An introduction to

enumerative
algebraic combinatorics
bijective

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IMSc

January-March 2016

Chapter 2

The Catalan garden

(3)

IMSc

2 February 2016

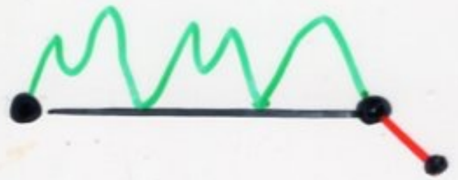
the cyclic lemma

- Definition $w, w' \in X^*$ are conjugate iff
  $w = uv \quad w' = vu$ equivalence relation

- primitive word $w = u^p \Rightarrow p = 1$
  $\Updownarrow$ all conjugates are distincts

- labelled conjugate $w = x_1 \cdots x_n \quad , x_i \in X$
  $(i, w_i) \quad w_i = x_i \cdots x_n \, x_1 \cdots x_{i-1}$

$A = D\bar{x} \quad \subseteq \{x, \bar{x}\}^* \qquad D$ Dyck words



Proposition (cyclic lemma)

Let $w \in \{x, \bar{x}\}^*$, $p > 0$, with $\delta(w) = -p$

$\delta : X^* \to \mathbb{Z}_+ \qquad \delta(x) = 1, \ \delta(\bar{x}) = -1$

$$\delta(w) = |w|_x - |w|_{\bar{x}}$$

There are exactly p labelled conjugates (i, w_i) such that $w_i \in A^*$

Dvoretzky, Motzkin (1947)
Raney (1960)

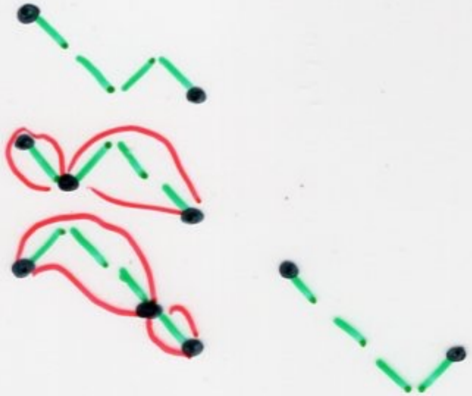
$w = \bar{x}\,\bar{x}\,x\,\bar{x}$

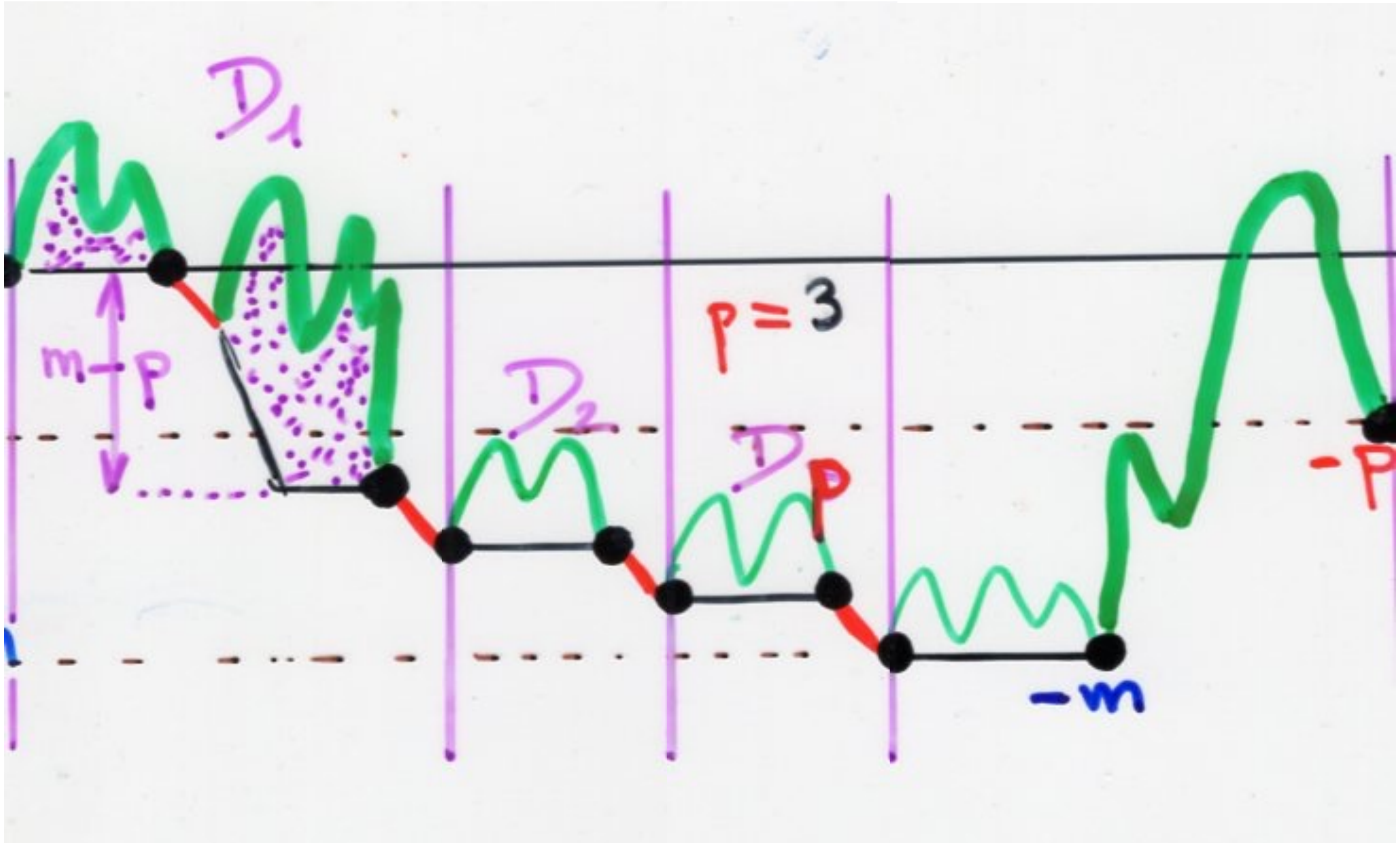
$(1, w_1)$ $\quad \bar{x}\,\bar{x}\,x\,\bar{x}$

$(2, w_2)$ $\quad \bar{x}\,x\,\bar{x}\,\bar{x}$

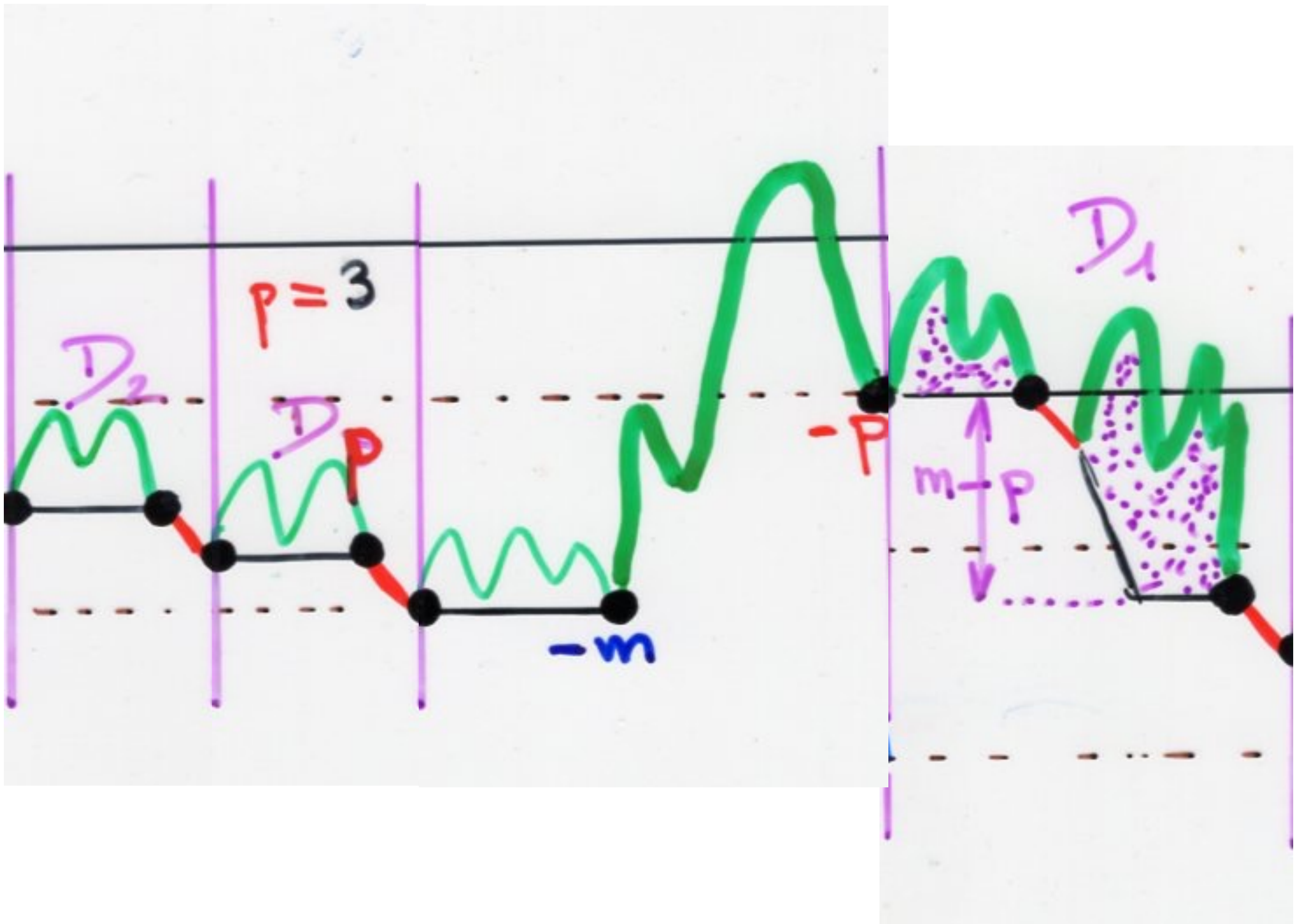
$(3, w_3)$ $\quad x\,\bar{x}\,\bar{x}\,\bar{x}$

