An introduction to

enumerative
algebraic combinatorics
bijective

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Chapter 3

exponential structures
and
exponential generating functions
(1)

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Species and structures

"naive" definition

U finite set

combinatorial structure

construction α

U underlying set with

α constructed on U,

supported by U

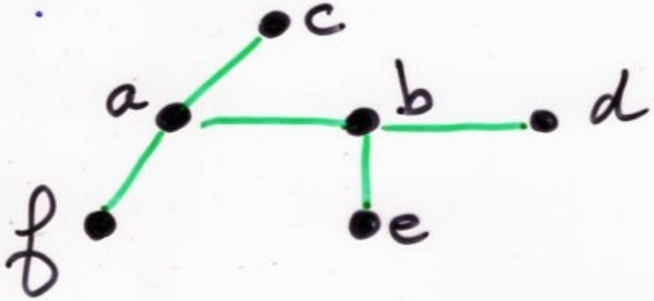
species F

structures of type F

set $F[U]$

F-structure $\alpha \in F[U]$

__example__ Tree (= graph having no cycle)



Permutations,
(set) Partitions,
Graphs,
Endofunctions, ...

Transport of structures

$$U \xrightarrow[\text{bijection}]{f} V$$

$$F[U] \xrightarrow{F[f]} F[V]$$

transport along f

example trees $U = \{a, b, c\}$ $V = \{1, 2, 3\}$

$$\begin{array}{rcl} U &=& \{a, b, c\} \\ f\downarrow && \ \downarrow\ \ \downarrow\ \ \downarrow \\ V &=& \{1, 2, 3\} \end{array} \qquad F[f]\Big\downarrow \begin{array}{l} \{a - b - c,\ b - a - c,\ a - c - b\} \\ \{1 - 2 - 3,\ 2 - 1 - 3,\ 1 - 3 - 2\} \end{array}$$

coherent transport

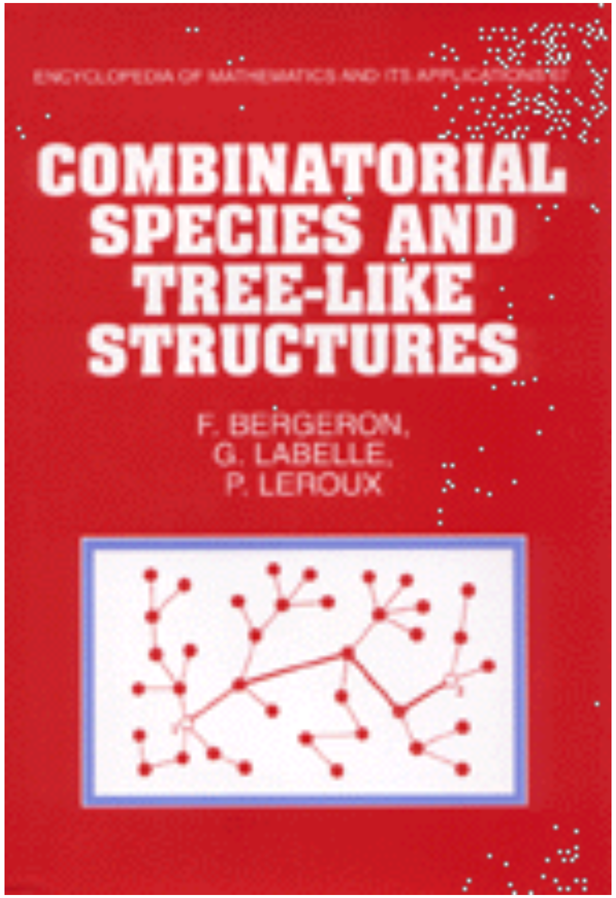
$$\mathbf{F}[f \circ g] = \mathbf{F}[f] \circ \mathbf{F}[g] \qquad \mathbf{F}[Id_U] = Id_{\mathbf{F}[U]}$$

Combinatorial model
for exponential generating function

$$f(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}$$

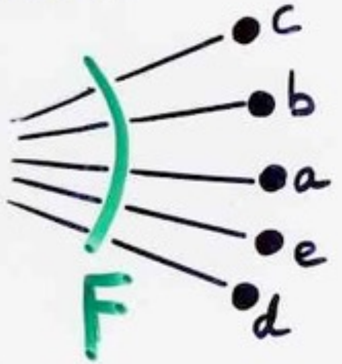
Species
(combinatorial)
structures

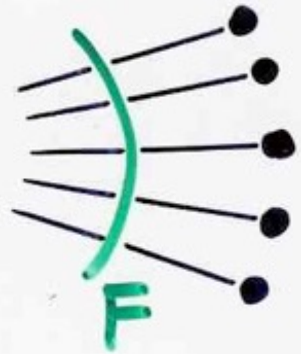
A. Joyal, G. Labelle
P. Leroux, F. Bergeron, ...
(UQAM, LACIM, Montréal)



Encyclopedia of Mathematics
and its Applications
Cambridge University Press (1977)

Convention.





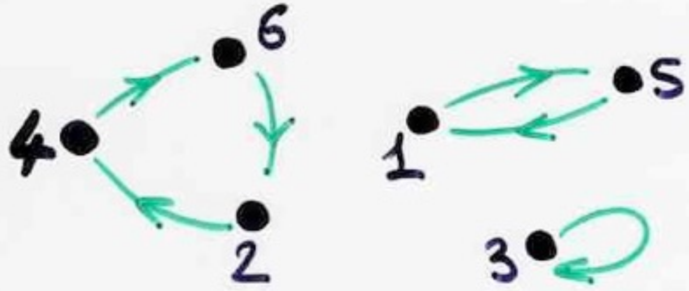
enumeration $a_n = \left| F[\{1, 2, \ldots, n\}] \right|$

<u>Definition</u> generating function of the species F

$$F(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}$$

Examples

1. Permutations S $\quad a_n = n!$ $\quad S(t) = \frac{1}{1-t}$



$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 6 & 1 & 2 \end{pmatrix}$$

$$\sigma = 543612$$

2. Total order L $\quad a_n = n!$ $\quad L(t) = \frac{1}{1-t}$

3. Involutions $\mathbf{I}$ $\sigma^2 = Id$

$$\mathbf{I}(t) = \exp\left(t + \frac{t^2}{2}\right)$$

4. Involutions with no fixed points $\mathbf{J}$

$$\mathbf{J}(t) = \exp\left(\frac{t^2}{2}\right)$$

5. Derangements D

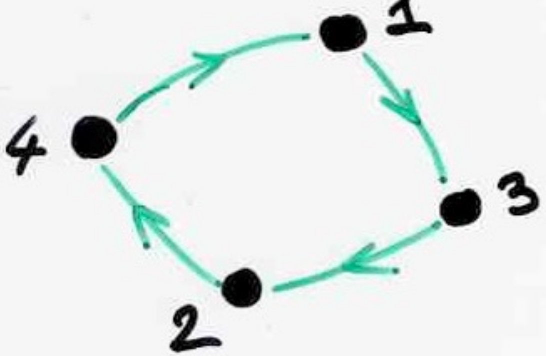
$$D(t) = \frac{e^{-t}}{1-t}$$

permutations with no fixed points

6. Cycle C

$a_n = (n-1)!$

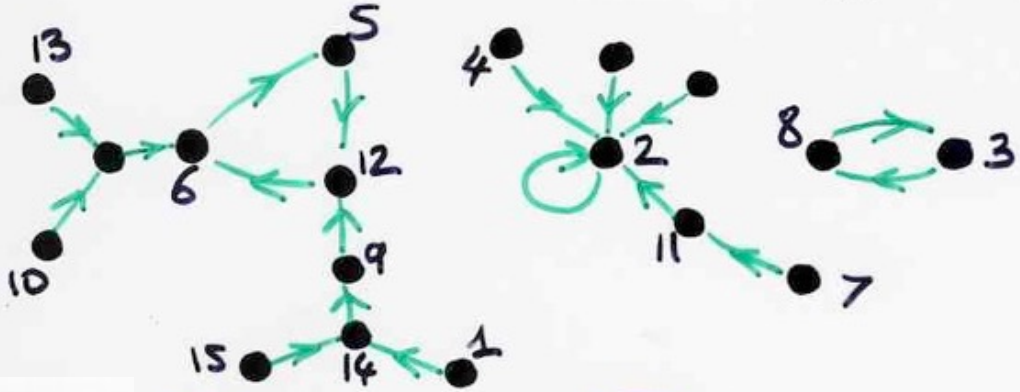
$$C(t) = \sum_{n \geq 1} \frac{t^n}{n} = \log(1-t)^{-1}$$



