n(x) \in \mathbb{K}[x]$

sequence of **Sheffer** polynomials

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = g(t)\ \exp(x f(t))$$

$f(t),\ g(t) \in \mathbb{K}[[t]]$, $\quad f(0)=0,\ f'(0)\neq 0,\ g(0)\neq 0$

$\Rightarrow \quad \deg(P_n(x)) = n$

Def- binomial type polynomials

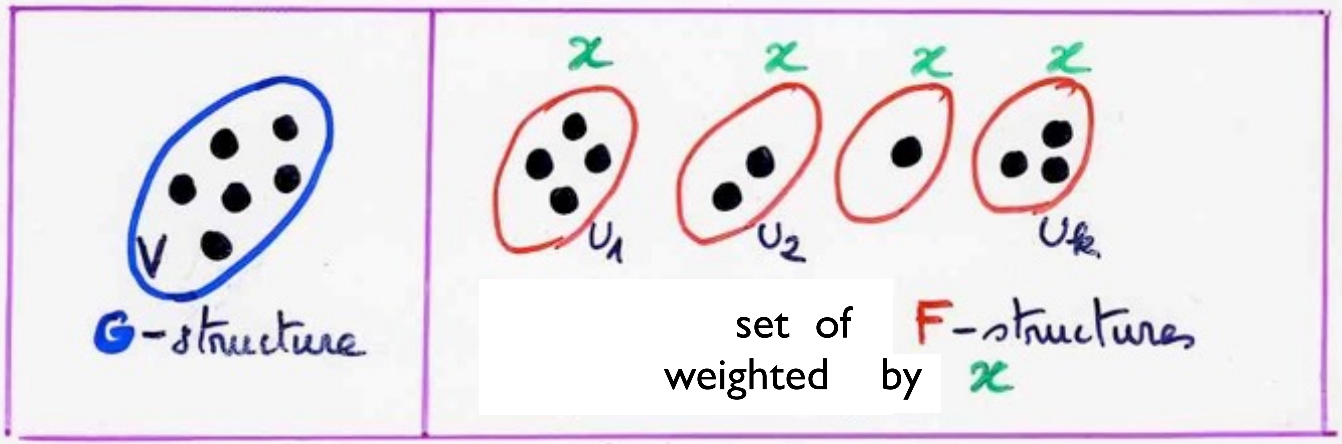
$$g(t) = 1$$

combinatorial interpretation

$\mathbf{F}$ $\quad$ $\mathbf{G}$ $\quad$ $\mathbf{H} = \mathbf{G} \cdot (\mathbf{E} \circ \mathbf{F})$

$f(t)$ $\quad$ $g(t)$

$$P_n(x) = \sum_{0 \leq k \leq n} a_{n,k}\, x^k$$



$\mathbf{H}_V$-structure

$\mathbf{F}_{V_1}$ $\quad$ $\mathbf{G}_{V_2}$ $\quad$ $\mathbb{K}$

ex– Stirling numbers first kind $s_{n,k}$

$$s_n(x) = \sum_{1 \le k \le n} s_{n,k}\, x^k$$

cycles

$$\sum_{n \geq 0} s_n(x) \frac{t^n}{n!} = \exp\left(x \log(1-t)^{-1}\right)$$

$$\sum_{n \geq 0} s_n(x) \frac{t^n}{n!} = (1-t)^{-x}$$

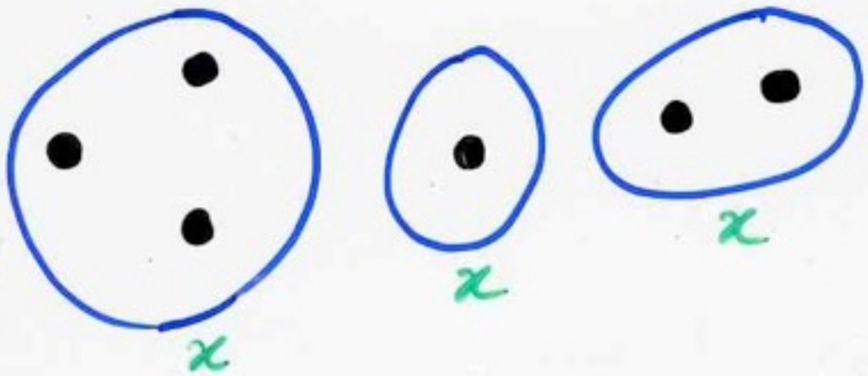
$$s_n(x) = x(x+1)\dots(x+n-1)$$

ex. Stirling numbers second kind $S_{n,k}$

$$S_n(x) = \sum_{1 \leq k \leq n} S_{n,k}\, x^k$$

partitions

$$\sum_{n \geq 0} S_n(x) \frac{t^n}{n!} = \exp(x\,(e^t - 1))$$



$B = E \circ E^*$

$v(\beta) = x_n$ si $|\beta| = n$

$$\sum_{n \geq 0} Y_n(x_1, x_2, \dots) \frac{t^n}{n!} = \exp\left(\sum_{n \geq 1} x_n \frac{t^n}{n!}\right)$$

exponential polynomials

exercise

binomial type polynomials

$$P_n(x+y) = \sum_k \binom{n}{k} P_k(x)\, P_{n-k}(y)$$

Sheffer type polynomials

$$s_n(x+y) = \sum_k \binom{n}{k} P_k(x)\, s_{n-k}(y)$$

Apell type polynomials

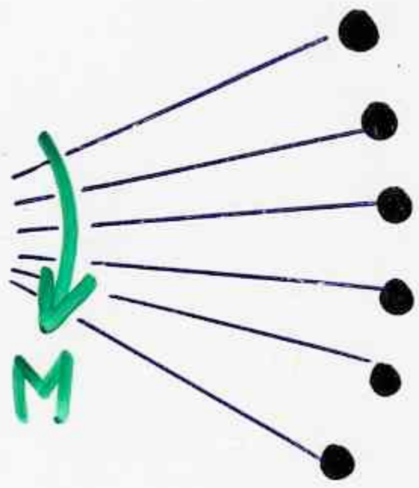
$$s_n(x+y) = \sum_k \binom{n}{k} x^k s_{n-k}(y)$$

Linear species

(or L-species)

$\mathbb{B}$ - species

$\mathbb{L}$ - species



$M[U]$

U totally ordered

$\mathbb{L}$ category $\begin{cases} \text{finite totally ordered sets} \\ \text{increasing bijections} \end{cases}$

$\mathbb{E}\mathrm{ns}$ category $\begin{cases} \text{finite sets} \\ \text{funitions} \end{cases}$

