Chapter 3

exponential structures and exponential generating functions (2)

complements

IMSc

11 February 2016

complements 1

combinatorial methods
in
control theory

differential equations with forcing terms

$$y' = f(y,t) + u(t)$$

M. Fliess

non commutative variables

Volterra kernels

Chern iterated integrals

M. Fliess (1981, 1983)

control system

$$y(t) = \sum_{w \in X^*} c(w)\, \mathcal{I}(w)$$

output

coefficient
$c(w) \in \mathbb{R}$

iterated integral

Fliess expansion

iterated integral

$$\underset{\text{word}}{w} \in \{\underbrace{x_0, x_1}_{\substack{X \\ \text{alphabet}}}\}^* \longrightarrow \mathcal{J}(w)$$

K. T. Chen

iterated path integral 1977

$$\int_0^t d\tau_5 \int_0^{\tau_5} u(\tau_4)\, d\tau_4 \int_0^{\tau_4} d\tau_3 \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} u(\tau_1)\, d\tau_1$$

$$x_0 \quad x_1 \quad x_0 \quad x_0 \quad x_1$$

Non commutative species $\longrightarrow$ non commutative generating function

Y_w $\rightarrow g = \sum_{w \in X^*} c(w)\, w$

$Y \cdot Z$

product

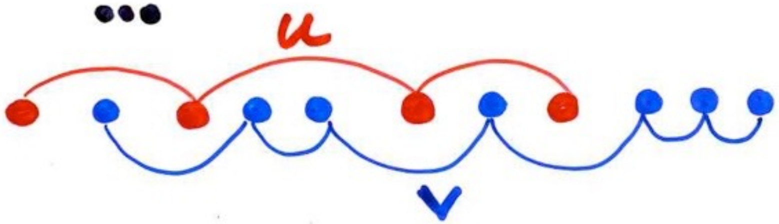
$g \,\underline{ш}\, h$

shuffle product

Shuffle product

$$u \sqcup\!\sqcup\, v = \sum_{\dots} w$$

words X^*

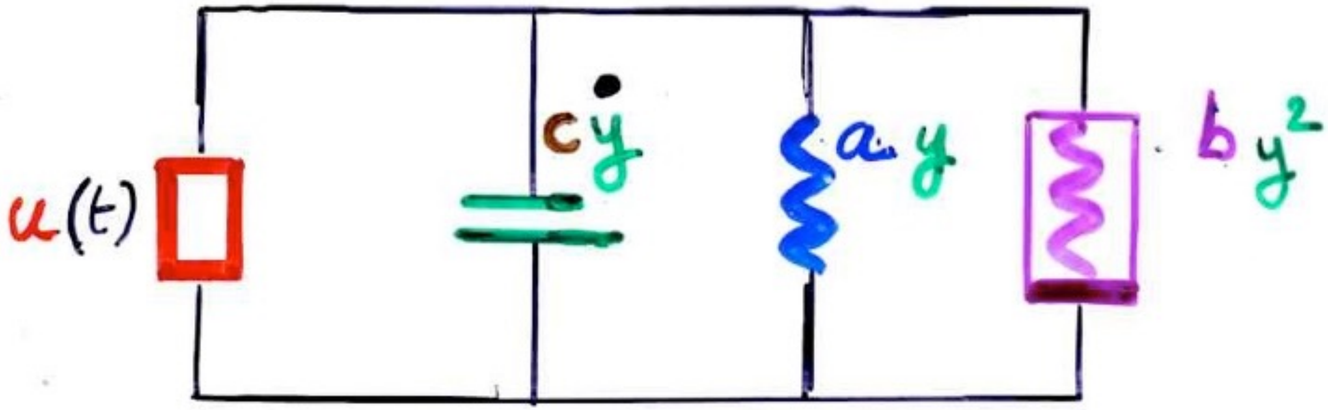


non-commutative series

$\mathbb{K}\langle\langle X \rangle\rangle$

an example

A simple nonlinear circuit



$$\frac{dy}{dt} = \alpha y + \beta y^2 + u(t)$$

$$\alpha = -\frac{a}{c} \qquad \beta = -\frac{b}{c}$$

J. Bussgang, L. Ehrman, J. Graham (1974)

M. Lamnabhi, F. Lamnabhi-Lagarrigue (1980, 1982)

M. Fliess, ", " (1983)

IEEE Trans. Circuits & Systems

$$\frac{dy}{dt} = \alpha y + \beta y^2 + u(t)$$

$$y(t) = \alpha \int_0^t y(\tau)d\tau + \beta \int_0^t y^2(\tau)d\tau + \int_0^t u(\tau)d\tau + \gamma$$

$$\mathbb{K} = \mathbb{Z}[\alpha, \beta, \gamma]$$

Let $H \in \mathbb{K}\langle\langle x_0, x_1 \rangle\rangle$ be the unique solution of the equation :

$$H = \alpha x_0 H + \beta x_0 H \sqcup\!\sqcup H + x_1 + \gamma$$

$$H = \sum_{w \in \{x_0, x_1\}^*} c(w)\, w$$

$$\frac{dy}{dt} = \alpha y + \beta y^2 + u(t)$$

$$y(t) = \alpha \int_0^t y(\tau)d\tau + \beta \int_0^t y^2(\tau)d\tau + \int_0^t u(\tau)d\tau + \gamma$$

Then y, solution of the differential equation, is given by:

$$y = \sum_{w \in \{x_0, x_1\}^*} c(w) \, J(w)$$

iterated integral

combinatorial resolution

combinatorial theory
of differential equations
and integral calculus

P. Leroux, X.V.

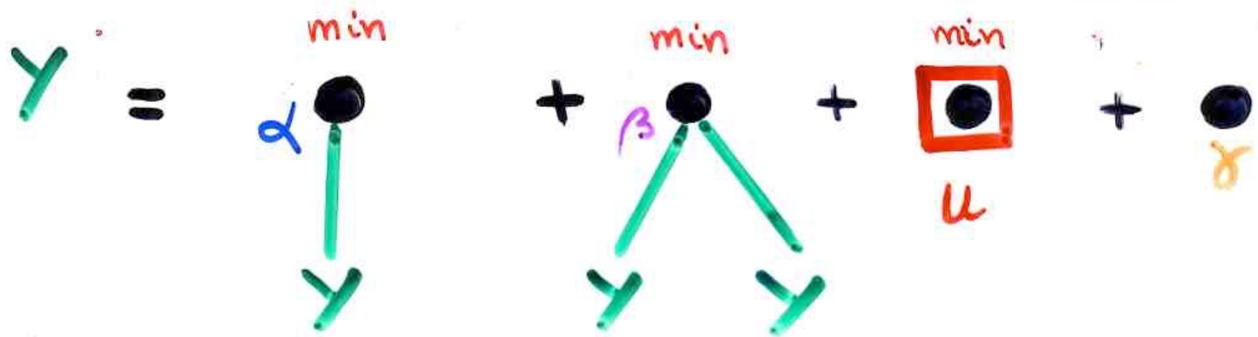
control theory
non-linear

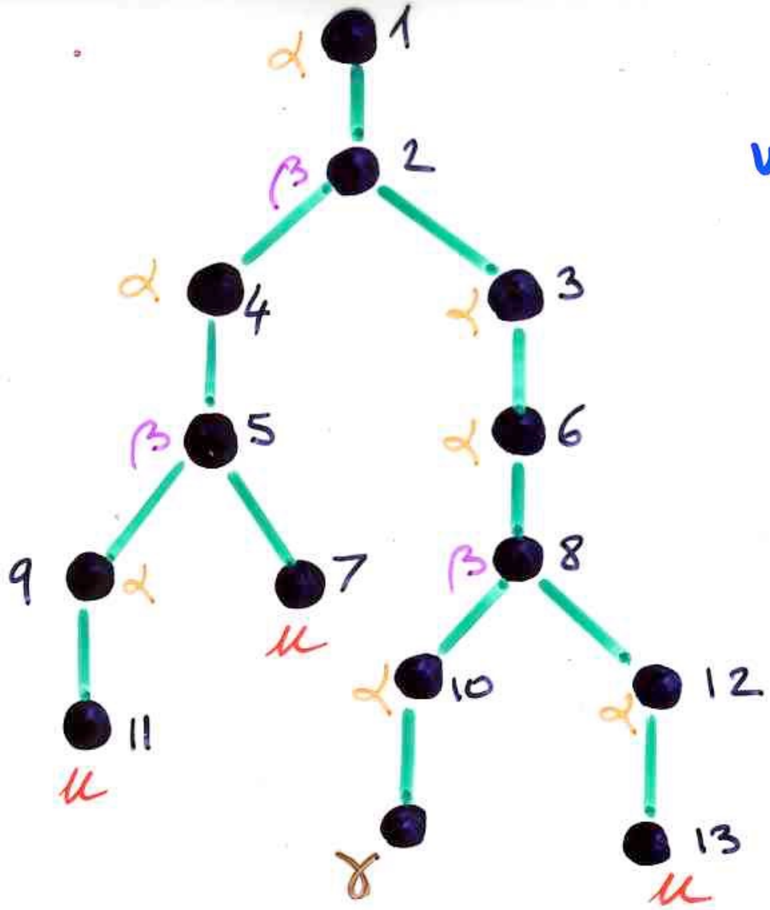
$$y' = y^2 + u(t)$$

differential equations
with forced terms

$$\frac{dy}{dt} = \alpha y + \beta y^2 + u(t)$$

$$y(t) = \alpha \int_0^t y(\tau)\,d\tau + \beta \int_0^t y^2(\tau)\,d\tau + \int_0^t u(\tau)\,d\tau + \gamma$$





$$\overset{1}{x_0}\ \overset{2}{x_0}\ \overset{3}{x_0}\ \overset{4}{x_0}\ \overset{5}{x_0}\ \overset{6}{x_0}\ \overset{7}{x_1}\ \overset{8}{x_0}\ \overset{9}{x_0}\ \overset{10}{x_0}\ \overset{11}{x_1}\ \overset{12}{x_0}\ \overset{13}{x_1} = w$$

word w

weight $\alpha^7 \beta^3 \gamma$

Prop. $$\frac{dy}{dt} = \alpha y + \beta y^2 + u(t) \qquad y(0) = \gamma$$

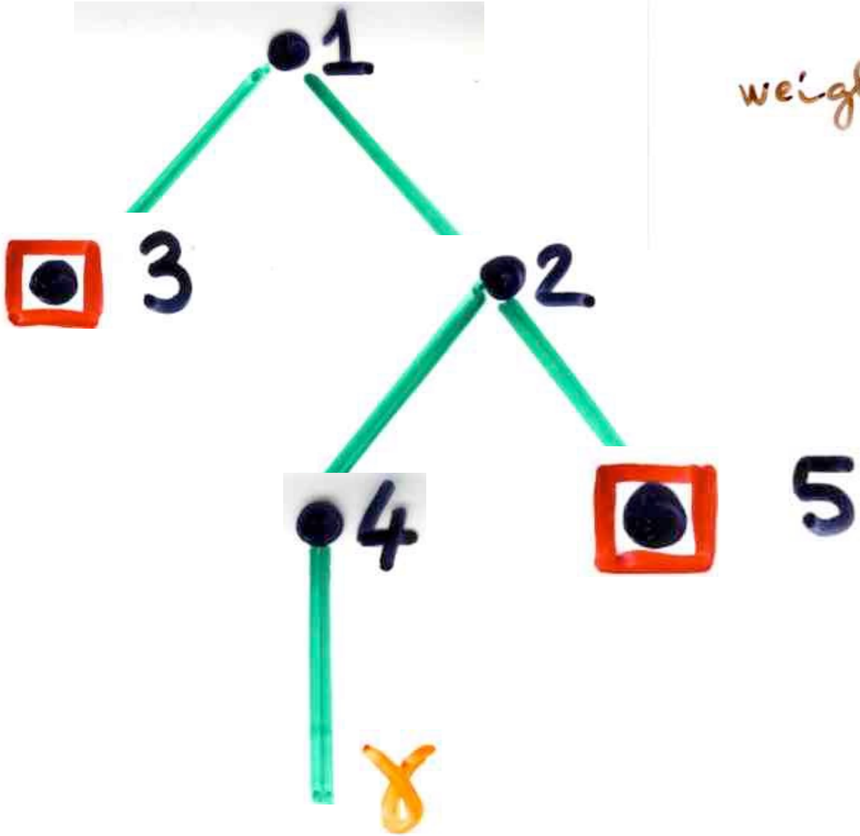
solution

$$y(t) = \sum_{\substack{T \\ \text{1-2 increasing} \\ \text{trees}}} v(T)\, J(w(T))$$

Second combinatorial interpretation :

