An introduction to

enumerative
algebraic combinatorics
bijective

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IMSc
January-March 2016

Chapter 2

The Catalan garden

(2)

IMSc

28 January 2016

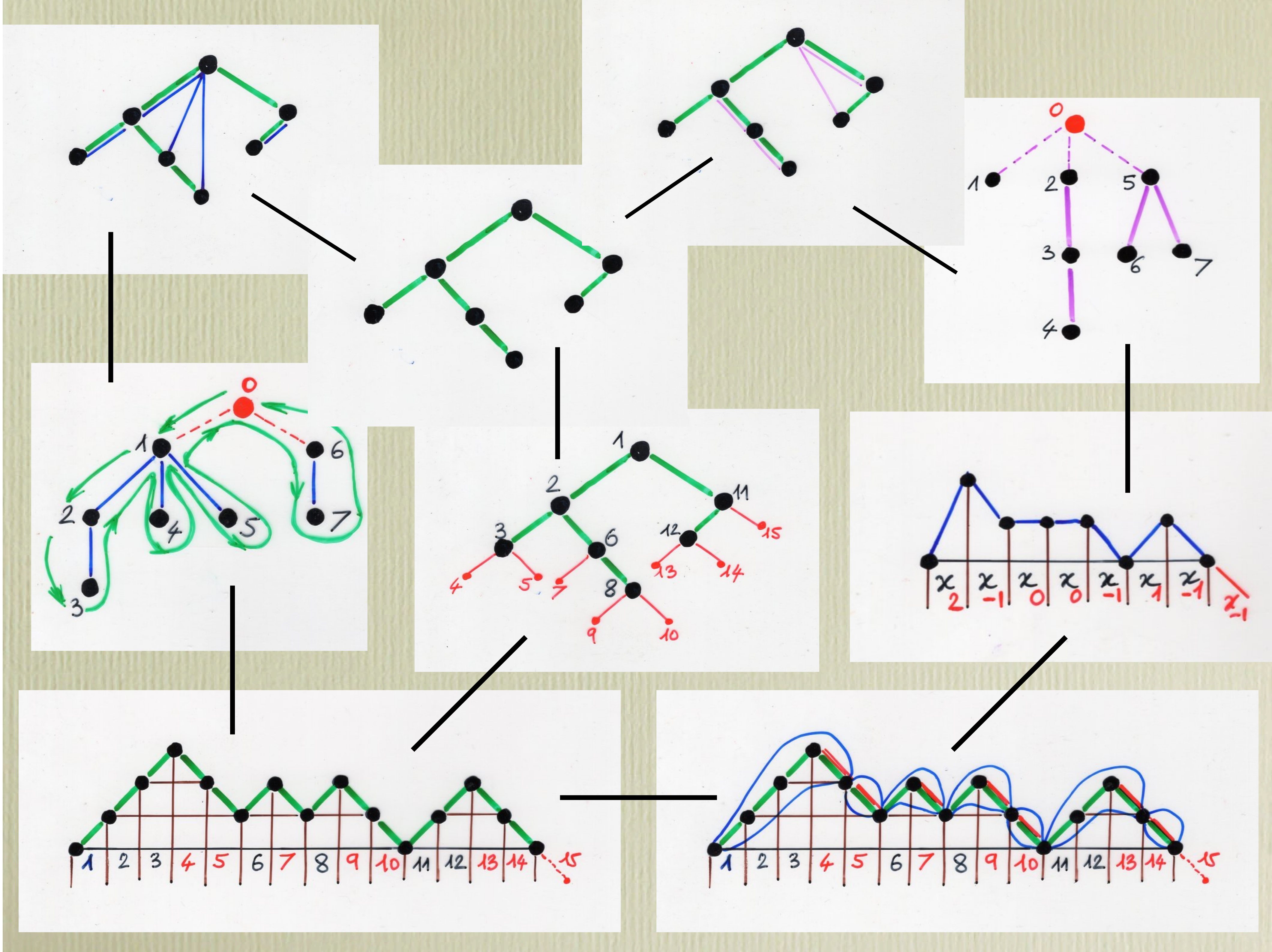
- binary trees
- complete binary trees
- planar trees

- Dyck paths
- 2-colored Motzkin paths
- Łukasiewicz paths

- triangulations (of a convex polygon)
  L. Euler

- Staircase polygons

- non-crossing partitions



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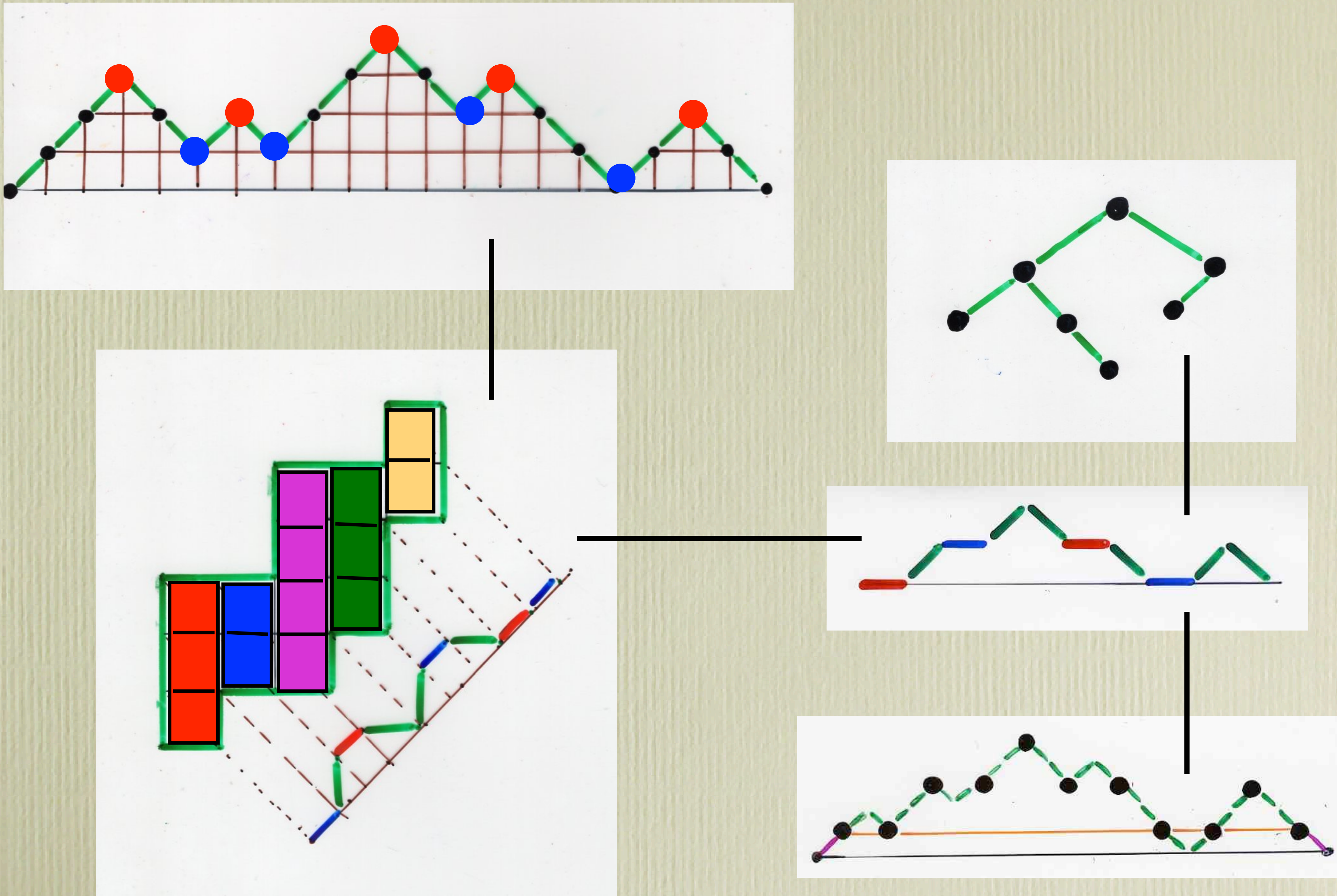


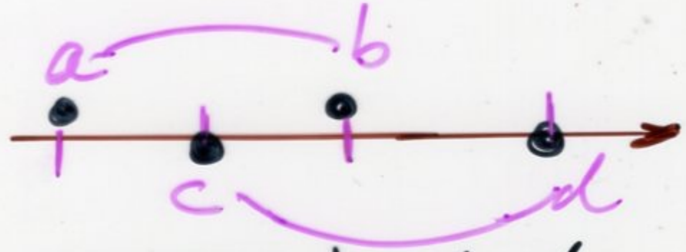
non-crossing partitions

Definition

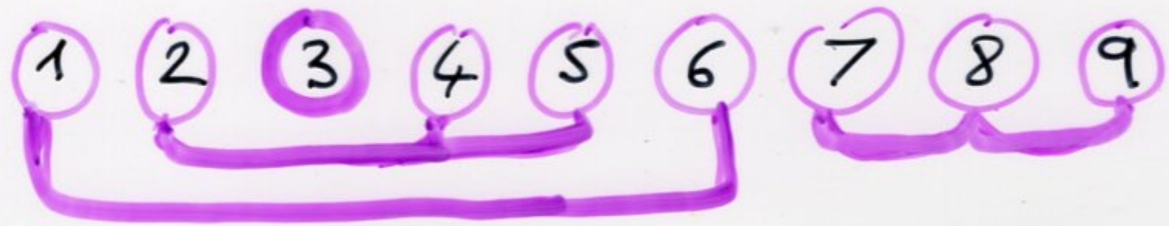
non-crossing partition of $\{1, 2, n\}$

$\{B_1, \ldots, B_k\}$
k blocks

if $a, b \in B_i$
$c, d \in B_j$ $(i \neq j)$



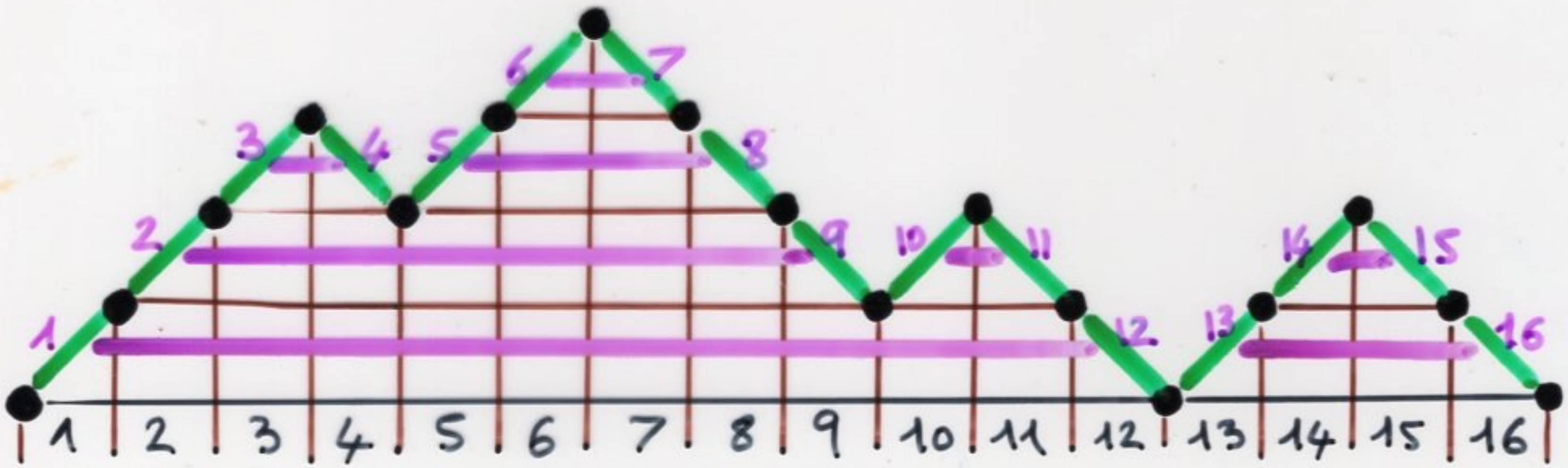
$a < c < b < d$
impossible



bijection

non-crossing partitions

Dyck paths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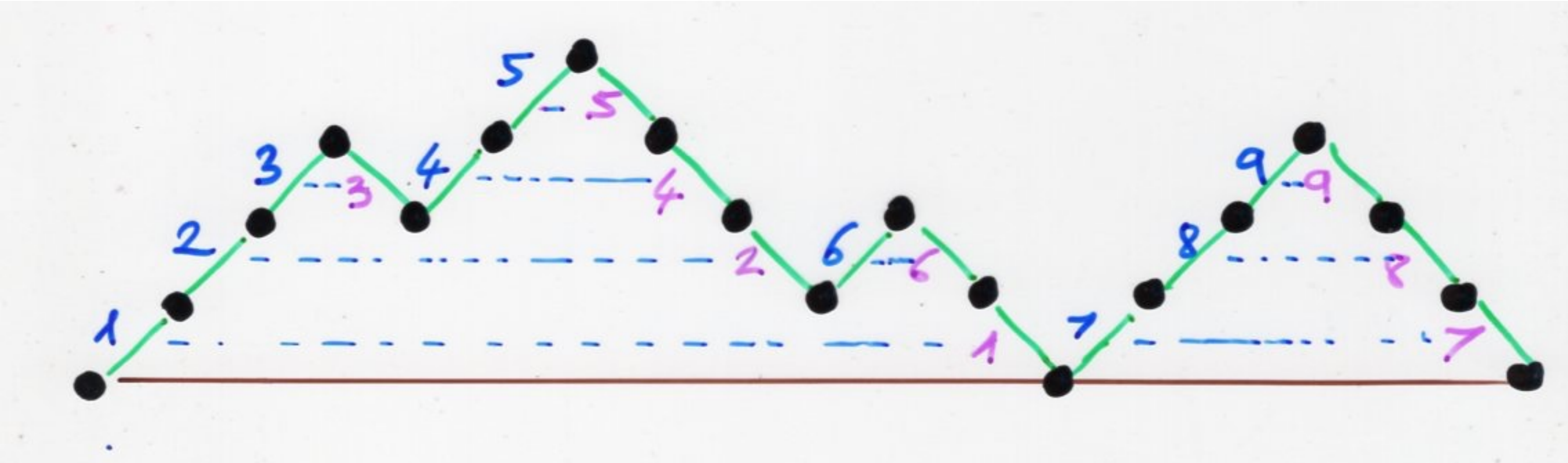

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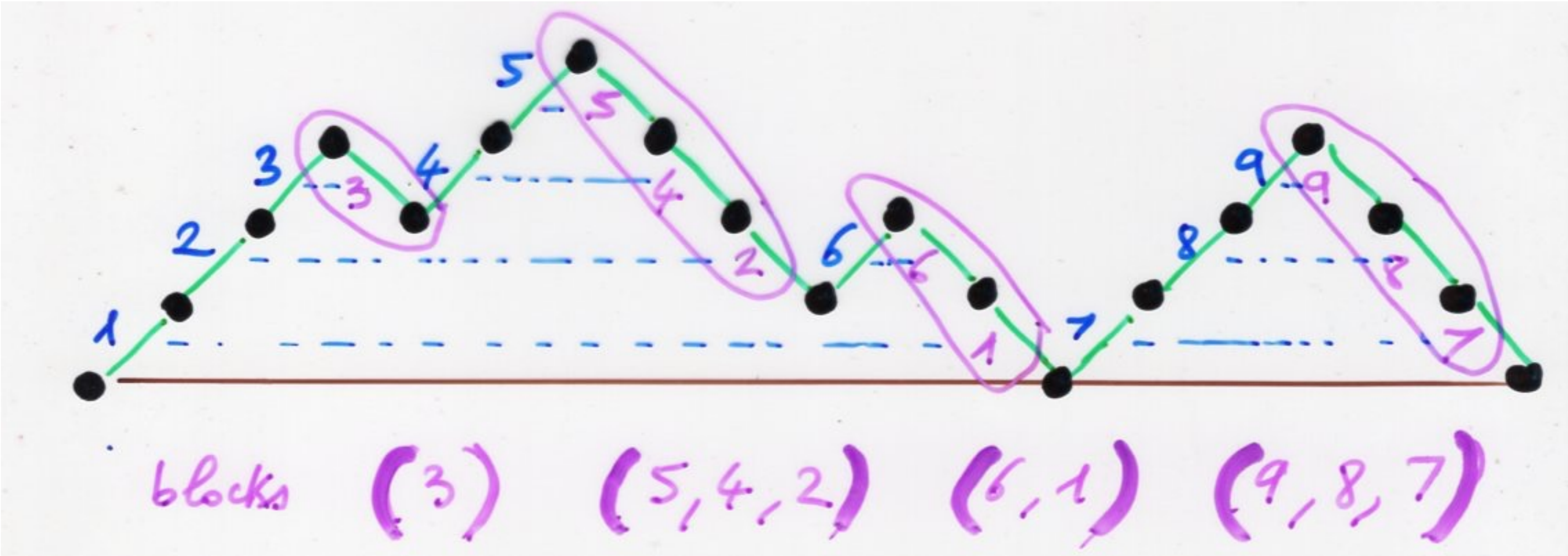
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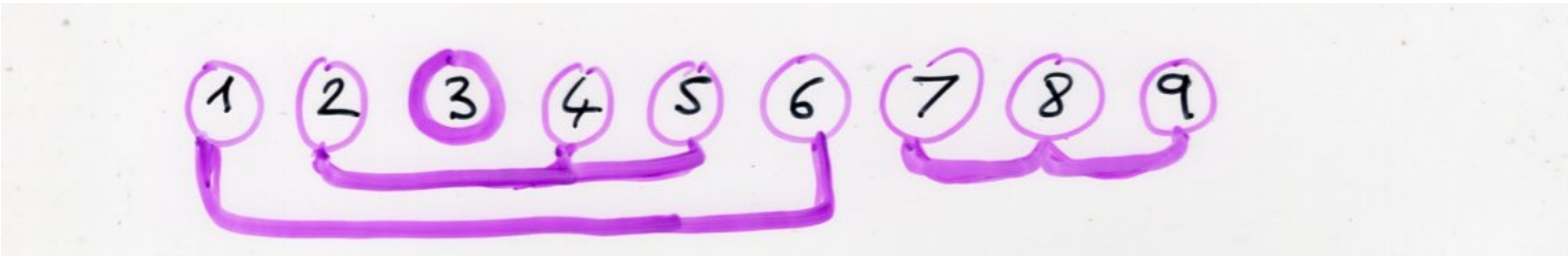
blocks (3) (5,4,2) (6,1) (9,8,7)