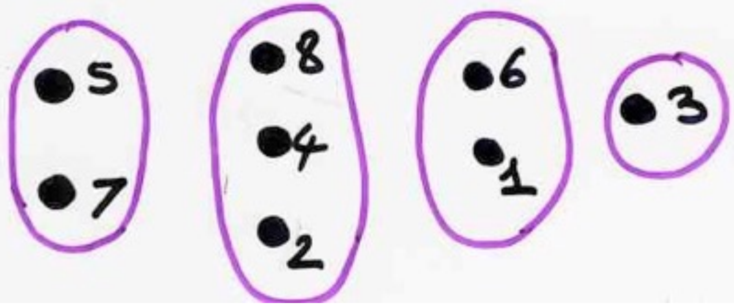
circular permutations

7. Endofonction End

$$End(t) = \sum_{n \geq 0} n^n \frac{t^n}{n!}$$



8. Partition B



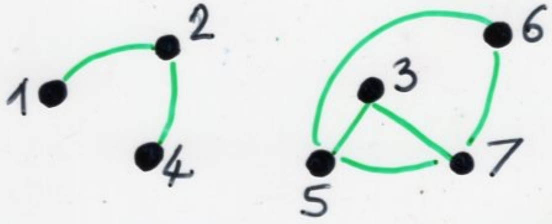
Bell number

$a_n = B_n$ nombre de Bell

$$B(t) = \exp(e^t - 1)$$

9. Graph G

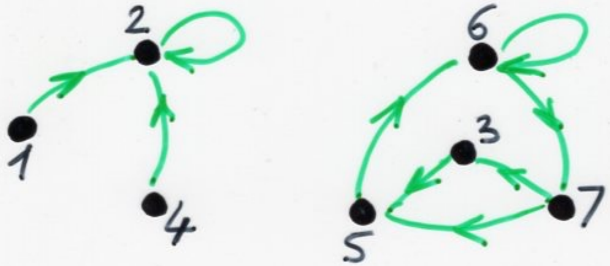
$a_n = 2^{\binom{n}{2}}$



connected graph G_c

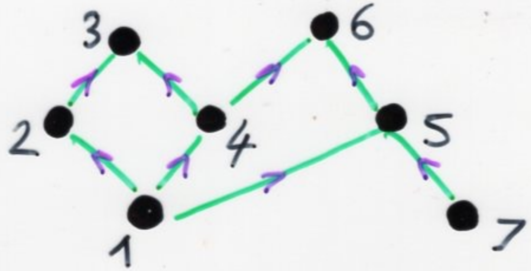
10. oriented graph $\vec{G}$ (possible loops)

$a_n = 2^{n^2}$



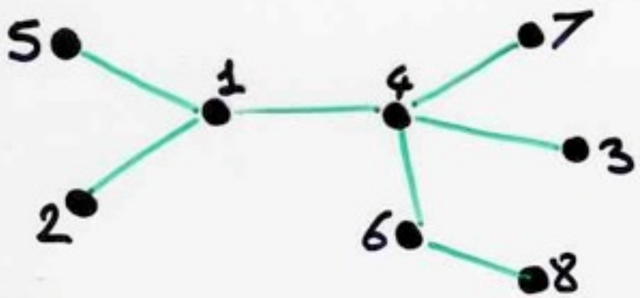
11. Poset (partially ordered set)

Hasse diagram



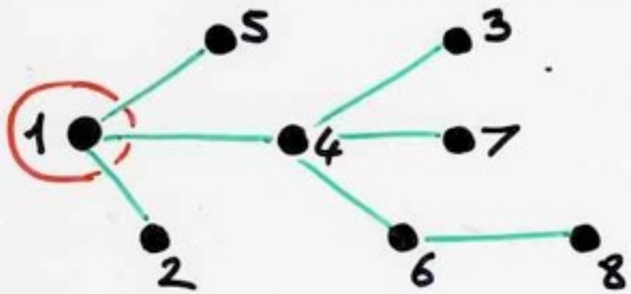
$1 \prec 2 \prec 3$
$1 \prec 4 \prec 3$
$4 \prec 6$
$1 \prec 5 \prec 6$
$7 \prec 5$

12. (Cayley) tree α



$a_n = n^{n-2}$

13. Arborescence A



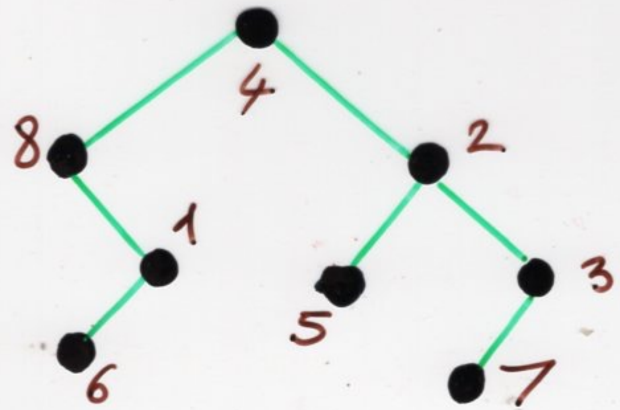
pointed vertex (root)

$a_n = n^{n-1}$

14.

(labeled) binary tree

$a_n = n!\, C_n$



15.

(labeled) planar tree

16. set E (ensemble) $E[t] = e^t$

uniform species

17. even set EP $EP[t] = ch(t)$

odd set EI $EI[t] = sh(t)$

18. subset P $a_n = 2^n$ $P[t] = e^{2t}$

19. Singleton T X

$T[U] = \emptyset$ si $|U| \neq 1$

$\quad = \{U\}$ si $|U| = 1$

$T[t] = t$

Ensemble vide 1

$1[U] = \emptyset$ si $|U| \neq 0$

$\quad = \{\emptyset\}$ si $U = \emptyset$

$1[t] = 1$

Espèce vide 0

$0[U] = \emptyset$

$0[t] = 0$

complements

formalisation of species

categories

$\mathbb{B}$ category objects: finite sets
arrows: bijections

$\mathbb{E}\mathrm{ns}$ category objects: finite sets
arrows: functions

Def. species F

functor $F : \mathbb{B} \longrightarrow \mathbb{E}\mathrm{ns}$

- U finite set $\longrightarrow$ $F[U]$ set of F-structures
- $U \xrightarrow{f} V$ bijection $\quad F[U] \xrightarrow{F[f]} F[V]$

F-transport along f

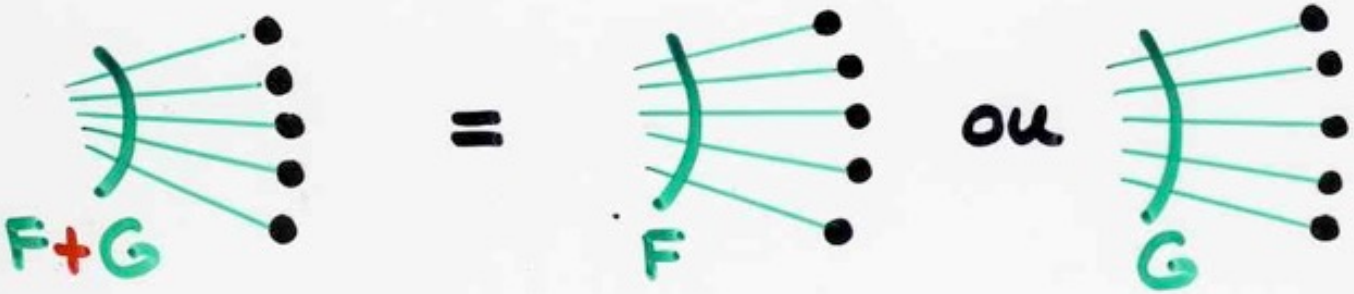
(i) $F[f \circ g] = F[f] \circ F[g]$

(ii) $F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]}$

Operations on species

Def. sum

$$(F+G)[U] = F[U] + G[U] \quad \text{(disjoint union)}$$



Rem. $(F+G)[f]$ $\qquad f : U \to V$

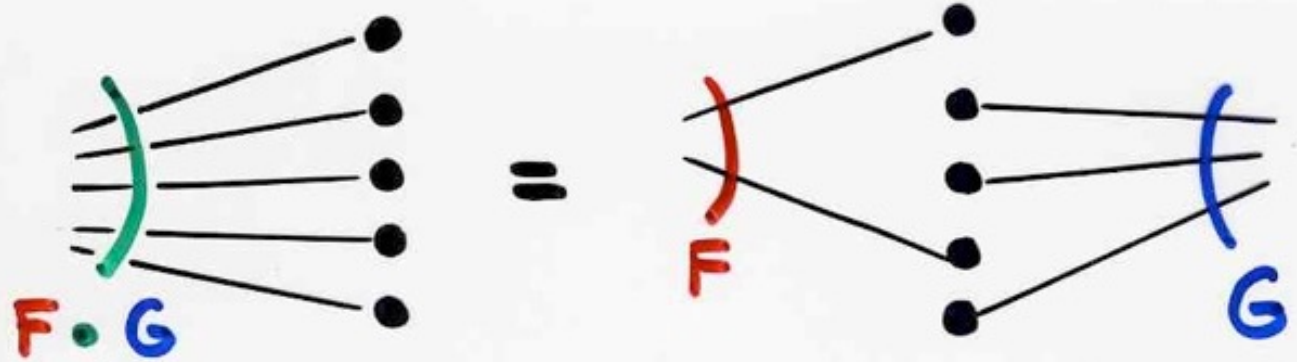
Prop. $(F+G)(t) = F(t) + G(t)$

$$c_n = a_n + b_n$$

ex. $E = EP + EI$

$$e^t = \operatorname{ch} t + \operatorname{sh} t$$

Déf. product $F \cdot G$



$\gamma \in F \cdot G[U]$ $\gamma = (U_1, U_2, \alpha, \beta)$ $\{U_1, U_2\}$ partition of U

$\alpha \in F[U_1]$ $\beta \in G[U_2]$

Prop.