Paths

Histories



weighted tree

$\Updownarrow$

weighted history $\Longleftrightarrow$ weighted path

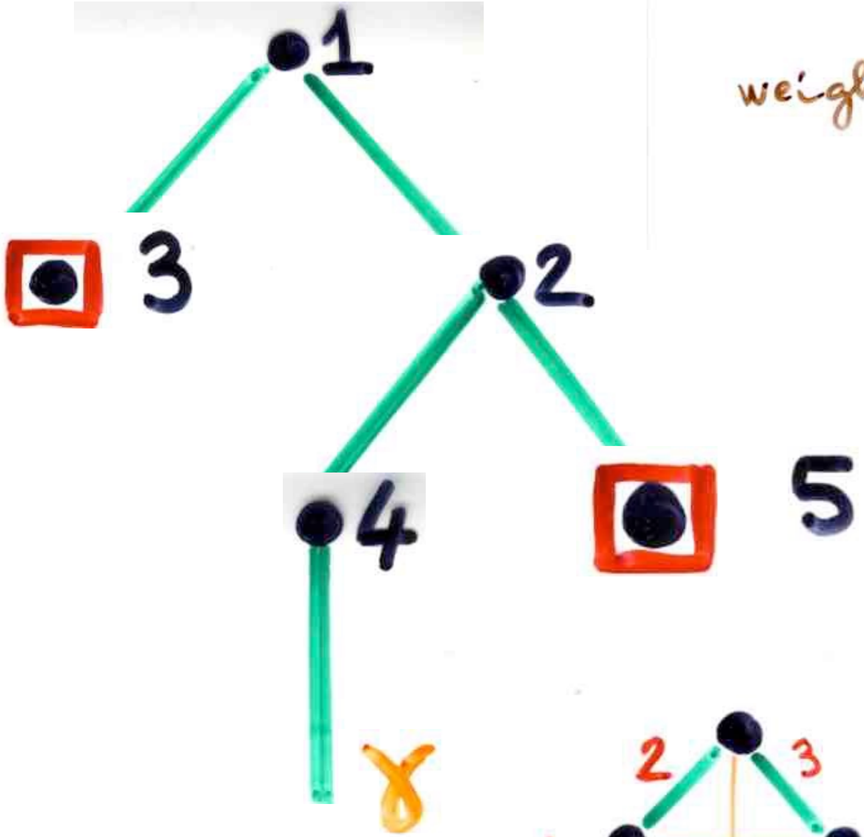
$\alpha^i \beta^j \gamma^k$

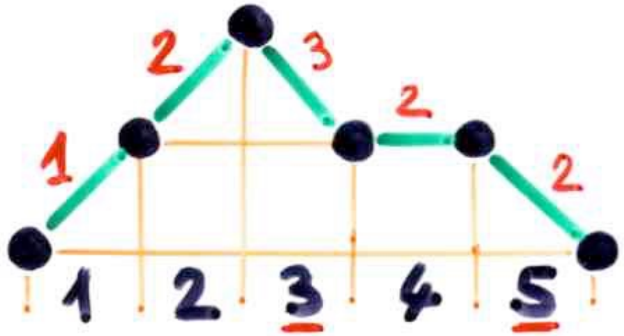
weighted tree

$\Updownarrow$

weighted history $\Longleftrightarrow$ weighted path

$\alpha^i \beta^j \gamma^k$



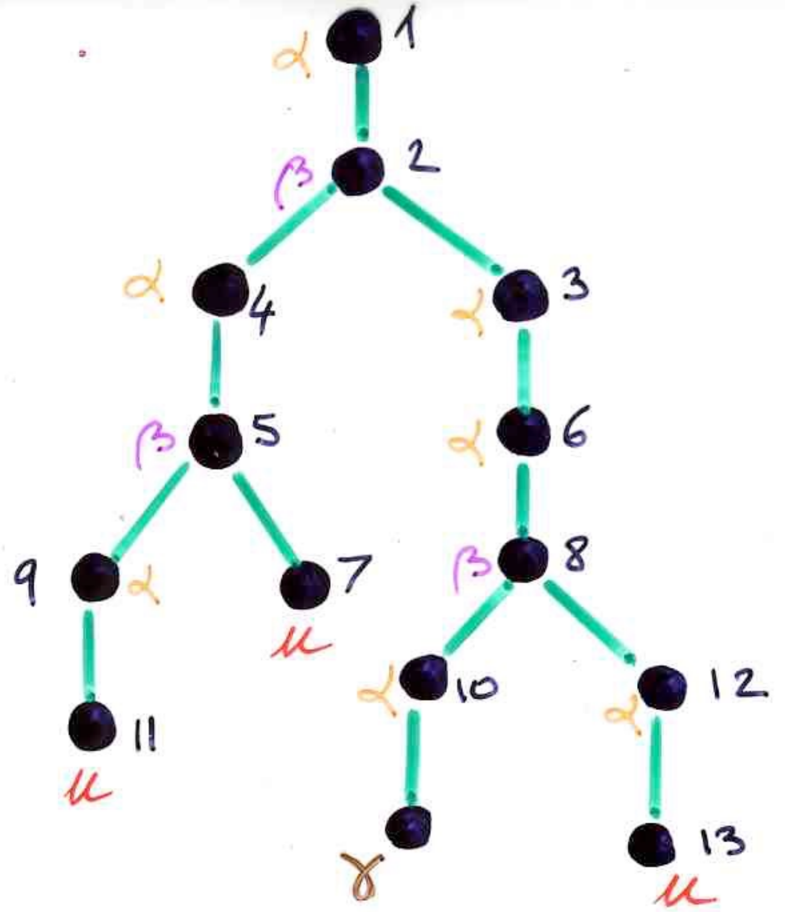


$$v(\omega) = 2^3\, 3 = 24$$

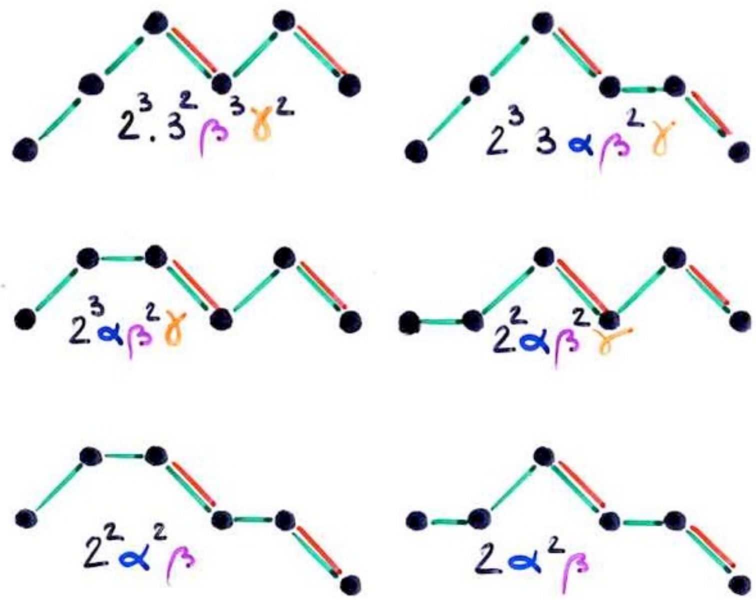
$$v^*(\omega) = 24\, \alpha \beta^2 \gamma$$

$$w(\omega) = x_0 x_0 x_1 x_0 x_1$$

History $\quad h = (\omega;\ \underset{1}{1}, \underset{2}{2}, \underset{3}{2}, \underset{2}{1}, \underset{2}{1})$