<u>Definition</u> linear species $\mathbf{M}$

($\mathbb{L}$-species)

functor $\mathbb{L} \longrightarrow \mathbb{E}\mathrm{ns}$

U totally ordered $\longrightarrow \mathbf{M}[U]$

$U \xrightarrow[f]{} V$ $\qquad \mathbf{M}[U] \xrightarrow{M[f]} \mathbf{M}[V]$

increasing bijection

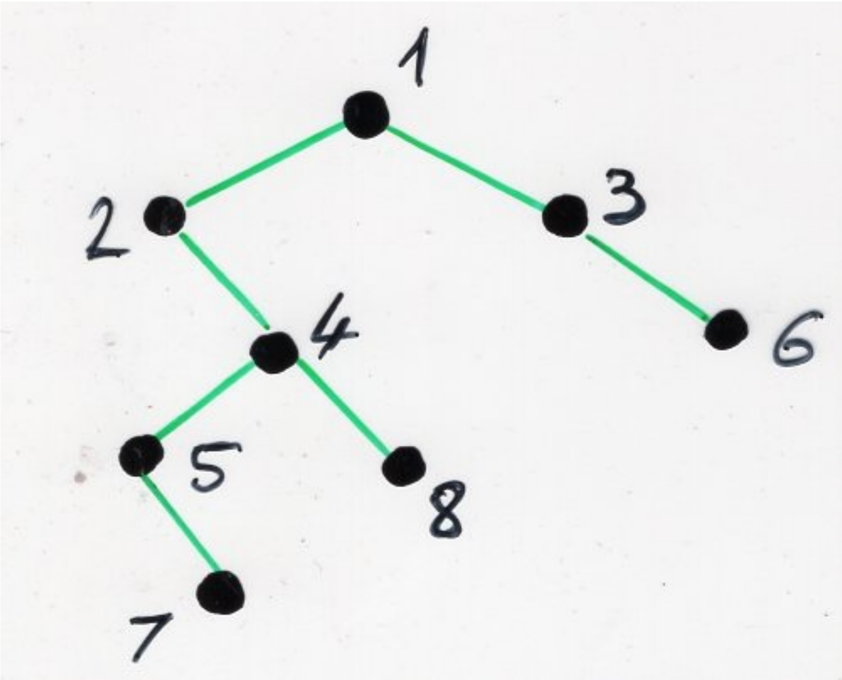
coherence of $\mathbf{M}$-transports

example of L-species

increasing binary trees