1 2 3 4 5 6 7 8 9

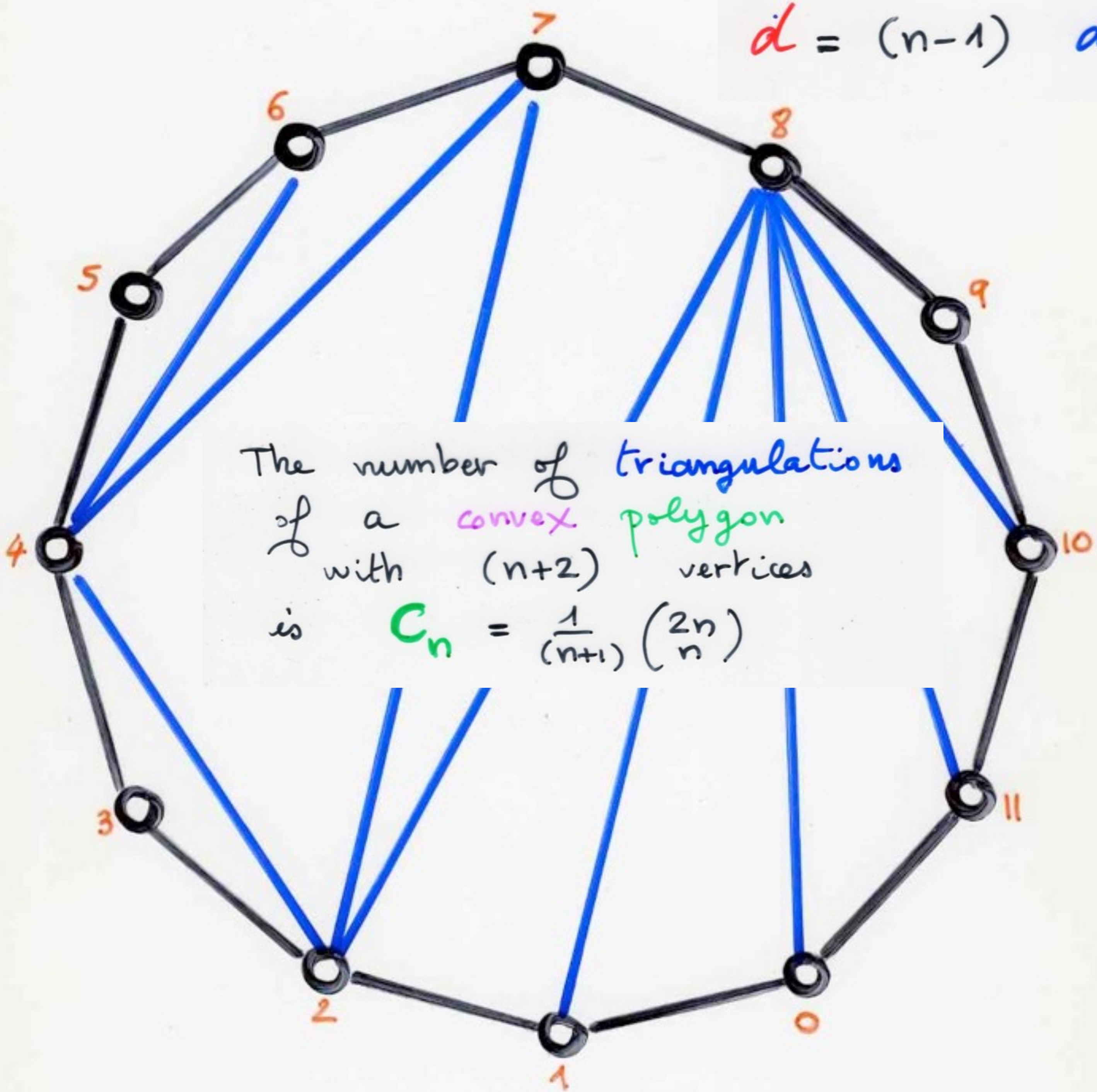
exercise

- prove this gives a non-crossing partition
- prove that it is a bijection

Dyck path$_{2n}$ $\longleftrightarrow$ non-crossing partitions of $\{1,2,\dots,n\}$

Triangulations of a convex polygon

$d = (n-1)$ diagonals



The number of triangulations of a convex polygon with $(n+2)$ vertices is $C_n = \frac{1}{(n+1)}\binom{2n}{n}$

A letter from Leonhard Euler to Christian Goldbach ....

Berlin, 4 September 1751

Leonhard
Euler
1707 - 1783

unter welchen summa 5 ... aa+bb+cc+dd so beschaffen
werden kan.

Da ich die quatuor quadrata aa+bb+cc+dd
so, dass $a+b+c+d=2$; so ist

$$aa+bb+cc+dd=(a+b-1)^2+(a+c-1)^2+(b+c-1)^2+1$$

folglich ist $aa+bb+cc+dd-1$ in 3 quadrata resolubel, wann nun selbst
in 3 quadrata resolubel.

Da nun $8n+3$ in 3 quadrata so wird

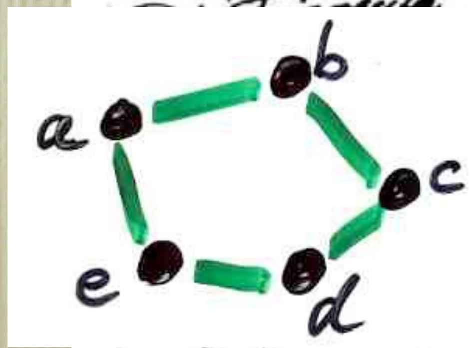
$$8n+3=(a+b-1)^2+(a+c-1)^2+(b+c-1)^2$$

$$8n+4=aa+bb+cc+dd$$ dergestalt, dass $a+b+c+d=2$

Also dieses ist das schöne Theorema welches ... aus dem Theoremate Fermatiano hergeleitet haben.

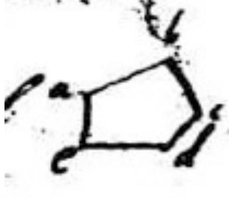
Es ist mir neulich auf eine Betrachtung gefallen, welche mir nicht wenig merckwürdig vorkommt. Dieselbe betrifft, auf wie vielerley Arten ein gegebenes Polygonum durch Diagonal Linien in Triangula zerschnitten werden kann.

Ein Trapezium ... durch die Diagonalen ... Triangula ...

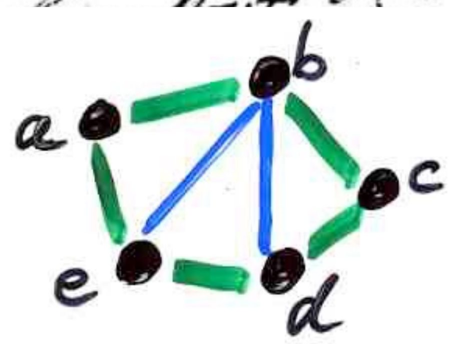


... quadrilaterum 

... bd, und also auf ...

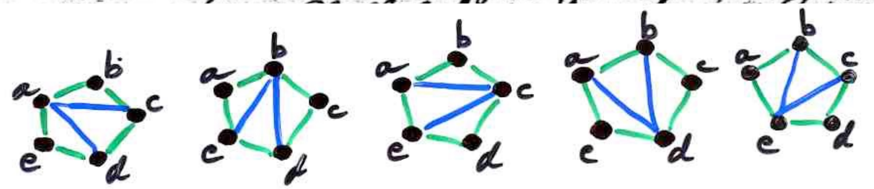


wird durch ...



Triangula ...

Triangula ...



so habe ich per inductionem gefunden

wenn $n = 3.\ 4.\ 5.\ 6.\ 7.\ 8.\ 9.\ 10$

so ist $x = 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430$

Hieraus habe ich nun den Schluss gemacht, dass generaliter sey

$$x = \frac{2.6.10.14.18.22.\ \ldots\ (4n-10)}{2.3.4.5.6.7\ \ldots\ (n-1)} = \frac{(2n)!}{(n+1)!\,n!}$$

$1 = \frac{2}{2};\ 2 = 1.\frac{6}{3};\ 5 = 2.\frac{10}{4};\ 14 = 5.\frac{14}{5},\ 42 =$

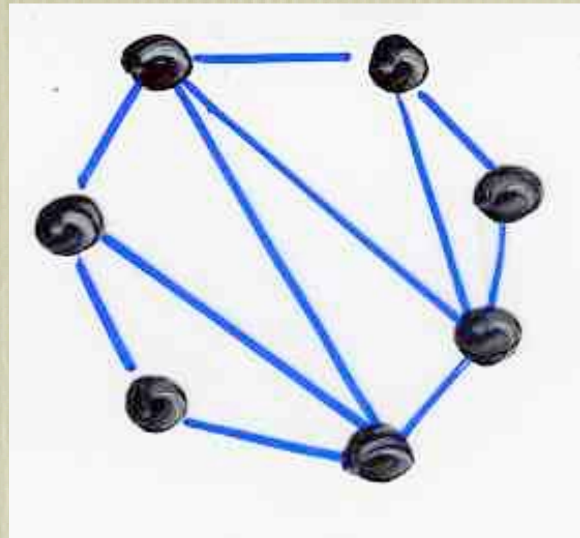
$1, 2, 5, 14, 42, 132$, etc. habe ich auch diese Eigenschaft angemerkt, dass

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + etc. = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

also wenn $a = \frac{1}{4}$ so ist $1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + etc = 4.$

Berlin d. 4. Sept 1751.

L. Euler



getheilet, und solches kann auf 5 verschiedene Arten geschehen, nehmlich durch die Diagonales I. $\frac{ac}{ad}$; II. $\frac{bd}{be}$; III. $\frac{ca}{ce}$; IV. $\frac{db}{da}$; V. $\frac{ec}{eb}$.

Ferner kann ein Sechseck durch 3 Diagonales in 4 Triangula zerschnitten, und dieses kann auf 14 verschiedene Arten geschehen. Nun ist die Frage generaliter, da ein Polygonum von n Seiten durch $n-3$ Diagonales in $n-2$ Triangula zerschnitten wird, auf wie vielerley verschiedene Arten solches geschehen könne. Setze ich nun die Anzahl dieser verschiedenen Arten $= x$ ... gefunden

so habe ich

wenn $n =$ ... $1, 2, 5, 14, 42, 132, 429, 1430, \ldots$

so ist $x = 1, 2, 5, 14, 42, 132, 429, 1430$

Hieraus habe ich nun den Schluss gemacht, dass generaliter sey

$$x = \frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdots (4n-10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots (n-1)} = \frac{(2n)!}{(n+1)!\,n!}$$

... $5 = 2 \cdot \frac{10}{4}$; $14 = 5 \cdot \frac{14}{5}$; $42 = 14 \cdot \frac{18}{6}$; ...

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

$$\frac{1-2a-\sqrt{1-4a}}{2a^2}$$

$1+2a+5a^2+14a^3+42a^4+132a^5+$ etc

$$1+2a+5a^2+14a^3+42a^4+132a^5+\text{etc} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

$$1+2a+5a^2+14a^3+42a^4+\text{etc} = 4.$$

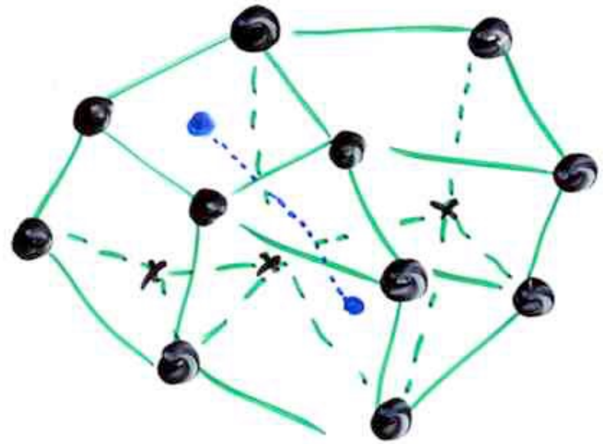
$a=\frac{1}{4}$ … $1+\frac{2}{4}+\frac{5}{4^2}+\frac{14}{4^3}+\frac{42}{4^4}+$ etc

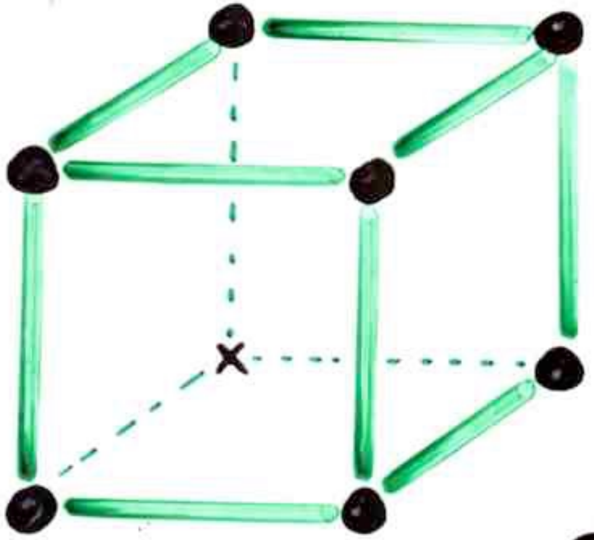
… 4. Sept 1751.