$W = x_0 x_0 x_1 x_0 x_1$



$C(w) = 36\,\beta^3\gamma^2 + 36\,\alpha\beta^2\gamma + 6\,\alpha^2\beta$

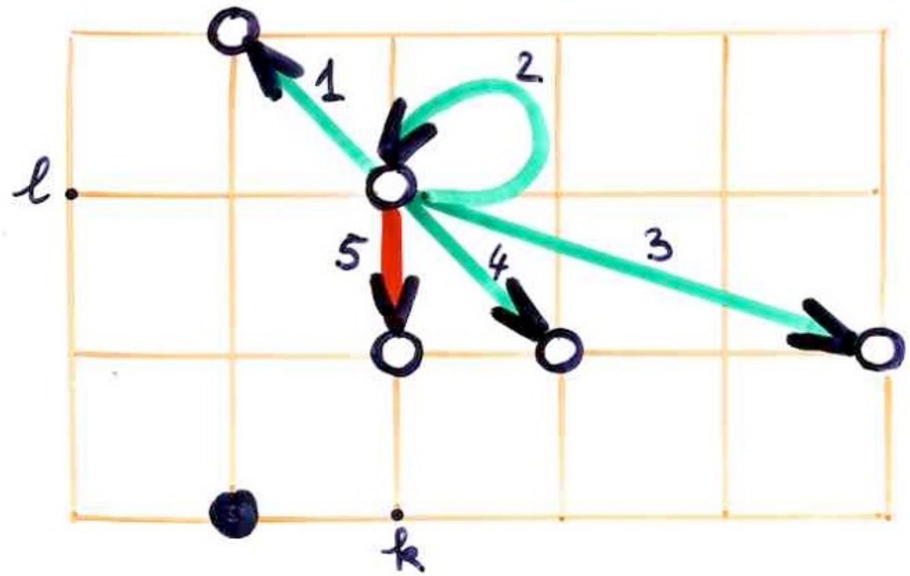
$$\overset{1}{x_0}\,\overset{2}{x_0}\,\overset{3}{x_0}\,\overset{4}{x_0}\,\overset{5}{x_0}\,\overset{6}{x_0}\,\overset{7}{x_1}\,\overset{8}{x_0}\,\overset{9}{x_0}\,\overset{10}{x_0}\,\overset{11}{x_1}\,\overset{12}{x_0}\,\overset{13}{x_1} = w$$

word w

weight $\alpha^7\beta^3\gamma$

Equation de Duffing

$$y'' = a\,y' + y + b\,y^3 + u(t)$$



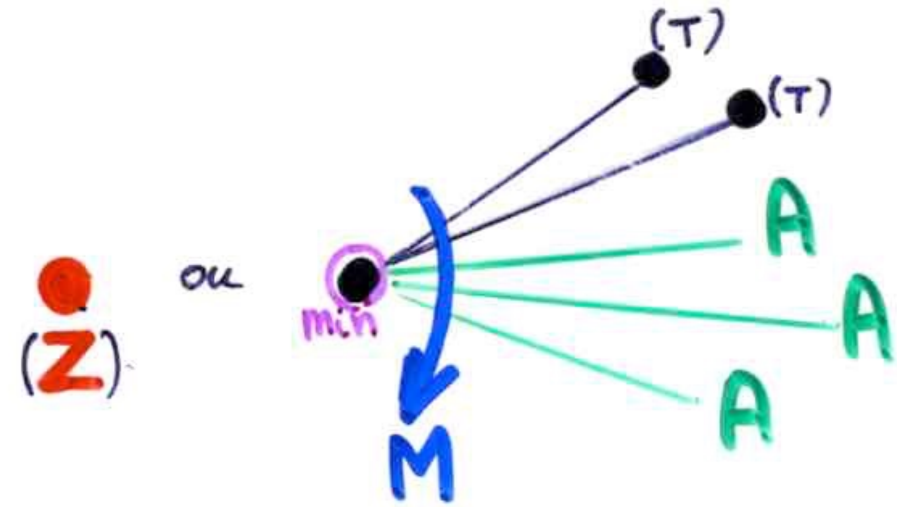
Duffing equation

complements 2

combinatorial resolution of ordinary differential equations with species

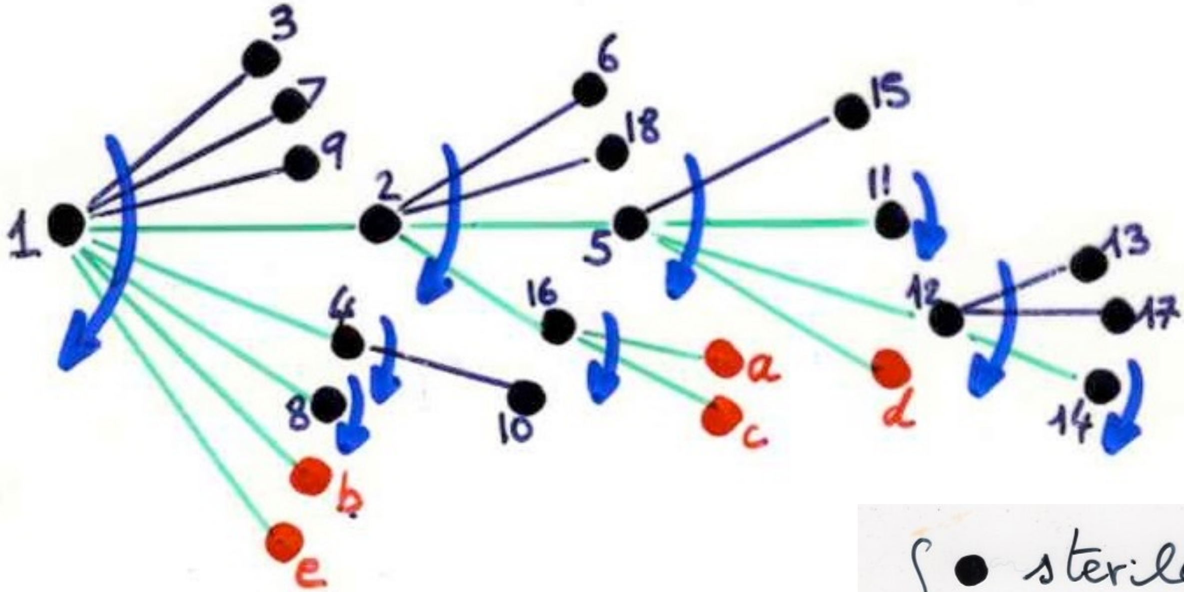
$$Y' = M(T, Y) \qquad Y(0) = Z$$

$$A(T, Z) = Z + \int_0^T M(X, A(X, Z))\, dX$$

$A =$ 

M-enriched increasing arborescences

$A_M(T, Z)$



{ ● sterile (T)
{ ● fertile

● bud (Z)