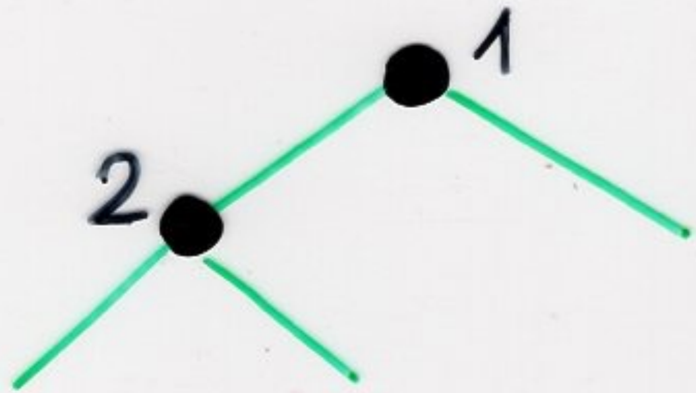
example

increasing binary trees



$n!$


1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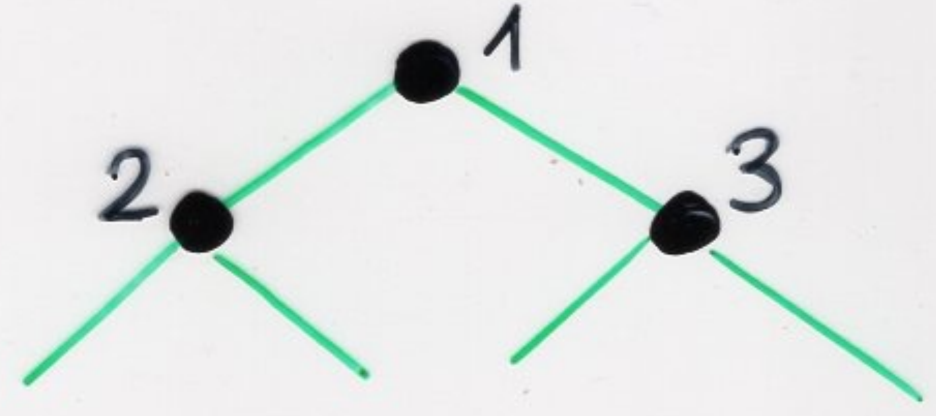
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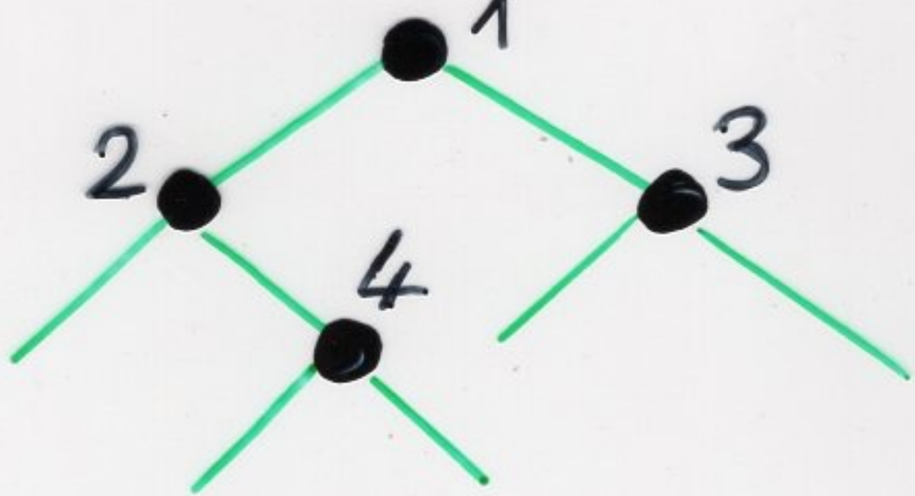
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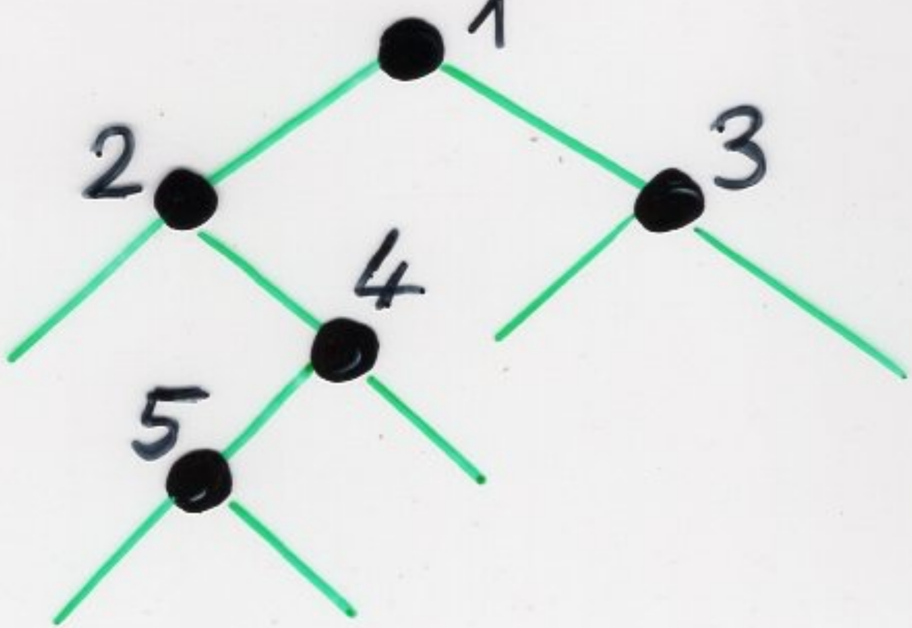