4 Sept 1751
Berlin

L. Euler

convex polyhedron





Descartes formula

$$\sum \text{defects} = (2\pi) \times (S - A + F)$$

$$S - A + F =$$

$$8 - 12 + 6 = 2$$

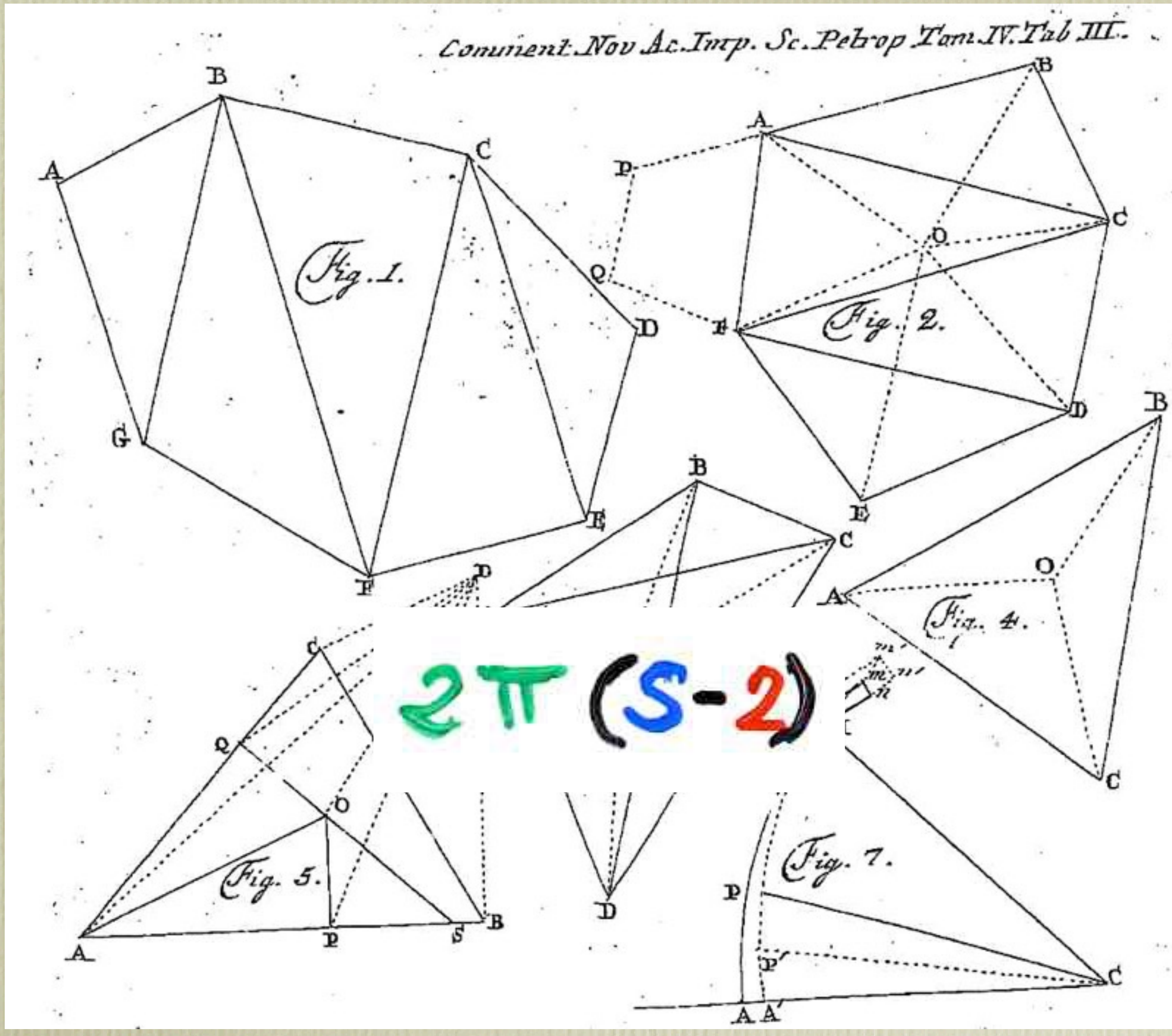

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Fig. 1.
Fig. 2.
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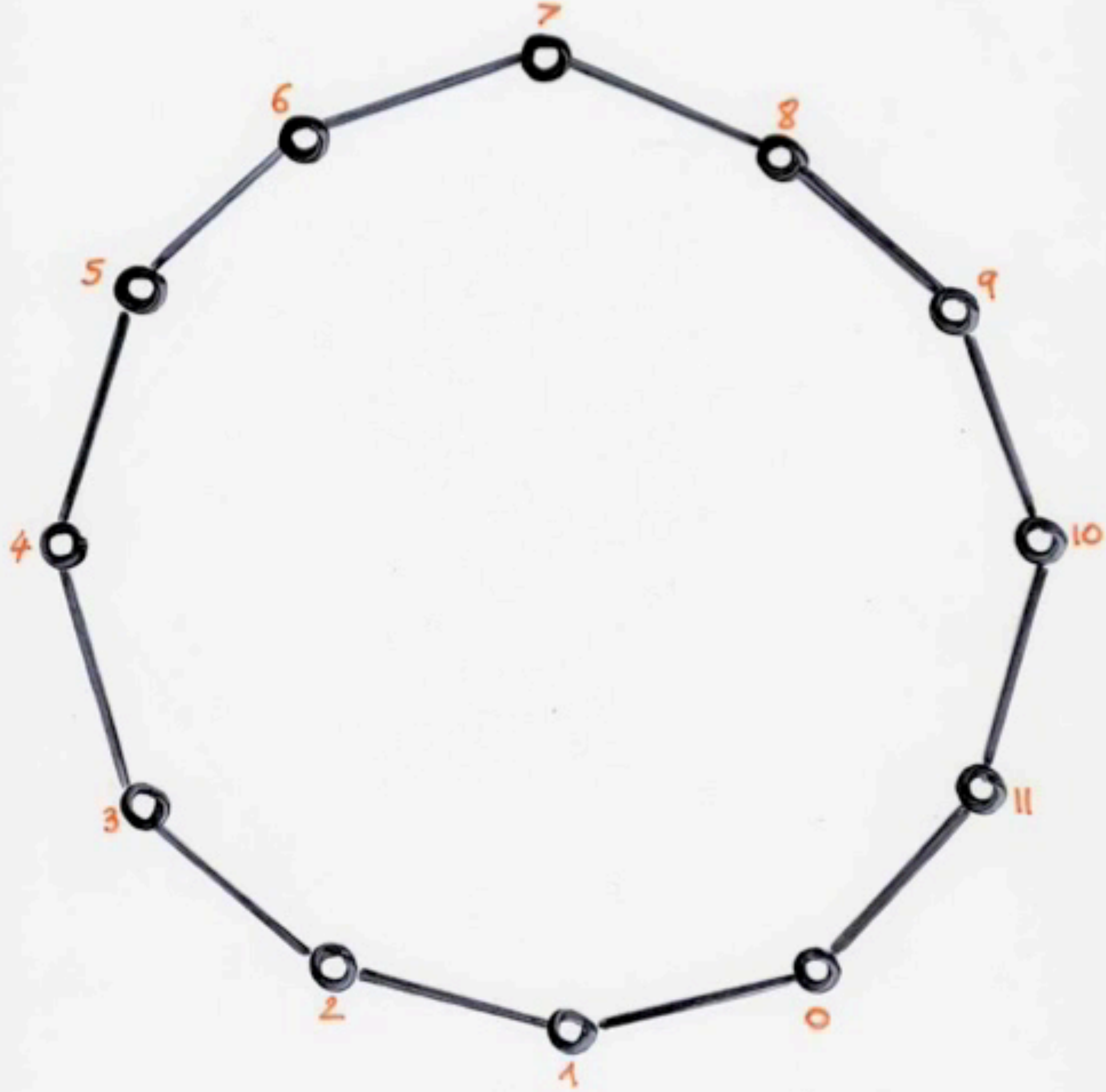
Fig. 7.
2π (S-2)

bijection

triangulations

(complete) binary trees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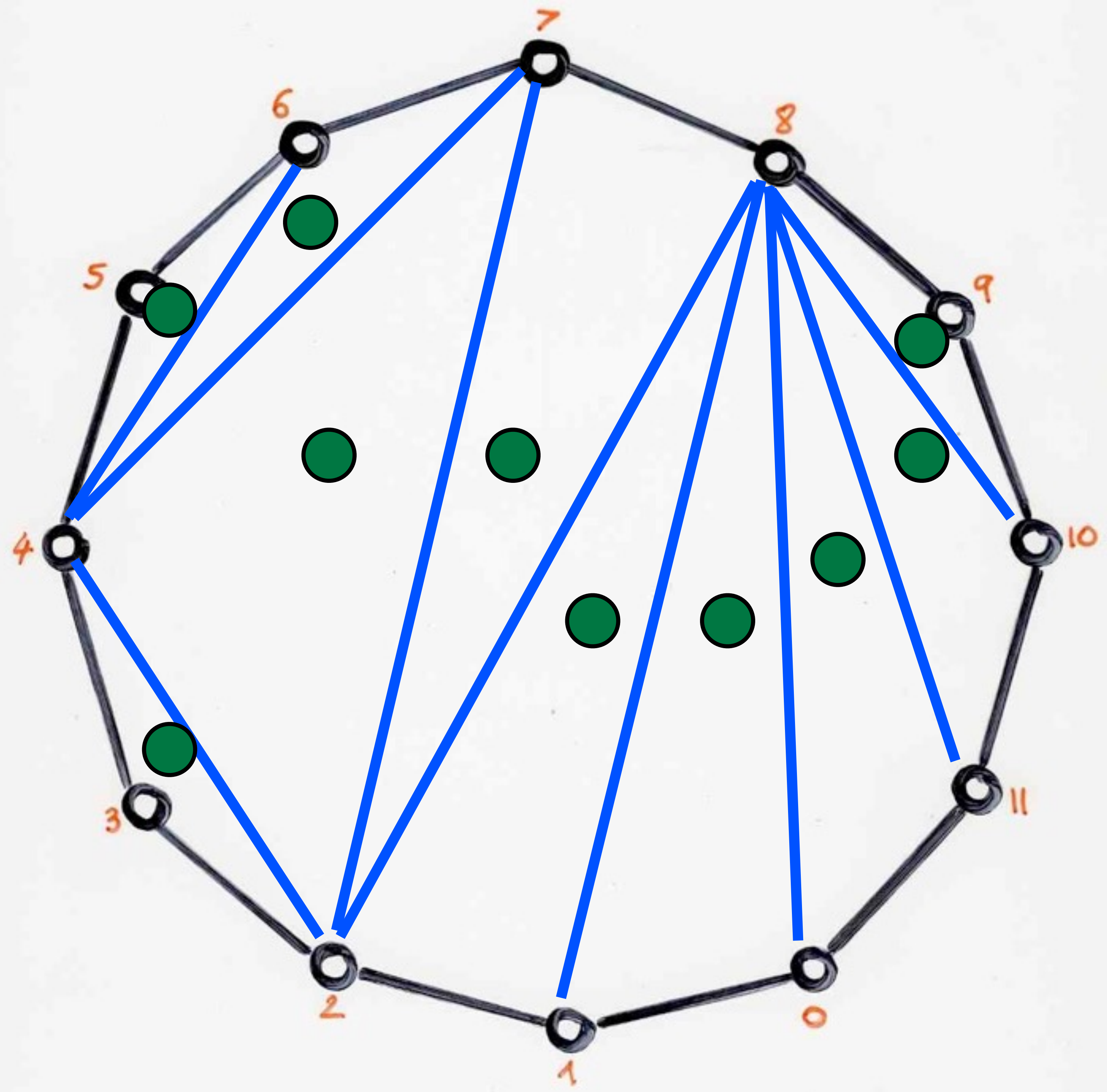
violins:

Mariette Freudentheil

Gérard H.E. Duchamp

Association
Cont'Science

Atelier audiovisuel
Université Bordeaux 1
Yves Descubes
Franck Marmisse



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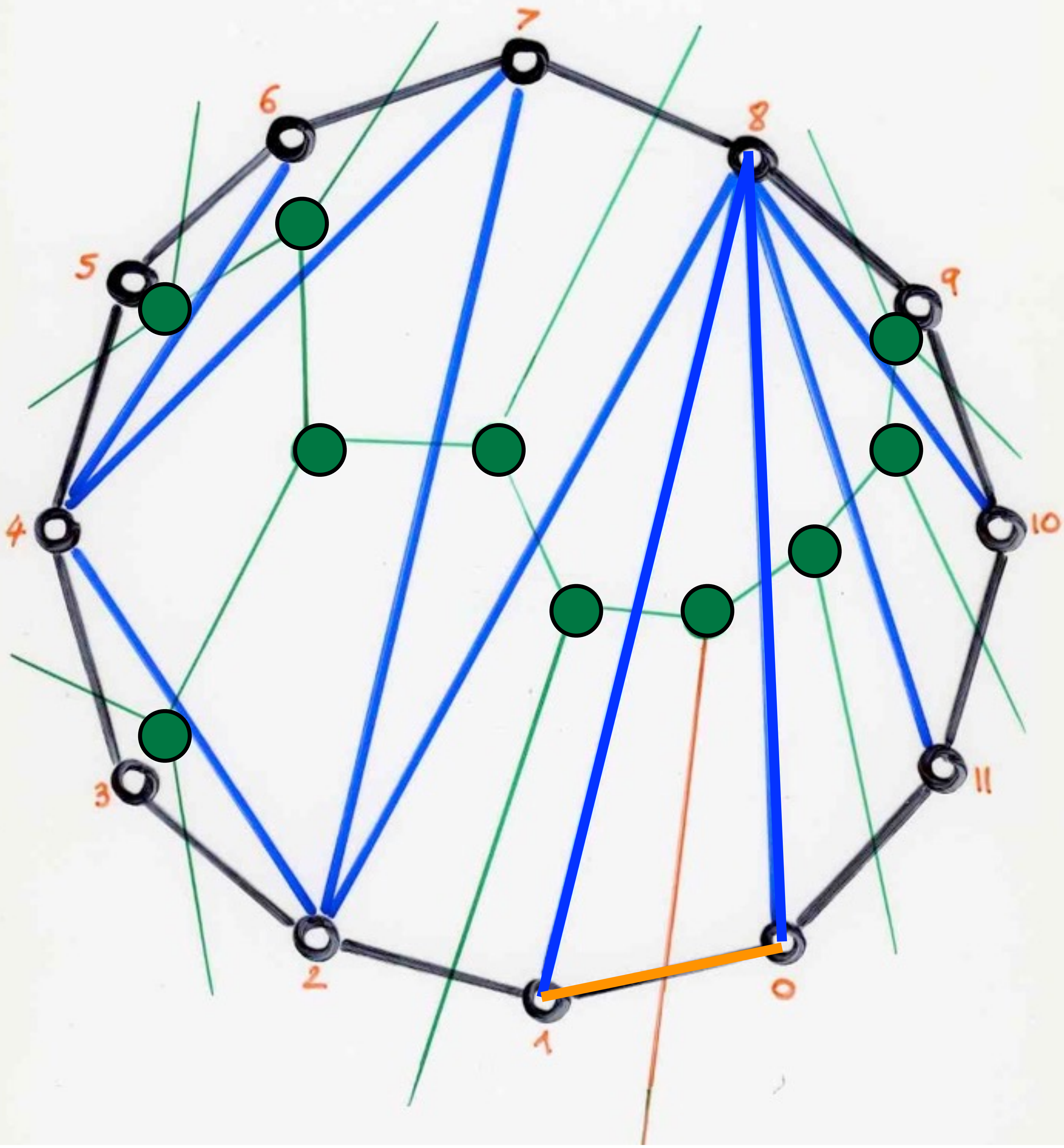
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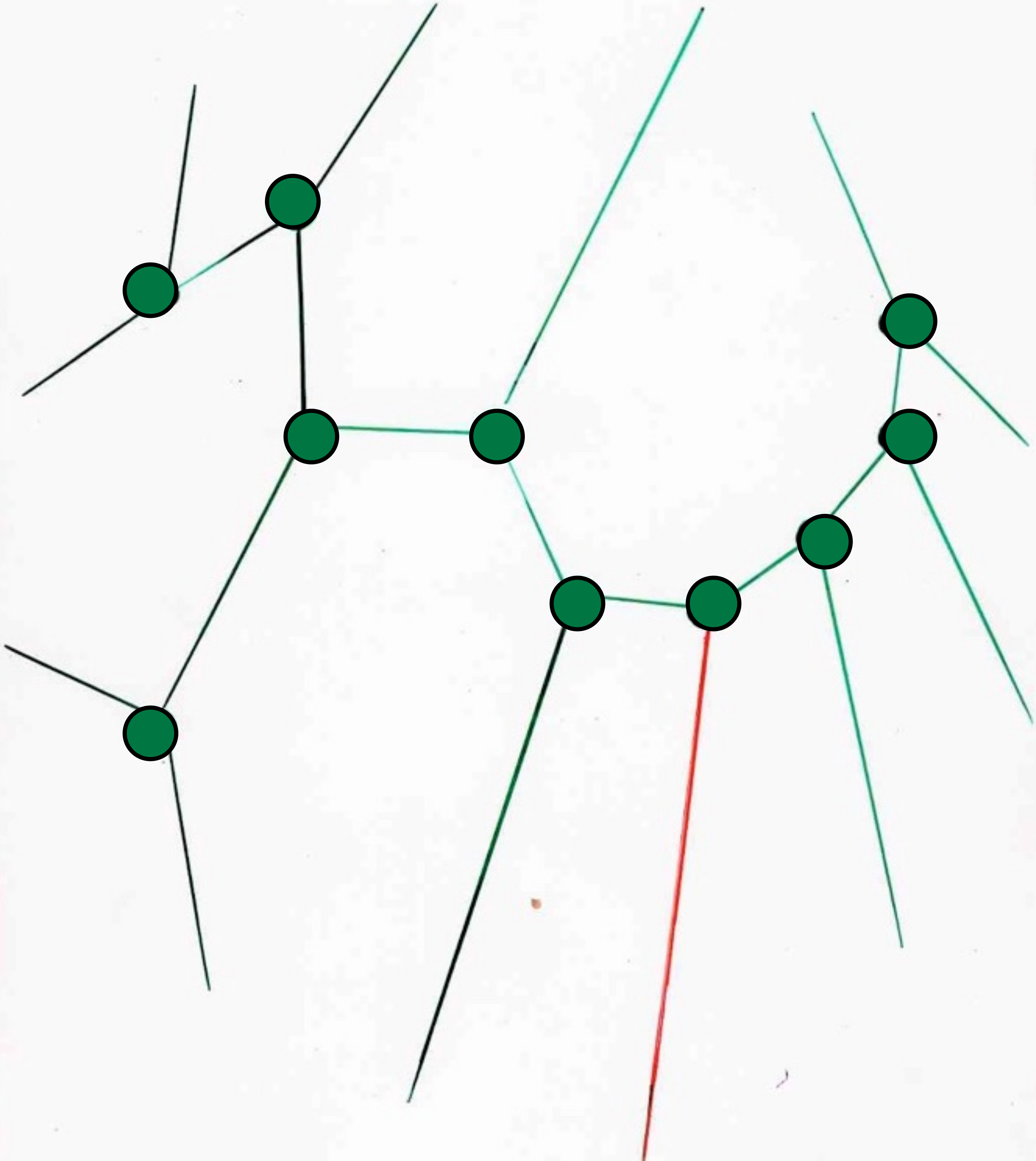
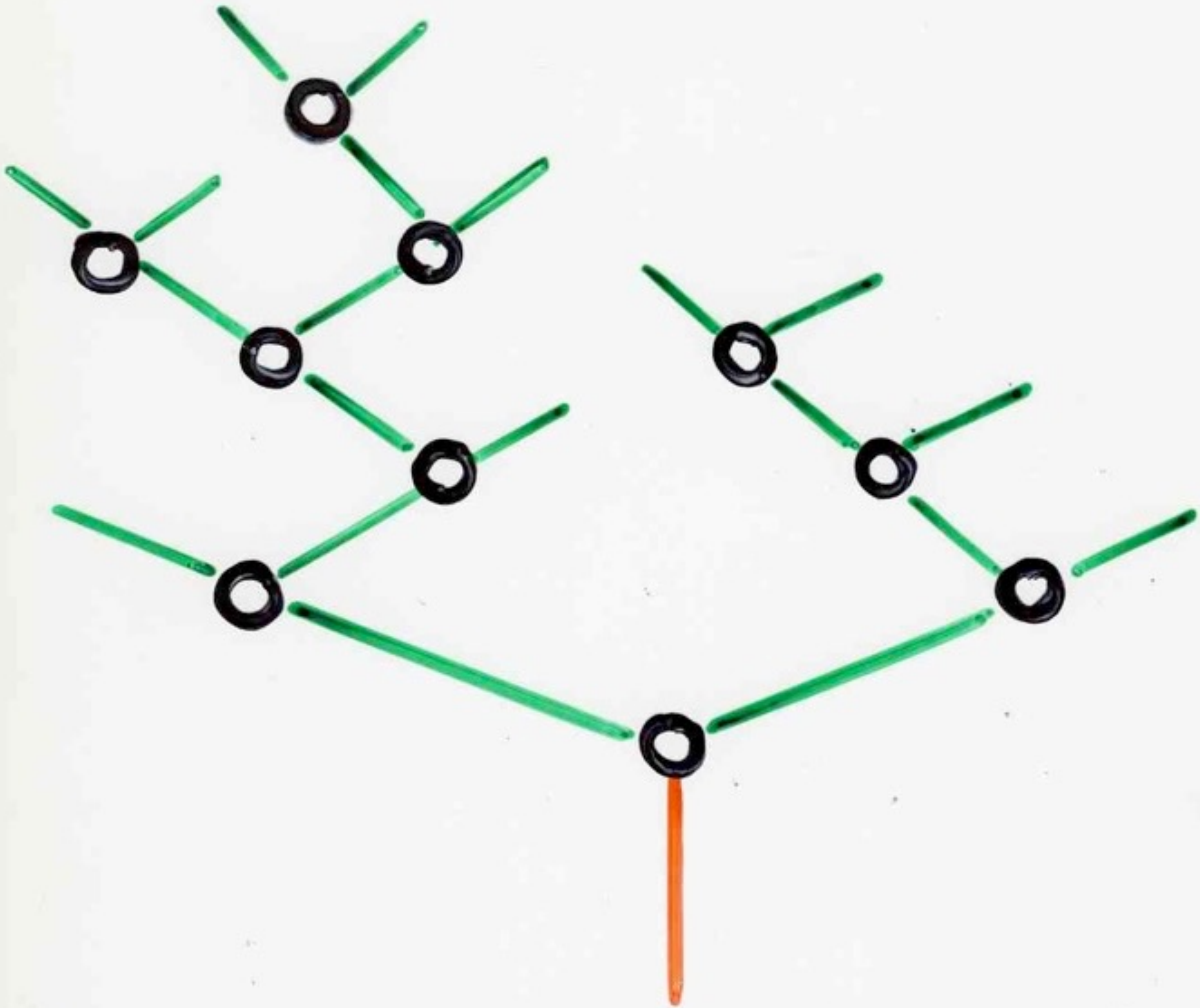
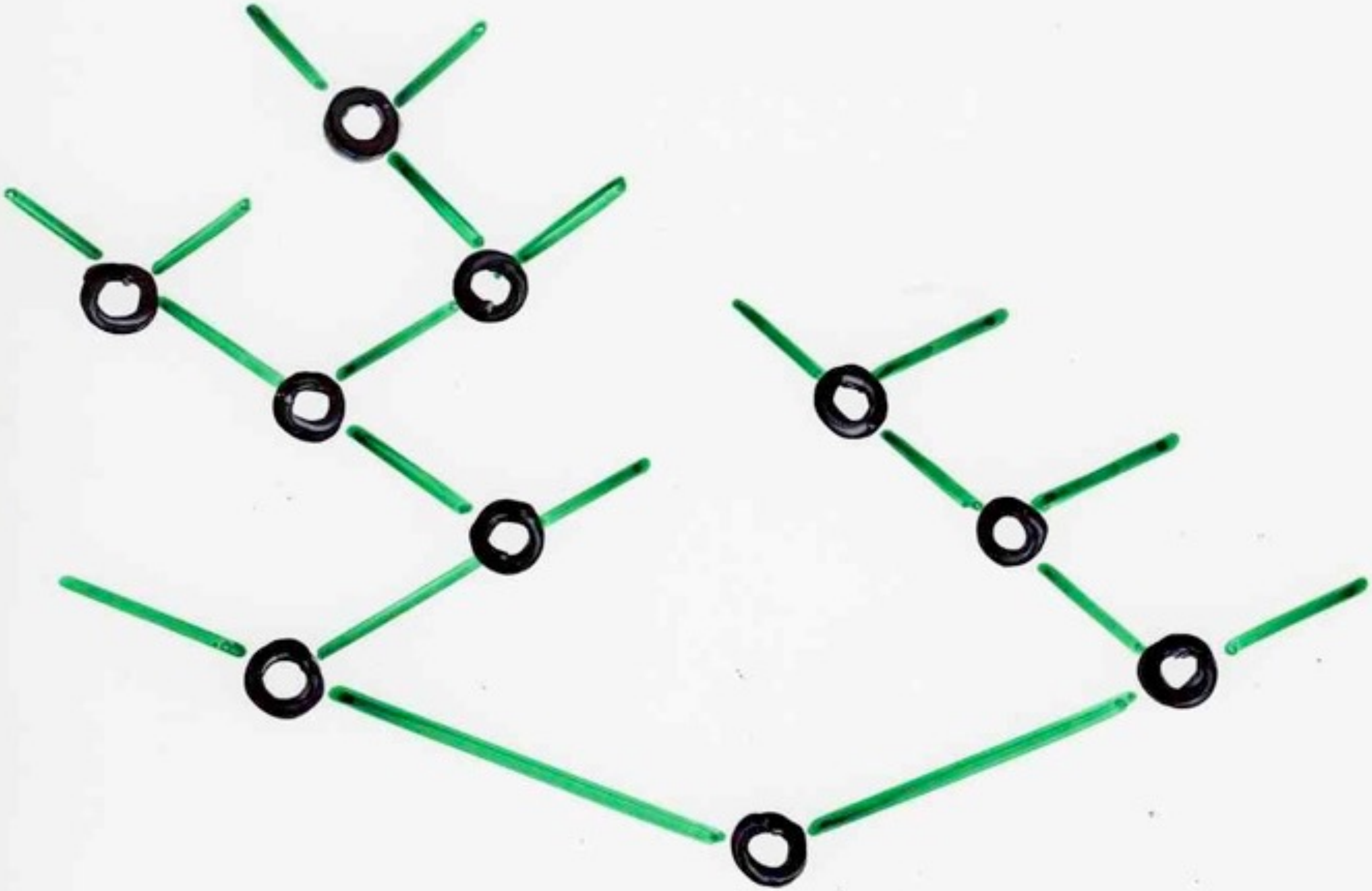
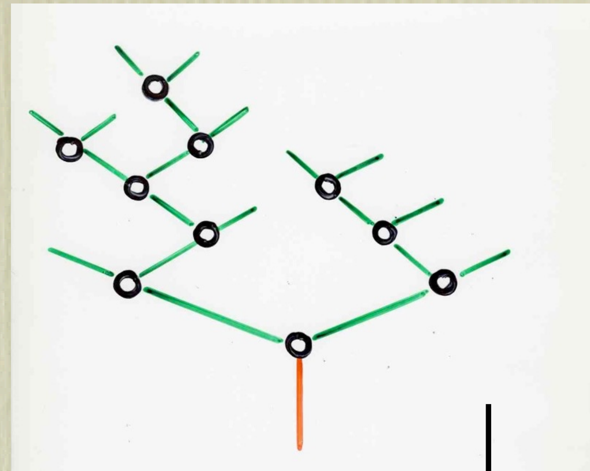