$(4, w_4)$ $\quad \bar{x}\,\bar{x}\,\bar{x}\,x$

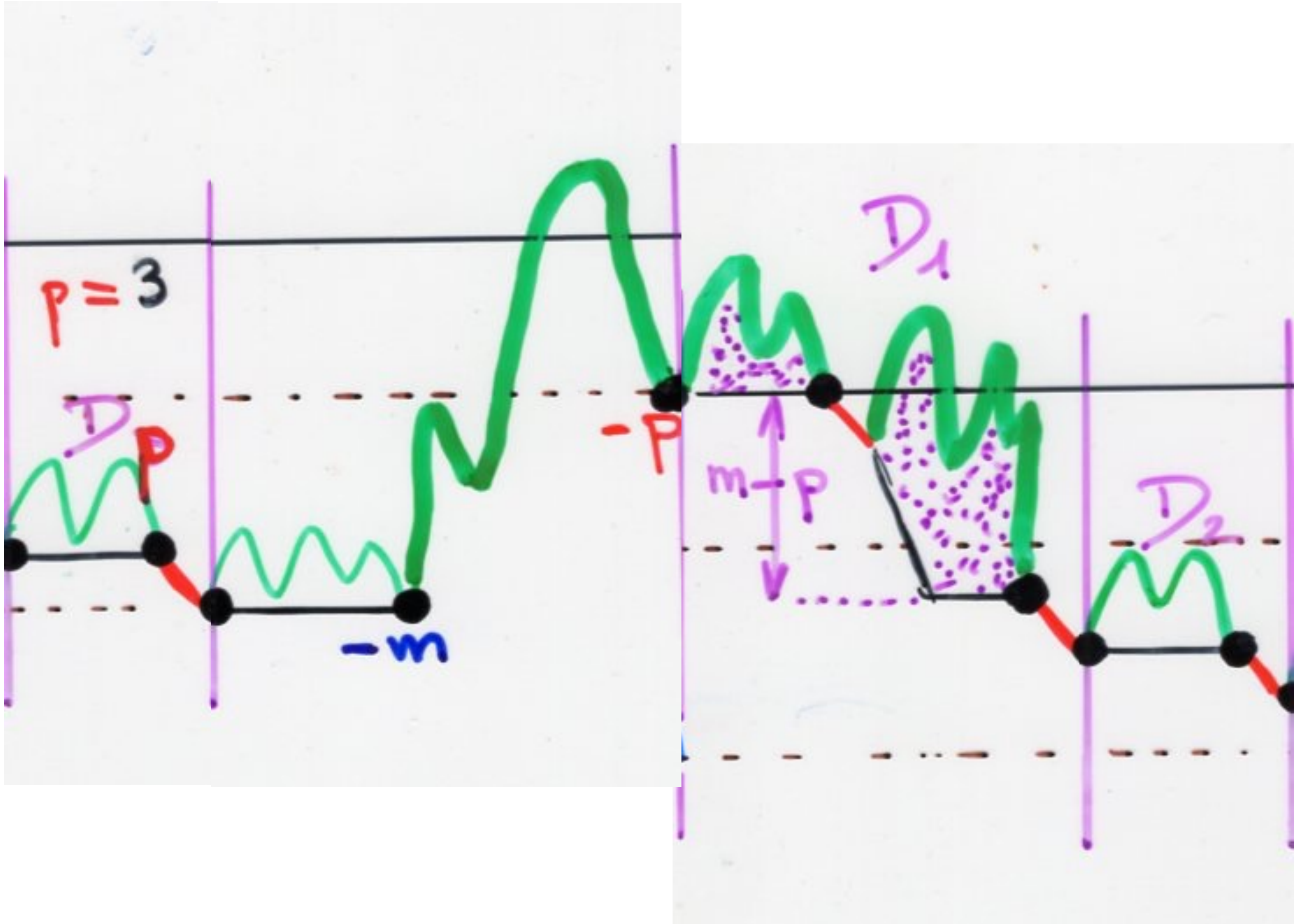




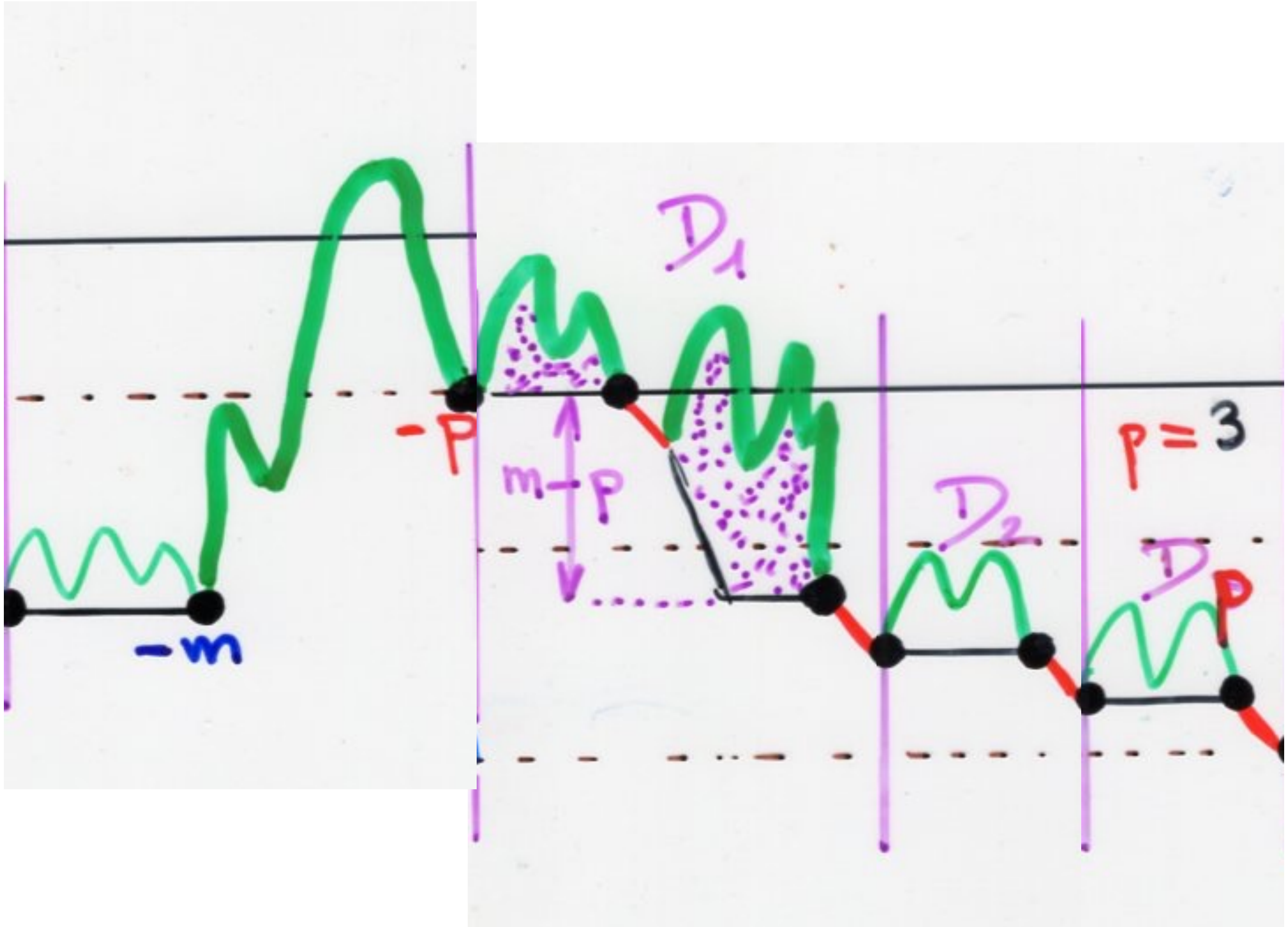
D1
m - p
D2
p=3
D
p
-m
-p



p = 3
D_2
D
p
$-m$
$-p$
D_1
$m-p$



p = 3
D
p
-m
-p
D1
m - p
D2



D_1
$-p$
$m-p$
D_2
$p = 3$
D_p
$-m$

$$\binom{2n+1}{n} = \left| X^{2n+1} \cap \delta^{-1}(-1) \right|$$

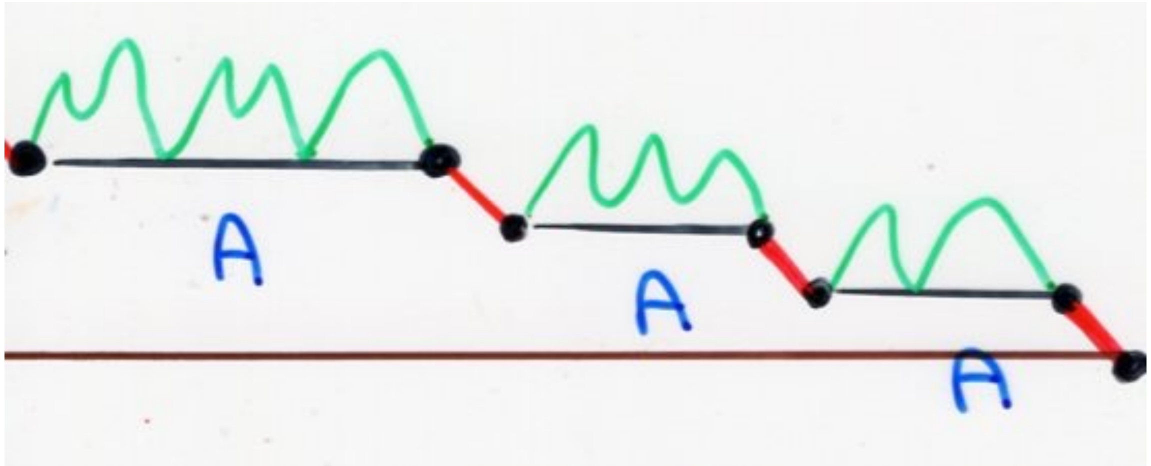
$$C_n = \left| X^{2n+1} \cap A \right|$$

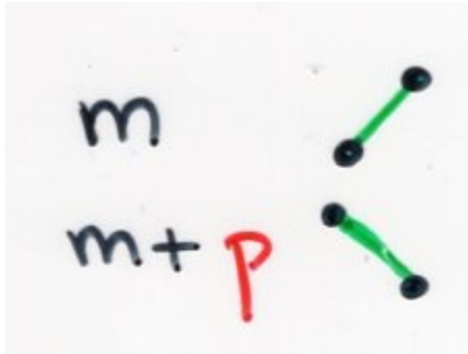
Corollary

$$C_n = \frac{1}{(2n+1)} \binom{2n+1}{n}$$

Corollary

$$\left| A^{p} \cap X^{2m+p} \right| = \frac{p}{2m+p} \binom{2m+p}{m}$$

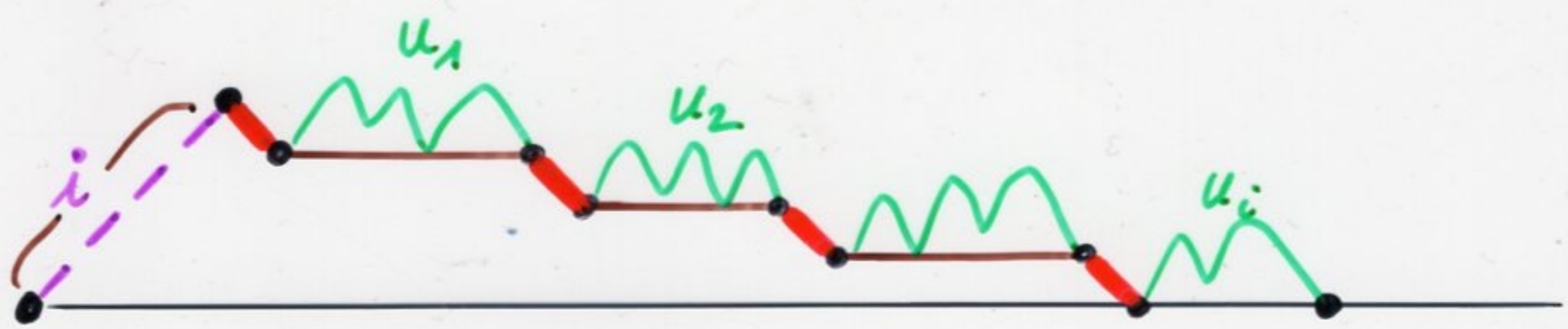


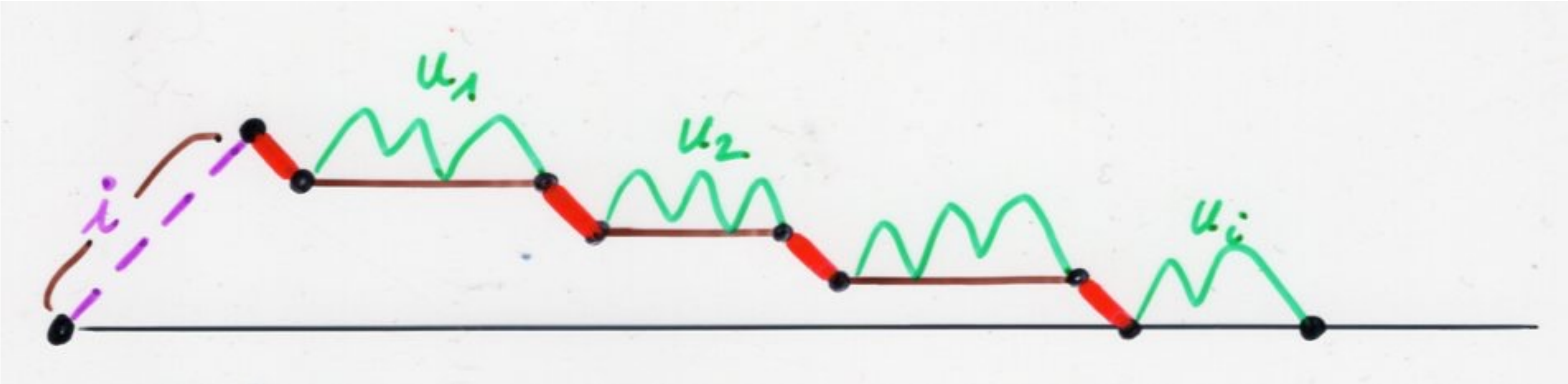


Corollary

(α) - distribution

The number of Dyck words of length $2n$ with i occurrences of x at the beginning is $\dfrac{i}{2n-i}\dbinom{2n-i}{n}$





$$w = x^{i}\,\bar{x}\,u_1\,\bar{x}\;\dots\;u_{i-1}\,\bar{x}\,u_i \qquad |w| = 2n$$

$$v = u_1\,\bar{x} \qquad u_{i-1}\,\bar{x}\,u_i\,\bar{x} \qquad |v| = 2n-i$$

$$v \in A^{i}$$

$$i = p \qquad \begin{aligned} 2m+p &= 2n-i \\ m &= n-i \end{aligned}$$

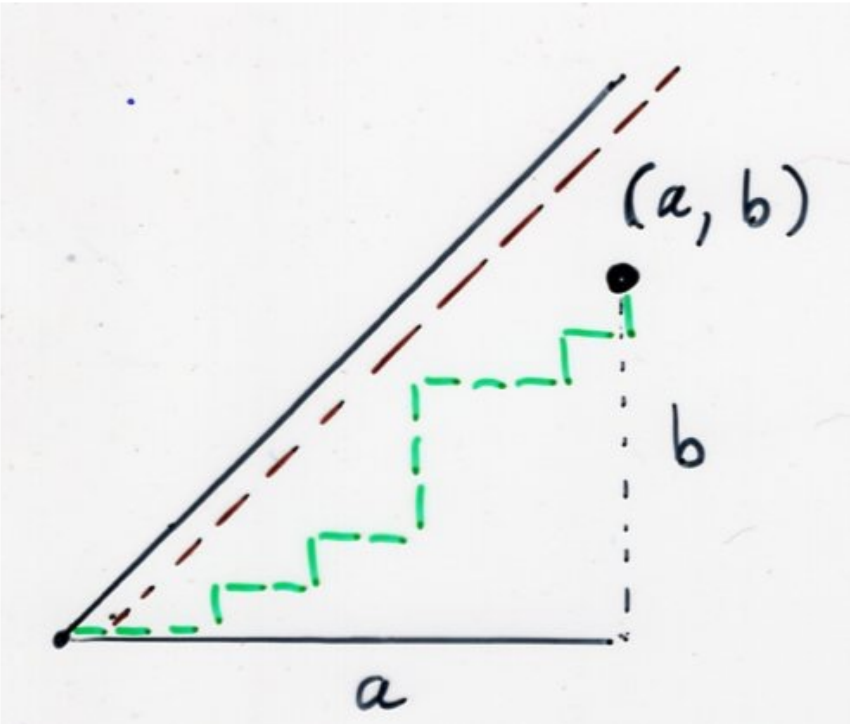
$$\frac{p}{2m+p}\binom{2m+p}{m} = \frac{i}{2n-i}\binom{2n-i}{n}$$

ballot problem

$a = m + p$
$b = m$

$p = a - b$

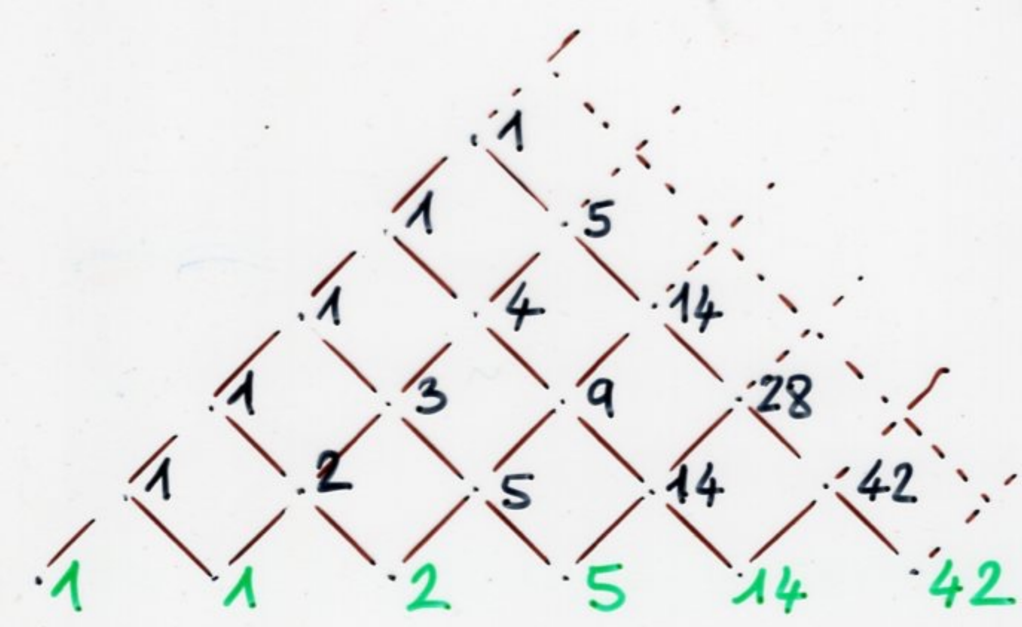
$$\frac{p}{2m+p}\binom{2m+p}{m} = \frac{a-b}{a+b}\binom{a+b}{a}$$

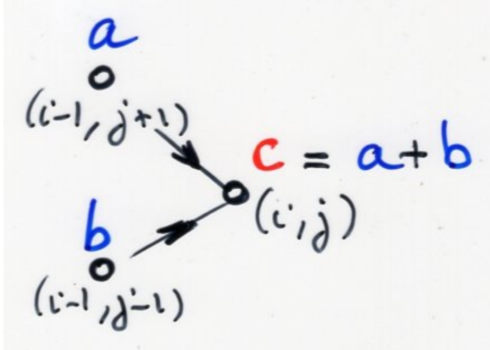


$$\text{proba} = \frac{a-b}{a+b}$$

W. Whitworth (1878)
J. Bertrand, D. André (1887)

a "Catalan triangle"





1					
1	1				
1	2	2			
1	3	5	5		
1	4	9	14	14	
1	5	14	28	42	42

laquelle, à cause que $\alpha = o$ et que sa seule

$$A_{m,-n} = \pm \beta'\{A_{0,0}.\beta^n \underset{c}{D}^{m-n}.\beta^{l-n-1} - A_{0,1}.\beta^{n+1}\underset{c}{D}^{m-n-1}.\beta^{l-n-2} + A_{0,2}.\beta^{n+2}\underset{c}{D}^{m-n-2}.\beta^{l-n-3} - \text{etc.}\}. \quad \ldots\ldots (5)$$

D'où il suit que $A_{m,-n}$ n'est zéro qu'autant que m est $< n$. Ainsi la série récurrente s'étend, sous forme de triangle, dans la quatrième région.

EXEMPLE VI.

246. Étant donné le commencement de la table suivante, où chaque terme est formé de la somme de celui qui le précède dans la même ligne horizontale et de celui qui le suit d'un rang dans la ligne horizontale immédiatement supérieure, avec la condition que chacun des termes de la première ligne horizontale soit égal à l'unité : on demande le terme général de cette table :

1	1	1	1	1	etc.
1	2	3	4	5	etc.
2	5	9	14	20	etc.
5	14	28	48	75	etc.
14	42	90	165	275	etc.
etc.	etc.	etc.	etc.	etc.	etc.