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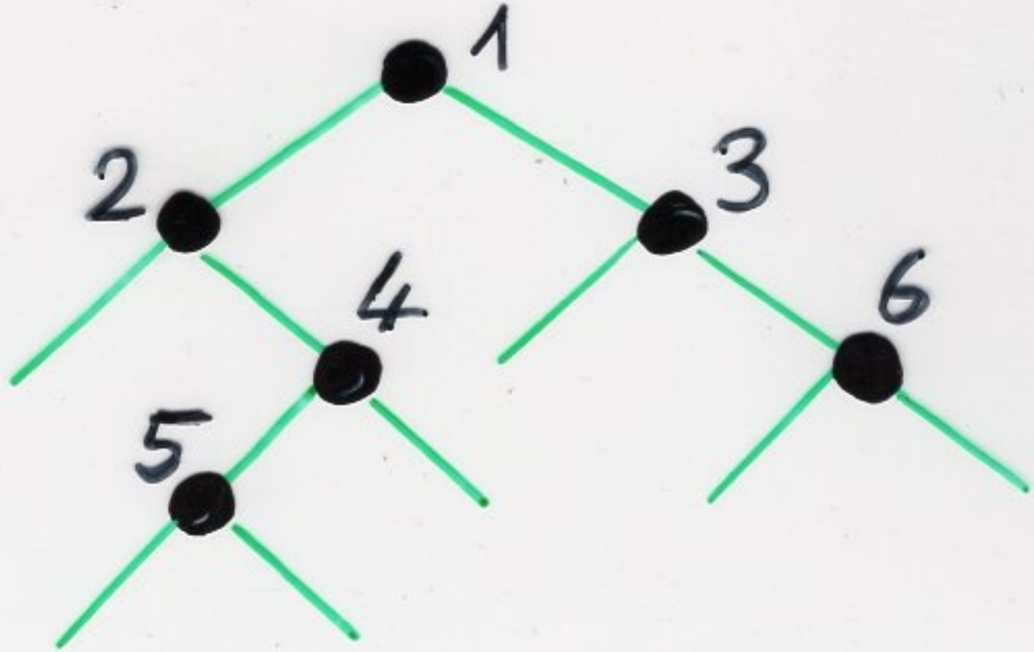
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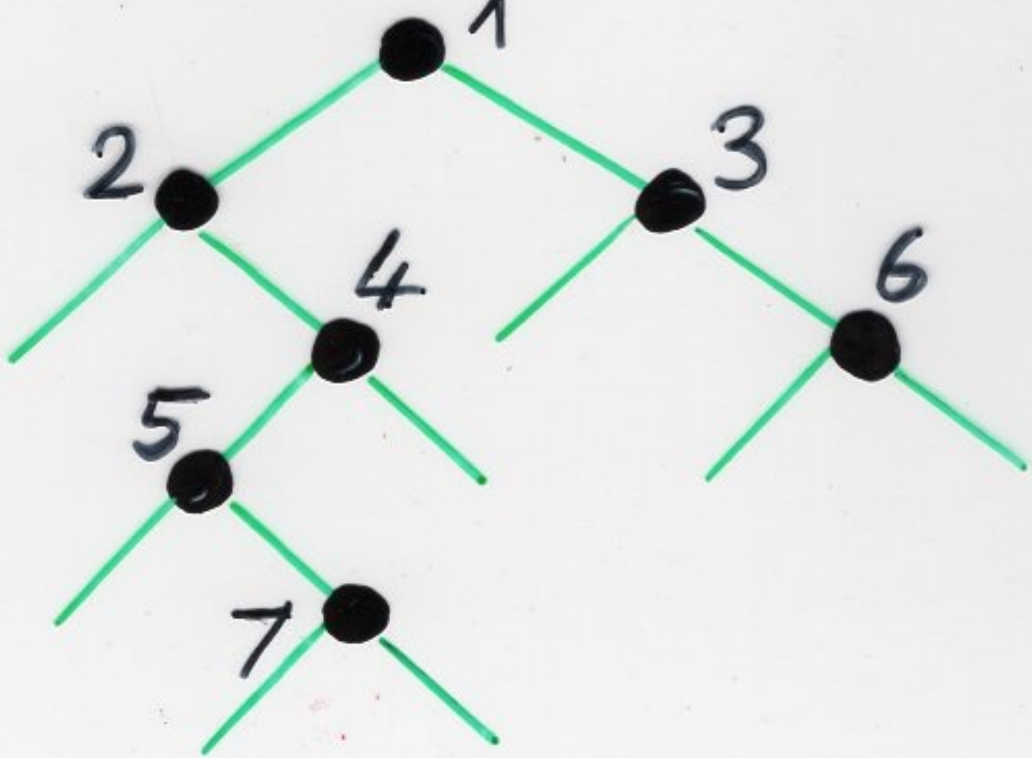
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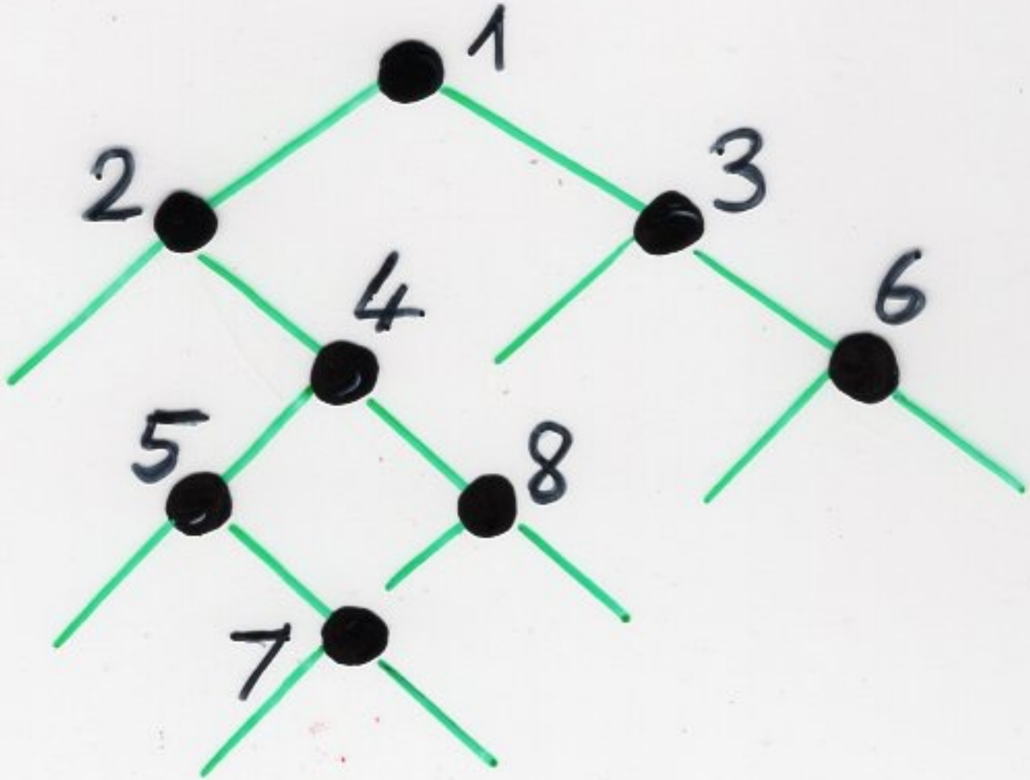
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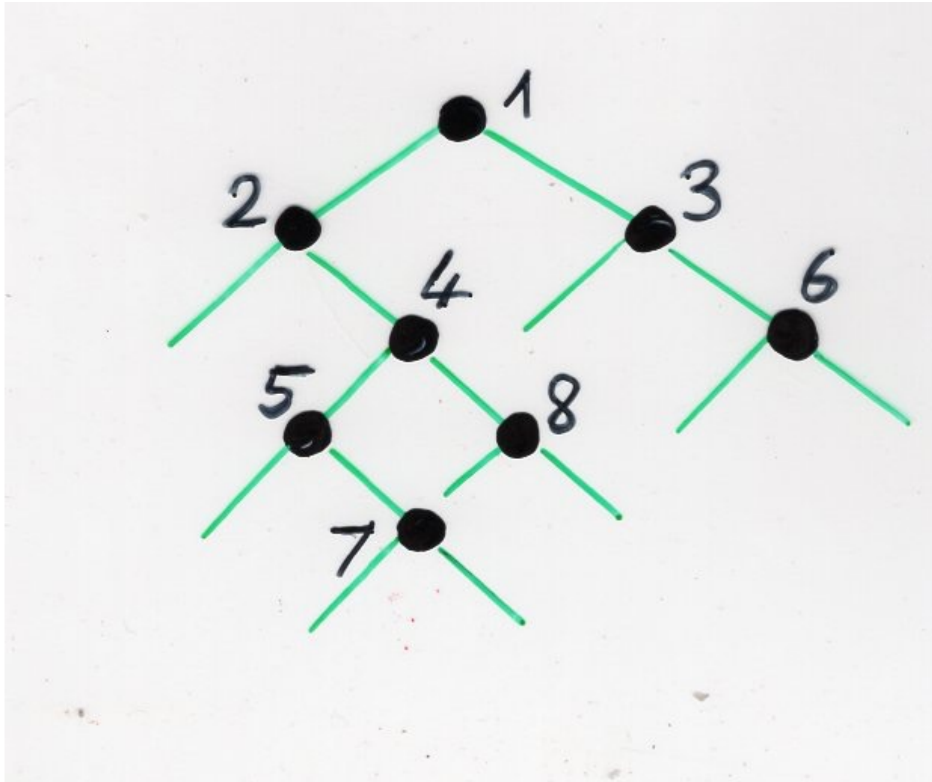
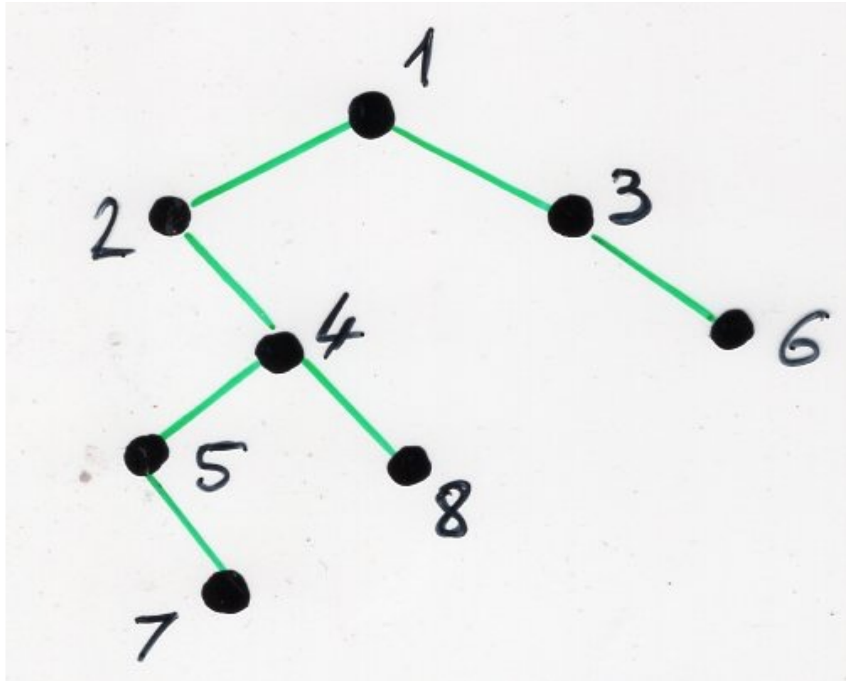