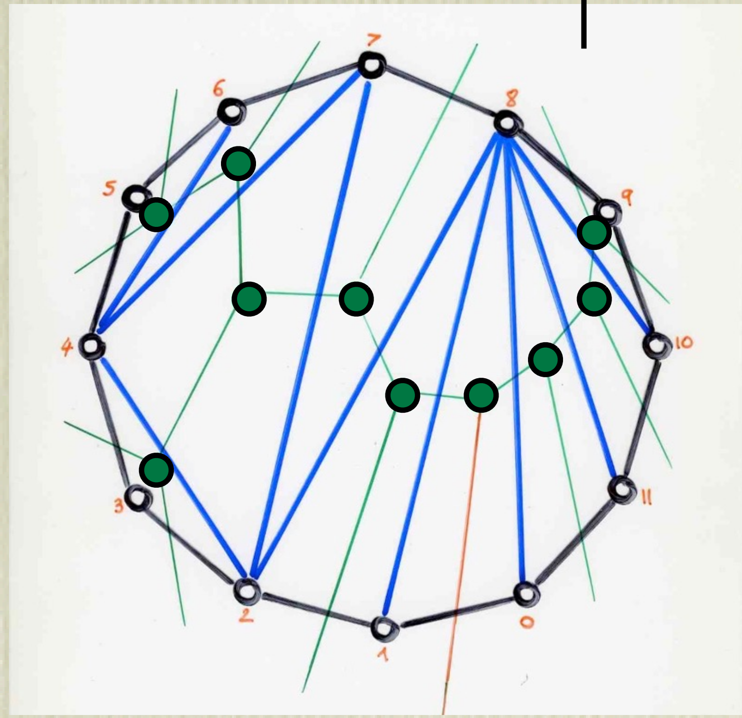


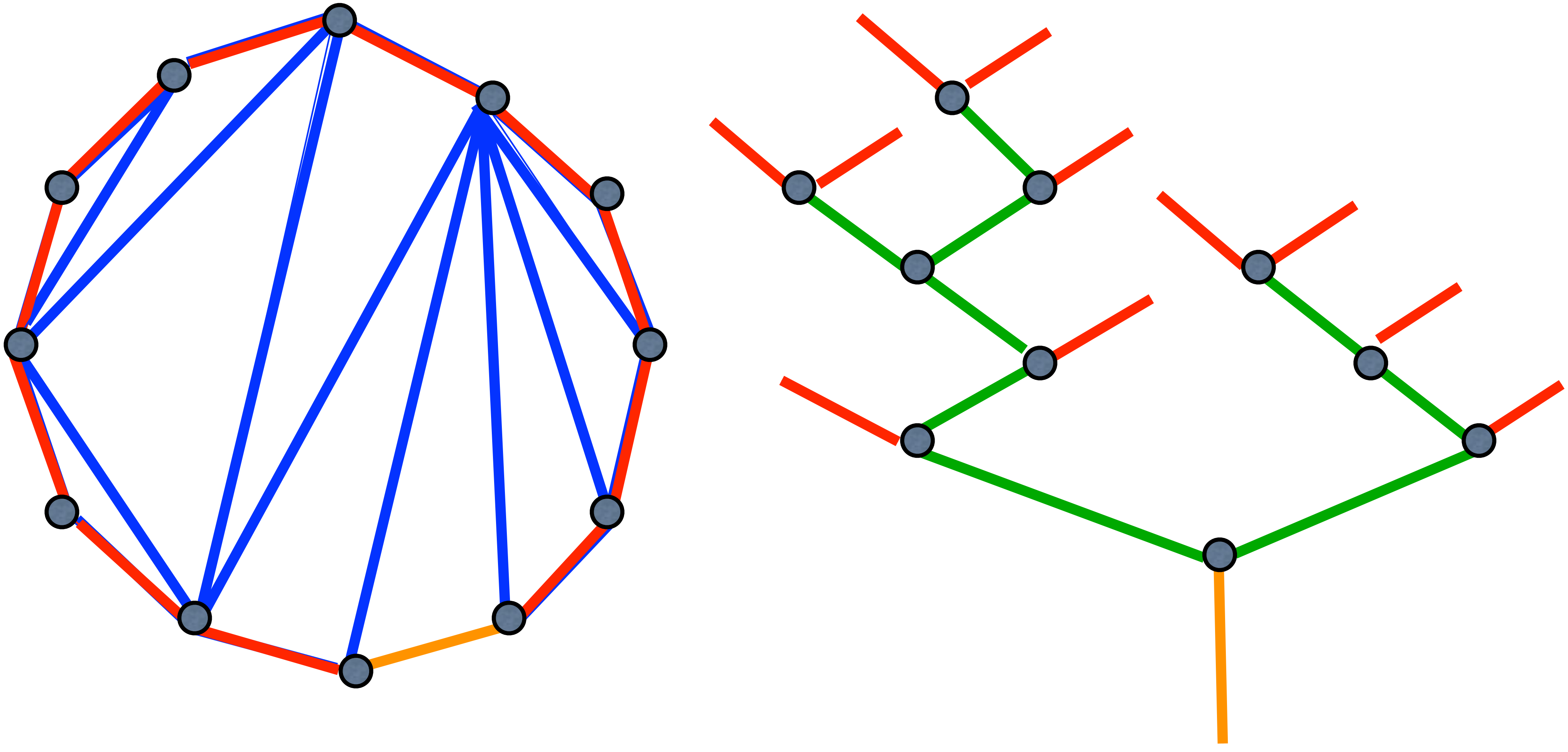


exercise

reciprocal bijection
from complete binary trees
to triangulations

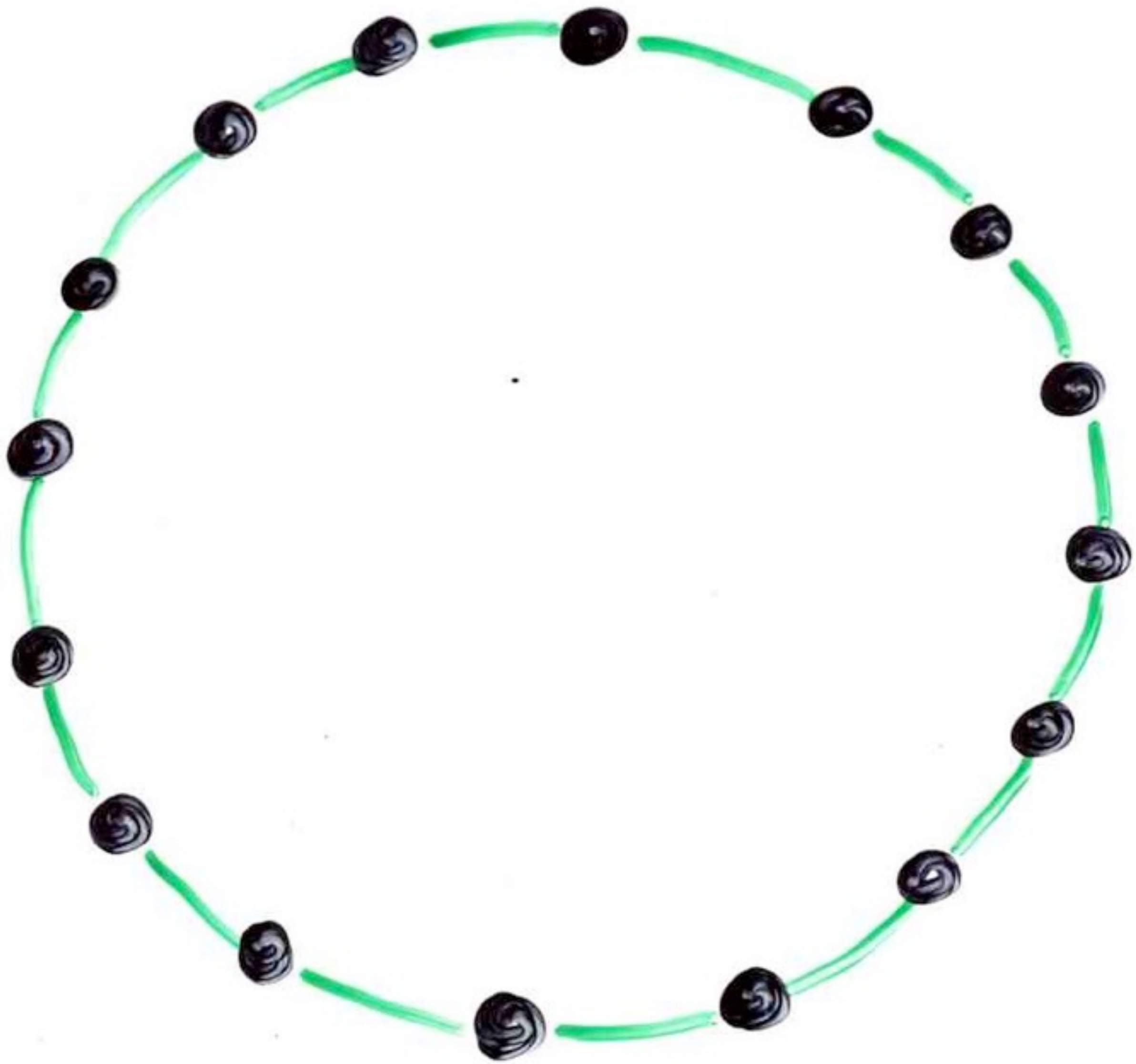


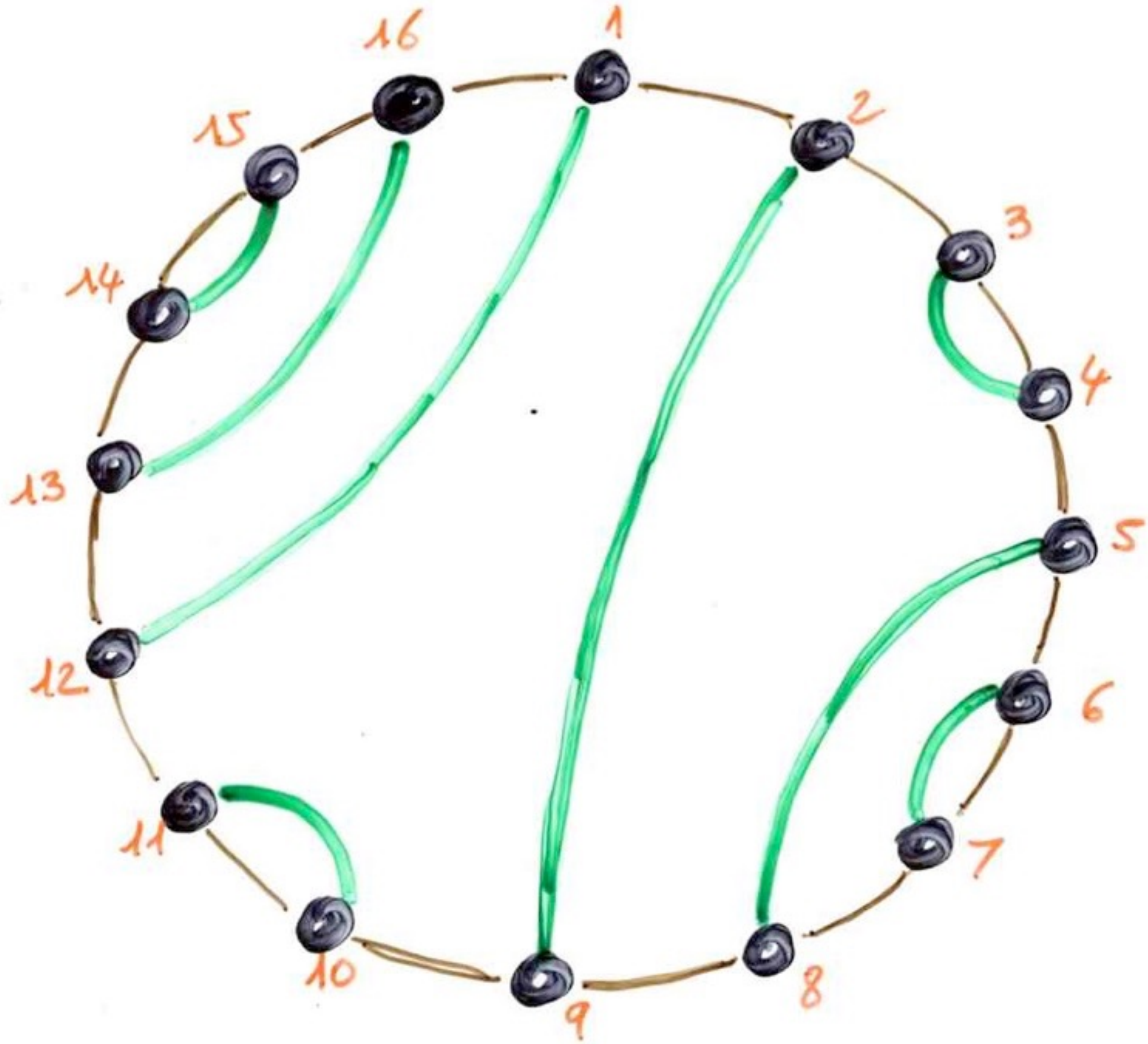




some other interpretations of Catalan numbers

Chord diagrams

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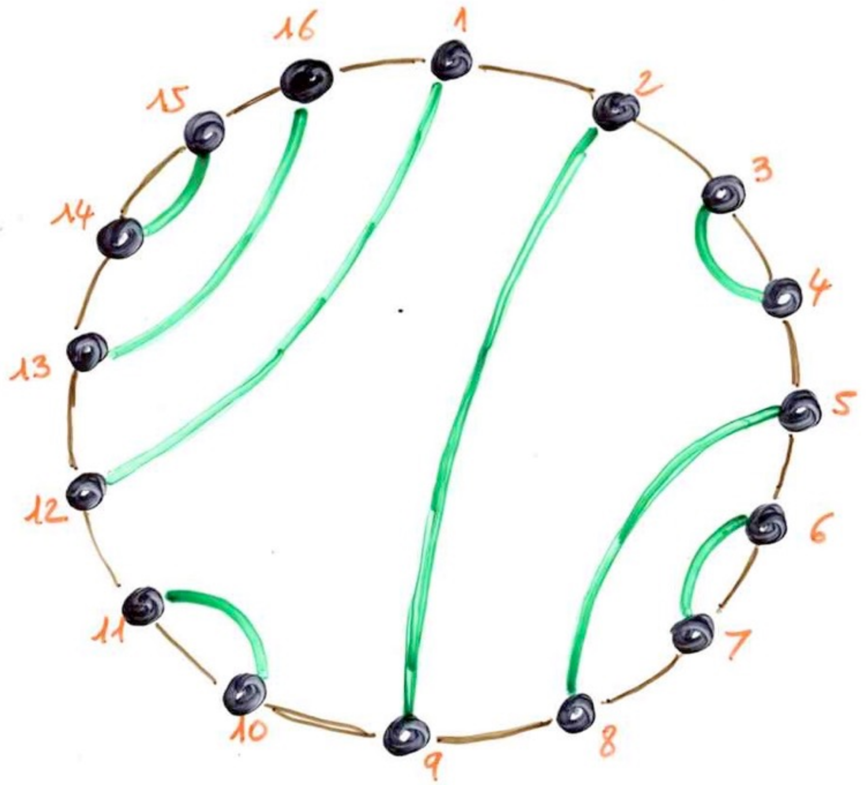
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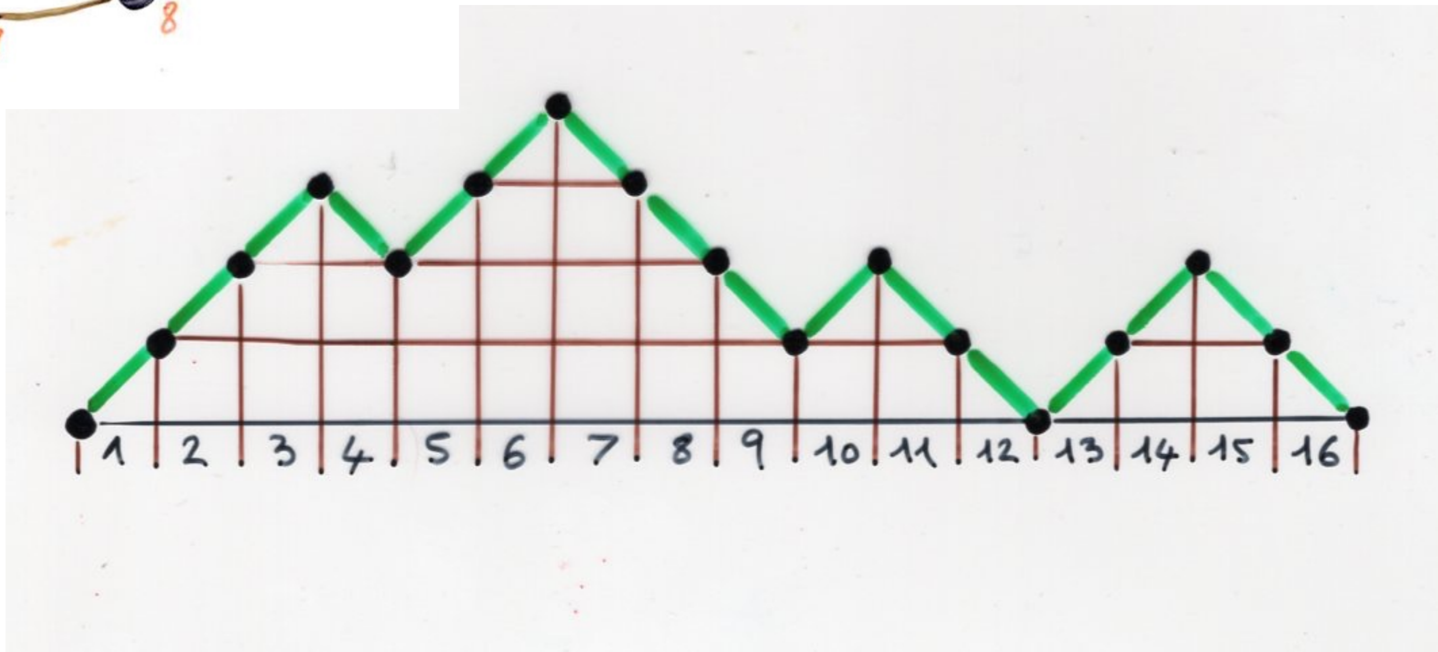
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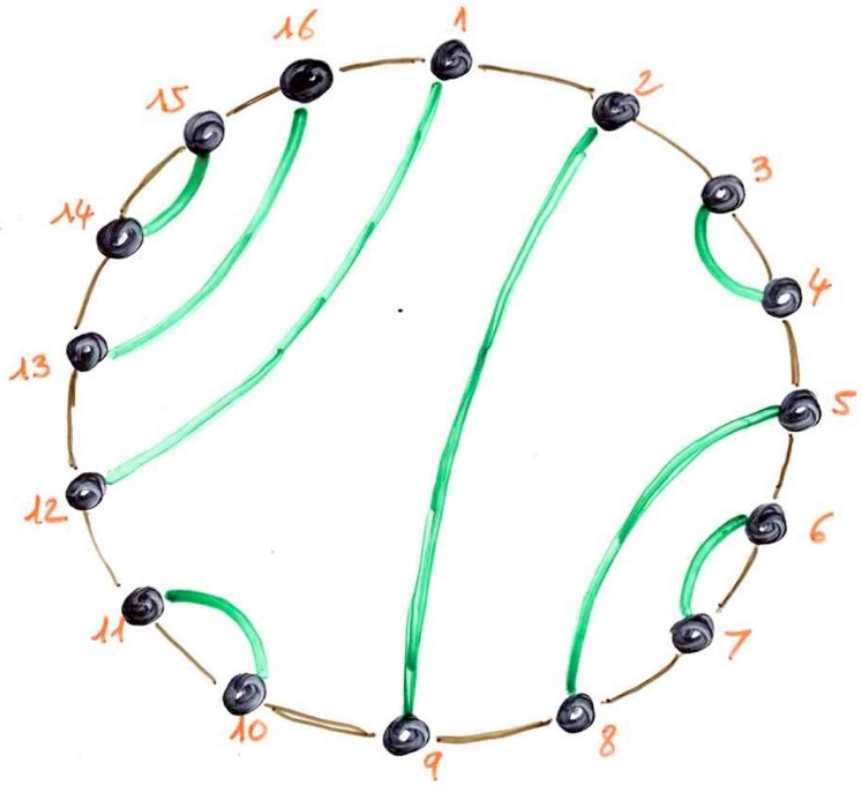
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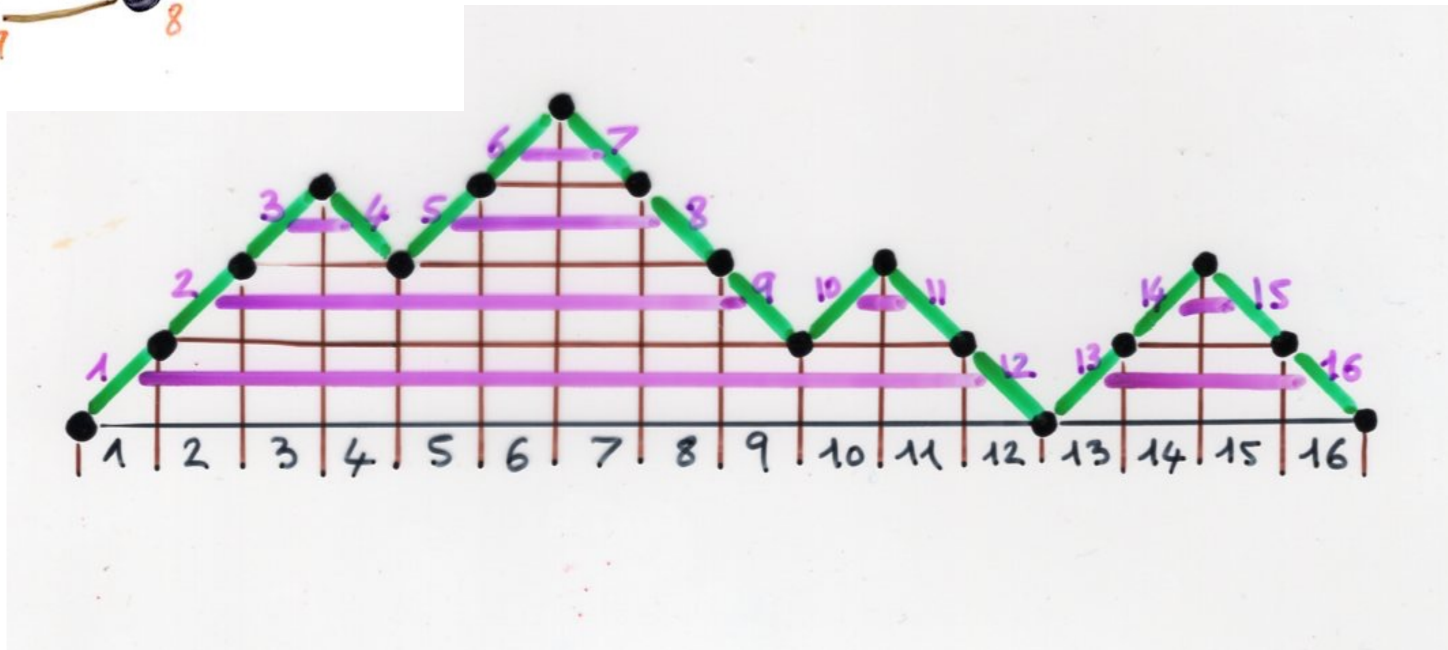
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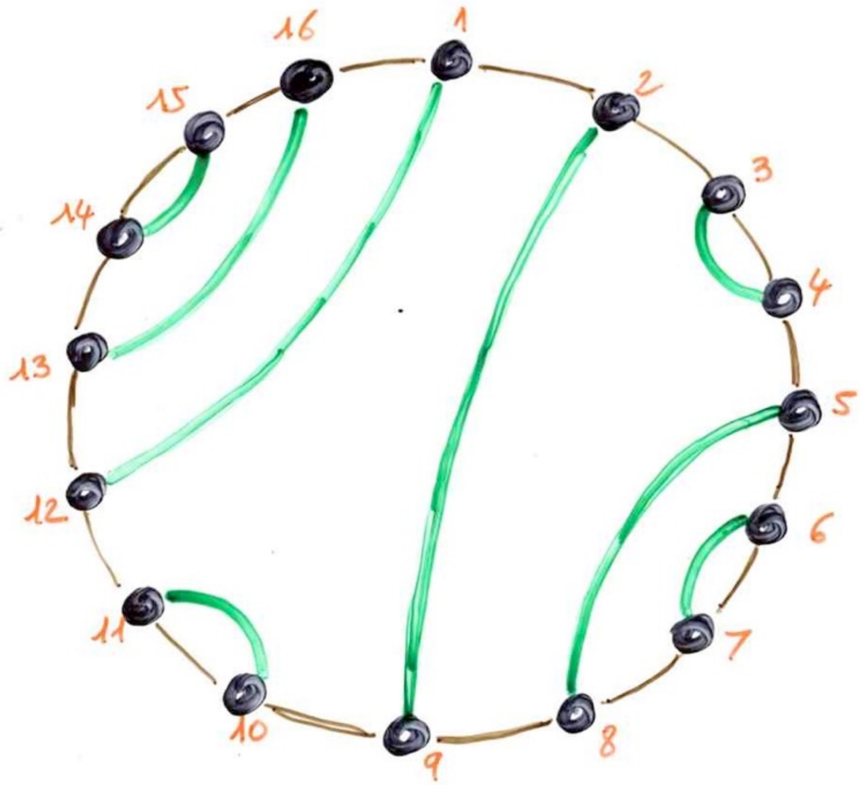
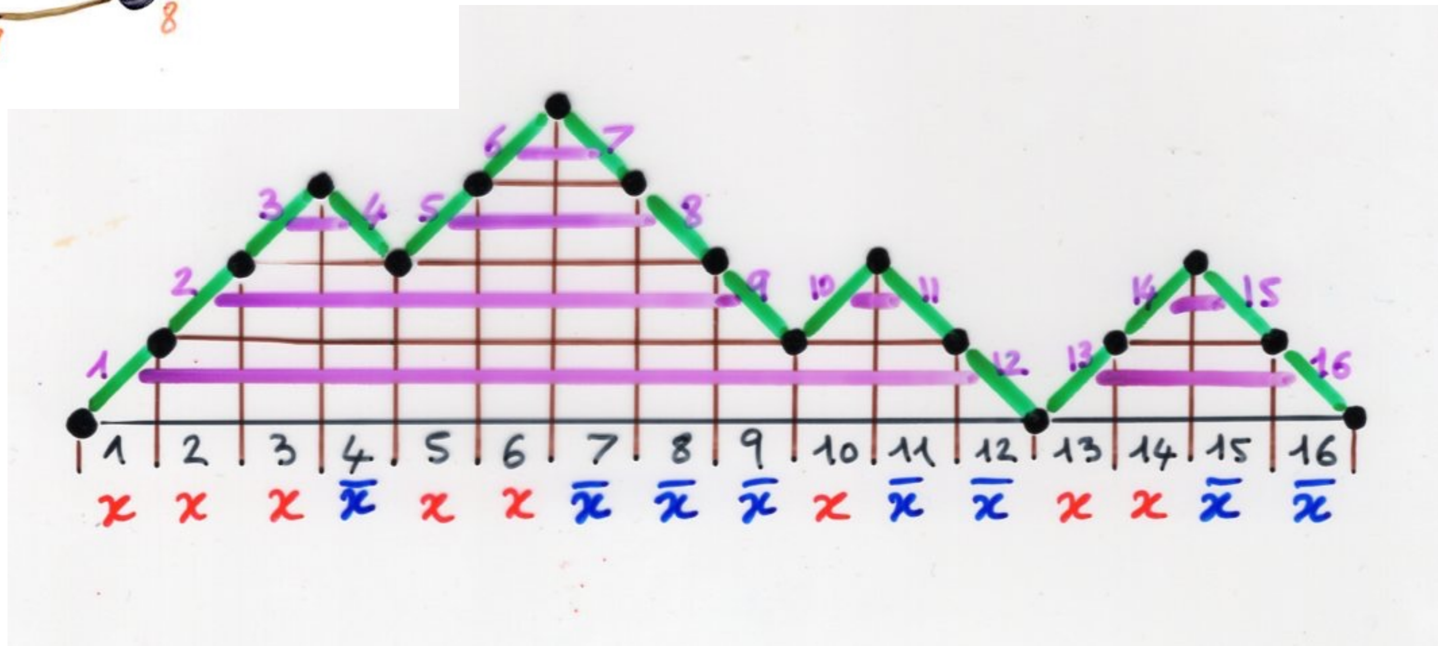
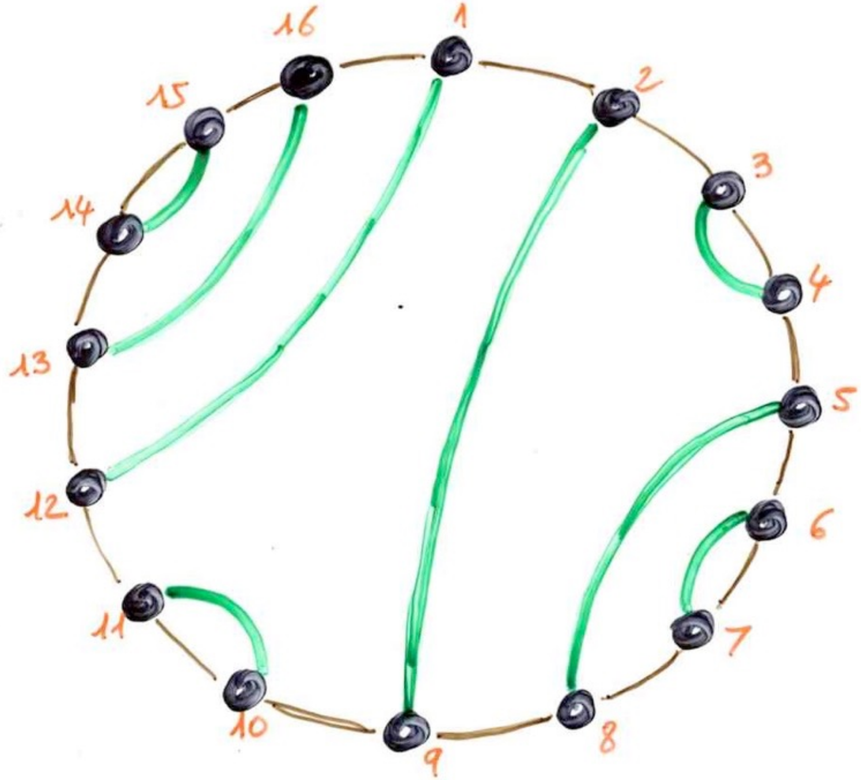