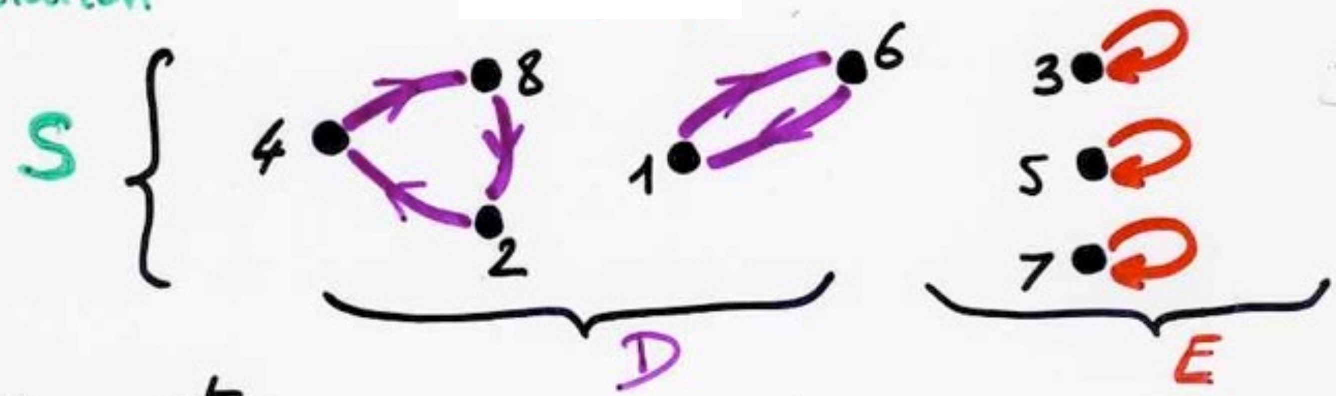
$$F \cdot G[t] = F[t]\, G[t]$$

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k\, b_{n-k}$$

ex. Derangements D set

$$S = D \cdot E$$

permutation



$$D[t] = \frac{e^{-t}}{1-t}$$

$$d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{(-1)^n}{n!}\right)$$

$$P_n = \frac{d_n}{n!} \longrightarrow \frac{1}{e}$$

$$S = D \cdot E$$

$$\Rightarrow \quad \frac{D}{S} = \frac{1}{E}$$

Ensemble (set species)
e
Euler

$$\Rightarrow \quad \lim_{n \to \infty} \frac{d_n}{n!} = \frac{1}{e}$$

I am joking !

May be not !

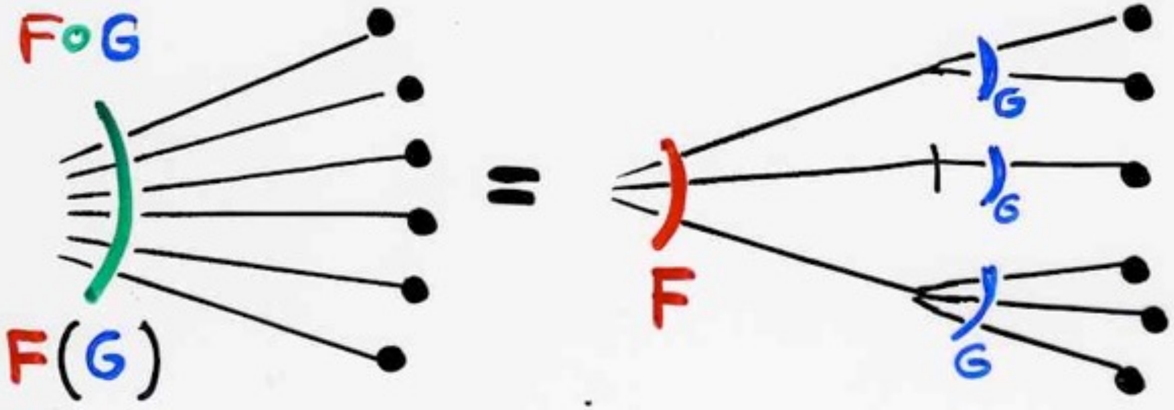
<u>exercise</u> prove the following recurrences

$d_{n+1} = n(d_n + d_{n-1})$ (bijective proof)

$d_{n+1} = (n+1)\,d_n + (-1)^{n+1}$ (more difficult)

Def. F G $G[\emptyset] = \emptyset$

substitution of G into F



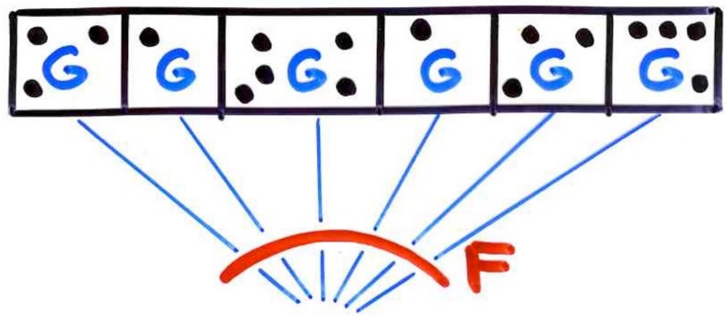
$\gamma \in F(G)[U]$

$\gamma \begin{cases} \bullet \text{ partition } \{U_1, \ldots, U_k\} \text{ de } U \\ \quad\quad \text{classes } \neq \emptyset \\ \bullet\ \beta_i \in G[U_i], \quad i = 1, \ldots, k \\ \bullet\ \alpha \in F[U/\equiv] \end{cases}$

F- "assemblée" of G-structures

ex- permutation = assemblee of cycles

$H =$



$$H = F(G)$$

Prop. $(F \circ G)(t) = F(G(t))$

$$c_n = \sum_{\substack{k=0 \\ n_1+\dots+n_k=n \\ n_1,\dots,n_k \geq 1}}^{n} \frac{n!}{k!\, n_1! \dots n_k!}\, a_k\, b_{n_1} \dots b_{n_k}$$

Cor $F = E$

$$(E \circ G)(t) = \exp(G(t))$$

assemblee of G-structures $\qquad E^G$

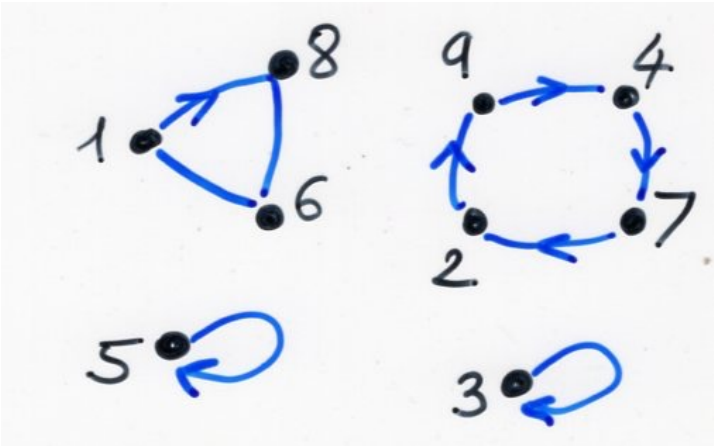
$H = \exp F$

$\{A_1, \ldots, A_k\}$ partition of $\{1, 2, \ldots, n\}$

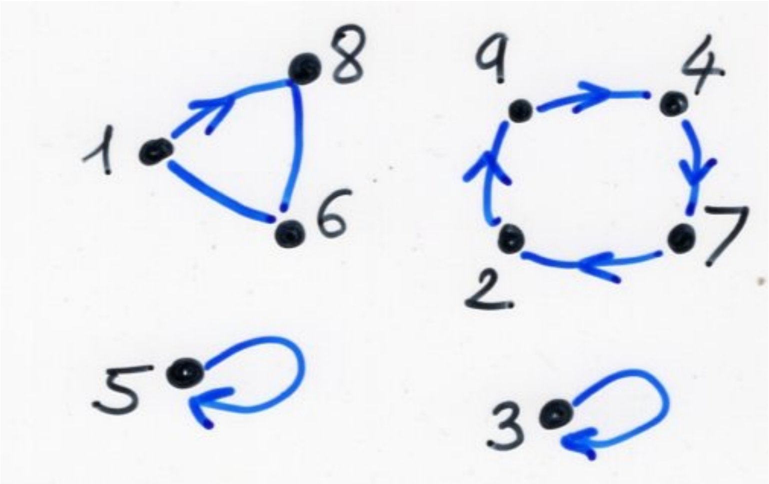
$\{\alpha_1, \ldots, \alpha_k\}$ α_i F-structure on A_i

log

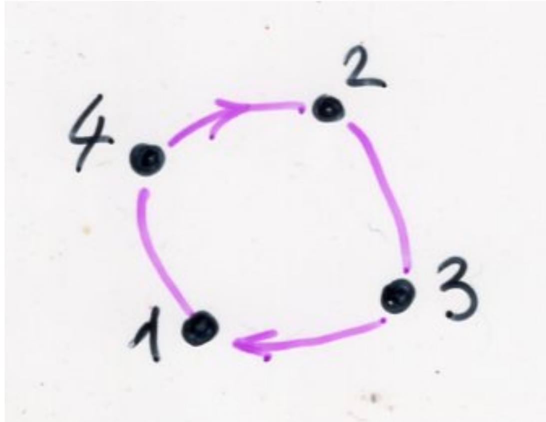
$h(t) = \exp(f(t))$



permutation = exp (cycle)



cyclic
permutation



$$\sum_{n\geq 0} n!\,\frac{t^n}{n!} = \frac{1}{1-t}$$

$$\sum_{n\geq 1} (n-1)!\,\frac{t^n}{n!} = \sum_{n\geq 1} \frac{t^n}{n}$$

$$= \log \frac{1}{1-t}$$

Let $I \subseteq \mathbb{N}$

S_I species of permutations such that all their cycles have length in I

$$S_I(t) = \exp\Big(\sum_{i \in I} \frac{t^i}{i}\Big)$$

Involutions $\quad I(t) = \exp\Big(t + \frac{t^2}{2}\Big)$

Involutions with no fixed points $\quad J(t) = \exp\Big(\frac{t^2}{2}\Big)$

$I = \{\text{even numbers}\}$

$$S_I(t) = \exp\left(\sum_{n\geq 1} \frac{t^{2n}}{2n}\right)$$

$$= \frac{1}{\sqrt{1-t^2}}$$

<u>exercise</u>

$a_{2n} = 1^2 \times 3^2 \times \dots \times (2n-1)^2$ (bijective proof)

ex. • Permutations $\quad S = E \circ C$ (set of cycles)

$$\log(1-t)^{-1} = \sum_{n \geq 1} \frac{t^n}{n}$$

• Partitions $\quad B = E \circ E^*$ (set of non-empty blocks)