$$n!$$

$$Y(t) = \frac{1}{1-t}$$

Operations on $\mathbb{L}$-species

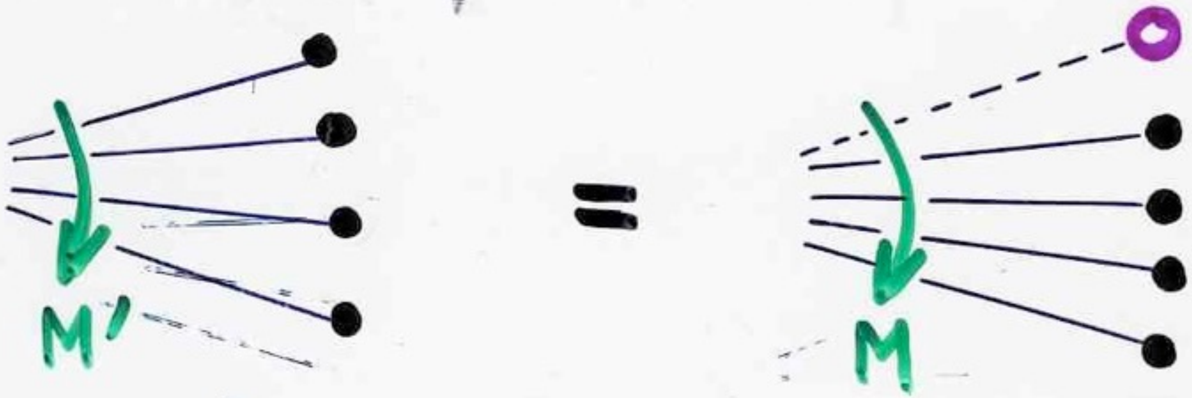
sum
product
substitution
pointed

derivative of an L-species

Definition

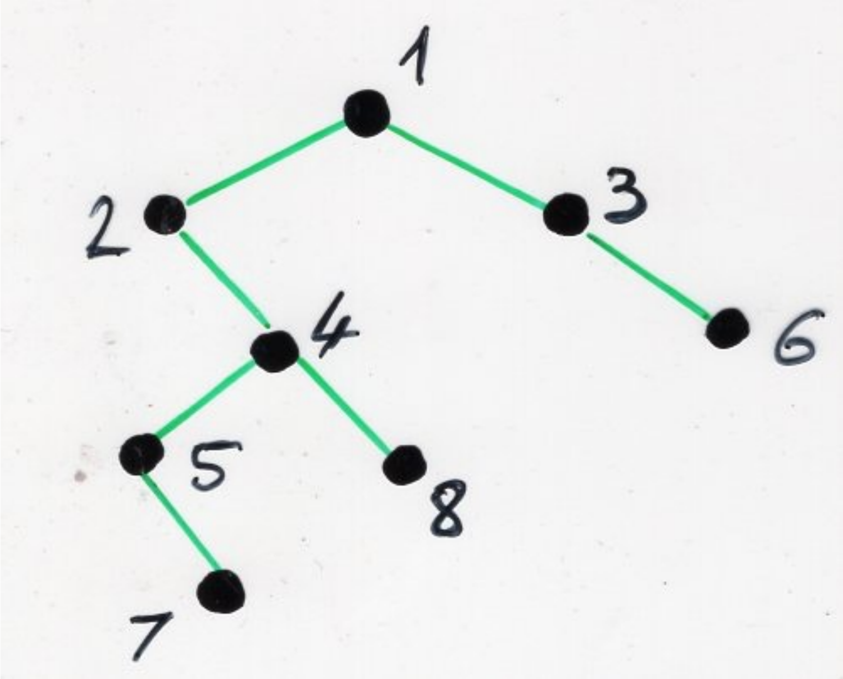
derivative of an $\mathbb{L}$-species M

$$M'[U] = M[1+U]$$

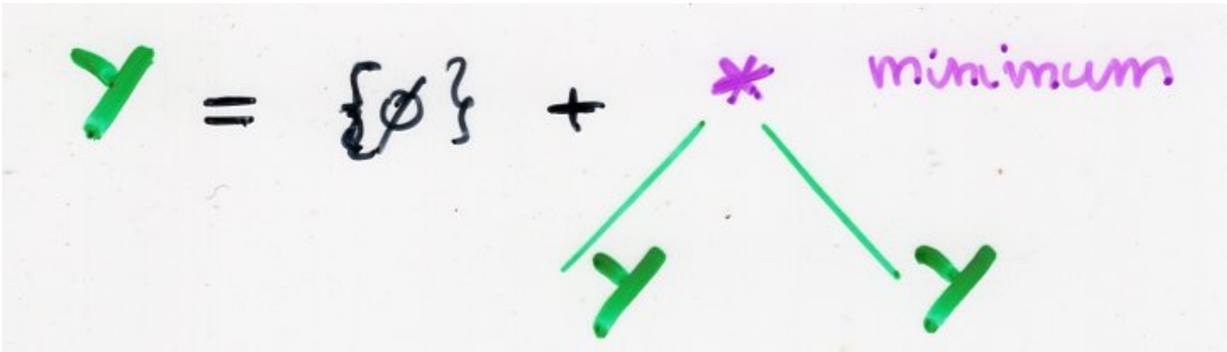


Proposition $M'(t) = \frac{d}{dt} M(t)$

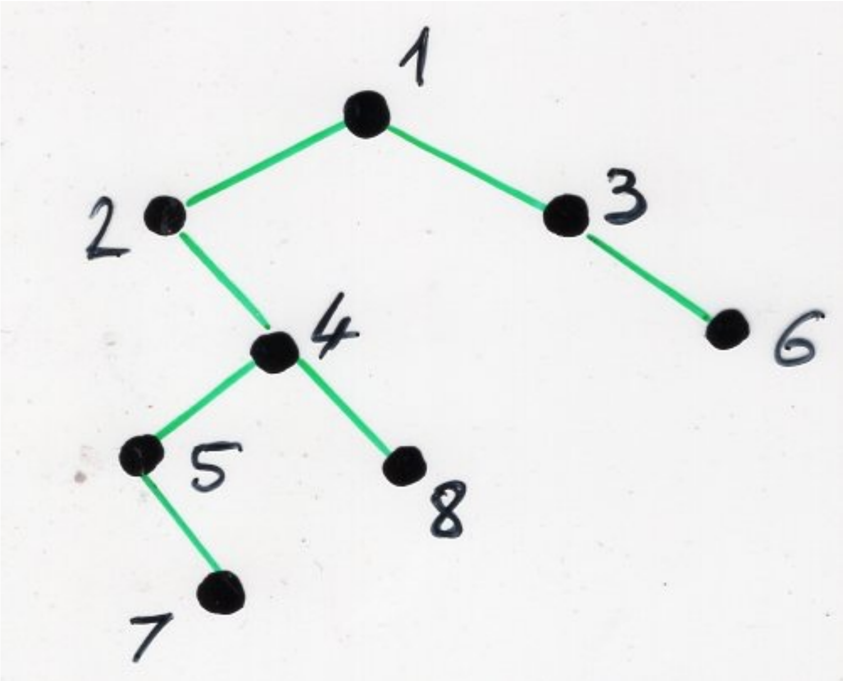
example Y increasing binary trees



$$Y' = Y^2, \qquad Y[\emptyset] = \{\emptyset\}$$



example $\mathcal{Y}$ increasing binary trees

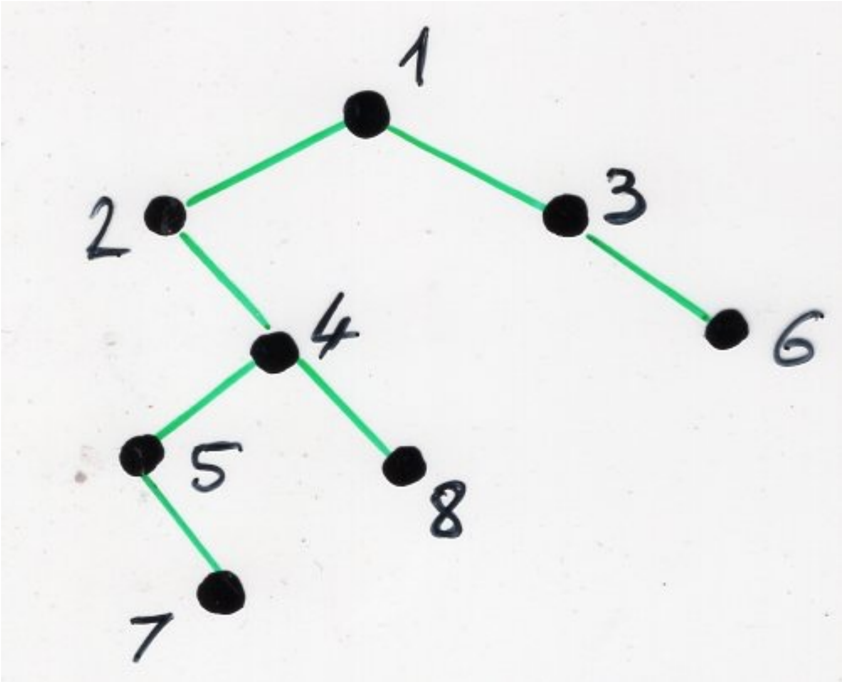


$\mathcal{Y}' = \mathcal{Y}^2$, $\mathcal{Y}[\emptyset] = \{\emptyset\}$

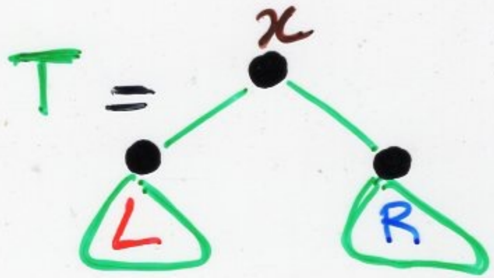
$y' = y^2$, $y(0) = 1$

$n!$

$Y(t) = \dfrac{1}{1-t}$



inorder



$$\pi(T) = \pi(L)\, x\, \pi(R)$$

$$\sigma = 2\ 5\ 7\ 4\ 8\ 1\ 3\ 6$$

bijection with permutations

→ Ch 4 The n! garden

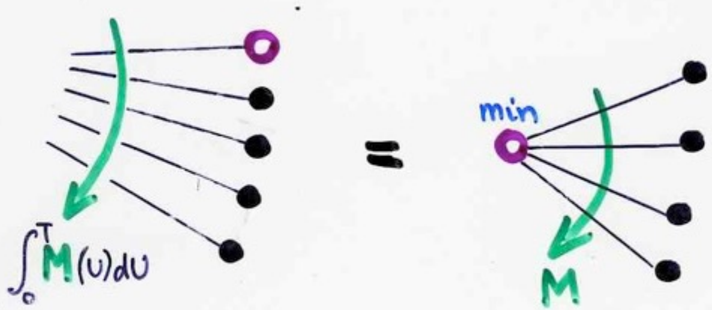
integral of an L-species

Definition Integral of an $\mathbb{L}$-species M

$$F = \int_0^T M(U)\,dU$$

$F[\emptyset] = \emptyset$

$F[U] = M[U \setminus \min(U)] \qquad U \neq \emptyset$



Proposition $$F(t) = \int_0^t M(u)\,du$$

$$y = \tan t$$

example

$$\tan t = t + 2\frac{t^3}{3!} + 16\frac{t^5}{5!} + 272\frac{t^7}{7!} + \dots$$

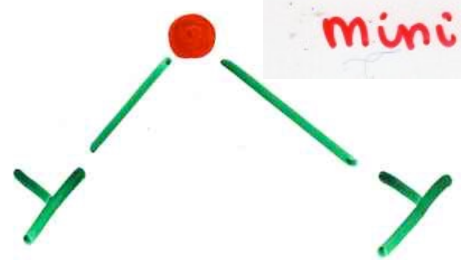
combinatorial interpretation ?