L'équation de relation est $A_{m,n} = A_{m-1,n} + A_{m+1,n-1}$; j'y mets $m-1$ au lieu de m, et elle devient

DU CALCUL
DES
DÉRIVATIONS;

PAR L. F. A. ARBOGAST,

De l'Institut national de France, Professeur de Mathématiques à Strasbourg.

A STRASBOURG,

DE L'IMPRIMERIE DE LEVRAULT, FRÈRES.

AN VIII (1800).

Donc on a enfin $$A_{m,n} = \qquad \ldots\ldots (3)$$

$$\pm \beta^{l-m}\{A_{0,n}.\gamma'^{m} + A_{0,n+1}.m\gamma'^{m-1}\beta + A_{0,n-2}.\frac{m.m-1}{1.\ 2}\gamma'^{m-2}\beta^{2} + \text{etc.} + A_{0,n+m-1}.m\gamma'\beta^{m-1} + A_{0,n+m}.\beta^{m}\},$$

le signe supérieur ou inférieur ayant lieu suivant que m est pair ou impair. Ce résultat s'accorde avec celui que l'on peut déduire d'une solution différente du même exemple, donnée par LAPLACE dans les mémoires de Paris, année 1779, n.° XVII, page 267.

Si l'on fait n négatif dans la formule (1) ci-dessus, on trouve, en rejetant les termes et celles de leurs parties où les indices de D sont négatifs et ceux de D' négatifs ou positifs > 0, que cette formule se réduit à la suivante :

$$A_{m,-n} = \qquad \ldots\ldots (4)$$

$$\pm \beta^{l}\{A_{0,0}\mathrm{D}_{\beta}^{m}.(\alpha^{n}.\beta^{l-n-1}) - A_{0,1}\mathrm{D}_{\beta}^{m}.(\alpha^{n+1}.\beta^{l-n-2}) + A_{0,2}\mathrm{D}_{\beta}^{m}.(\alpha^{n+2}.\beta^{l-n-3}) - \text{etc.}\}$$

laquelle, à cause que $\alpha = 0$ et que sa seule dérivée D est β, devient

$$A_{m,-n} = \qquad \ldots\ldots (5)$$

$$\pm \beta^{l}\{A_{0,0}.\beta^{n}\mathrm{D}_{\beta}^{m-n}.\beta^{l-n-1} - A_{0,1}.\beta^{n+1}\mathrm{D}_{\beta}^{m-n-1}.\beta^{l-n-2} + A_{0,2}.\beta^{n+2}\mathrm{D}_{\beta}^{m-n-2}.\beta^{l-n-3} - \text{etc.}\}.$$

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etc.	etc.	etc.	etc.	etc.	etc.

L'équation de relation est $A_{m,n} = A_{m-1,n} + A_{m+1,n-1}$; j'y mets $m - 1$ au lieu de m, et elle devient