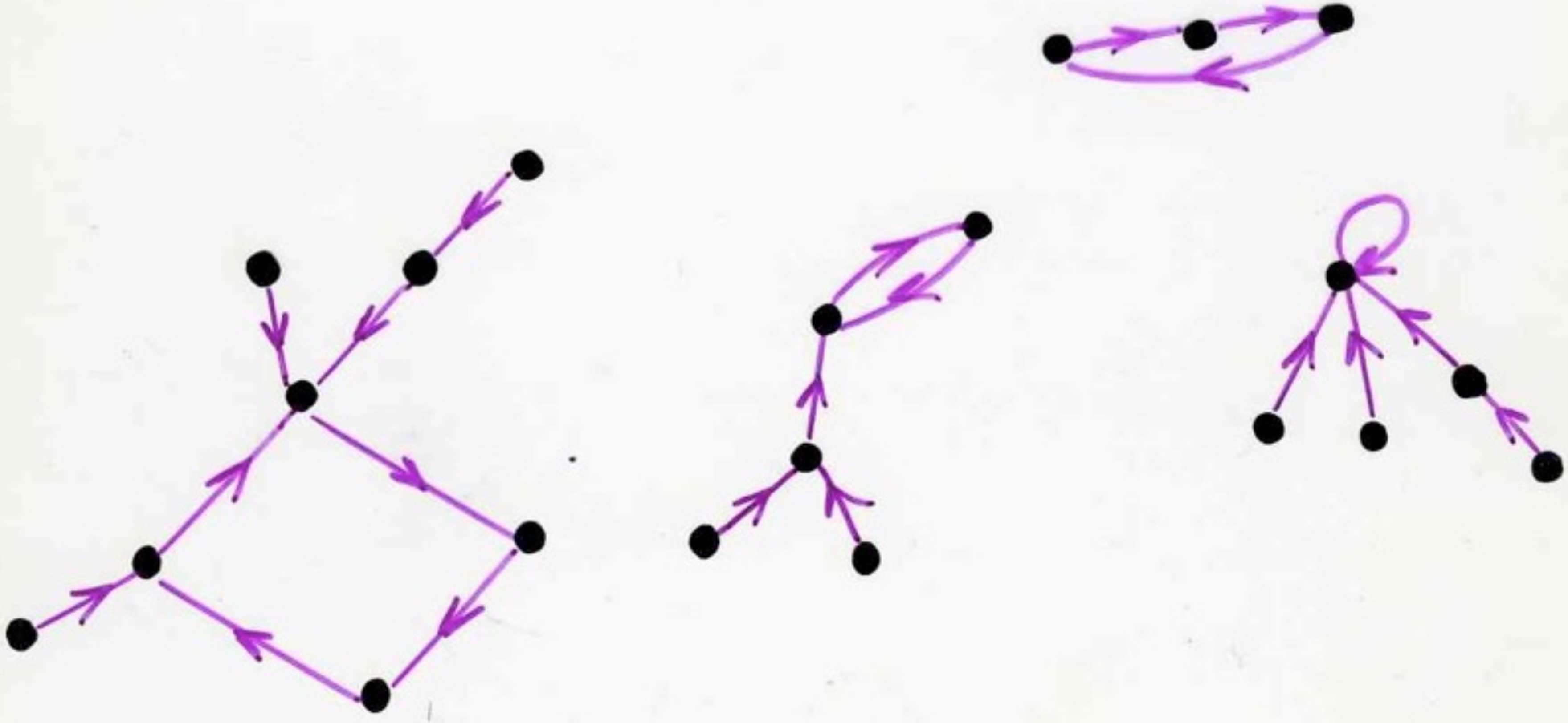
$$B(t) = \exp(e^t - 1)$$

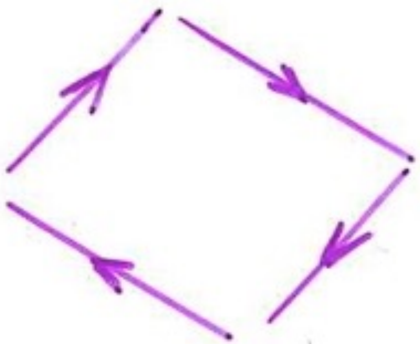
• Graphs $\quad G = E \circ GC$ (set of connected graphs)