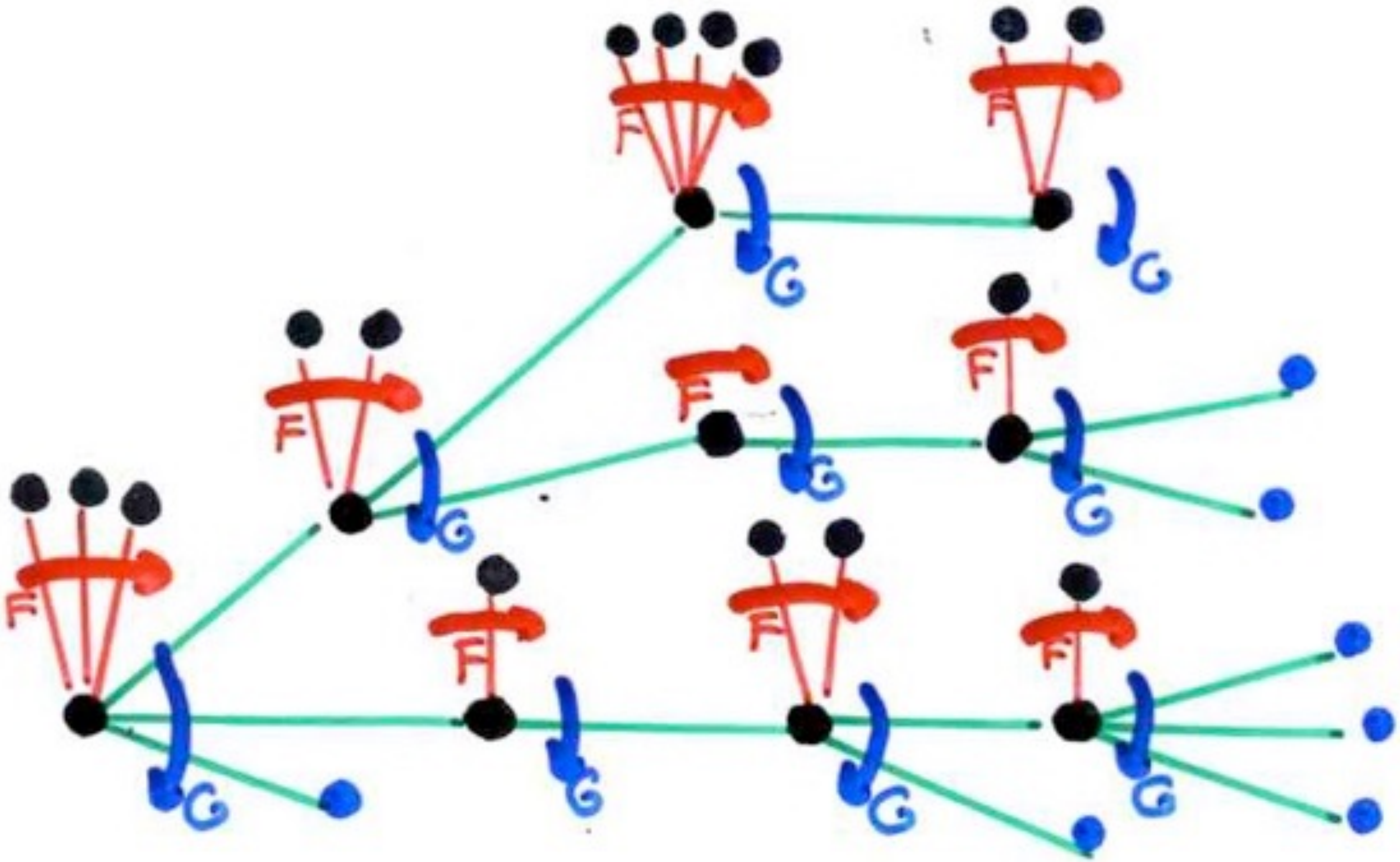
solution of the equation $Y' = M(T, Y)$

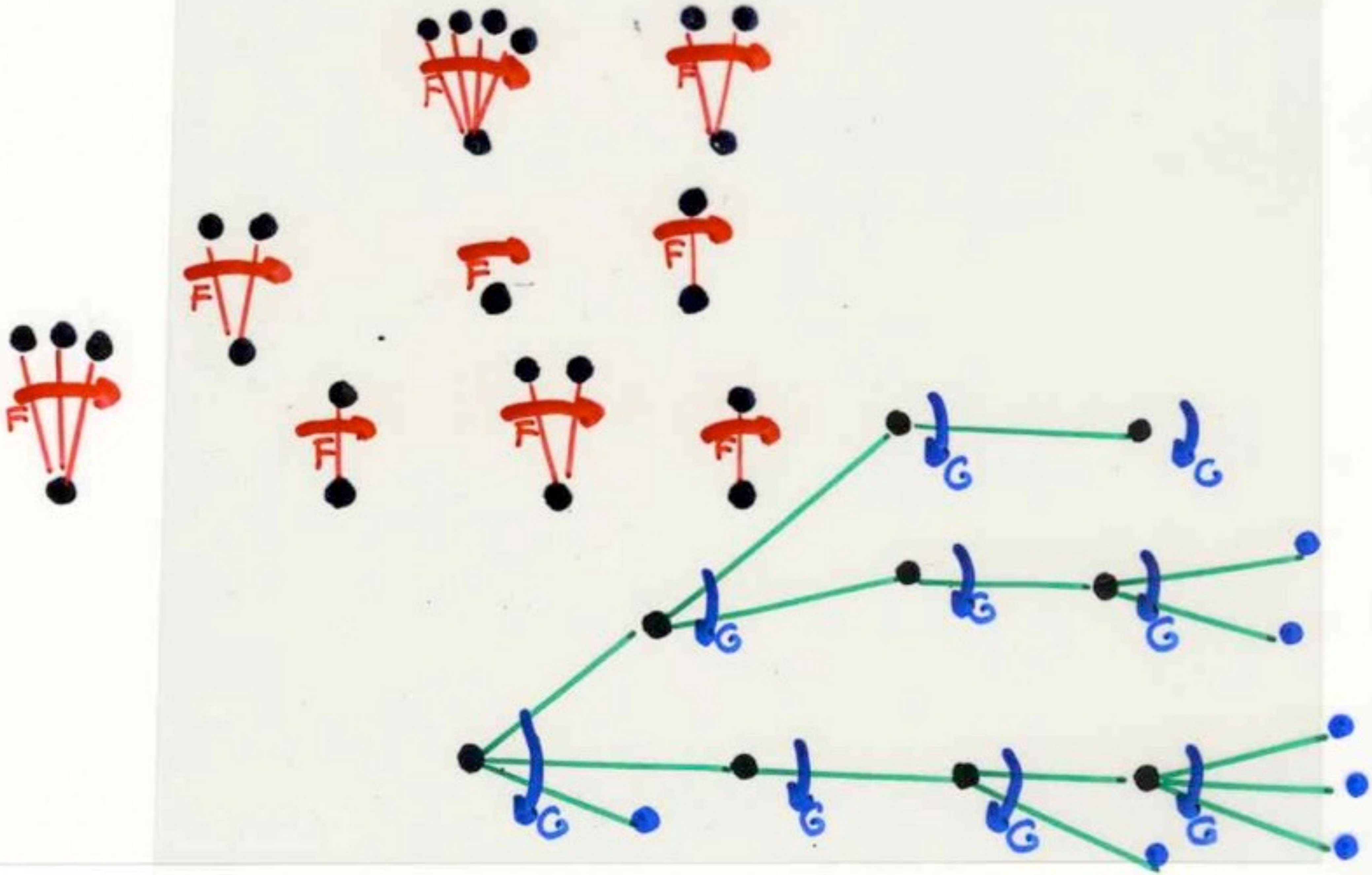
separation of variables

$$Y' = G(Y) \qquad Y(0) = Z$$

$$Y' = F(T)\,G(Y) \qquad Y(0) = Z$$