bijective proofs

for Catalan numbers

$$C_n = \frac{1}{(2n+1)} \binom{2n+1}{n}$$

$$2(2n+1)C_n = (n+2)C_{n+1}$$

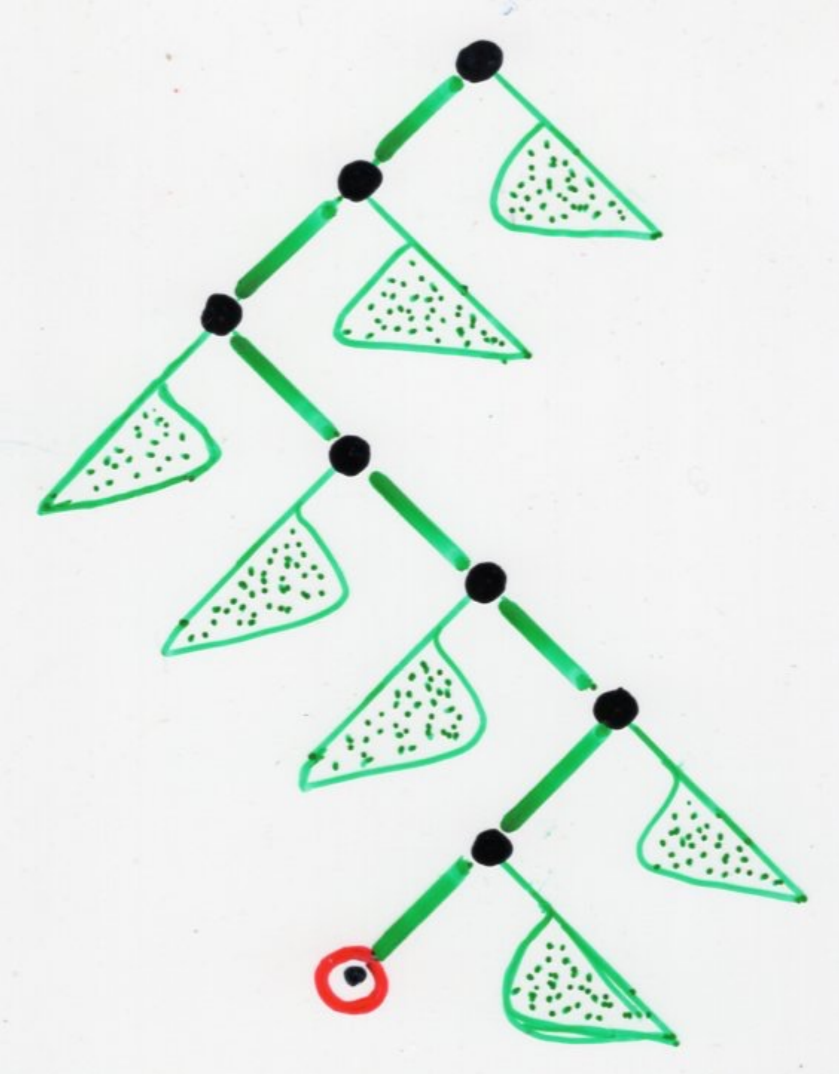
Catalan number

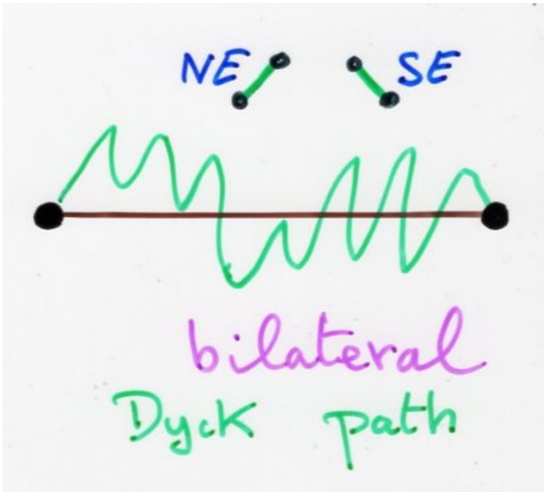
$$C_n = \frac{1}{(n+1)} \binom{2n}{n}$$

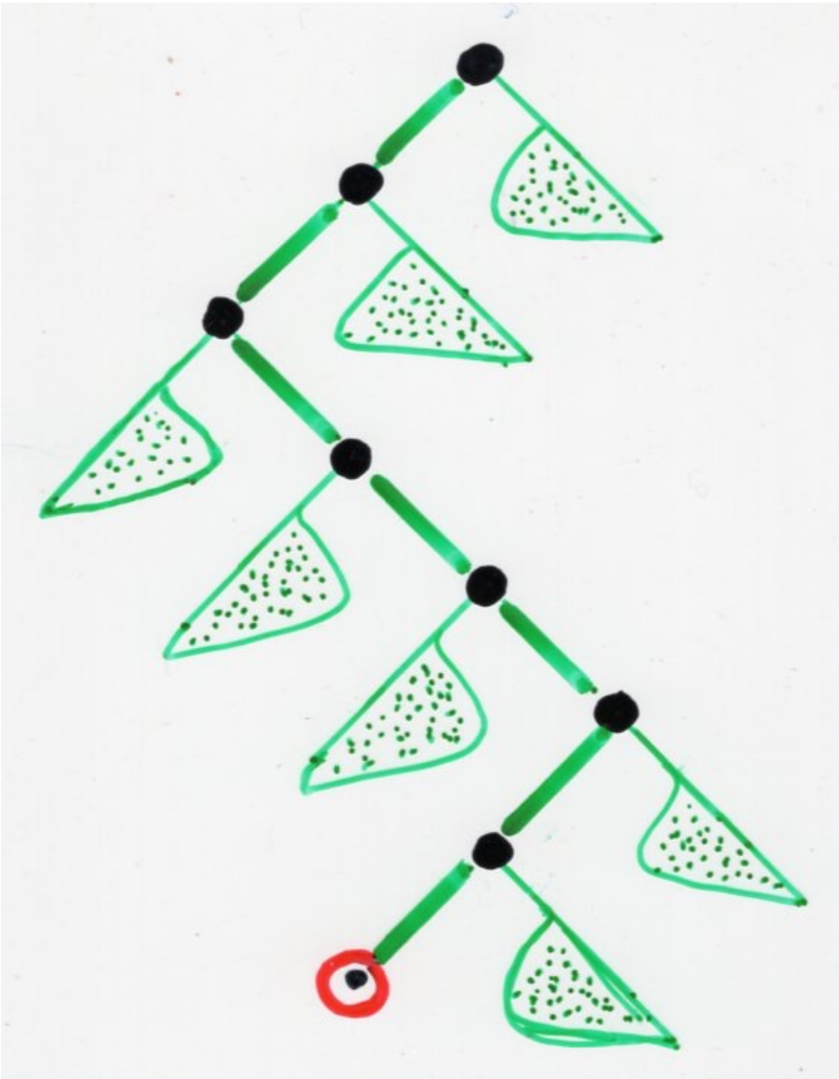
$$(n+1)C_n = \binom{2n}{n}$$

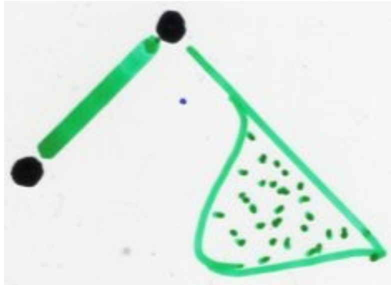
$$(n+1)C_n$$

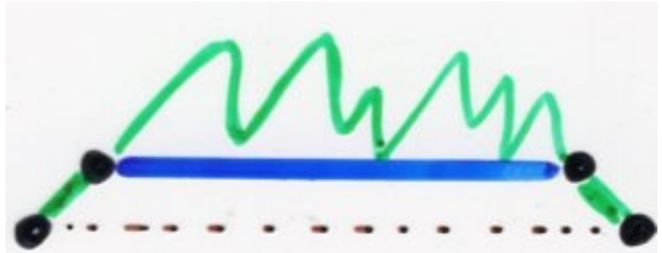
$$\binom{2n}{n}$$



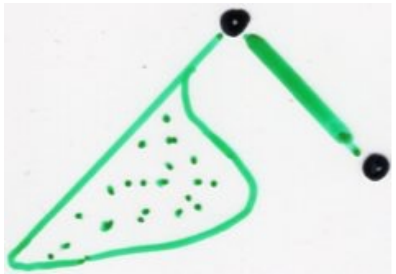


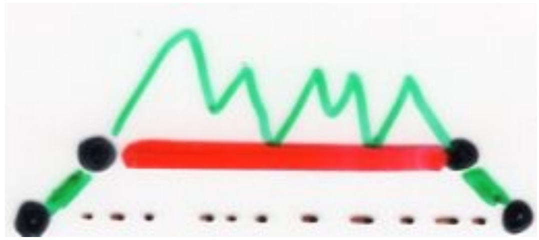


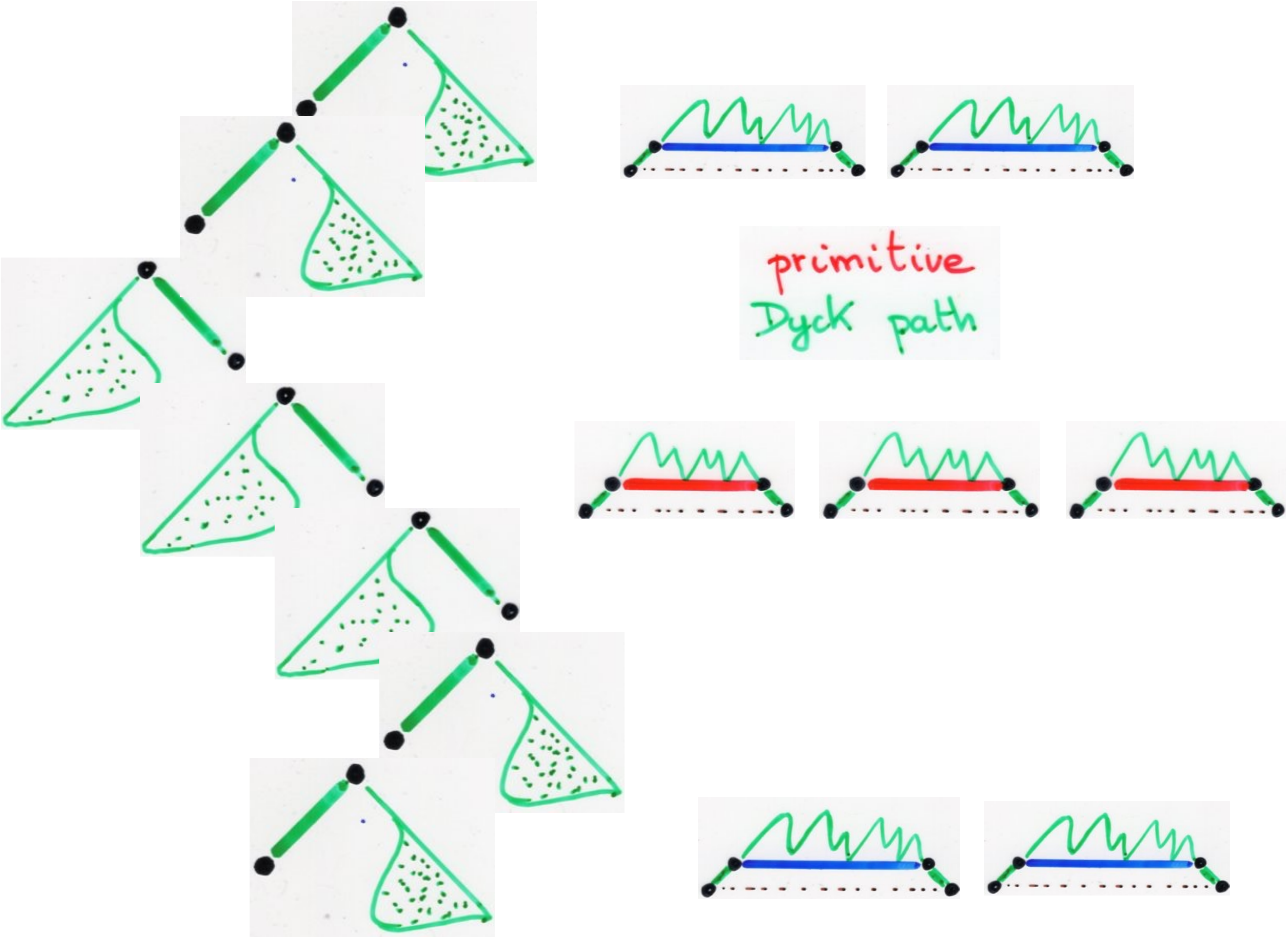


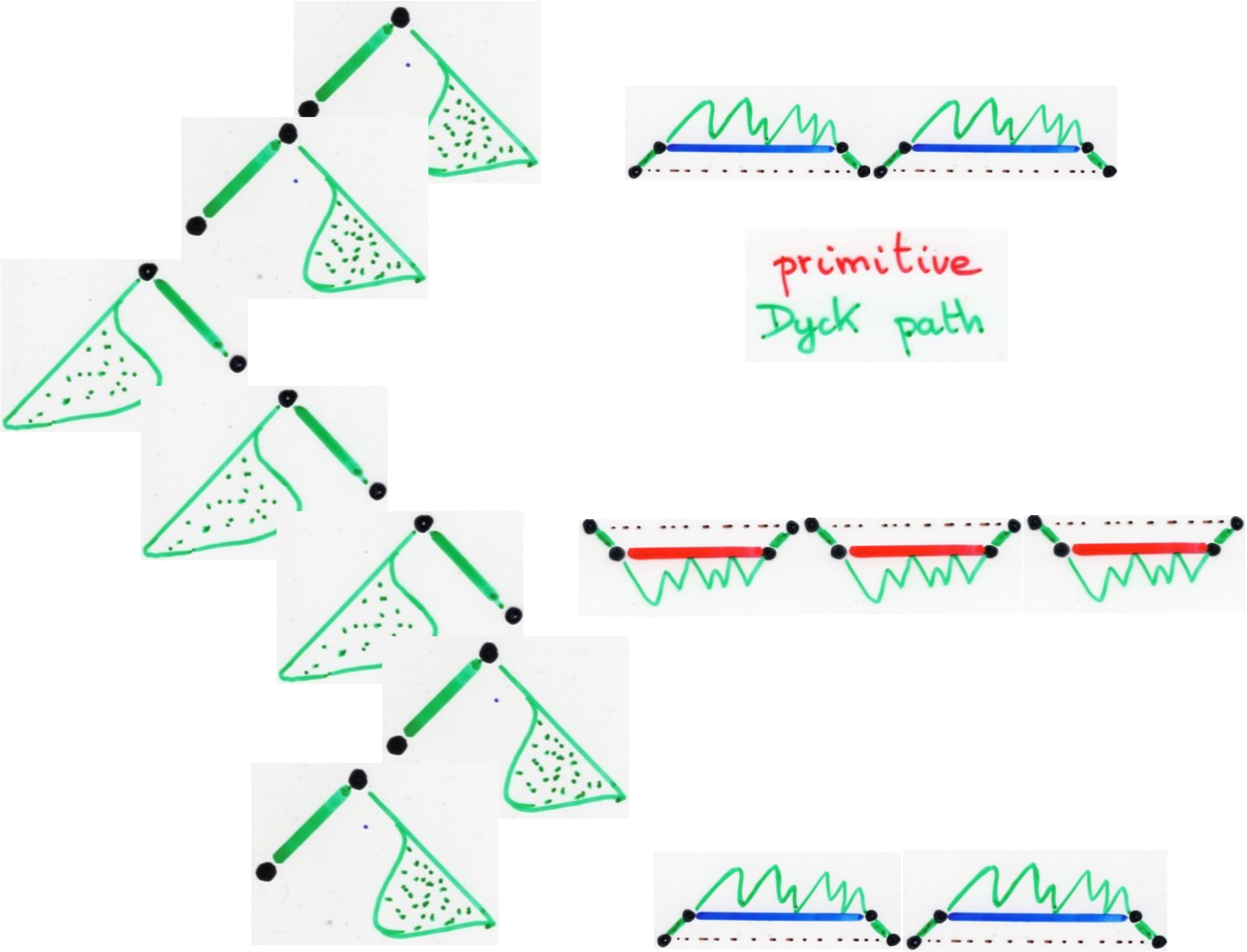


primitive
Dyck path

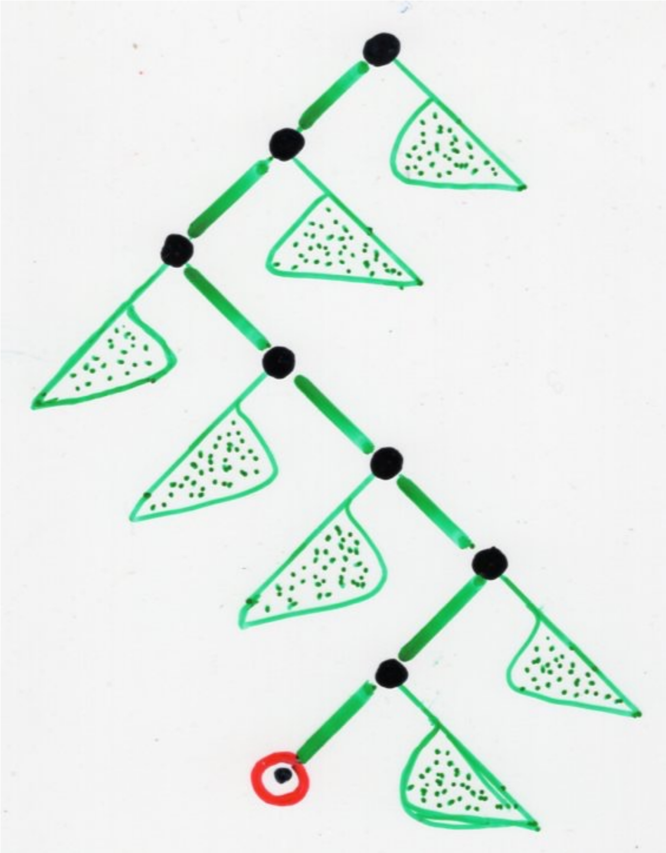


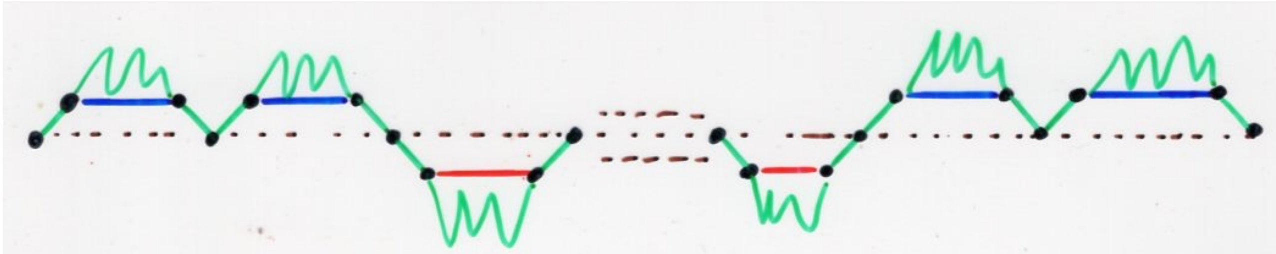



primitive
Dyck path


primitive
Dyck path

$$(n+1)\,C_n = \binom{2n}{n}$$





Lagrange inversion formula

and the cyclic lemma

Lagrange inversion formula

$g(t) = t$

$\varphi(t) = \sum_{n\geqslant 0} a_n t^n$ $\quad a_0 \neq 0$

y unique solution
(with $y(0)=0$)

$y = t\,\varphi(y)$

$$[t^n]\, y = \frac{1}{n} [t^{n-1}] \left(\varphi(t)^n \right)$$

Lukasiewicz path

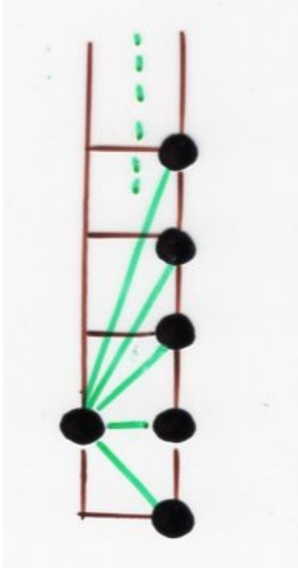
$\omega = (s_0, \dots, s_n)$

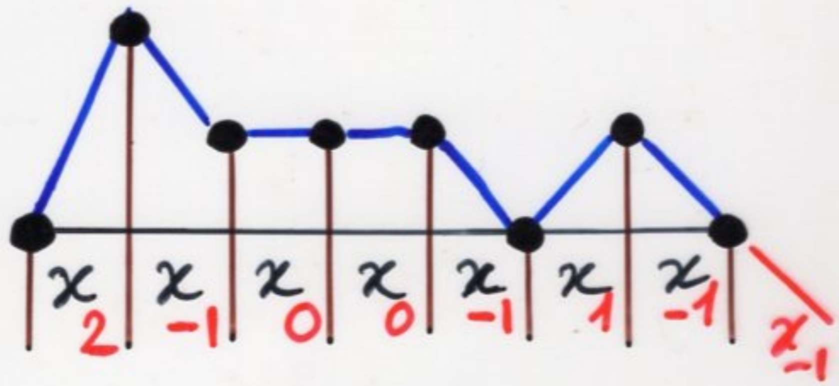
$s_0 = (0,0)\ ,\quad s_n = (n,0)$

elementary step $s_i = (x_i, y_i)\quad s_{i+1} = (x_{i+1}, y_{i+1})$

with

$x_{i+1} = 1 + x_i \qquad y_{i+1} \geq y_i - 1$





Lukasiewicz language $L \subseteq X^*$

$$X = \{ x_{-1}, x_0, \dots, x_p, \dots$$

$$\delta : X^* \longrightarrow \mathbb{Z}$$ monoid morphism

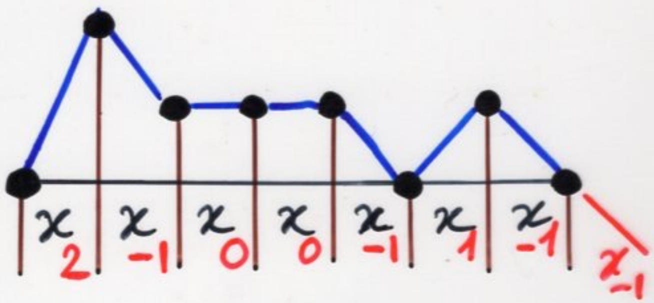
$$\delta(x_i) = i \qquad \delta(uv) = \delta(u) + \delta(v)$$

$w \in L$

iff (i) $\delta(w) = -1$

(ii) $\delta(u) \geqslant 0$ for every u left factor of w

(i.e. $w = uv$, $u, v \in X^*$)



$h : X^* \to \mathbb{K}[t]$ monoid morphism

$h(x_{n-1}) = a_n t , \quad n \geq 0$

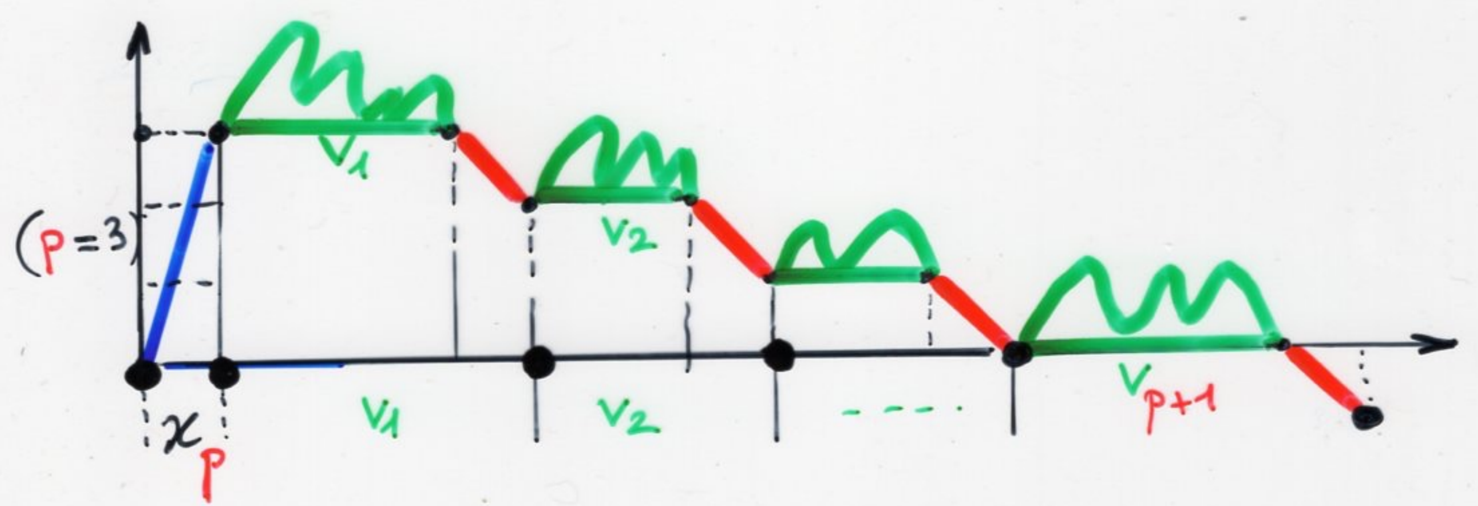
<u>Lemma</u> The unique solution of $y = t\varphi(y)$ is

$$y = \sum_{w \in L} h(w)$$

for $p \geq -1$, $L_p = \{ w \in L,\ w = x_p v \}$
(first letter of w is x_p)

every word $w \in L_p$ has a unique factorization

$$w = x_p v_1 \dots v_{p+1}, \quad v_i \in L, \quad 1 \leq i \leq p+1$$



$$\sum_{w \in L} h(w) = \sum_{p \geq -1} \sum_{w \in L_p} h(w)$$

$$\underbrace{a_{p+1} t\, h(v_1) \cdots h(v_{p+1})}$$

$$= \sum_{n \geq 0} a_n t \Big(\sum_{w \in L} h(w) \Big)^n$$

$$y = \sum_{n \geq 0} a_n t y^n$$

$$y = t \varphi(y)$$

The cyclic Lemma (Raney)

(for Łukasiewicz words)
$p = 1$

<u>Lemma</u> every word $w \in X^*$ with $\delta(w) = -1$
has a unique factorization