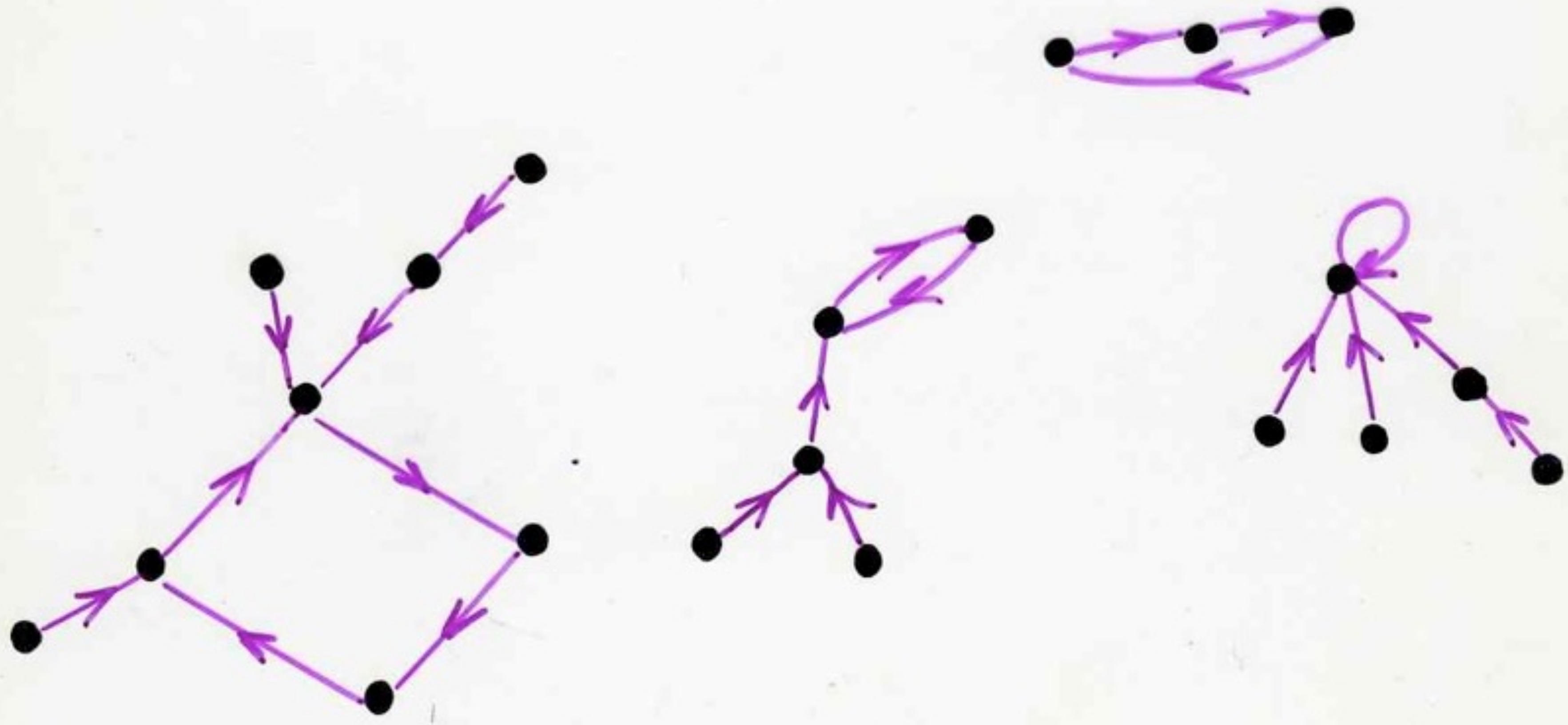
ex. Endofonctions

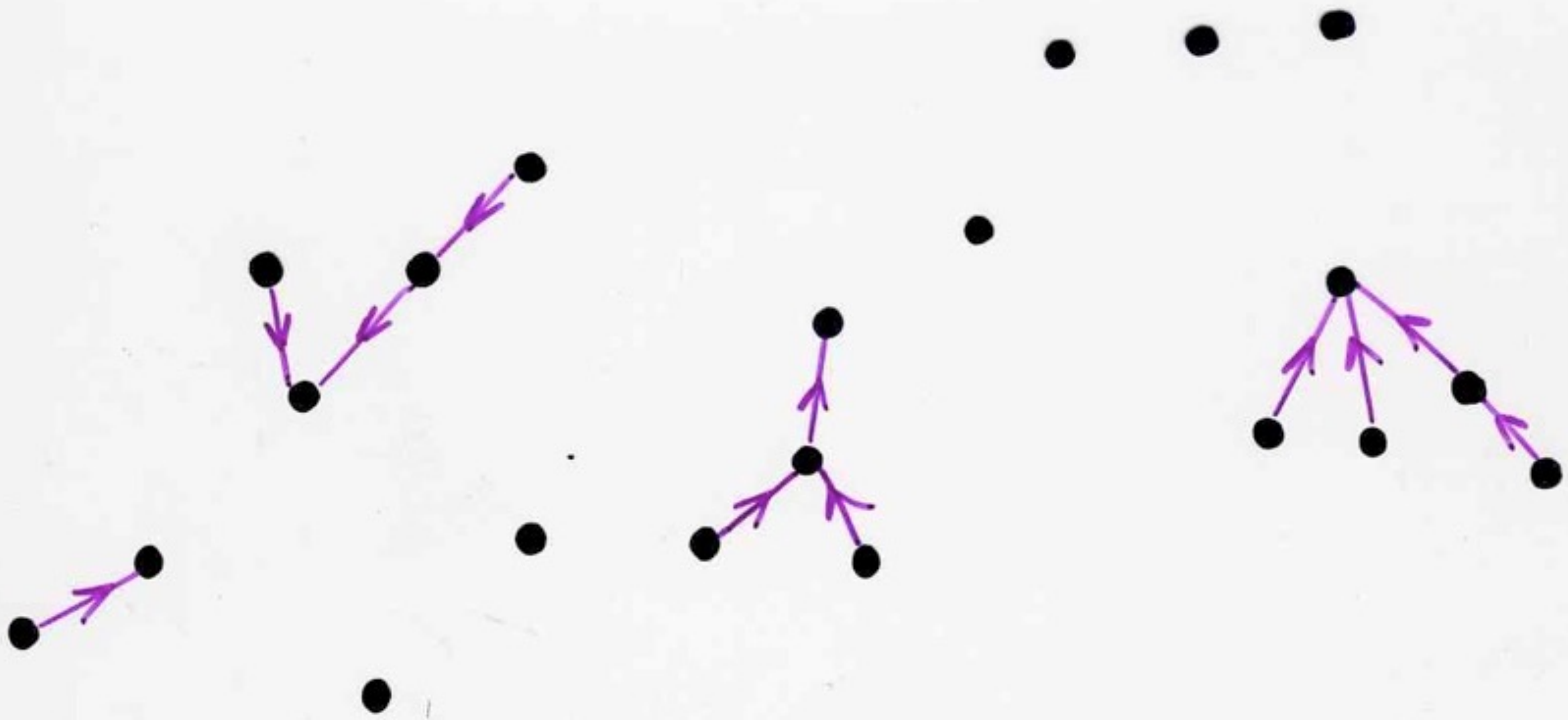
$$\mathrm{End} = S \circ A$$

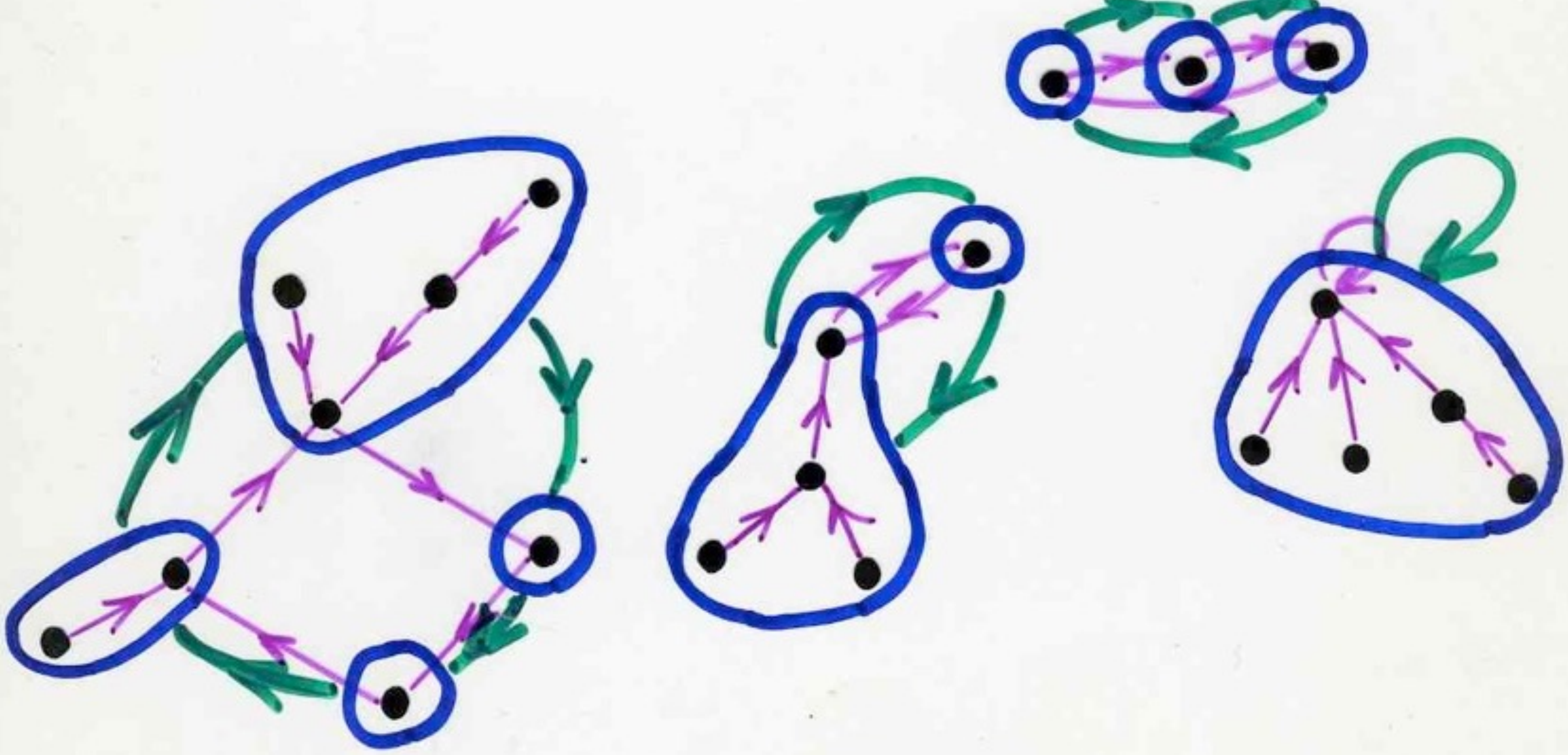
endofonctions — permutations — arborescence