$$Y_{FG} = Y_G\left(\int_0^T F(x)\,dX\right)$$


F
G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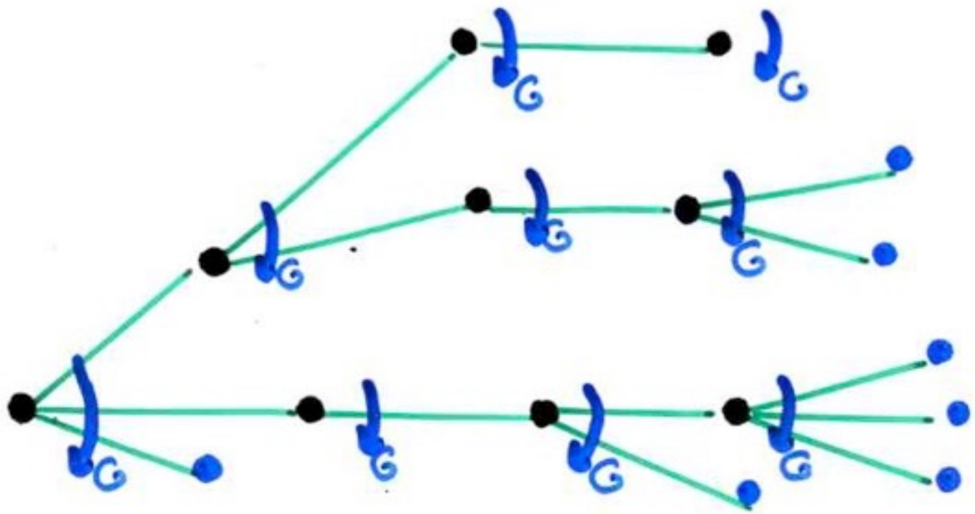
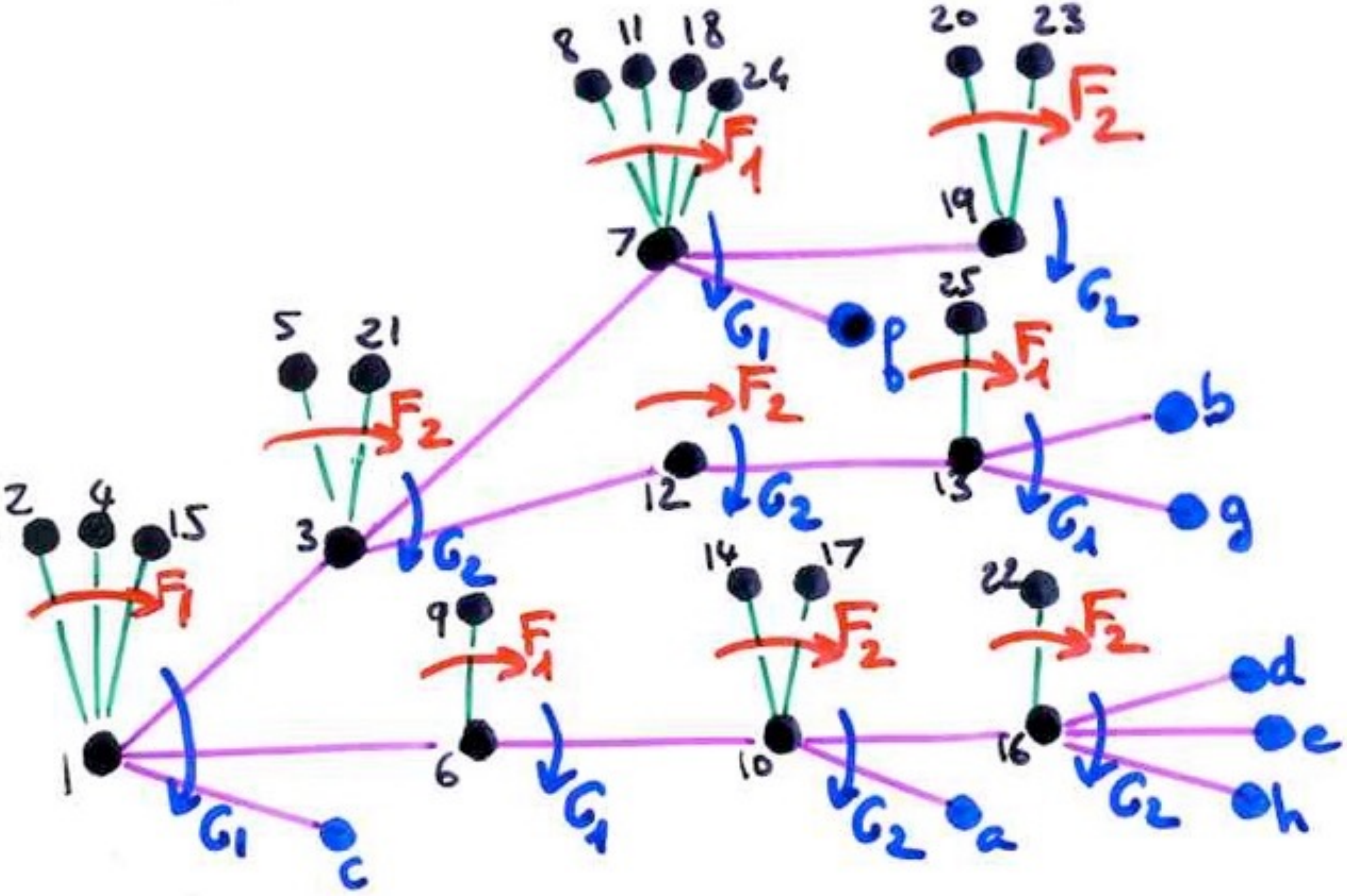
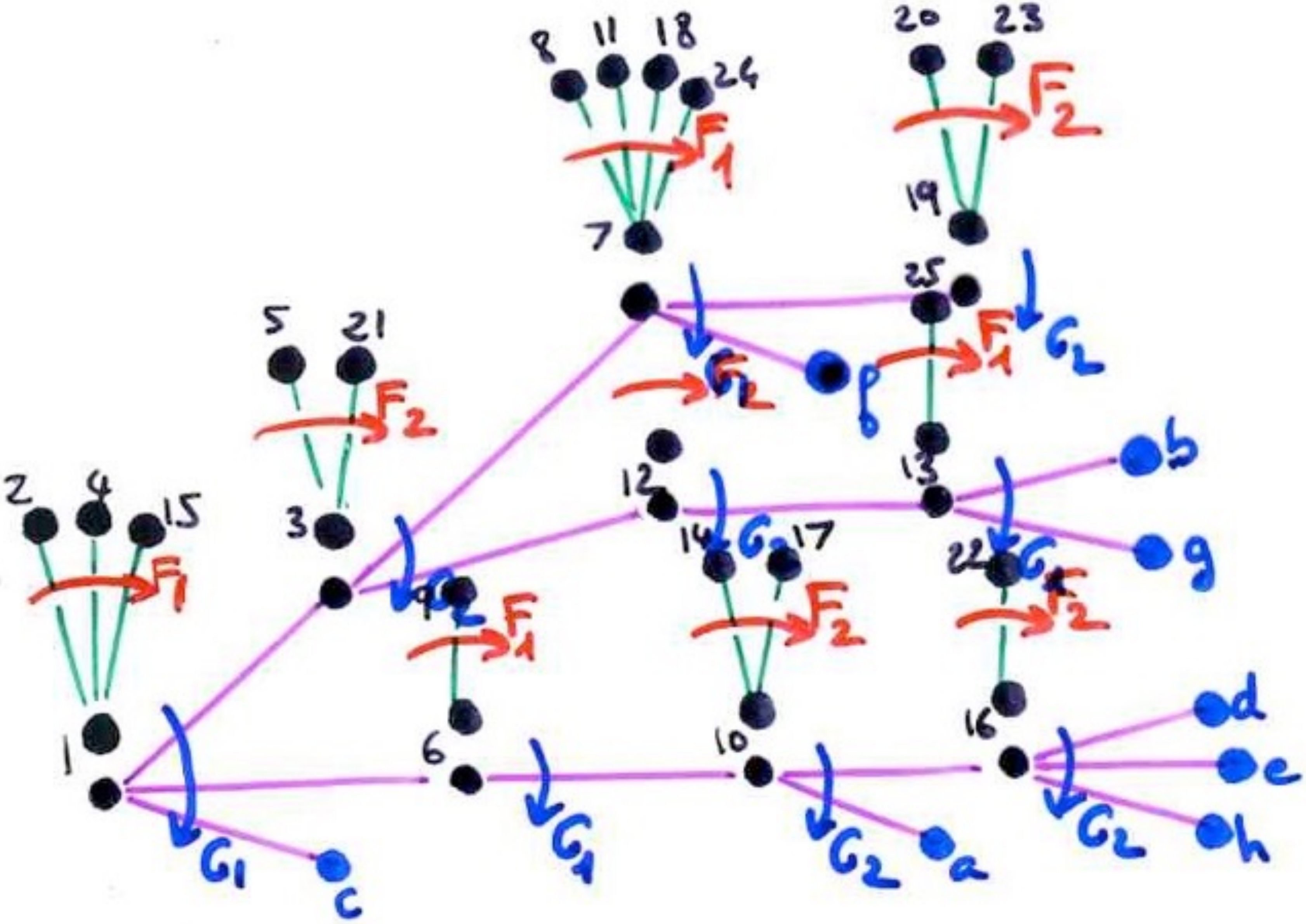