$w = uv$ with $vu \in L$

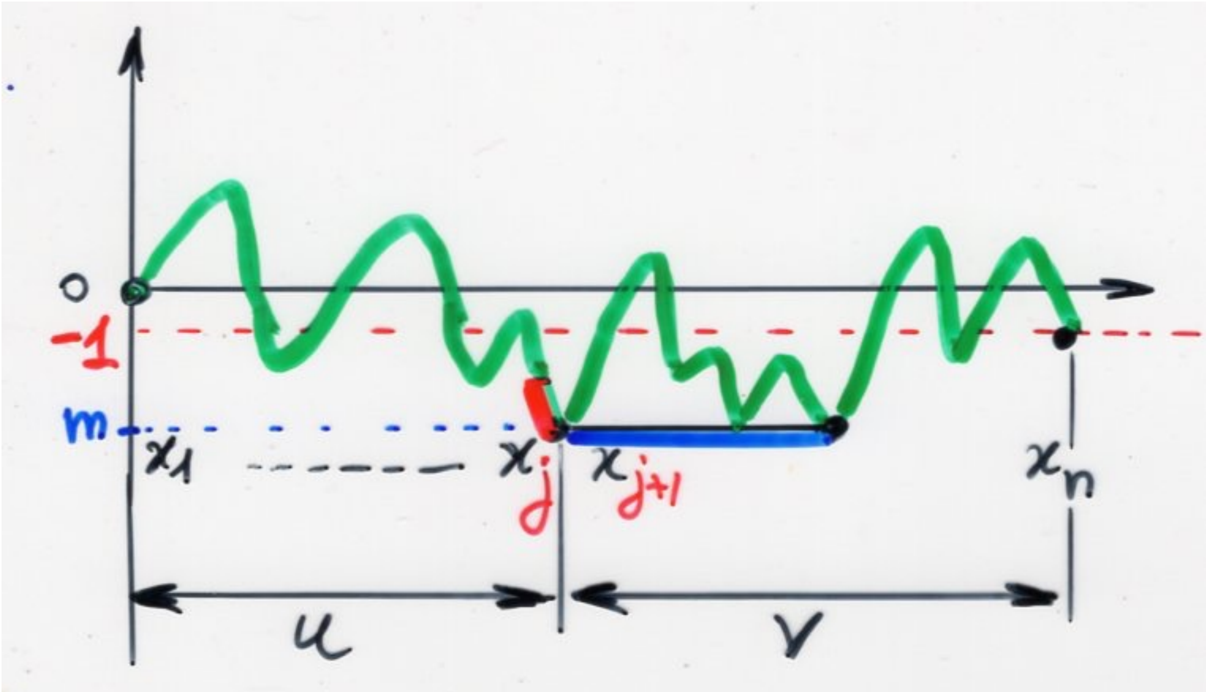
unique conjugate in L

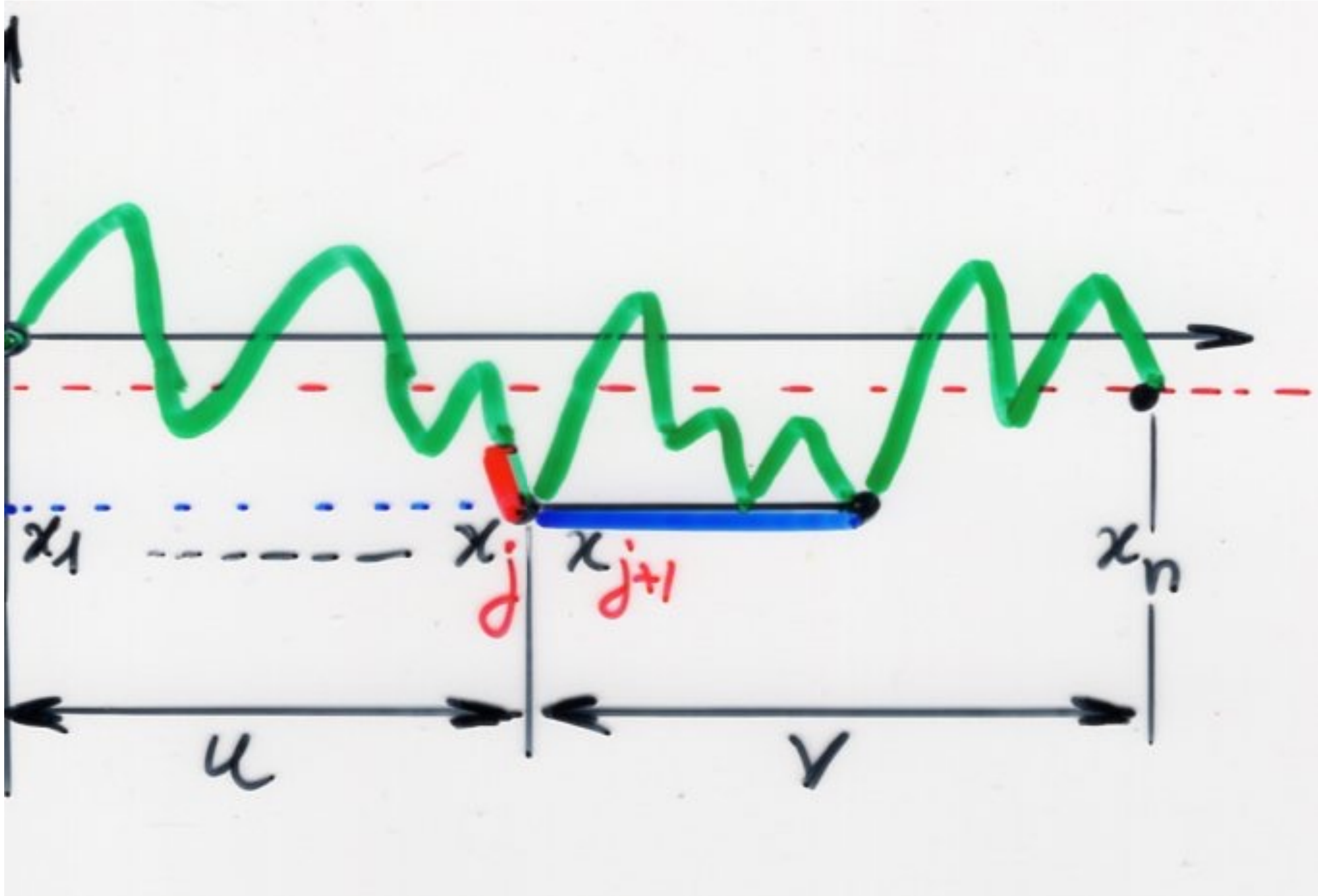
$w = x_1 \ldots x_n \;,\quad x_i \in X$

$$m = \min_{1 \le i \le n} \big(\delta(x_1 \ldots x_i) \big)$$

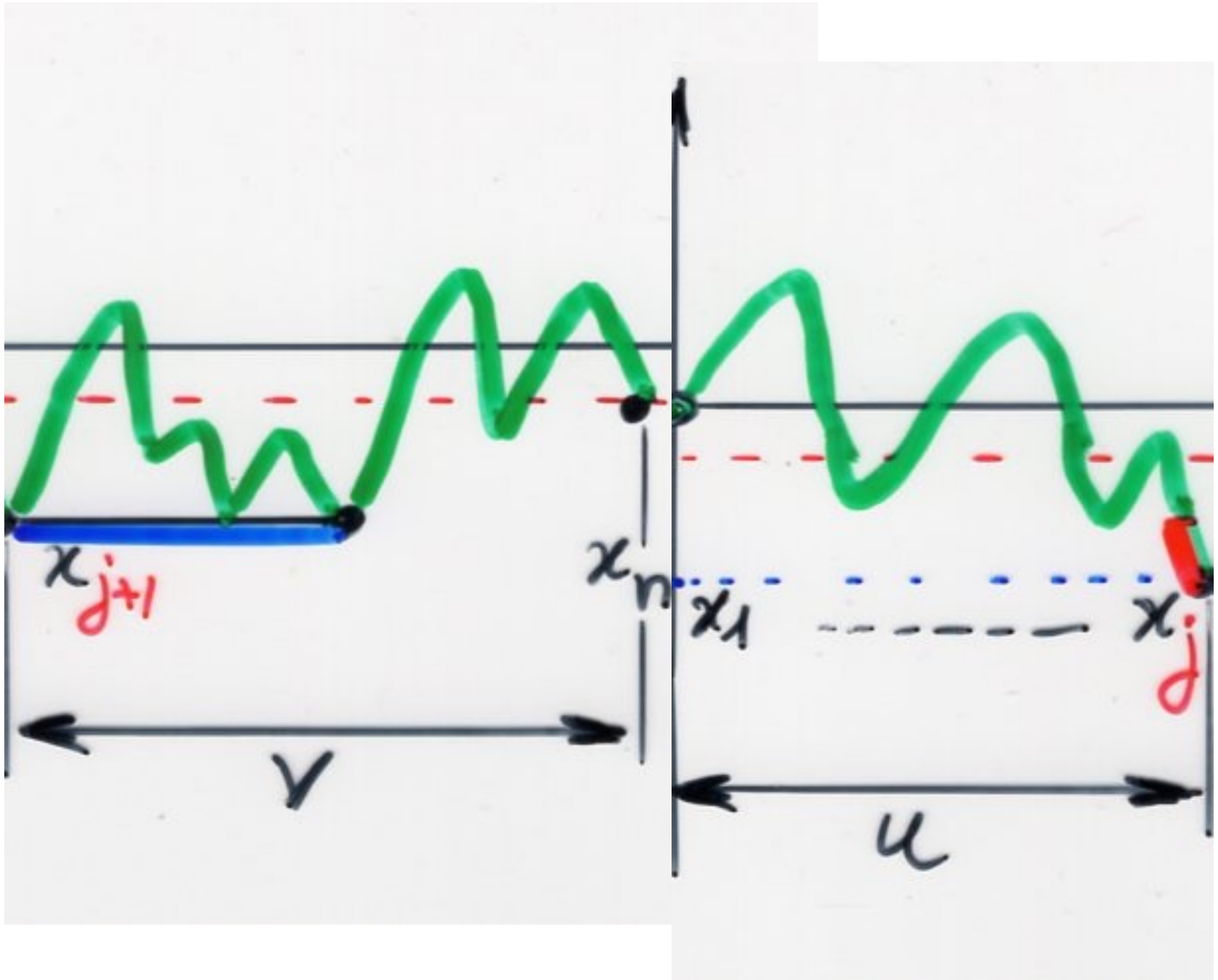
j smalest integer $1 \le j \le n$, $\delta(x_1 \ldots x_j) = m$

$w = uv$ with $u = x_1 \ldots x_j$, $v = x_{j+1} \ldots x_n$





x1
xj
j
xj+1
j+1
xn
u
v



x_{j+1}
x_n
x_1
x_j
v
u

$h_0 : X^* \to \mathbb{K}$

$h_0(x_{n-1}) = a_n$

$h(w) = h_0(w)\, t^{|w|}$

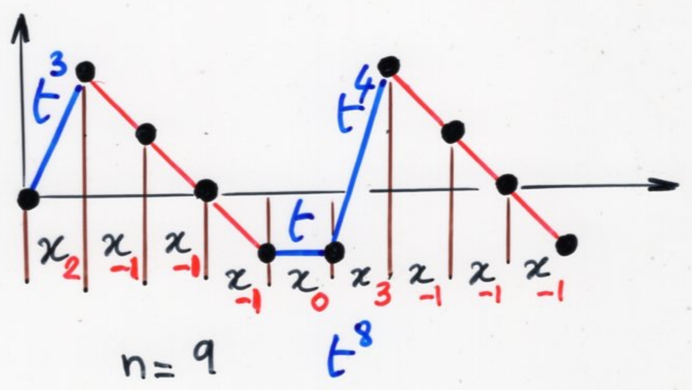
$h(x_{n-1}) = a_n t \,, \quad n \geq 0$

$$[t^n]\, y = \sum_{\substack{w \in L \\ |w|=n}} h_0(w)$$

(Lemma 1)

$$n \Bigg(\sum_{\substack{w \in L \\ |w|=n}} h_0(w) \Bigg) = \sum_{\substack{w \in \delta^{-1}(-1) \\ |w|=n}} h_0(w)$$

(cyclic Lemma)

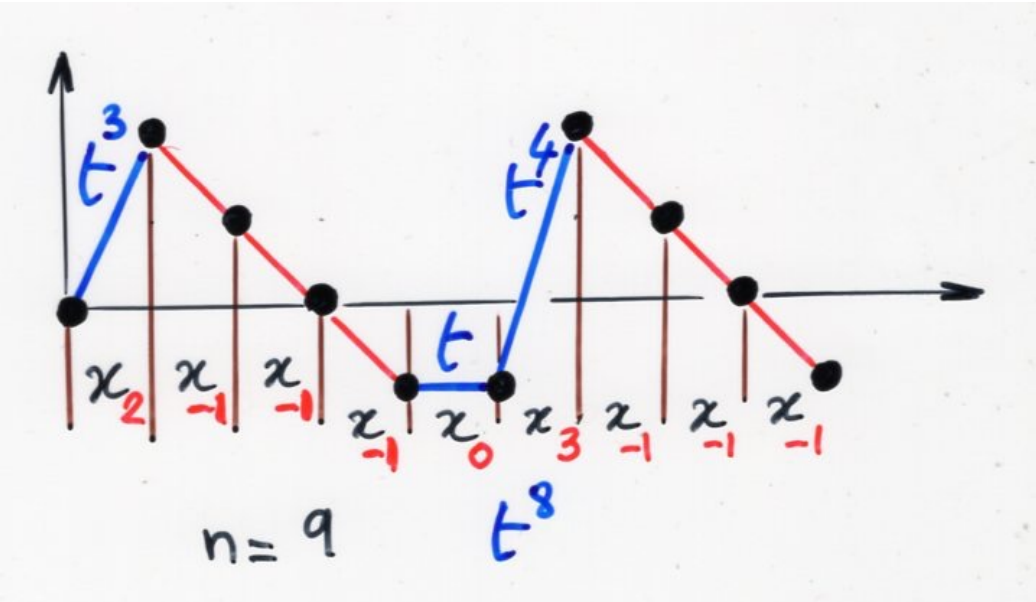


$\uparrow$

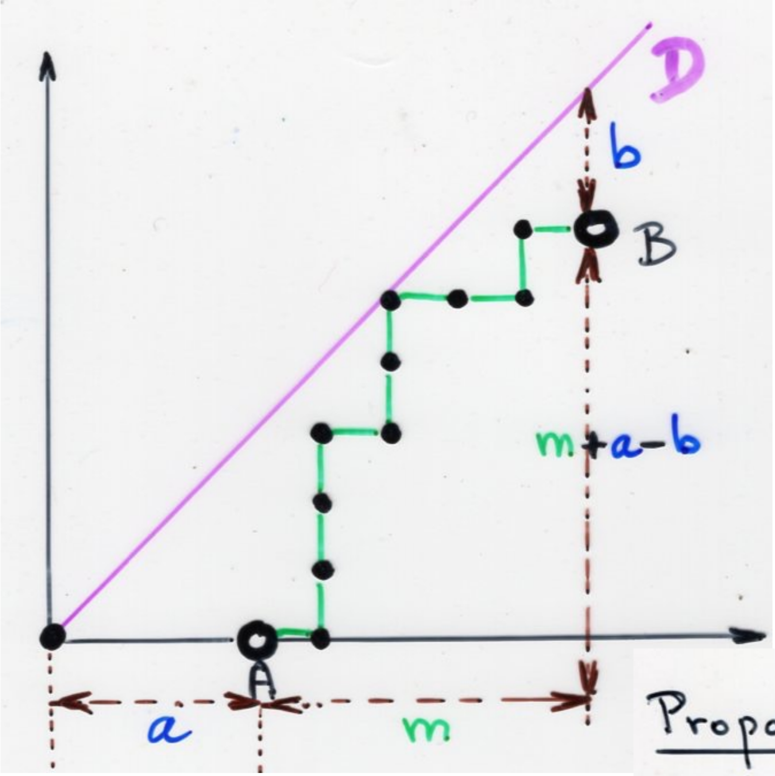
$$[t^{n-1}] \big(\varphi(t) \big)^n$$

$$n\left(\sum_{\substack{w\in L\\ |w|=n}} h_0(w)\right) = \sum_{\substack{w\in \delta^{-1}(-1)\\ |w|=n}} h_0(w)$$