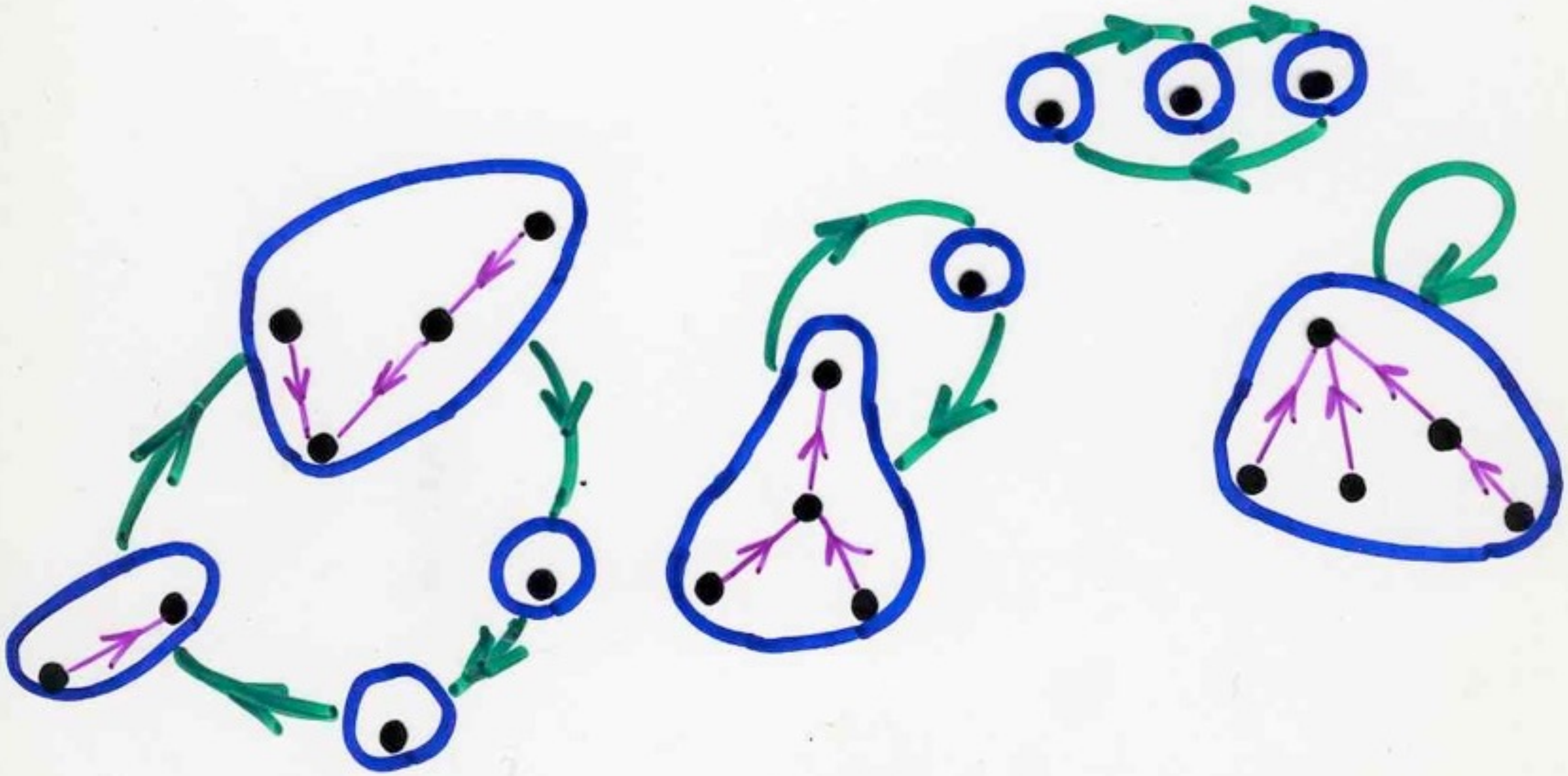




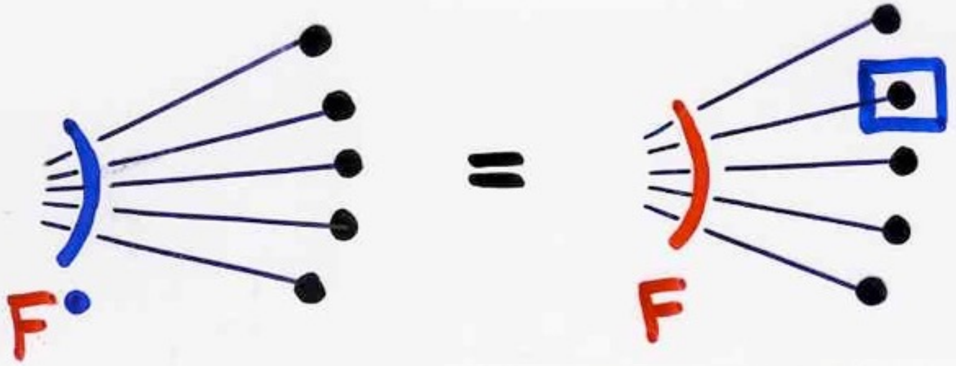








Def. Pointed species $F^\bullet$



$\alpha \in F[U]$ $x \in U$ $(\alpha, x) \in F^\bullet[U]$

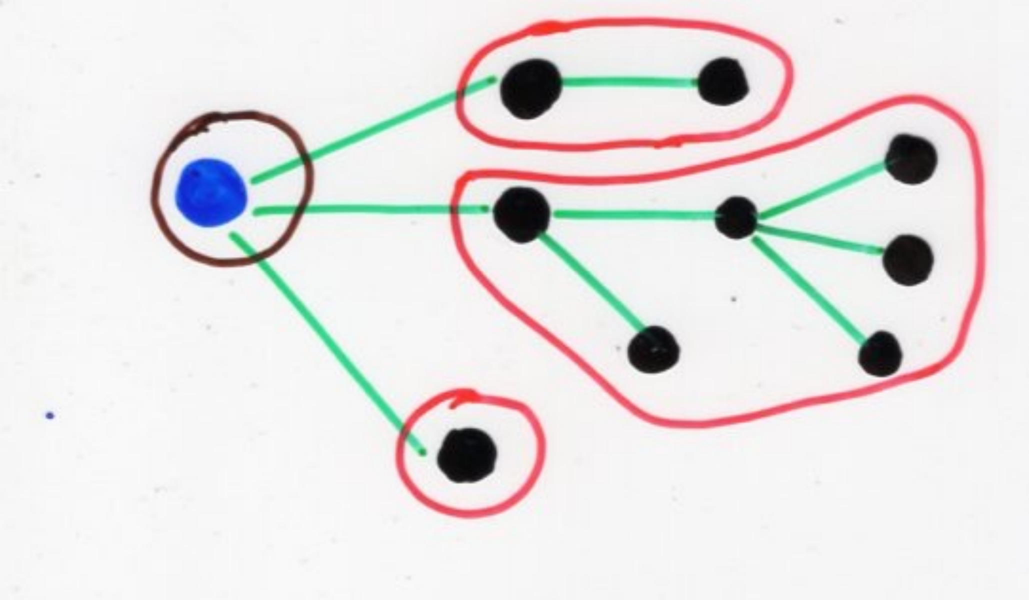
Prop $F^\bullet(t) = t \frac{d}{dt} F(t)$

example

A Arborescences species

$\mathcal{A}$ (Cayley) tree species

$$A = \mathcal{A}^{\bullet}$$



$$A = T \cdot (E \circ A)$$

$$y = t \exp(y)$$

$$y = \sum_{n\geq 1} a_n \frac{t^n}{n!}$$

$a_n =$ number of arborescences on n elements

$$y = t \exp(y)$$

$$\frac{a_n}{n!} = \frac{1}{n} [t^{n-1}] e^{nt}$$

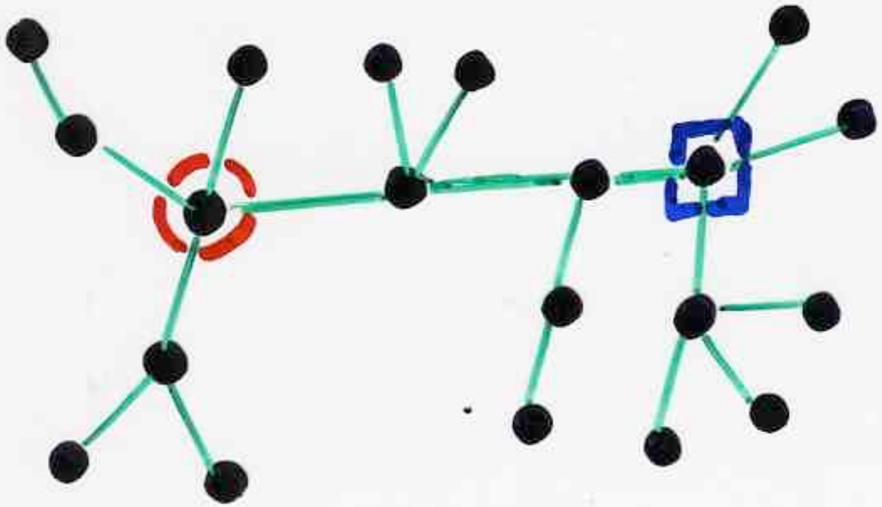
$$= \frac{1}{n} \frac{n^{(n-1)}}{(n-1)!}$$

$$a_n = n^{n-1}$$

$t_n =$ number of trees on n elements

$$t_n = n^{n-2}$$

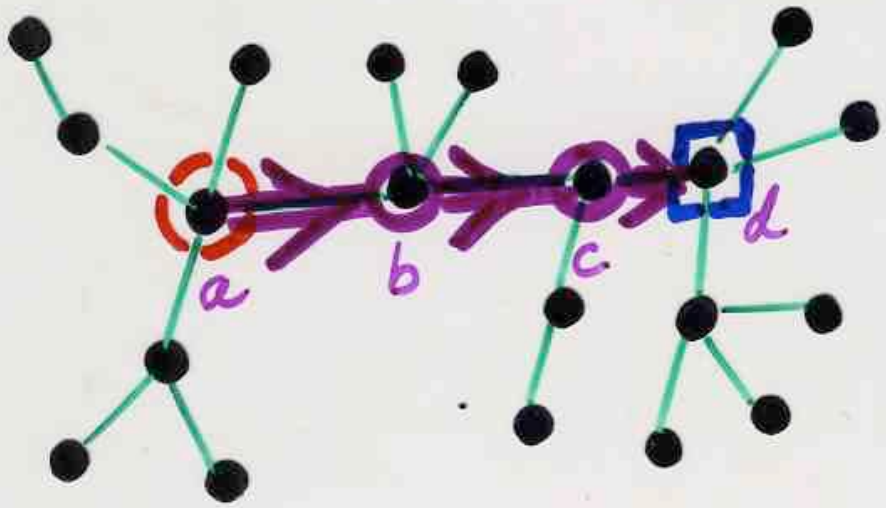
ex. Vertébrés $V = A^{\bullet} = (a^{\bullet})^{\bullet}$