separation of variables: extensions

$$Y' = G_1(Y)F_1(T) + G_2(Y)F_2(T)$$





1
2
4
15
F_1
G_1
c
3
5
21
F_2
G_2
6
9
F_1
G_1
7
8
11
18
24
F_1
G_2
f
10
12
14
17
G_2
F_2
G_2
a
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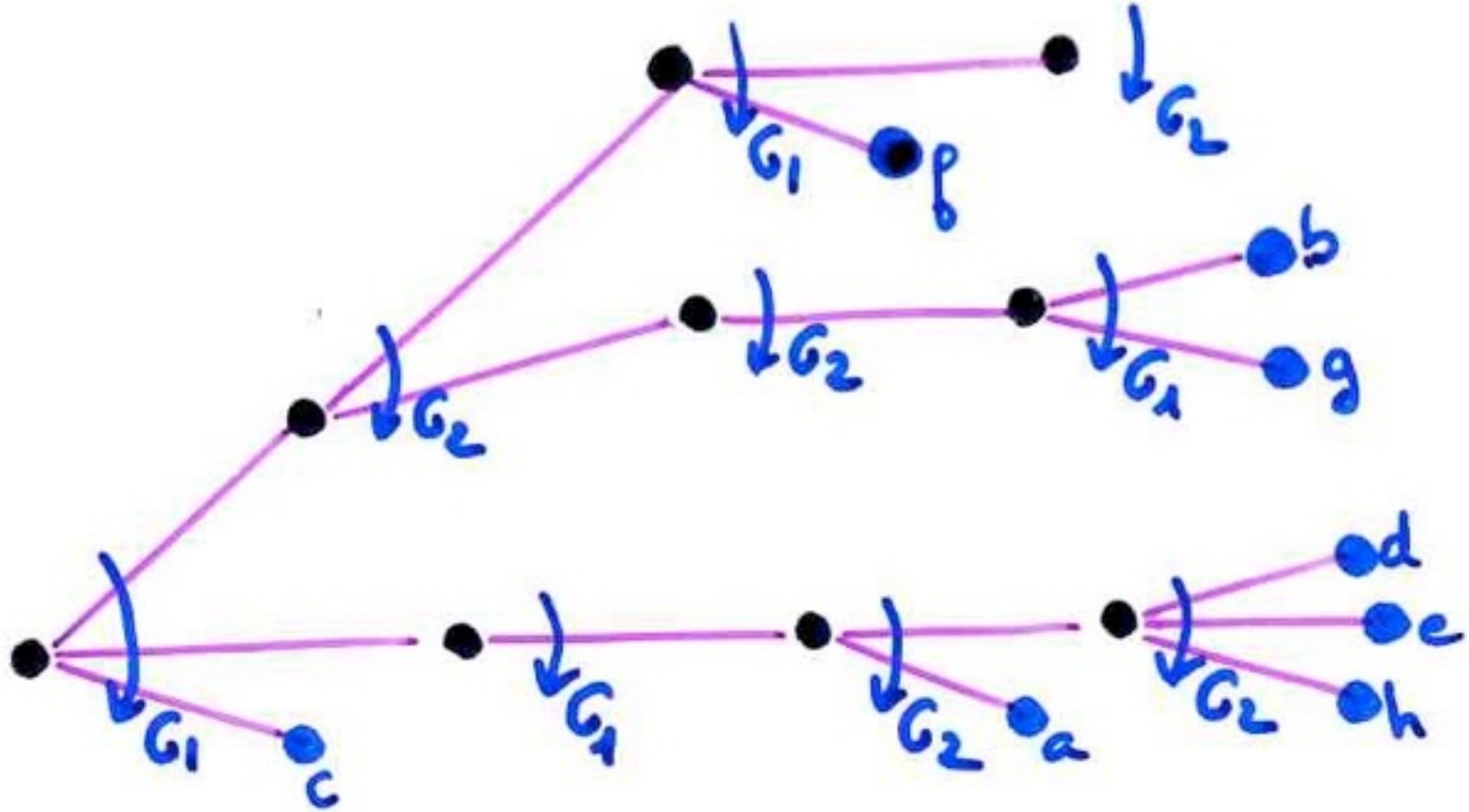