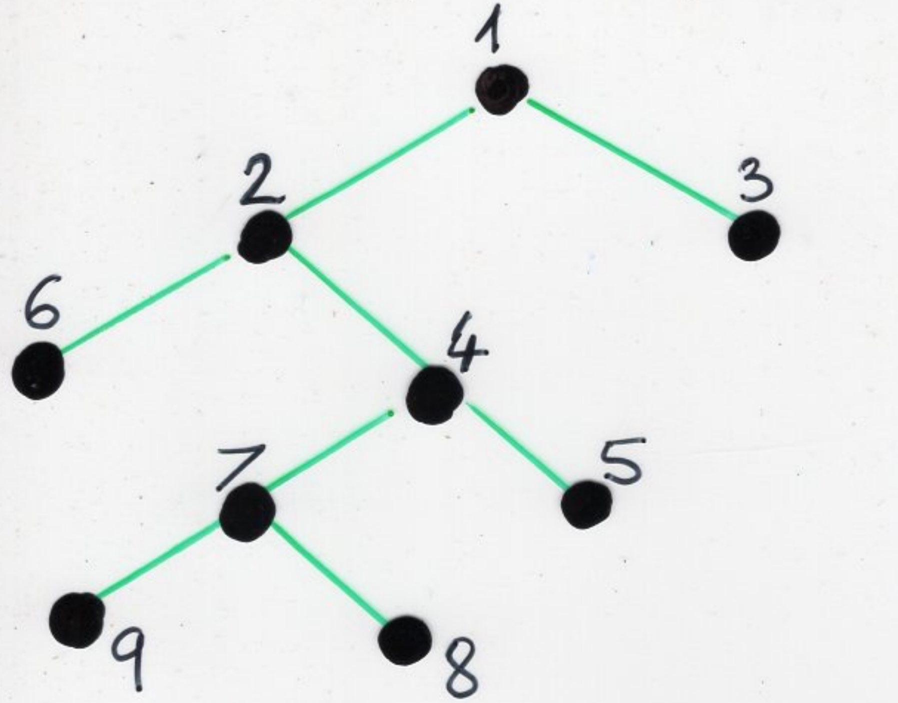
$$y' = 1 + y^2, \quad y(0) = 0$$

$$y = t + \int_0^t y^2(t)\,dt$$

$$Y = T + \int_0^T Y^2(T)\,dT$$

$$Y = T + \int_0^T Y^2(T)\,dT$$

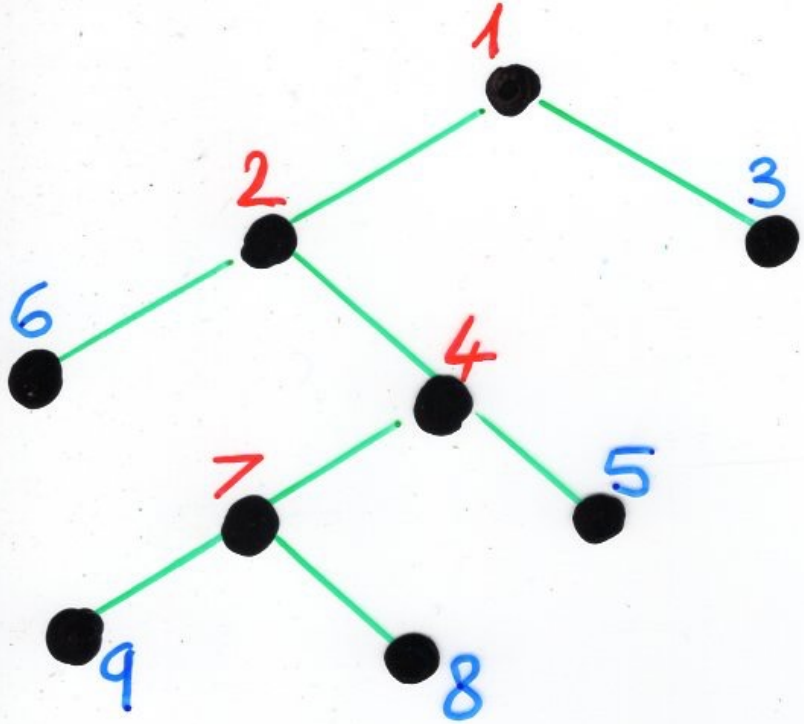
$Y = \bullet$ ou  minimum



$$Y' = \mathbb{1} + Y^2$$

$$Y[\emptyset] = \emptyset$$

complete increasing binary trees



$\sigma(i) < \sigma(i+1)$ rise

$\sigma(i) > \sigma(i+1)$ descent

D. André (1880)

$6 \searrow 2 \nearrow 9 \searrow 7 \nearrow 8 \searrow 4 \nearrow 5 \searrow 1 \nearrow 3$

alternating permutations

$$\tan t = \sum_{n \geq 0} T_{2n+1} \frac{t^{2n+1}}{(2n+1)!}$$

T_{2n+1} = number of alternating permutations on $[1, 2n+1]$

E_{2n} = number of alternating permutations on $[1, 2n]$

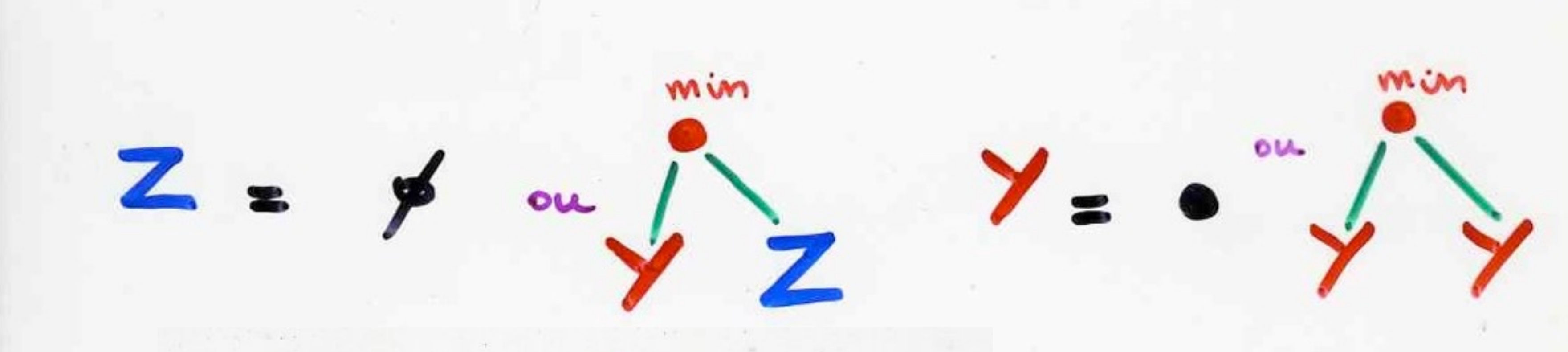
$$\frac{1}{\cos t} = 1 + 5\frac{t^2}{2!} + 61\frac{t^4}{4!} + 1385\frac{t^6}{6!} + \cdots$$