head □

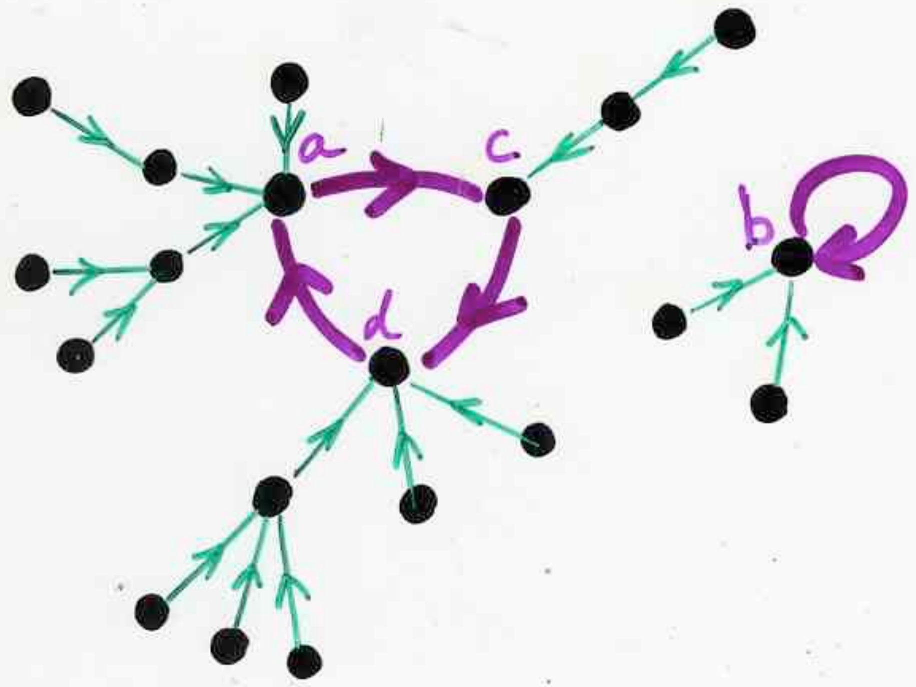
tail ○ spine

ex. Vertébrés $\quad V = A^{\bullet} = (a^{\bullet})^{\bullet}$



head $\square$
tail $\circ$ spine

$$V = L \circ A$$



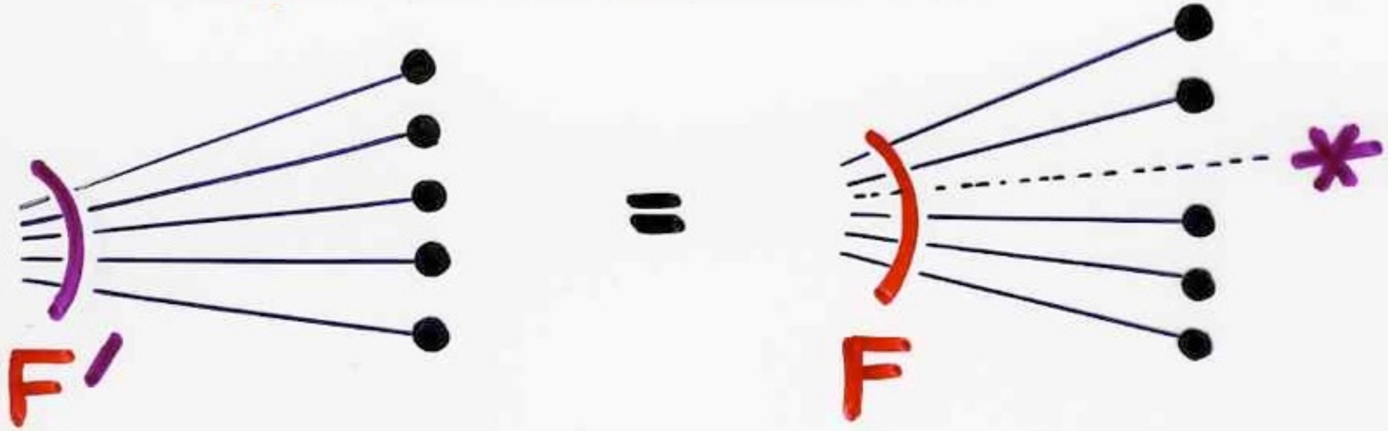
$$V(t) = S \circ A(t)$$

$$V(t) = End(t)$$

number of "vertébrés" $c_n = n^n$

number of trees $a_n = n^{n-2}$ Cayley formula

Def- derivative of species F'

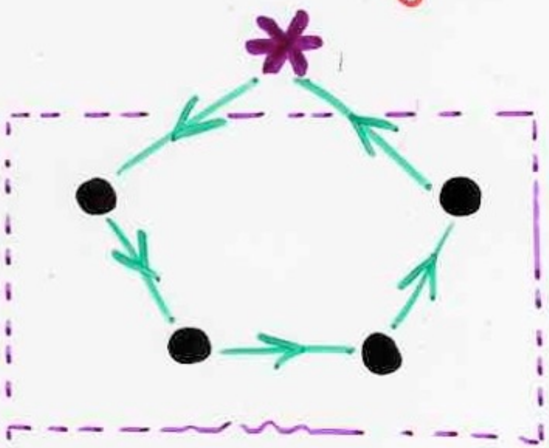


$U + \{*\}$

Prop- $\quad (F')(t) = \dfrac{d}{dt} F(t)$

ex- C cycles

C' = L



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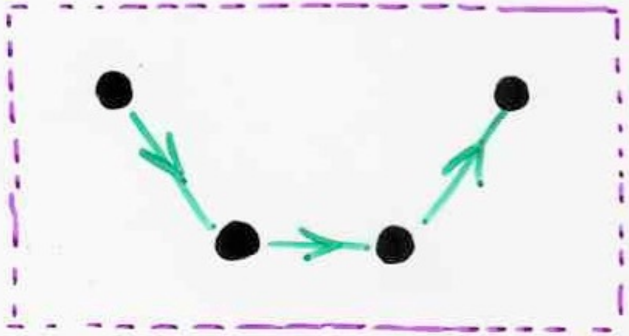


$$C(t) = \int_0^t L(u)\,du$$

$$\log(1-t)^{-1} = \int_0^t \frac{du}{1-u}$$

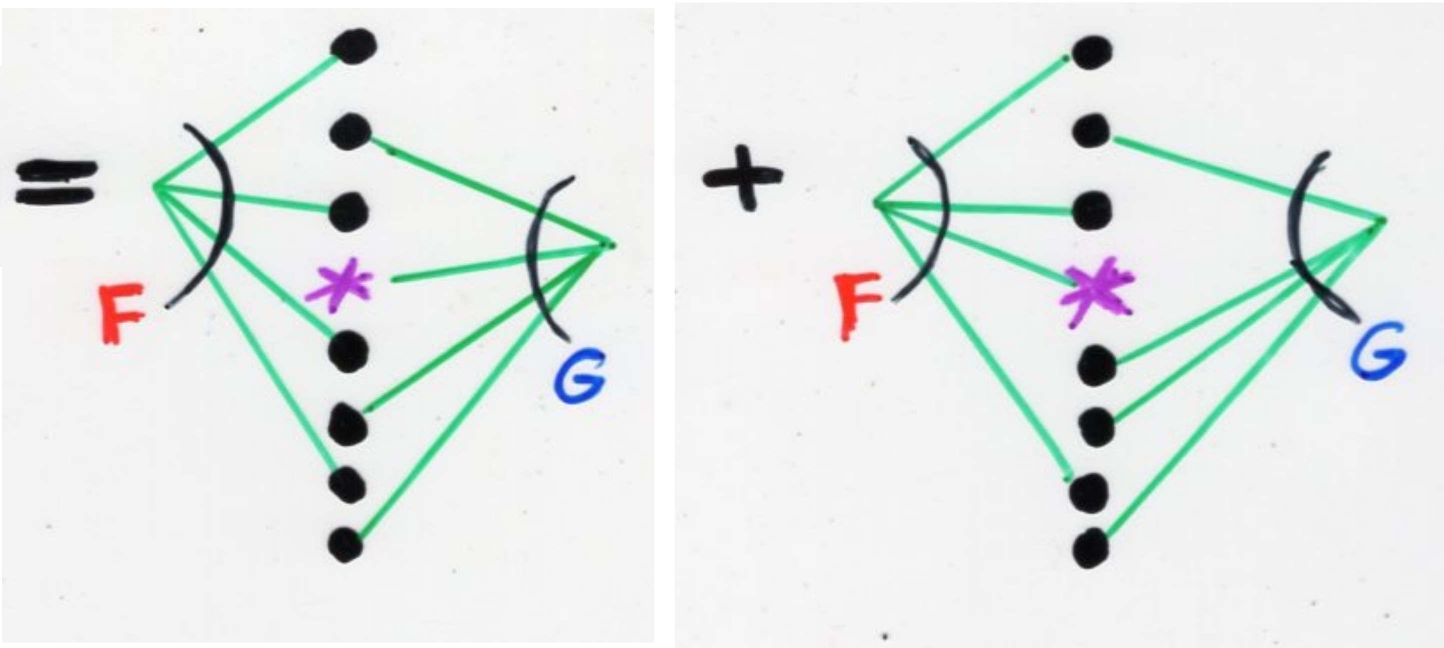
(well known!)