$$[t^{n-1}]\big(\varphi(t)\big)^n$$



The reflection principle

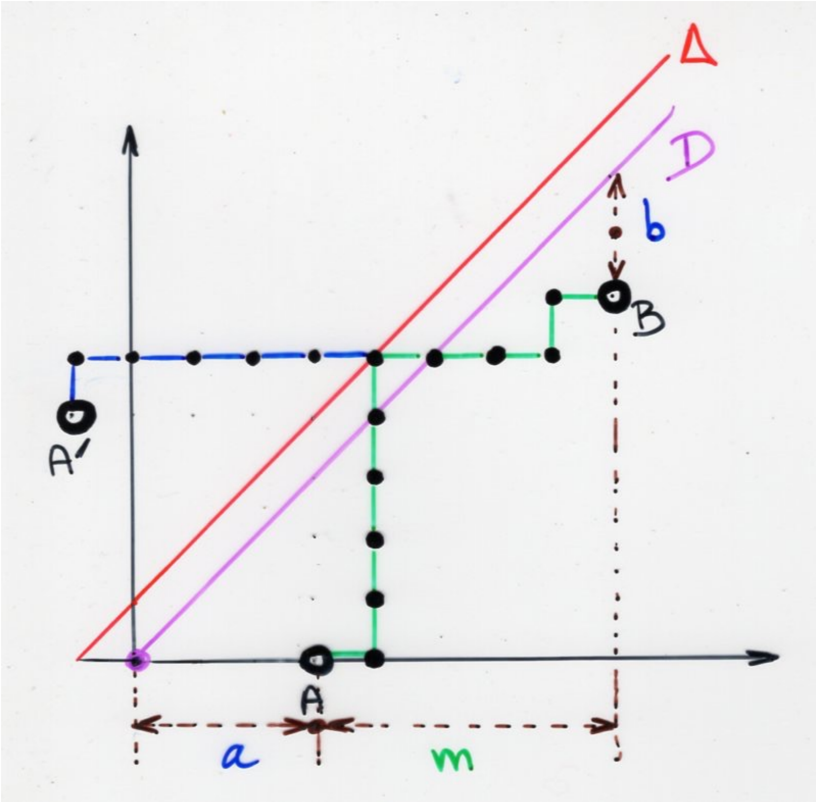


<u>Proposition</u> Let $A=(a,0)$, $B=(a+m, a+m-b)$

$a, b, m, a+m-b \geqslant 0$

$P_a^{\leq D}(A,B)=$ set of paths of $\mathbb{N}\times\mathbb{N}$ going from A to B, under the diagonal D (possibly touching D), with elementary steps North and East. The number of such paths is:

$$\left|P_a^{\leq D}(A,B)\right| = \binom{2m+a-b}{m} - \binom{2m+a-b}{m+a+1}$$



Δ line $y = x + 1$

$A' = (-1,\ a+1)$

symmetric of A / Δ

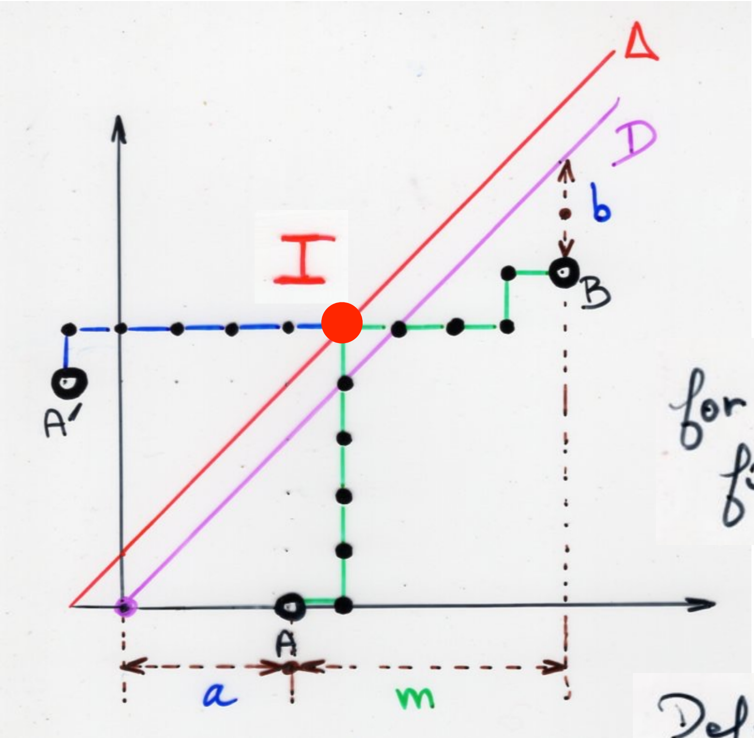
$$\left| Pa(A, B) \right| = \binom{2m + a - b}{m}$$

set of paths, going from A to B, with N, S elementary step

$$\left| Pa(A', B) \right| = \binom{2m + a - b}{m + a + 1}$$

$$\mathcal{P}a(A', B) \underset{\text{bijection}}{\longleftrightarrow} \mathcal{P}a^{\Delta}(A,B) = \mathcal{P}a(A,B) \setminus \mathcal{P}a^{\leqslant D}(A,B)$$

$\uparrow$ set of paths having a non-empty intersection with Δ



for such a path ω, define $I = (i, i+1)$ the first intersection of ω with Δ ($= i$ minimum)

Define ω' the path obtained by reflecting the portion of ω between A and I, and keeping invariant the portion between I and B.

$\omega \longrightarrow \omega'$ is the desired bijection

Proposition let $A = (a, 0)$, $B = (a+m, a+m-b)$

$a, b, m, a+m-b \geqslant 0$

$P_a^{\leq D}(A, B)$ = set of paths of $\mathbb{N} \times \mathbb{N}$ going from A to B, under the diagonal D (possibly touching D), with elementary steps North and East. The number of such paths is:

$$\left| P_a^{\leq D}(A, B) \right| = \binom{2m+a-b}{m} - \binom{2m+a-b}{m+a+1}$$

For $a = b = 0$ $\quad \mathcal{P}_a^{\leq D}(A, B)$ are in bijection with Dyck paths of length $2n$

$$\left|\mathcal{P}_a^{\leq D}(A, B)\right| = \binom{2n}{n} - \binom{2n}{n+1}$$

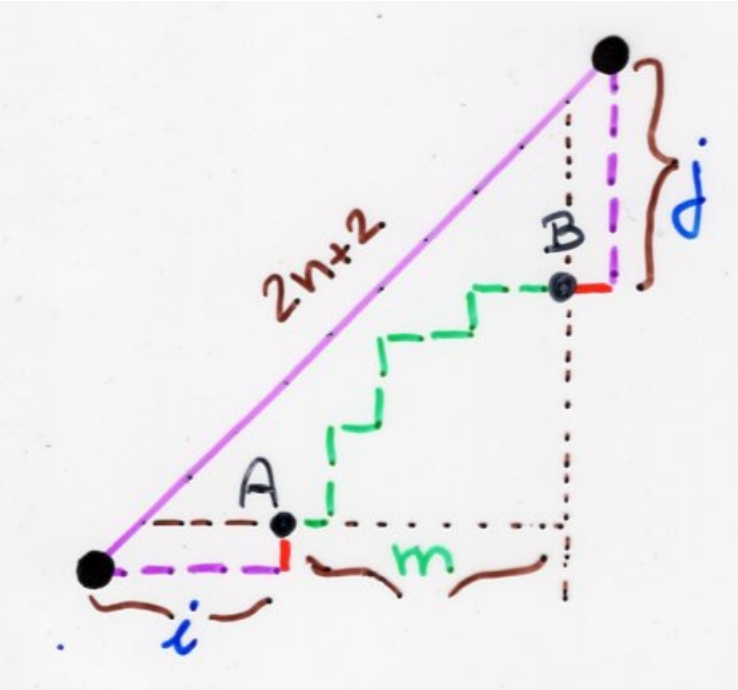
$$= C_n$$

Catalan

Corollary The number of Dyck words of length $2n+2$ having i (resp. j) occurrences of the letter x (resp. $\bar{x}$) at the beginning (resp. end) is

$$\alpha_{n+1,i,j} = \binom{2n-i-j}{n-i} - \binom{2n-i-j}{n} \quad (1 \le i, j \le n)$$

$$\alpha_{n+1,\,n+1,\,n+1} = 1$$



$$a = i-1$$
$$b = j-1$$
$$m = n-i$$

For $j = 1$, we get the (α)-distribution of Catalan numbers:

$$\alpha_{n,i} = \binom{2n-i-1}{n-i} - \binom{2n-i-1}{n}$$

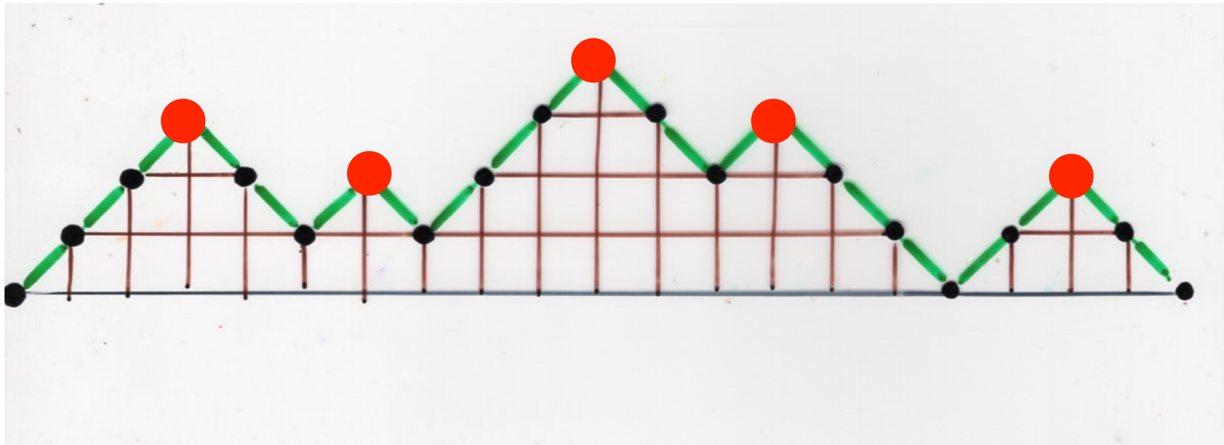
$$= \frac{i}{2n-i}\binom{2n-i}{n}$$

(check!)

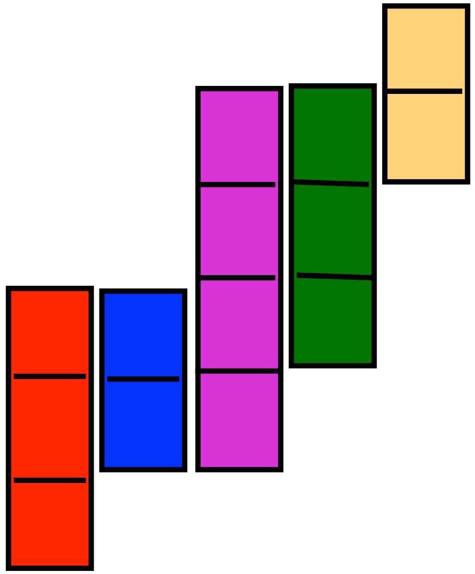
The reflection principle

(again)

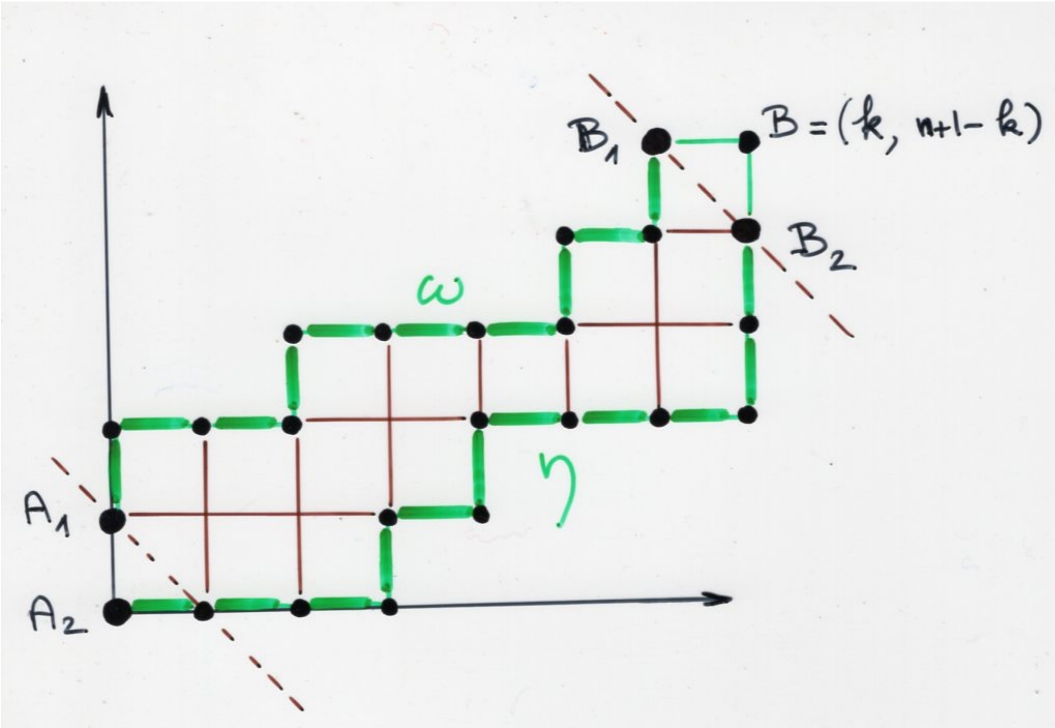
(β)- distribution on Catalan numbers



number of peaks
in Dyck paths



number of columns
in staircase polygons



$A_2 = (0, 0)$ $\quad$ $A_1 = (0, 1)$

$B_2 = (k, n-k)$ $\quad$ $B_1 = (k-1, n+1-k)$

$a_{ij} = \left| \mathrm{Pa}(A_i, B_j) \right| \qquad 1 \leq i, j \leq 2$

number of paths $A_i \rightsquigarrow B_j$
with elementary N, E steps

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \binom{n-1}{k-1} & \binom{n-1}{k} \\ \binom{n}{k-1} & \binom{n}{k} \end{bmatrix}$$

<u>Proposition</u> The number of pair of paths (ω, η) such that:

(i) • ω, η : $(0,0) \rightsquigarrow B$, elementary steps N, S

(ii) • non-intersecting (except in $(0,0)$ and B)

is the determinant : $\det(A)$

and $\det(A) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$

Narayana numbers

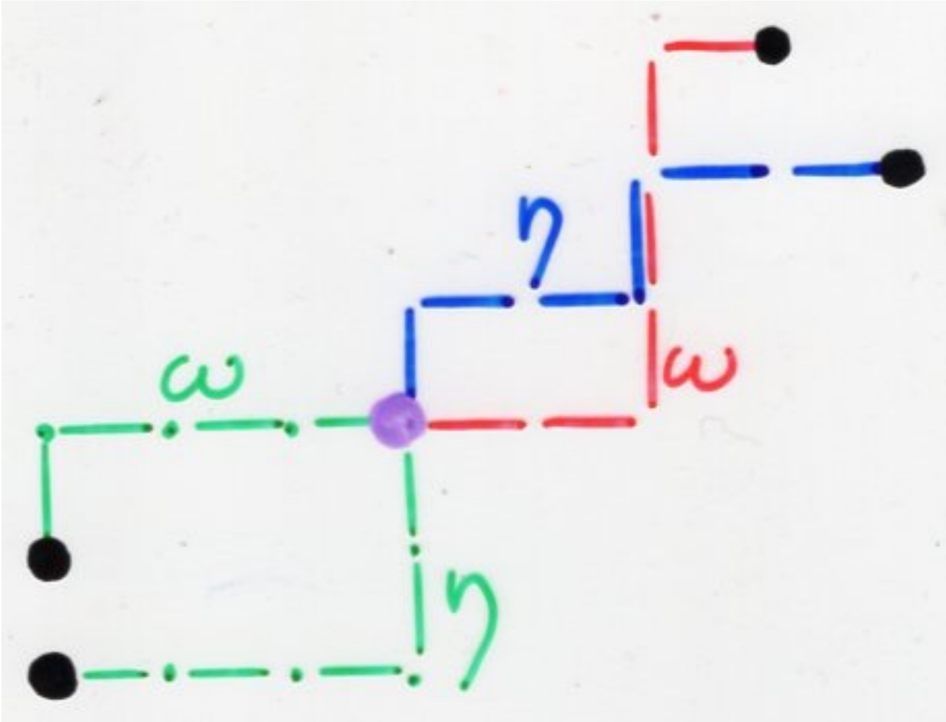
$a_{11} a_{22}$ = number of pairs (ω, η)
$\omega \in \mathcal{P}_a(A_1, B_1), \eta \in \mathcal{P}_a(A_2, B_2)$