ex. secant numbers

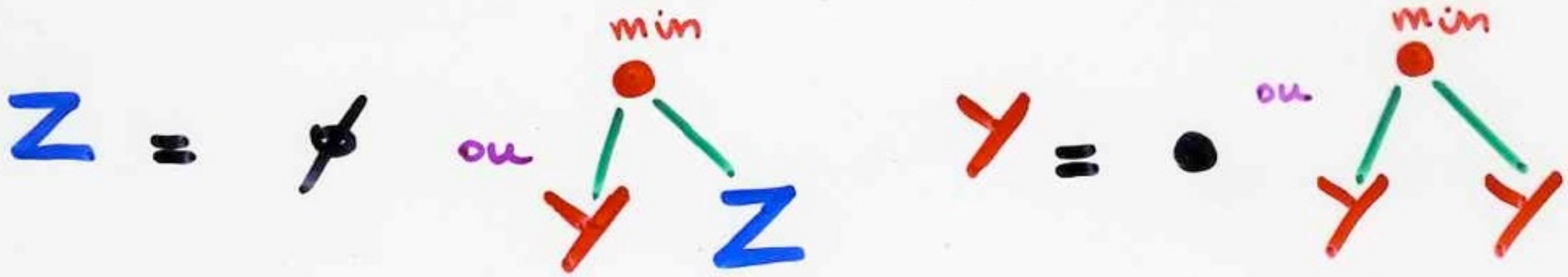
$$\frac{1}{\cos t} = \sum_{n \geq 0} E_{2n} \frac{t^{2n}}{(2n)!}$$

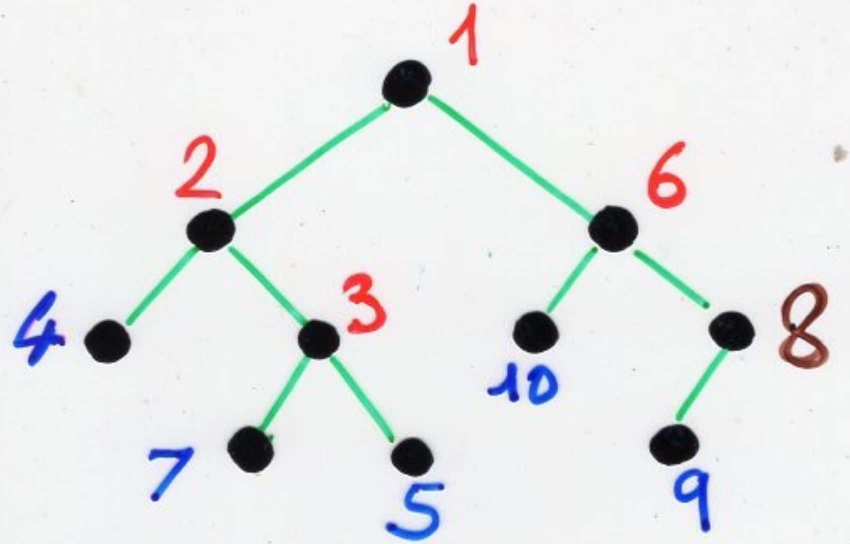
$$\begin{cases} z' = yz, & z(0) = 1 \\ y' = 1 + y^2, & y(0) = 0 \end{cases}$$

$$\begin{cases} Z' = YZ, & Z[\emptyset] = \{\emptyset\} \\ Y' = 1 + Y^2, & Y[\emptyset] = \emptyset \end{cases}$$



Z = ∅ ou
min
Y Z
Y = • ou
min
Y Y





4 — 2 — 7 — 3 — 5 — 1 — 10 — 6 — 9 — 8

weighted $\mathbb{L}$-species

integral ($\mathbb{L}$-species) $$F_v = \int_0^T M_{v_1}(U)\,dU$$

$$F = \int_0^T M(U)\,dU$$

$$\gamma \in M[U \setminus \min(U)]$$

$$v(\gamma) = v_1(\gamma)$$

some historical remarks

about tangent and secant numbers

$$\tan t = \sum_{n\geq 0} T_{2n+1} \frac{t^{2n+1}}{(2n+1)!}$$

D. André (1880)

$$\frac{1}{\cos t} = \sum_{n\geq 0} E_{2n} \frac{t^{2n}}{(2n)!}$$

$\sec t = \dfrac{1}{\cos t}$

E_{2n} $\{1, 5, 61, 1385, \ldots\}$

secant numbers (Euler numbers)

alternating permutations

T_{2n+1} $\{1, 2, 16, 272, 7936, \ldots\}$

tangent numbers

D. Foata
M.P. Schützenberger

"Théorie géométrique des polynômes Eulériens" (1970)

Leonhard
Euler
(1707-1783)

erit:

$\alpha = 1$	$\eta = 2702765$
$\beta = 1$	$\theta = 199360981$
$\gamma = 5$	$\iota = 19391512145$
$\delta = 61$	$\varkappa = 2404879661671$
$\varepsilon = 1385$	&c.
$\zeta = 50521$	

ex hisque valoribus obtinebitur:

$$\sec x = \alpha + \frac{\beta}{1.2}xx + \frac{\gamma}{1.2.3.4}x^4 + \frac{\delta}{1.2\ldots6}x^6 + \frac{\varepsilon}{1.2\ldots8}x^8 + \&c.$$

erit hanc seriem ab illa subtrahendo:

$$\operatorname{tg} x = \frac{2^2(2^2-1)\mathfrak{A}x}{1.2} + \frac{2^4(2^4-1)\mathfrak{B}x^3}{1.2.3.4} + \frac{2^6(2^6-1)\mathfrak{C}x^5}{1.2\ldots.6} + \frac{2^8(2^8-1)\mathfrak{D}x^7}{1.2\ldots.8} + \&c.$$

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1.2} - \frac{2^4\mathfrak{B}x^3}{1.2.3.4} - \frac{2^6\mathfrak{C}x^5}{1.2.3\ldots6} - \frac{2^8\mathfrak{D}x^7}{1.2\ldots.8} - \&c.$$

Si ergo hic introducantur numeri A, B, C, &c. §. 182. inventi,

erit: $\tan x = \frac{2Ax}{1.2} + \frac{2^3 Bx^3}{1.2.3.4} + \frac{2^5 Cx^5}{1.2\ldots6} + \frac{2^7 Dx^7}{1.2\ldots8} + \&c.$

Bernoulli numbers

B_{2n} $\quad \frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730}, \frac{7}{6}, \ldots$