$$(F\cdot G)' = (F'\cdot G) + (F\cdot G')$$

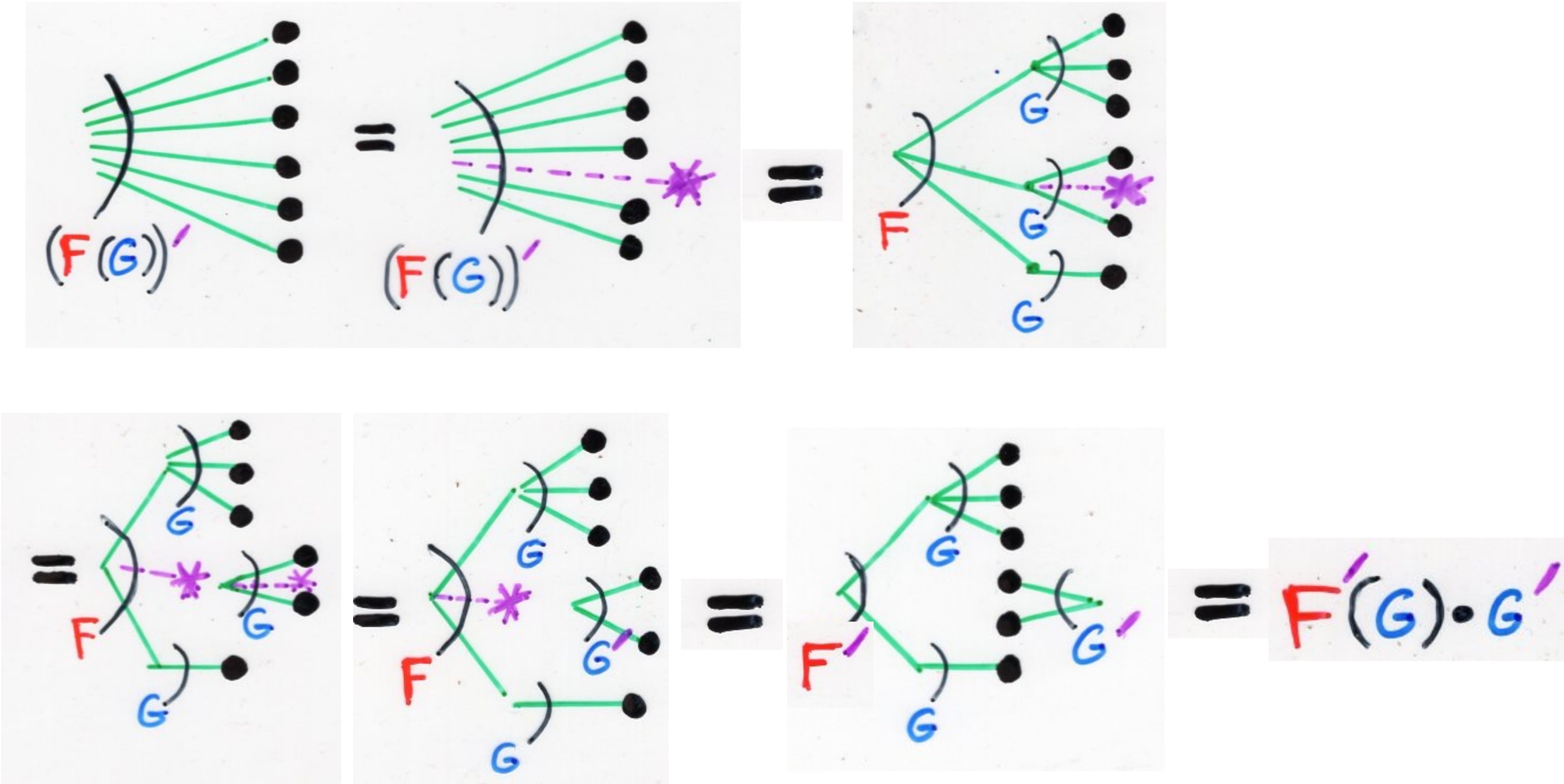
$$(F(G)) = F(G)\cdot G'$$

$$(F+G)' = F' + G'$$

$$(F\cdot G)' =$$



A typical "species proof"

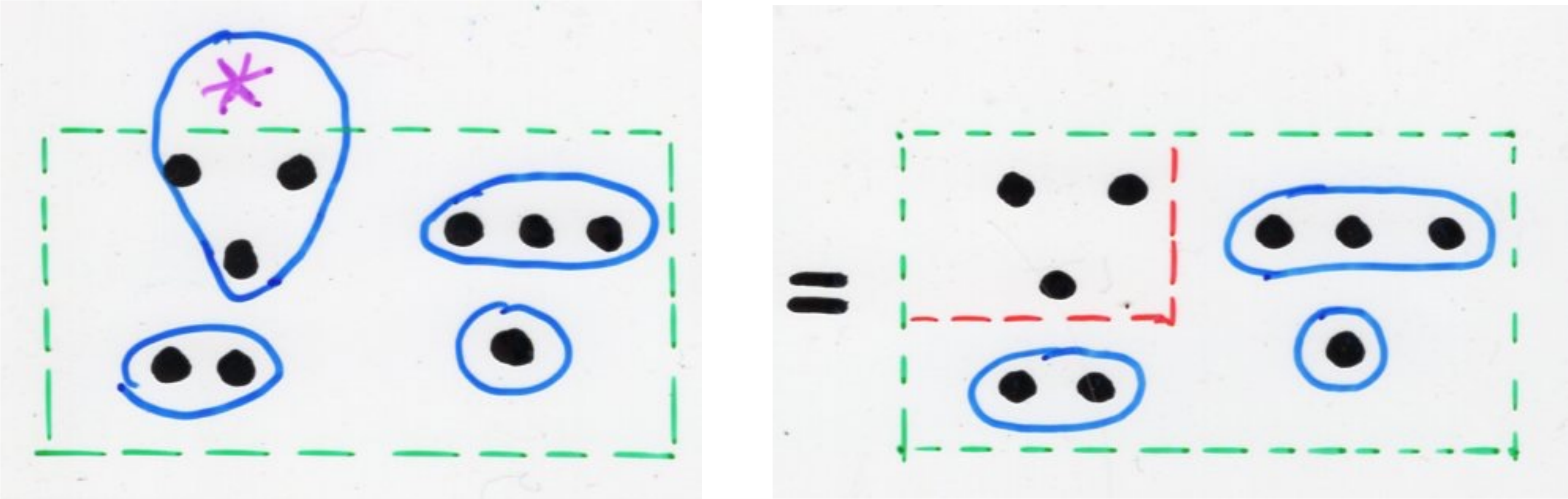


<u>example</u> $L' = L^2$

$$\frac{d}{dt}\left(\frac{1}{1-t}\right) = \left(\frac{1}{1-t}\right)^2$$

B partitions species

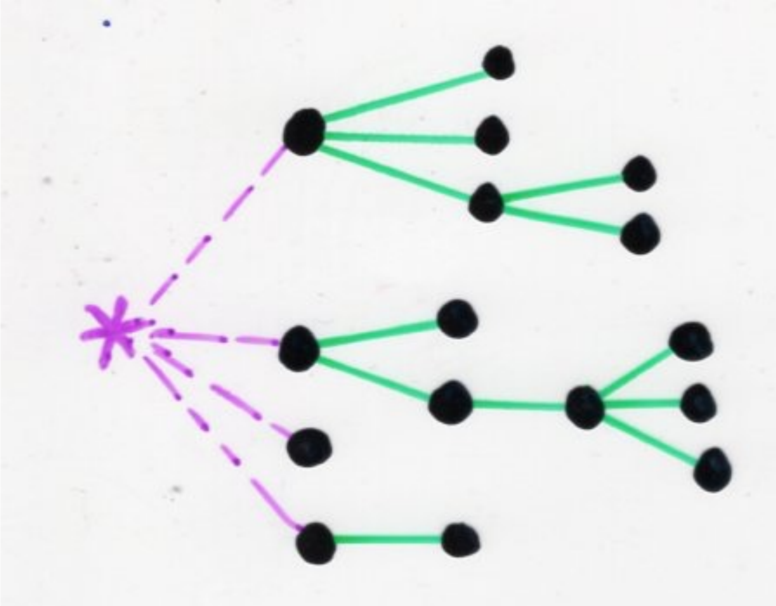
$$B' = E \cdot B$$



$$B_{n+1} = \sum_{0 \leq k \leq n} \binom{n}{k} B_k$$

A arborescences species

$\mathcal{A}$ trees species



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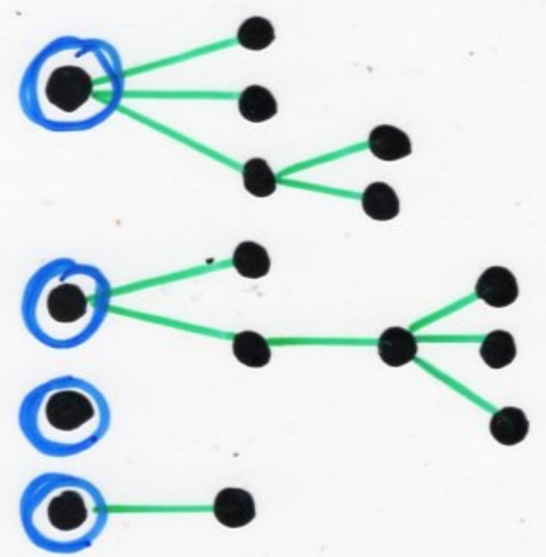

$\mathcal{F}$ assemblées of arborescences

$\mathcal{A}' = \mathcal{F}$

Primitive for graph species G

$$H' = G \quad ?$$

H species of graph such that every vertices has even degree (number of incident edges)

$$H' = G$$