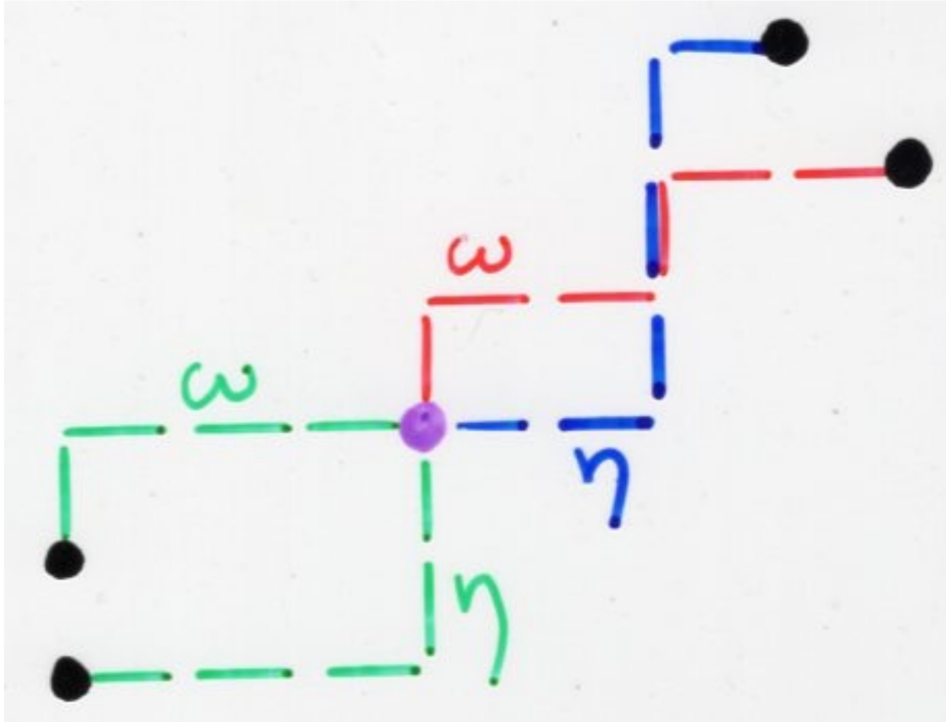
$a_{21} a_{12}$ = number of pairs (ω, η)
$\omega \in \mathcal{P}_a(A_2, B_1), \eta \in \mathcal{P}_a(A_1, B_2)$

$\mathcal{P}_a(A_2, B_1) \times \mathcal{P}_a(A_1, B_2)$

$\updownarrow$ bijection

$(\omega, \eta) \in \mathcal{P}_a(A_1, B_1) \times \mathcal{P}_a(A_2, B_2)$
intersecting





$$\mathcal{P}_a(A_2, B_1) \times \mathcal{P}_a(A_1, B_2)$$

$\updownarrow$ bijection

$$(\omega, \eta) \in \mathcal{P}_a(A_1, B_1) \times \mathcal{P}_a(A_2, B_2)$$

intersecting

3 distributions

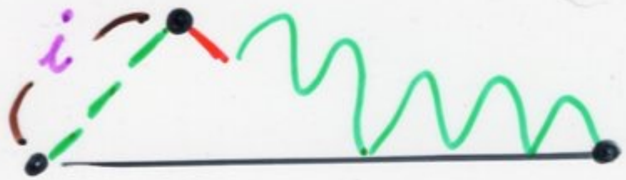
in the Catalan garden

(α)- distribution

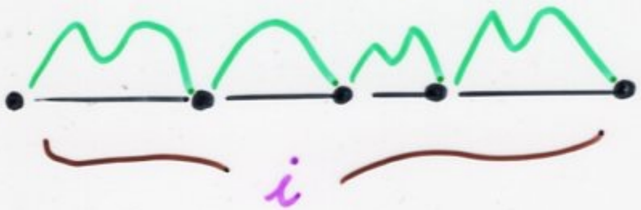
$$\frac{i}{2n-i}\binom{2n-i}{n}$$

(α)- distribution

$$\frac{i}{2n-i}\binom{2n-i}{n}$$

- 

Dyck paths

- 

- 

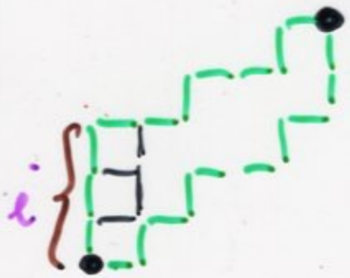
binary trees

- 

planar trees

(α)- distribution

$$\frac{i}{2n-i}\binom{2n-i}{n}$$

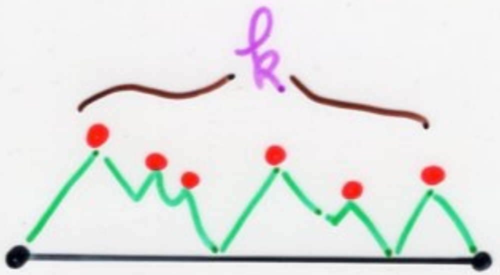
- 

staircase polygons

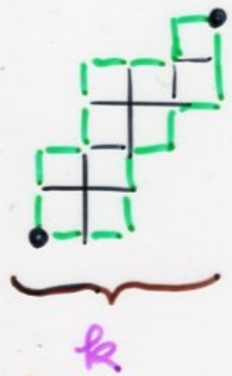
- $\left\{\begin{array}{l}\text{number of blocks} \\ \text{in non-crossing} \\ \text{partitions}\end{array}\right.$

(β)- distribution $\quad \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$

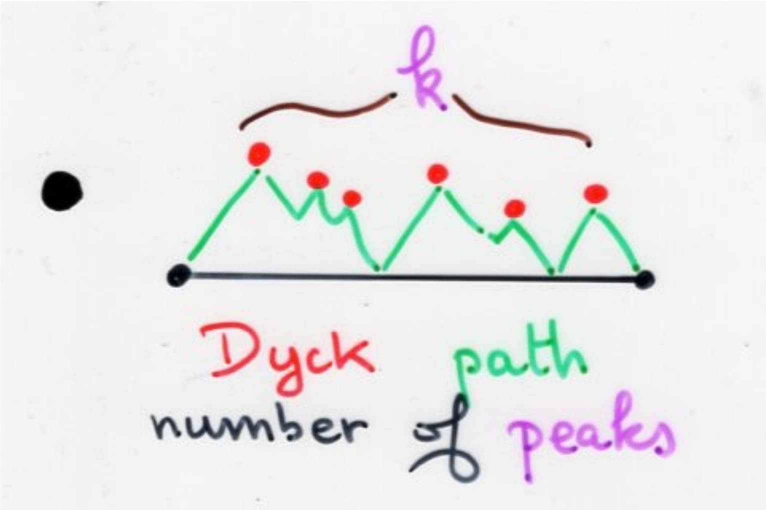
(β)- distribution $\quad \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$

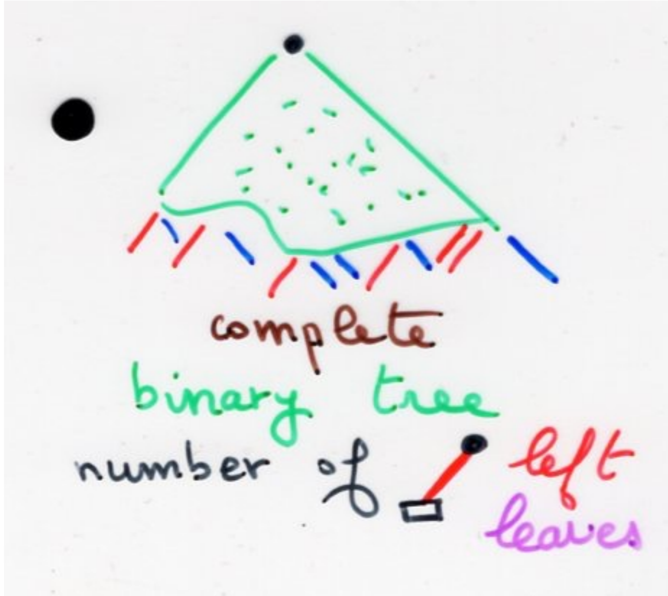
- 

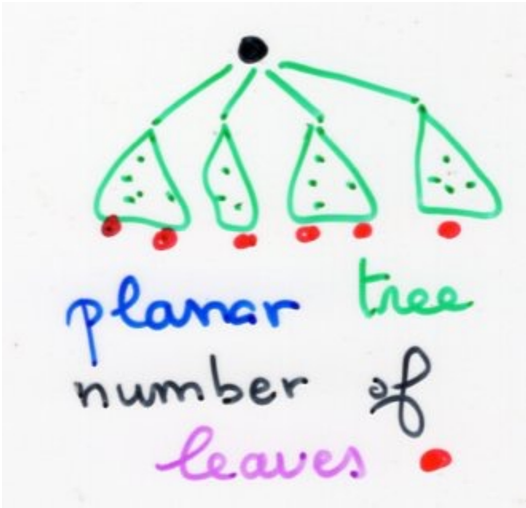
Dyck path
number of peaks

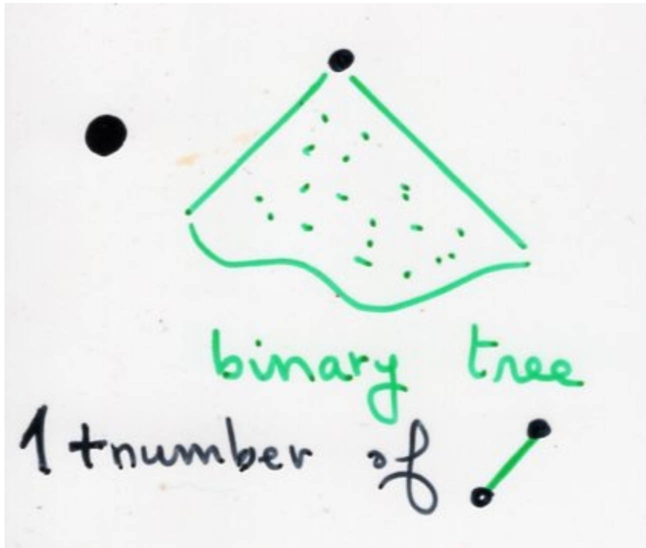
- 

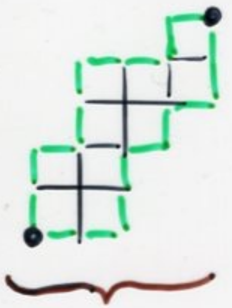
staircase polygons
number of columns


k
Dyck path
number of peaks


complete
binary tree
number of left
leaves

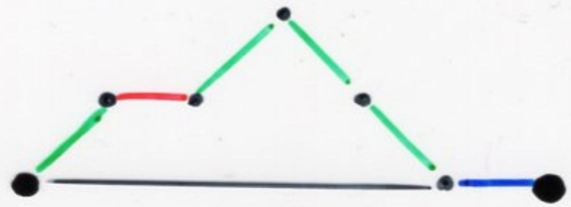

planar tree
number of
leaves


binary tree
1 + number of



- k staircase polygons
  number of columns

- Dyck path
  number of SE steps
  from odd to even level



- number of / or —
  in a 2-colored Motzkin path

(γ)- distribution

$$\frac{F_k(t)}{F_{k+1}(t)}$$

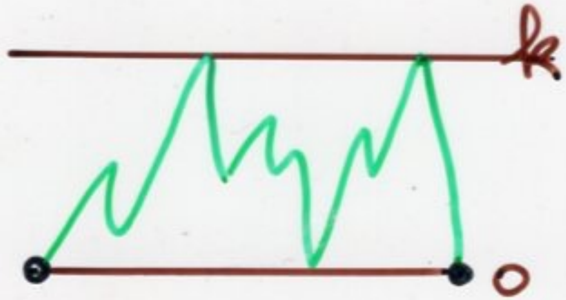
$(\ln \gamma)$- distribution

= Strahler distribution

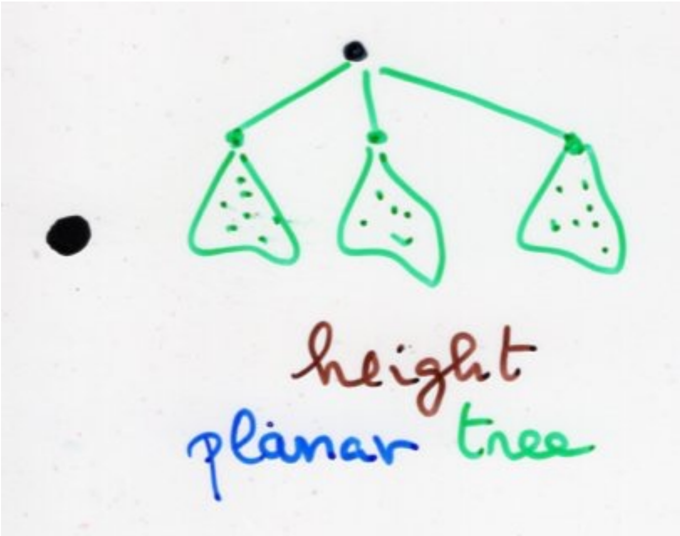
$$= \frac{t^{(2^{k-1}-1)}}{F_{2^k-1}(t)}$$