Genocchi numbers

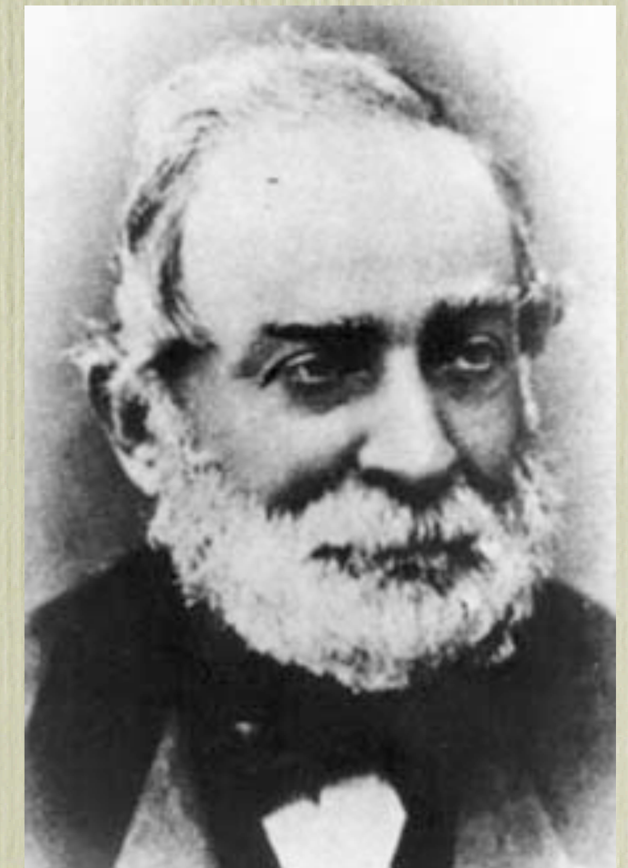
$$G_{2n} = 2(2^{2n}-1)\, B_{2n}$$

Bernoulli

$$2^{2n} G_{2n+2} = (n+1)\, T_{2n+1}$$

G_{2n} $\quad \{1, 1, 3, 17, 155, 2073, \ldots\}$

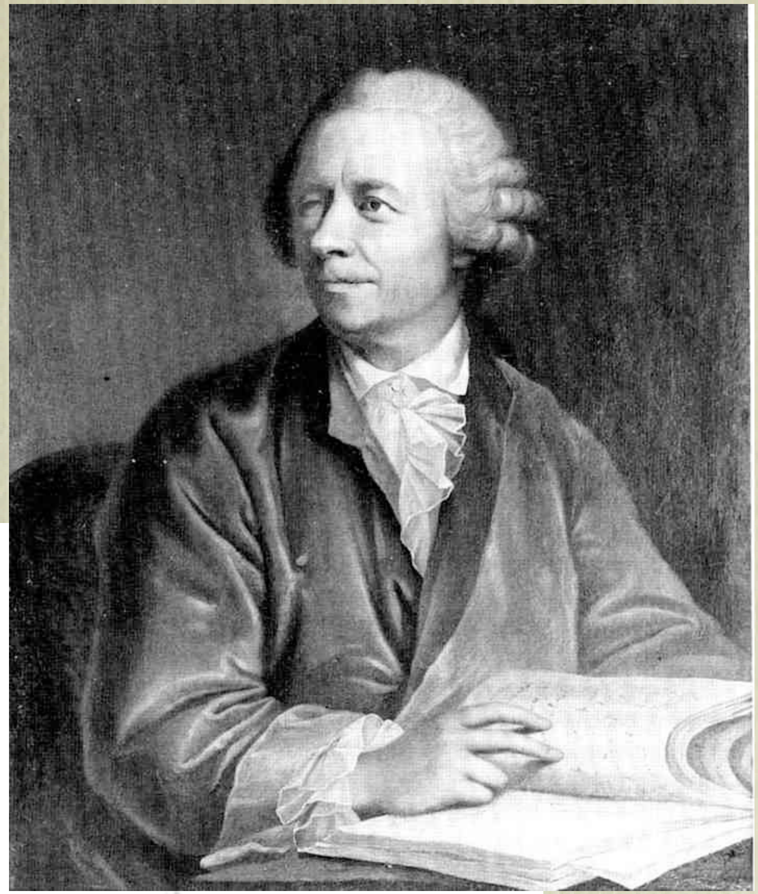
G_{2n} $\{1, 1, 3, 17, 155, 2073, \ldots\}$

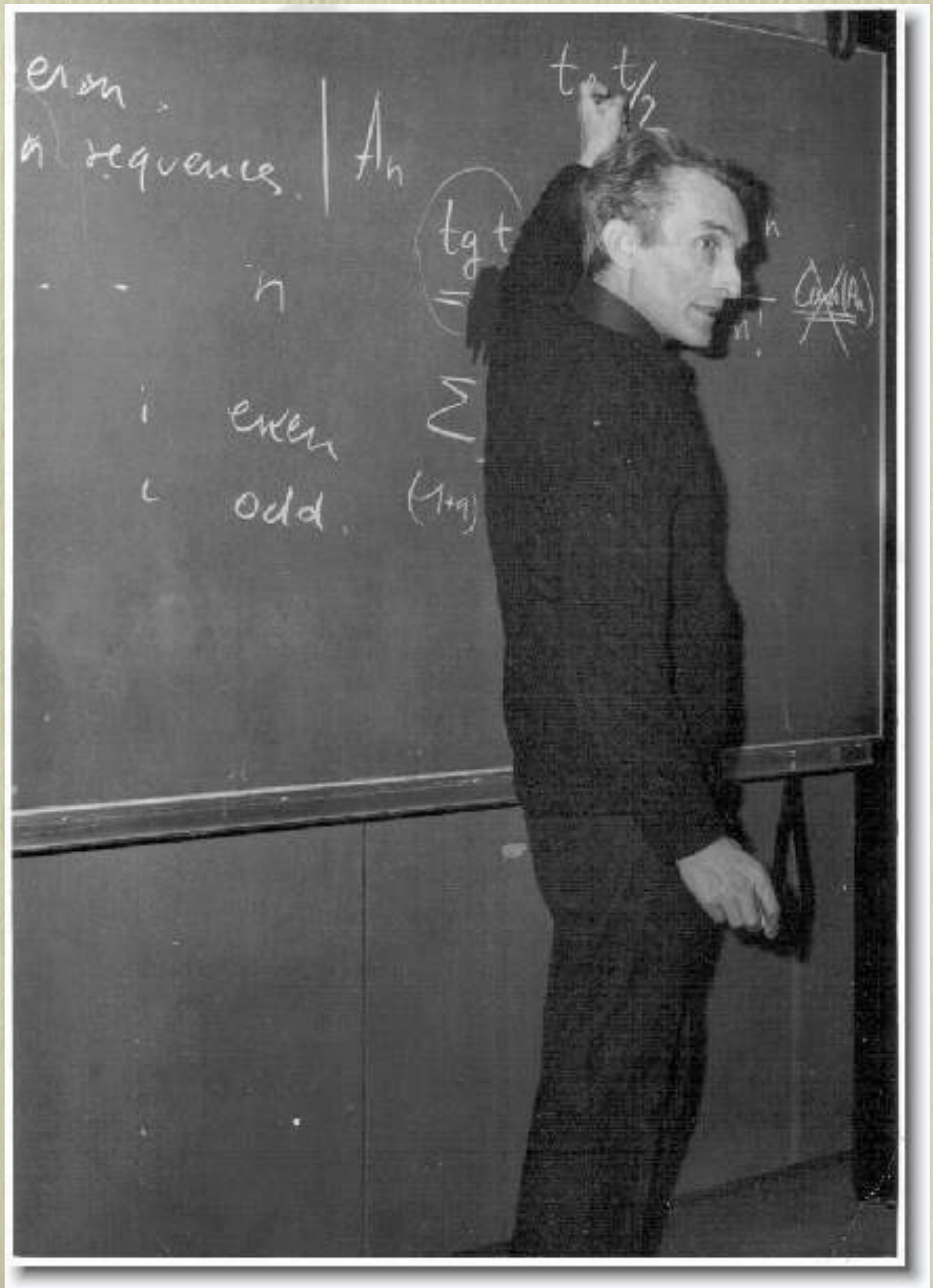


Angelo Genocchi
1817 - 1889

Hinc igitur calculo instituto reperietur:

$$\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= 3 \\
D &= 17 \\
E &= 155 = 5.31 \\
F &= 2073 = 691.3 \\
G &= 38227 = 7.5461 = 7.\frac{127.129}{3} \\
H &= 929569 = 3617.257 \\
I &= 28820619 = 43867.9.73 \quad \&c.
\end{aligned}$$





our Master

Marcel Paul Schützenberger

1920 - 1996

André permutations,

non-commutative differential equations

Jacobi elliptic functions

$$\begin{cases} sn' = cn \cdot dn, & sn(0) = 0 \\ cn' = -dn \cdot sn, & cn(0) = 1 \\ dn' = -k^2 sn \cdot cn, & dn(0) = 1 \end{cases}$$

Dumont, X.V., Flajolet 80's

3 different combinatorial interpretations