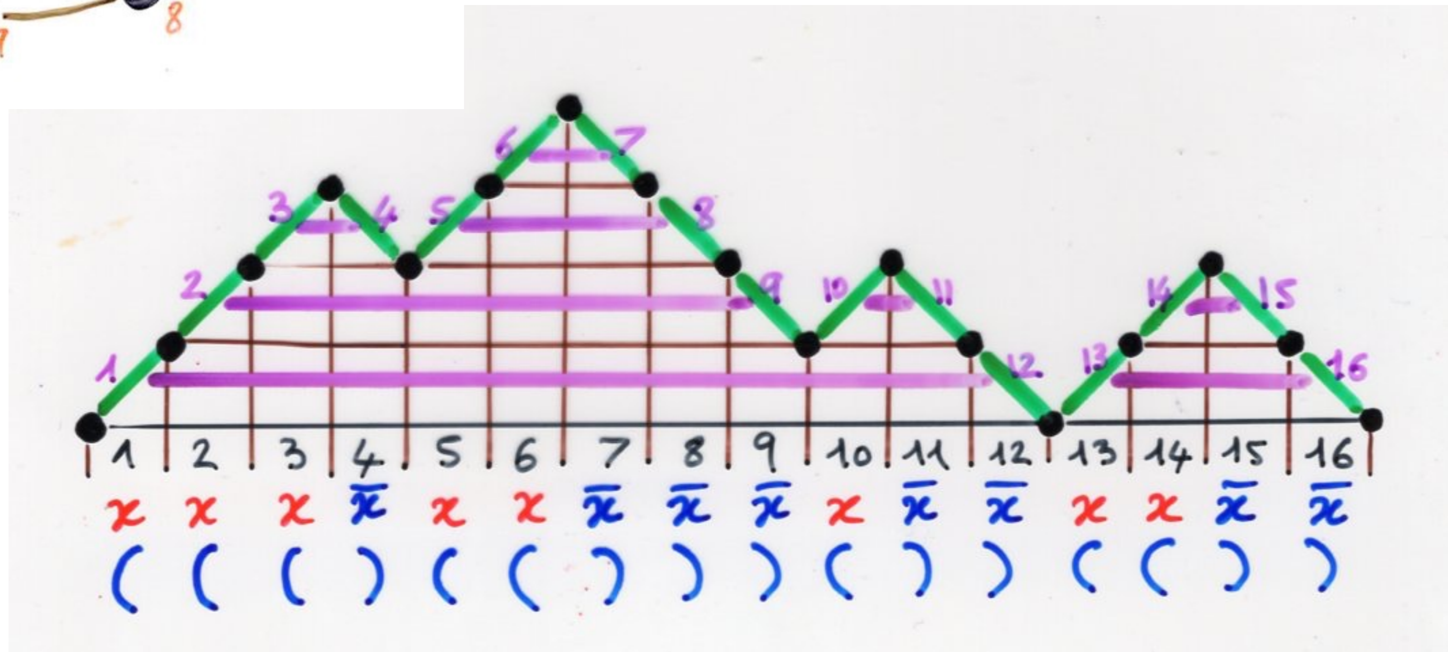
Dyck word

x, $\bar{x}$





system of
parenthesis
( )