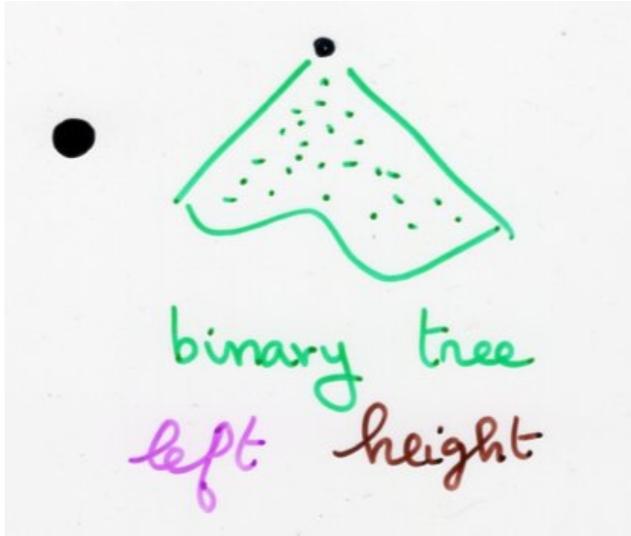
(γ)- distribution

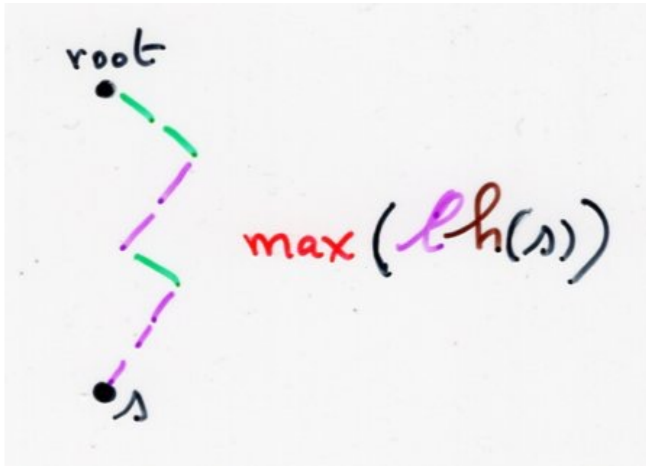
$$\frac{F_k(t)}{F_{k+1}(t)}$$

- 

Dyck path
height

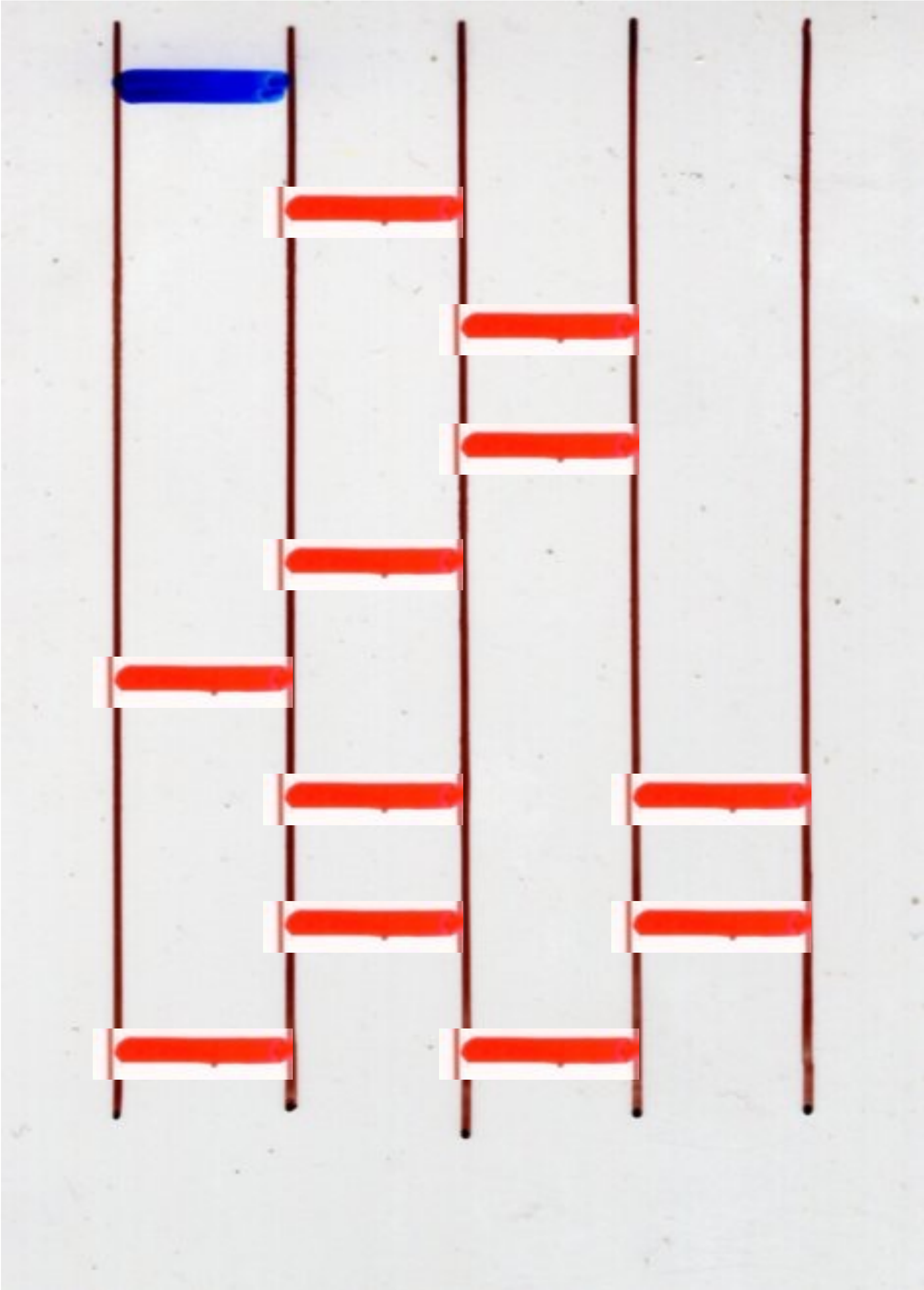
- 
height
planar tree

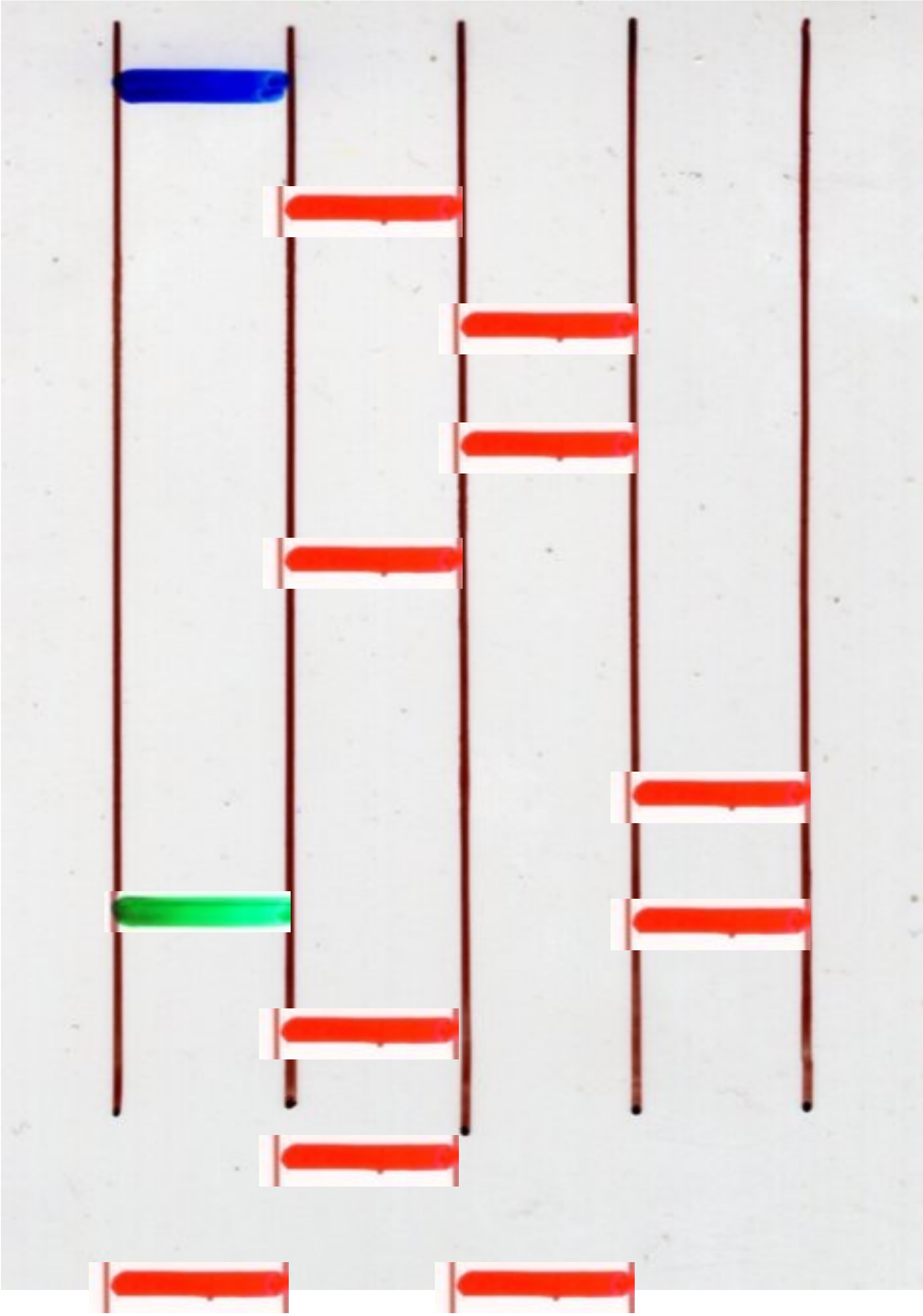
- 
binary tree
left height

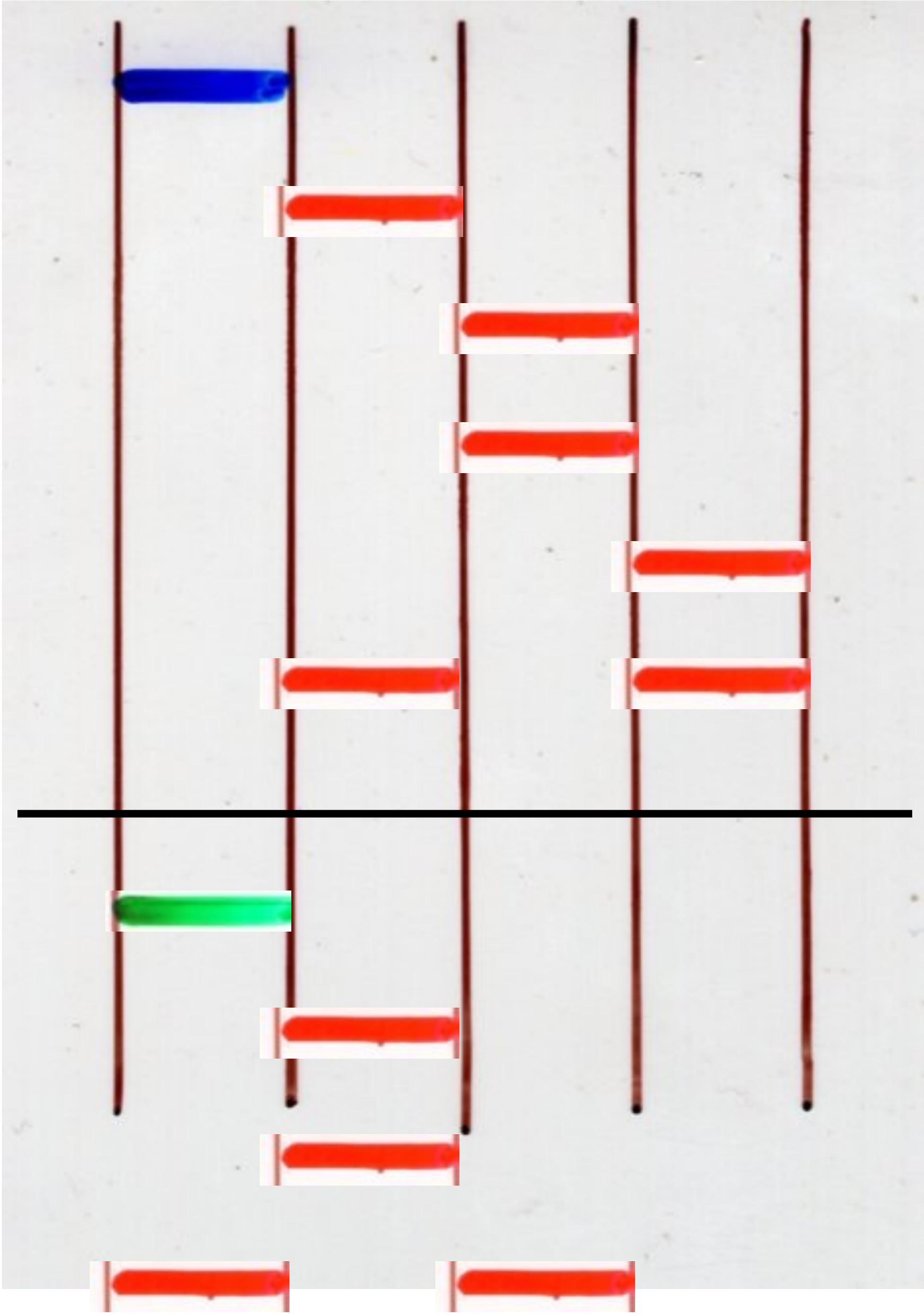

root
max(lh(s))
s

solution of exercise:

semi-pyramids of dimers







$$y = 1 + t\,y^2$$

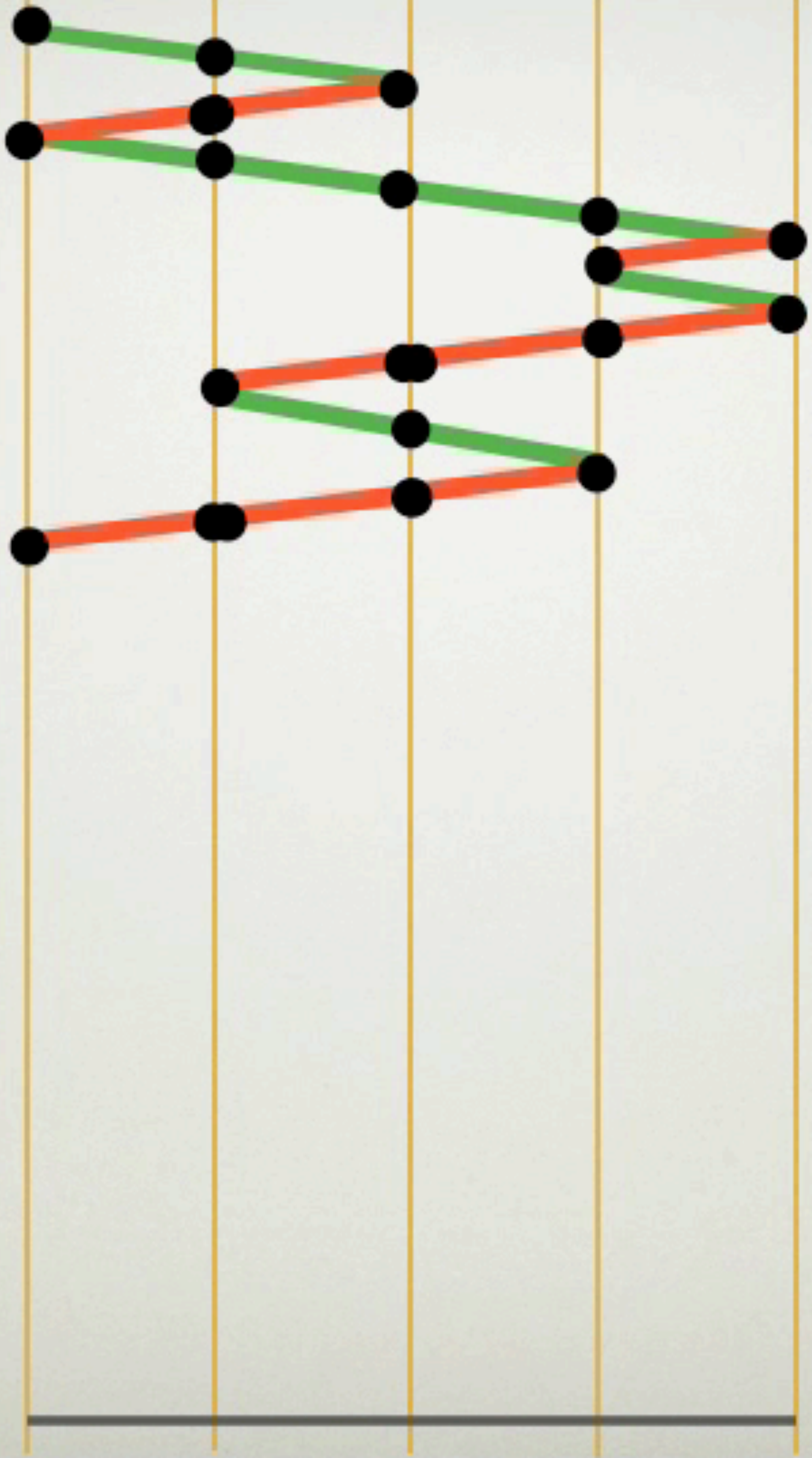
$$a_n = C_n$$

from Dyck paths

to

semi-pyramids of dimers

(video)



violin: Gérard Duchamp