G1
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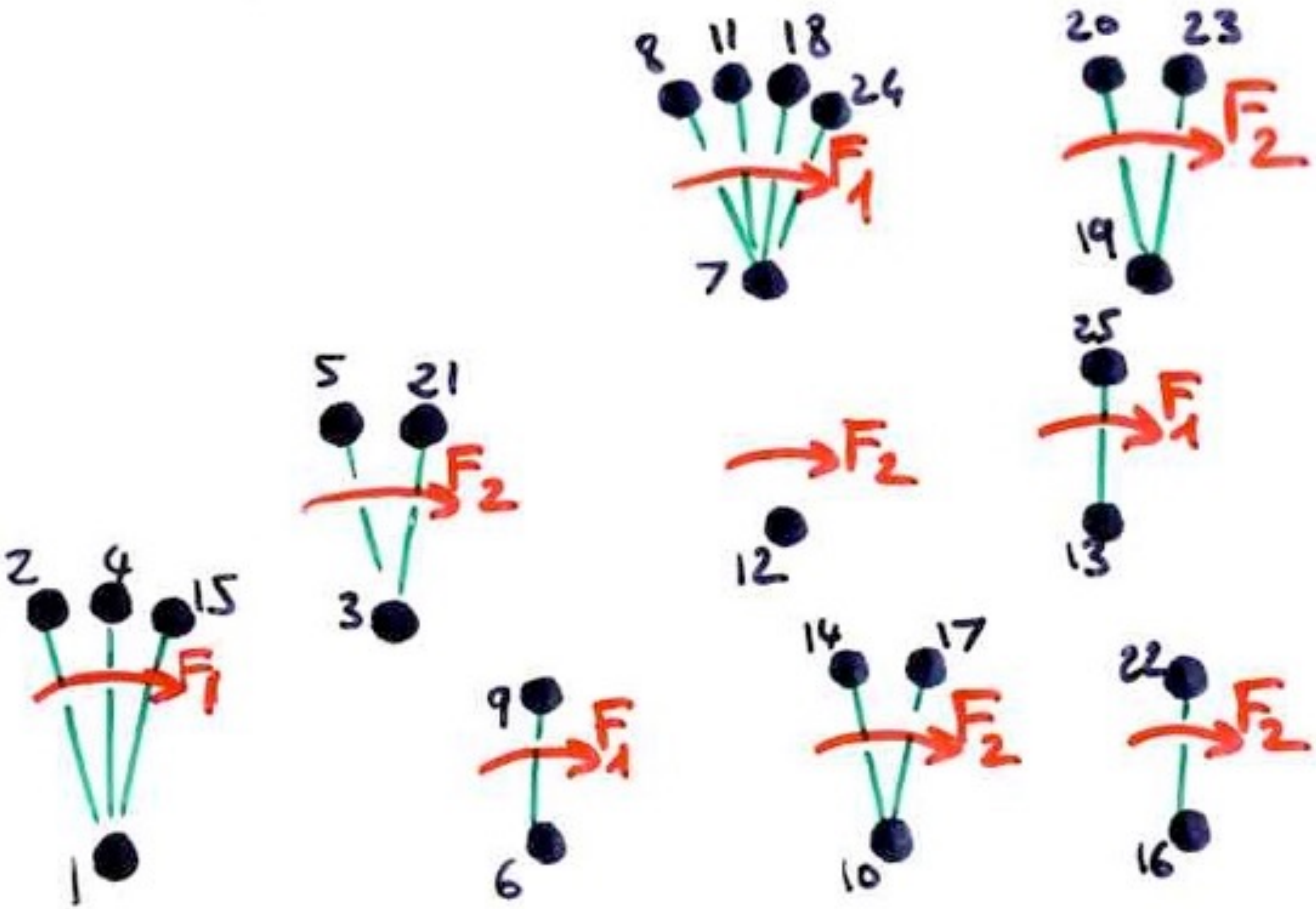

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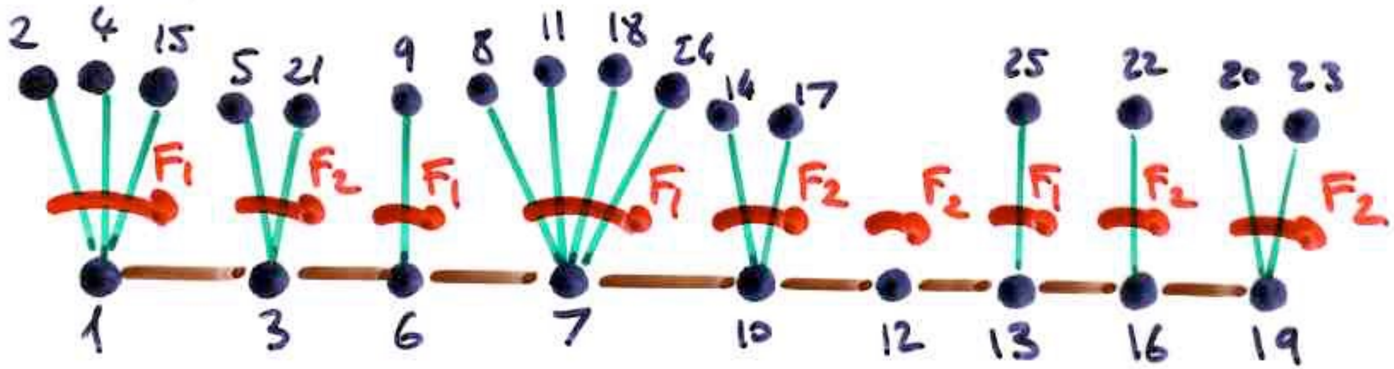
16



$$w = x_1\, x_2\, x_1\, x_1\, x_2\, x_2\, x_1\, x_2\, x_2$$

iterated integral

complement 3

elliptic and Dixon functions,
Polya urn model

Jacobi elliptic functions

$$\begin{cases} sn' = cn \cdot dn, & sn(0) = 0 \\ cn' = -dn \cdot sn, & cn(0) = 1 \\ dn' = -k^2 sn \cdot cn, & dn(0) = 1 \end{cases}$$

Dumont, X.V.; Flajolet 80's

3 different combinatorial interpretations

Pólya urn model

Polya-Eggenberger

Laplace (1812)

Théorie analytique des probabilités

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

$$\begin{array}{l} \to \text{red} \\ \to \text{blue} \end{array} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

Dixon (1890)

Fermat cubic

$$x^3+y^3=1$$

$$\begin{cases} sm' = cm^2, & sm(0)=0 \\ cm' = -sm^2, & cm(0)=1 \end{cases}$$

$$\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

Van Fossen Conrad, Flajolet (2006)

complement 4

(formal) orthogonal polynomials

→ course on combinatorics of orthogonal polynomials

Orthogonal polynomials

Def. $\{P_n(x)\}_{n\geq 0}$ $\quad P_n(x) \in \mathbb{K}[x]$

orthogonal iff $\exists\ f : \mathbb{K}[x] \longrightarrow \mathbb{K}$

linear functional

$$\begin{cases} \text{(i)} & \deg(P_n(x)) = n \quad (\forall n \geq 0) \\ \text{(ii)} & f(P_k P_l) = 0 \quad \text{for } k \neq l \geq 0 \\ \text{(iii)} & f(P_k^2) \neq 0 \quad \text{for } k \geq 0 \end{cases}$$

$$\mathcal{f}(x^n) = \mu_n \qquad (n \geqslant 0)$$

moments

$$\mathcal{f}(PQ) = \int_a^b P(x)\,Q(x)\,d\mu$$

mesure

moments
Hermite
polynomials

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of
involutions
no fixed point
on $\{1, 2, \ldots, 2n\}$

linear functional

$$f(P(x)) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} P(x)\, e^{-x^2/2}\, dx$$

orthogonal

polynomials

(binomial type)

Sheffer type

$$\sum P_n(x)\frac{t^n}{n!} = g(t)e^{xf(t)}$$

orthogonal

polynomials

(binomial type)

Sheffer type

$$\sum P_n(x)\frac{t^n}{n!} = g(t)e^{xf(t)}$$

- Hermite H_n
- Laguerre $L_n^{(\alpha)}$
- Charlier $C_n^{(a)}$
- Meixner I $M_n^{I\,(\alpha,\beta)}$
- Meixner II $M_n^{II\,(\delta,\eta)}$

Askey-Wilson