Catalan permutations

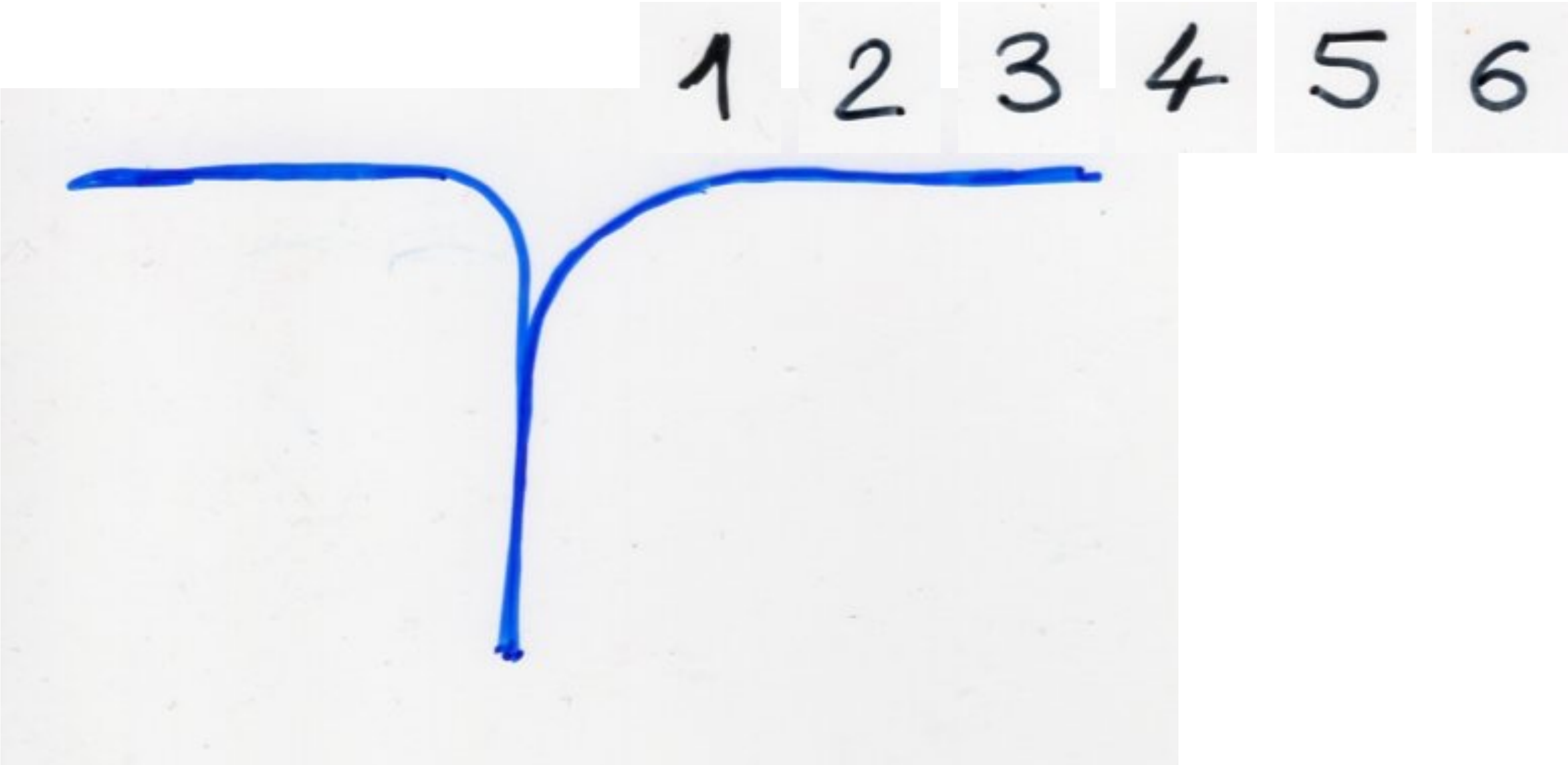
Definition

Catalan permutations

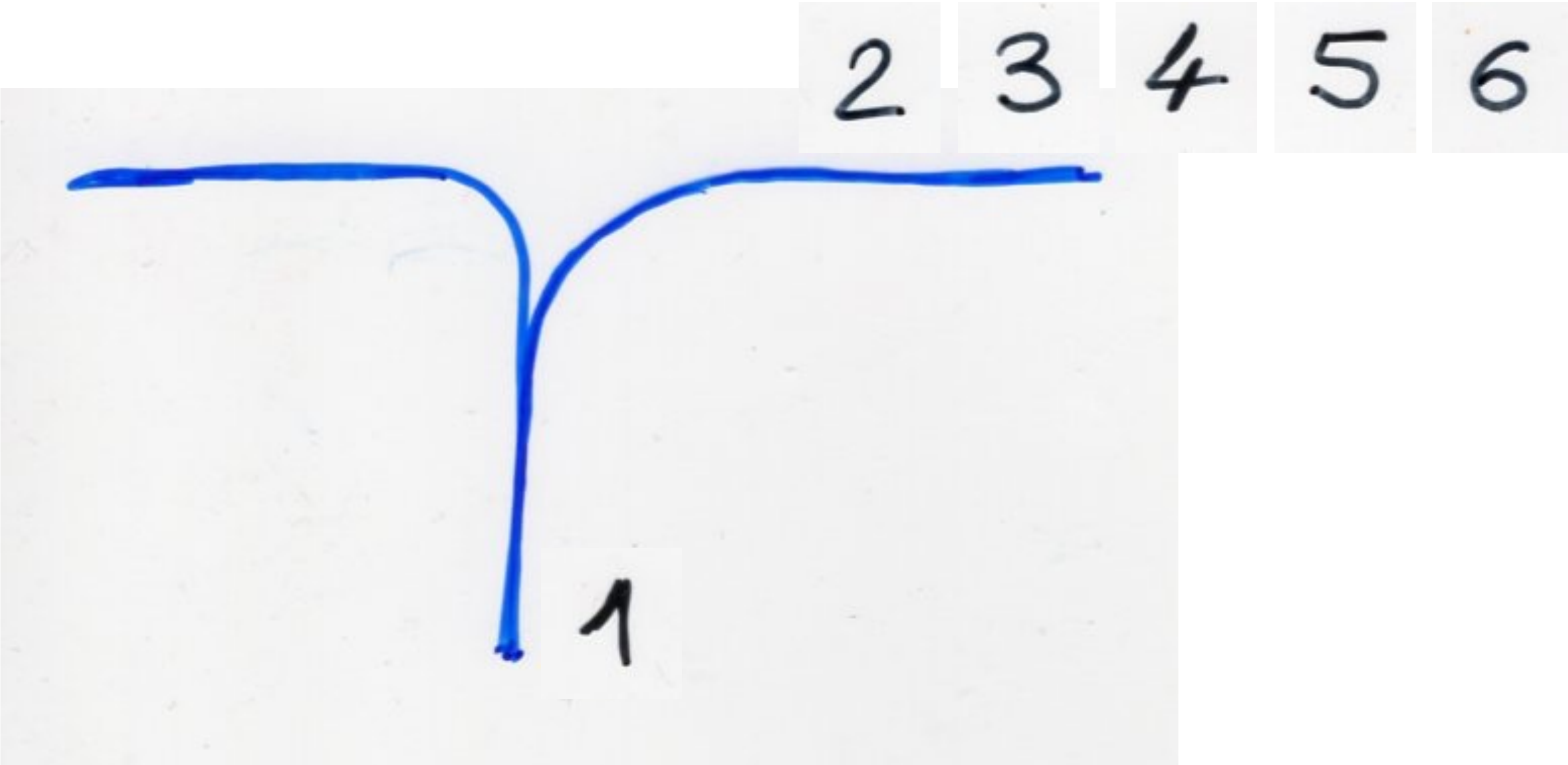
Permutations sortable with one stack

1 2 3 4 5 6

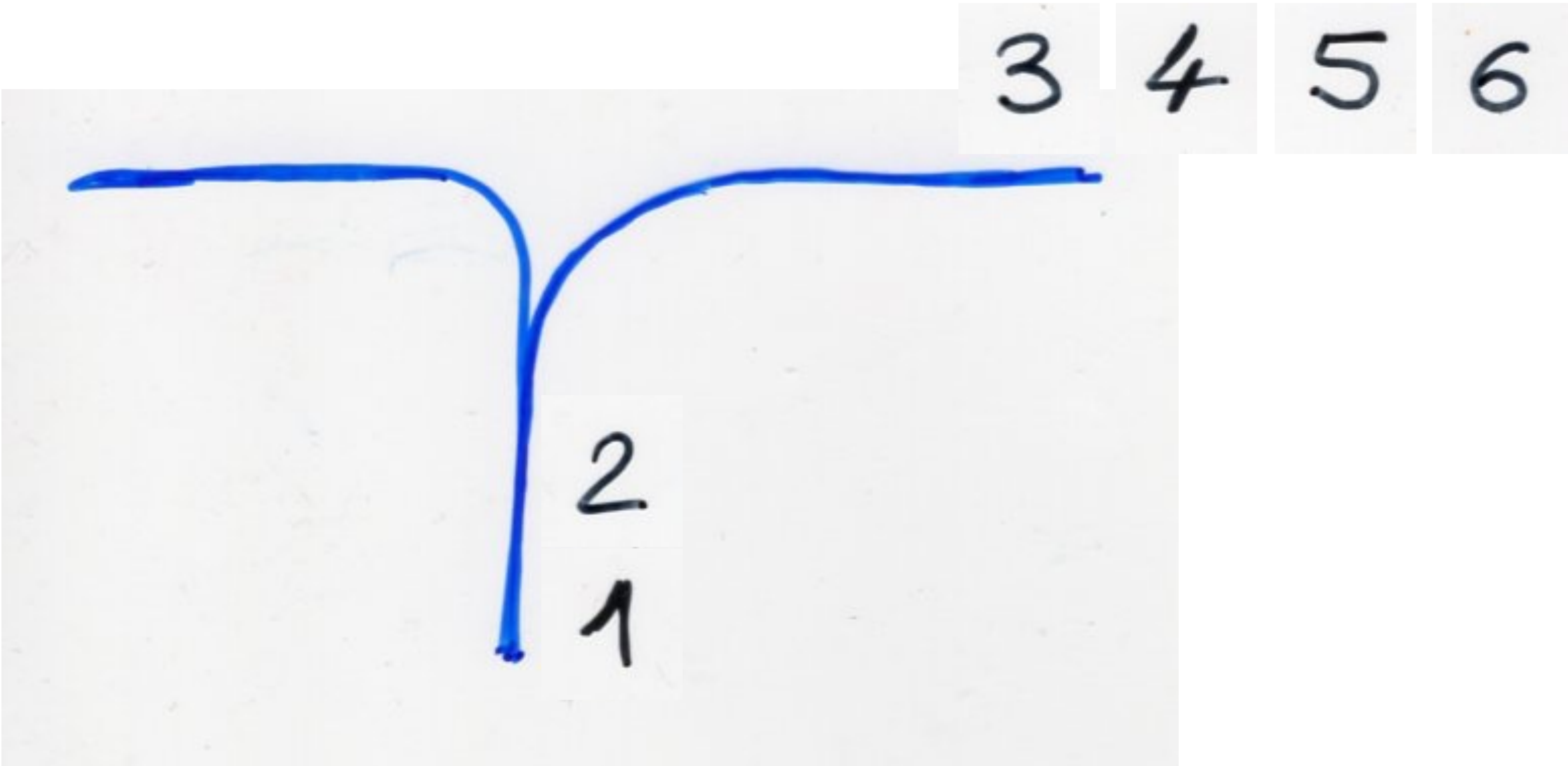
Permutations sortable with one stack



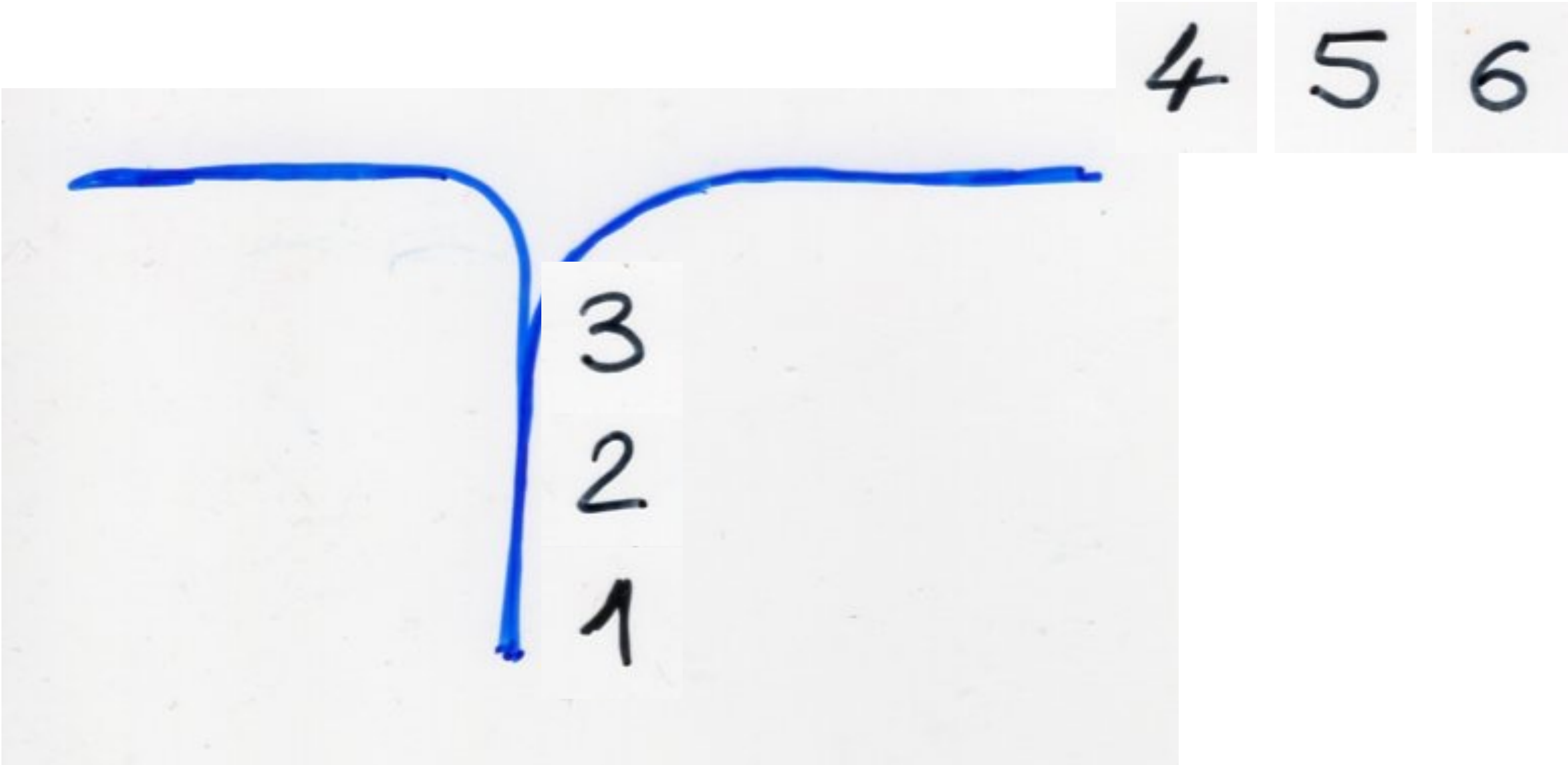
Permutations sortable with one stack



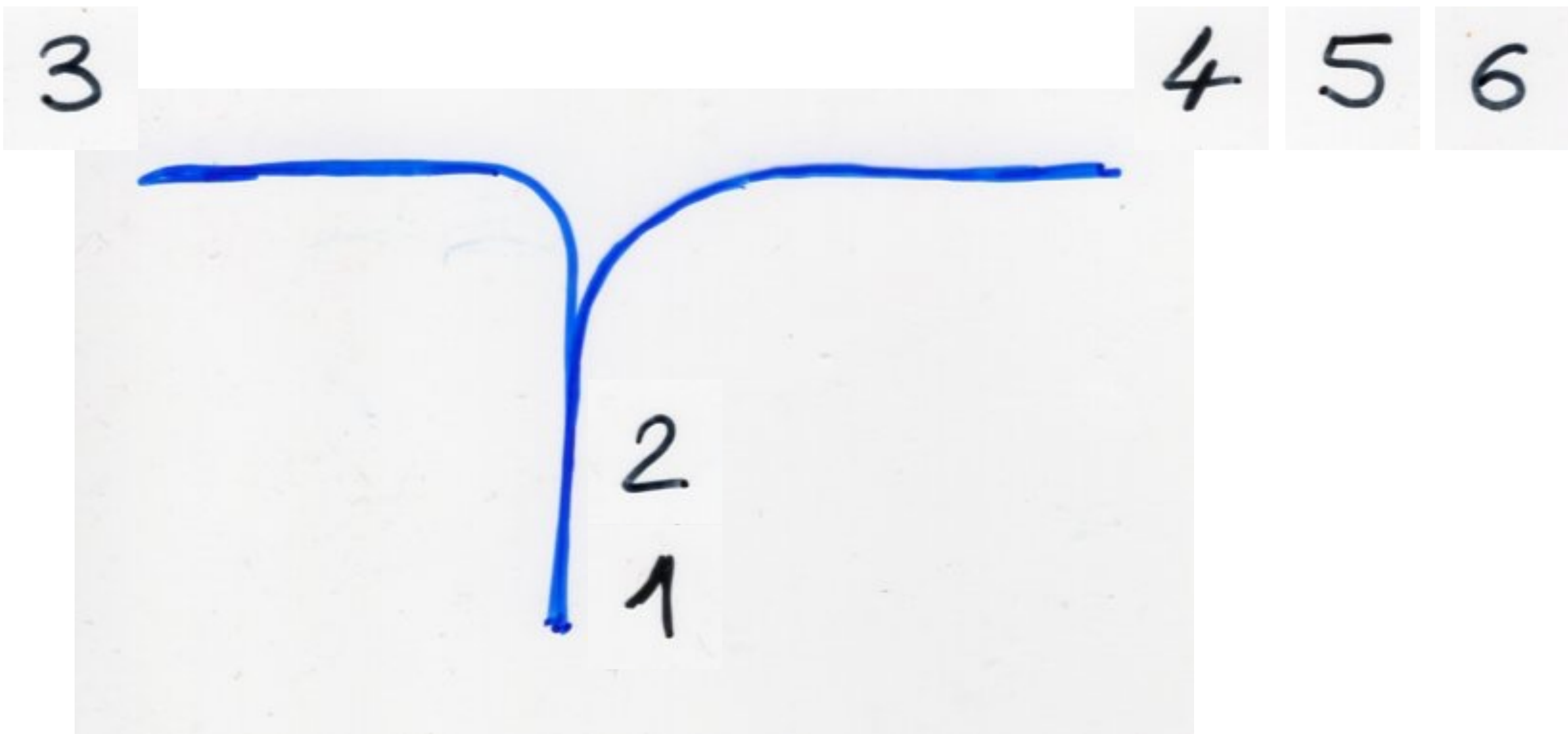
Permutations sortable with one stack



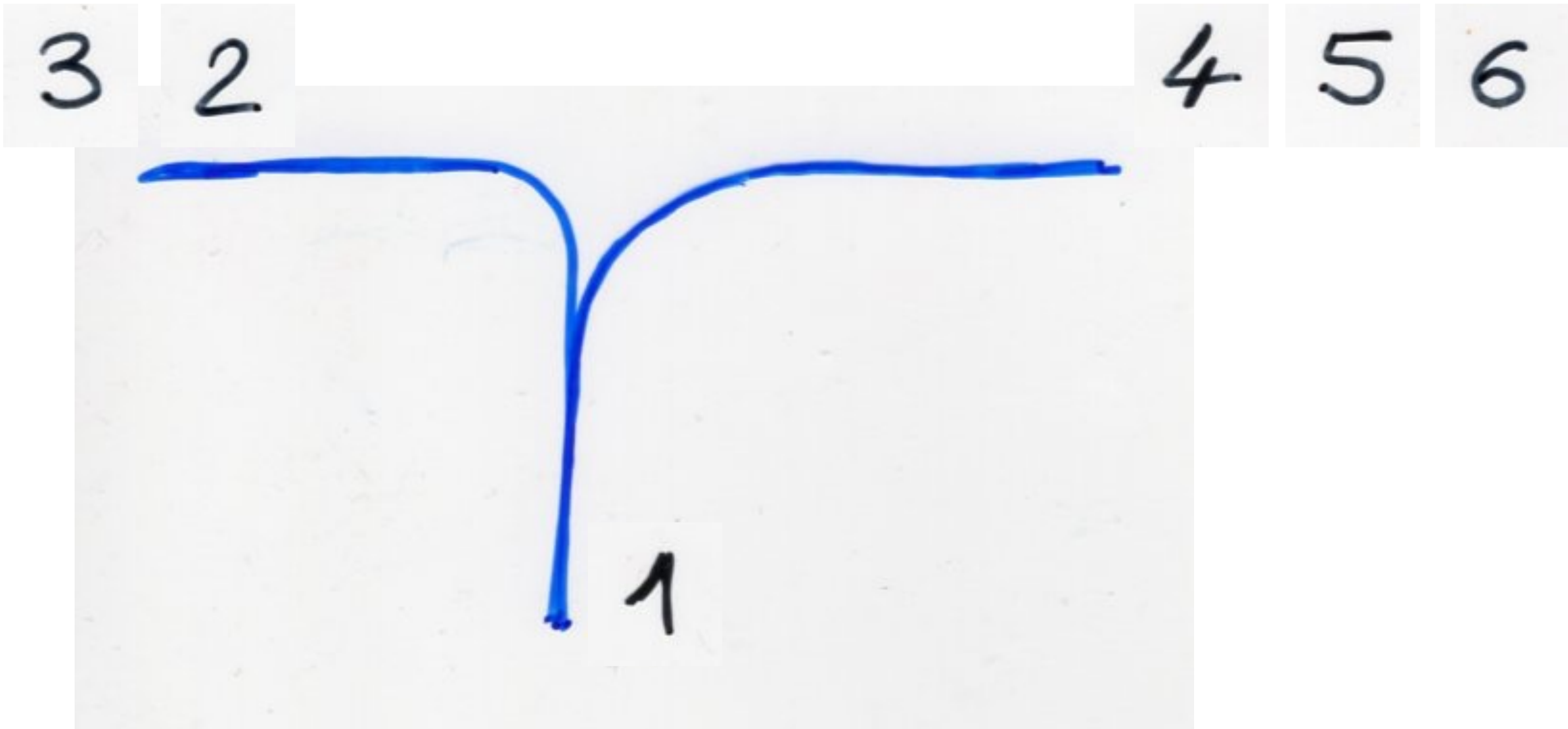
Permutations sortable with one stack



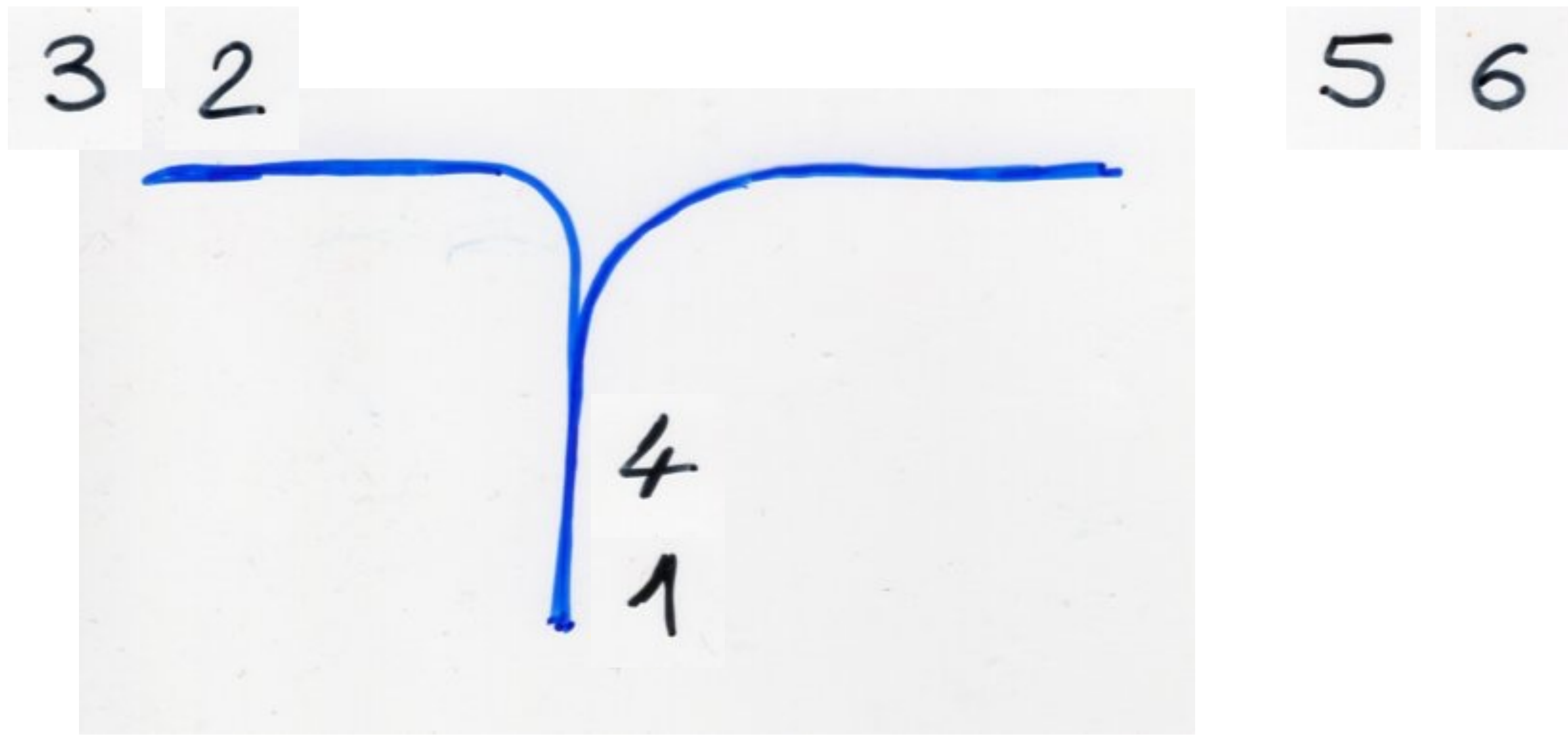
Permutations sortable with one stack



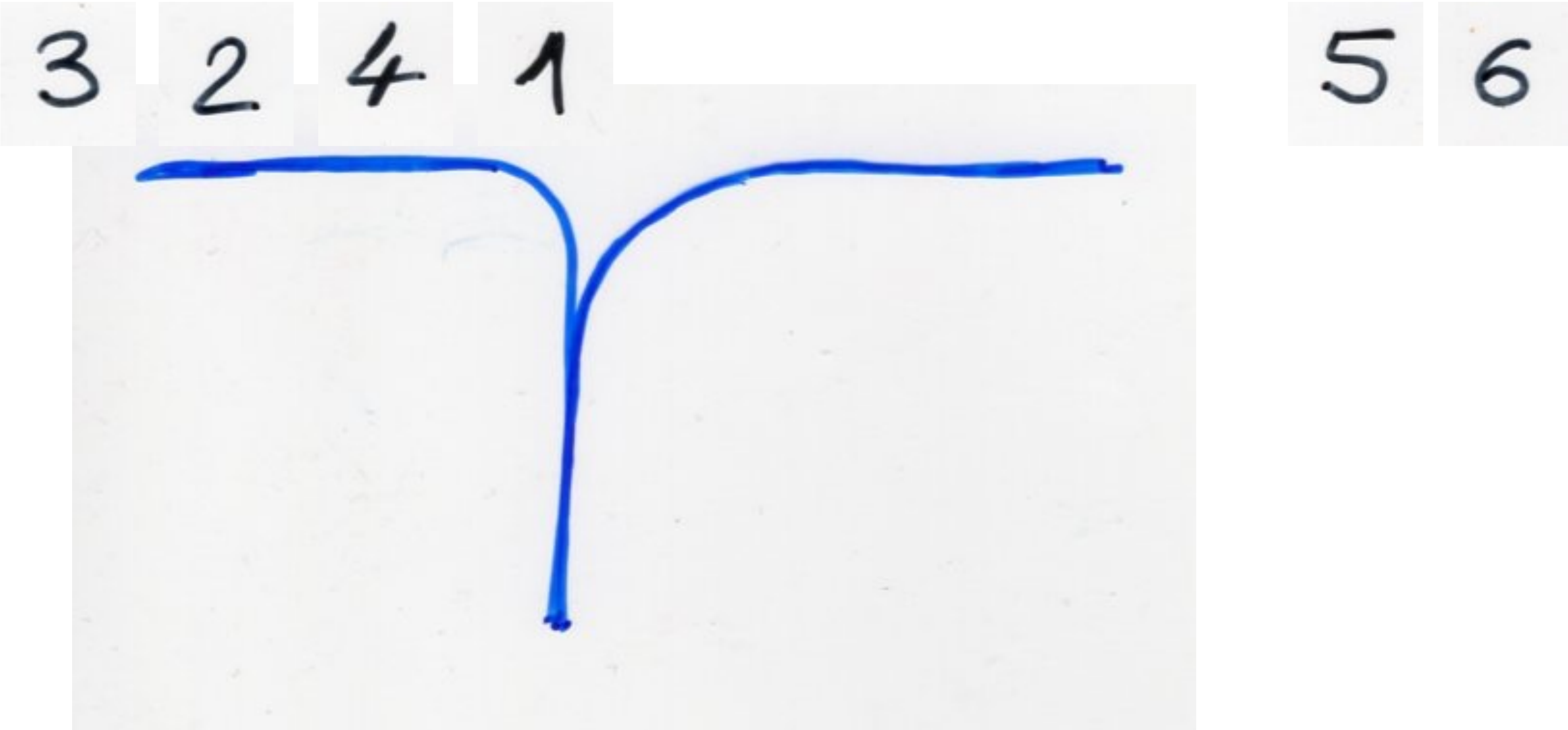
Permutations sortable with one stack



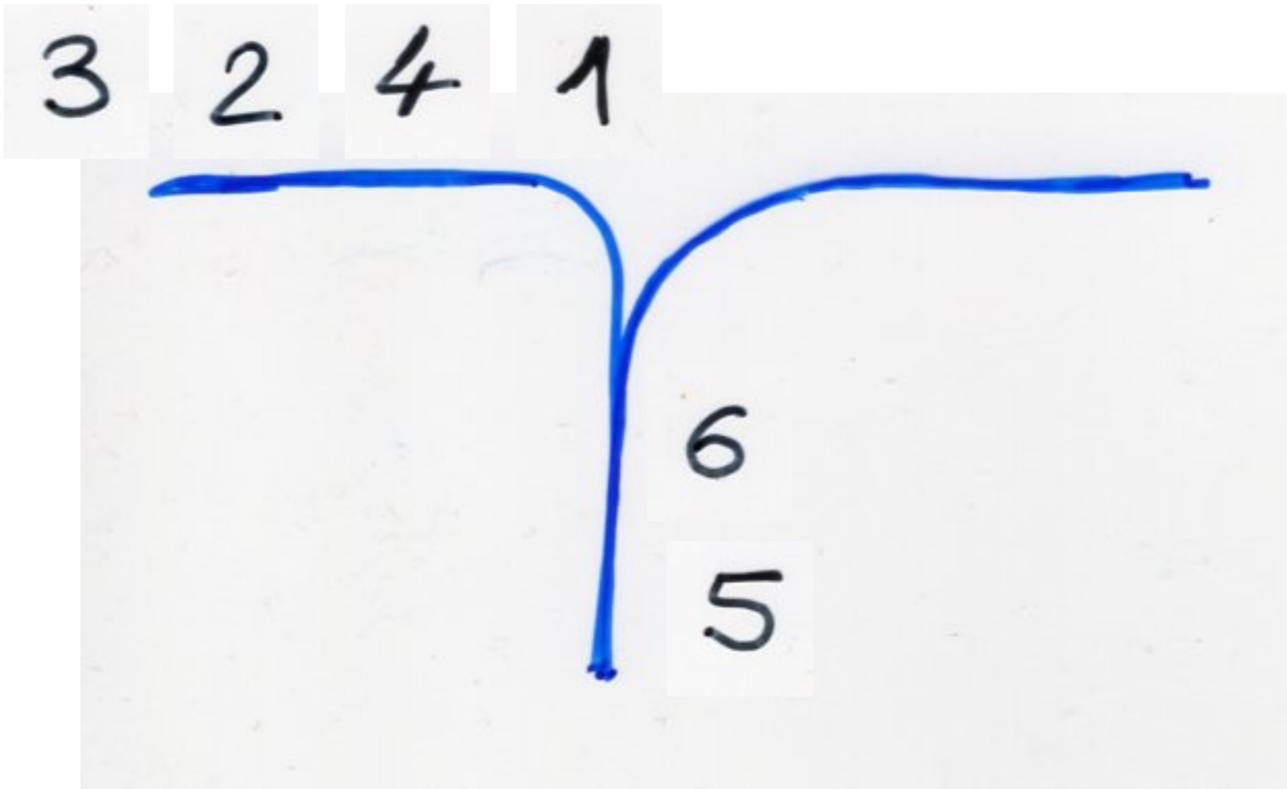
Permutations sortable with one stack



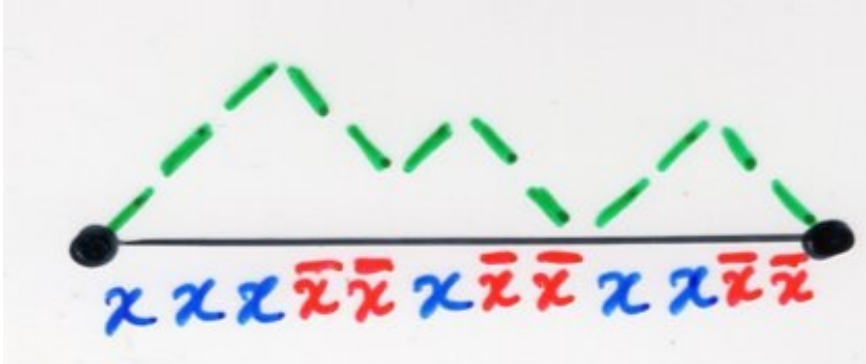
Permutations sortable with one stack

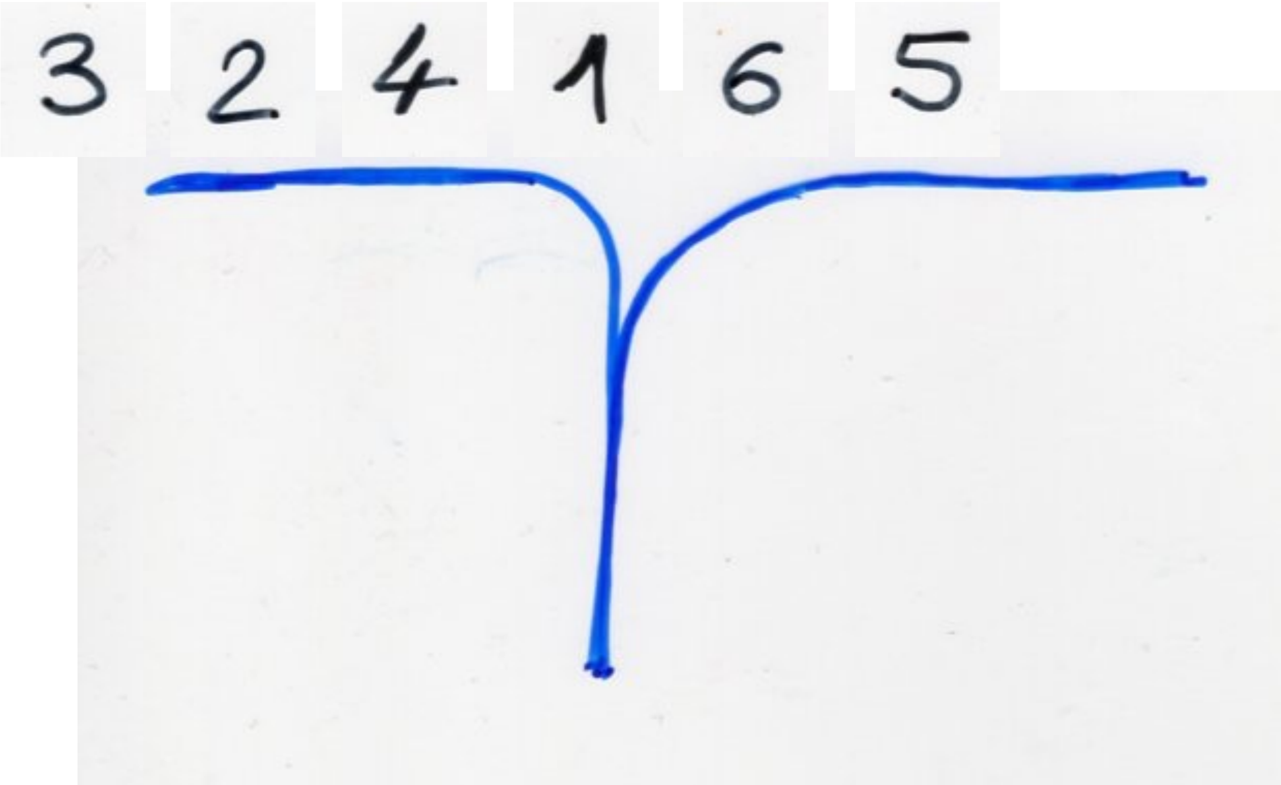


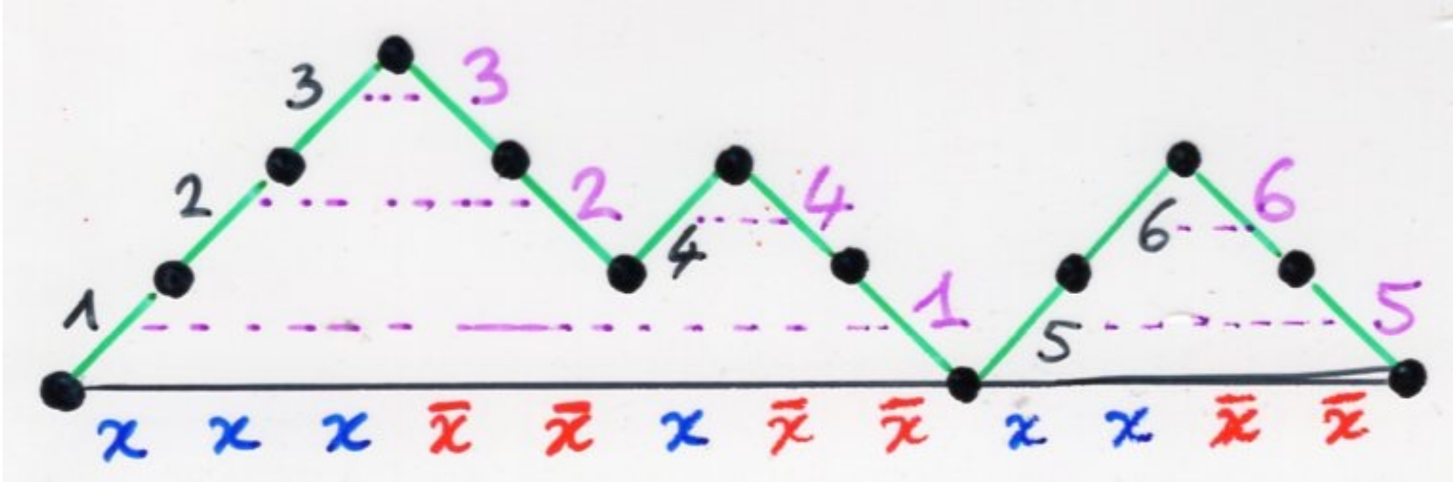
Permutations sortable with one stack



Permutations sortable with one stack



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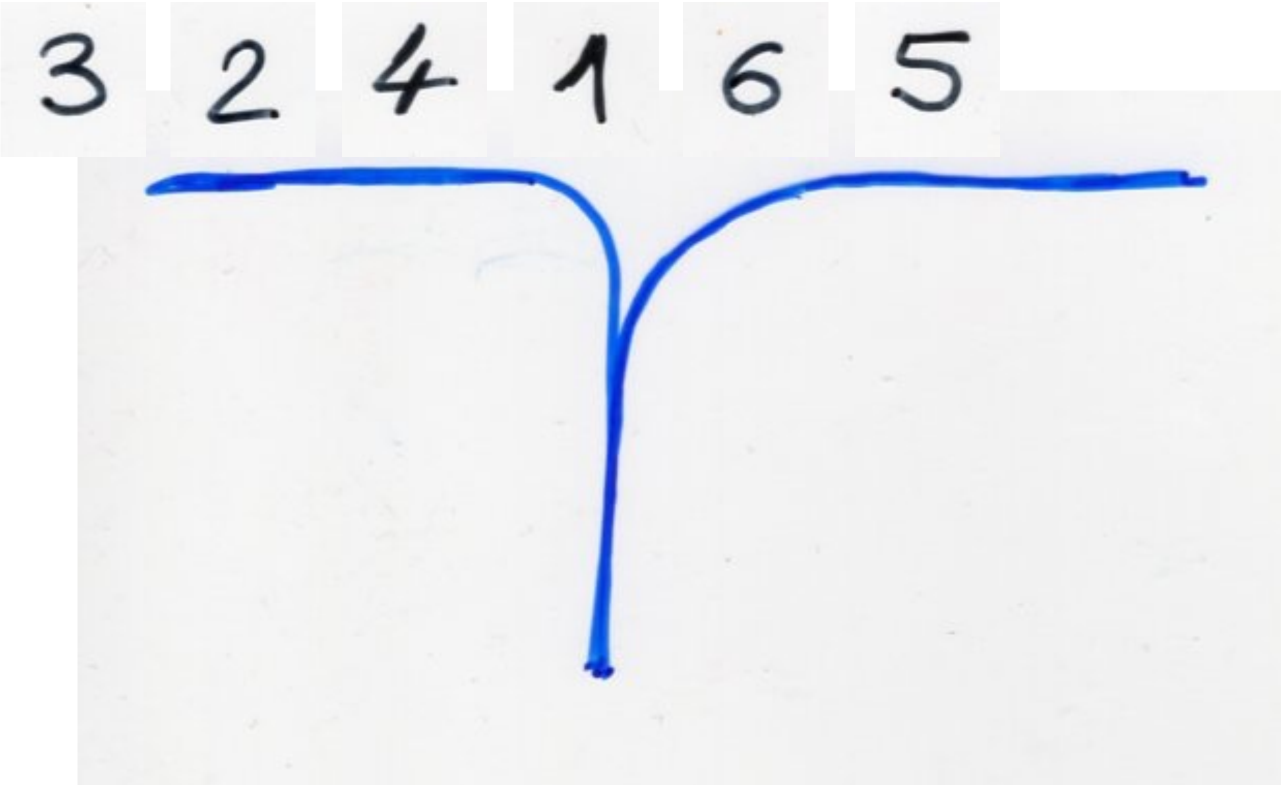
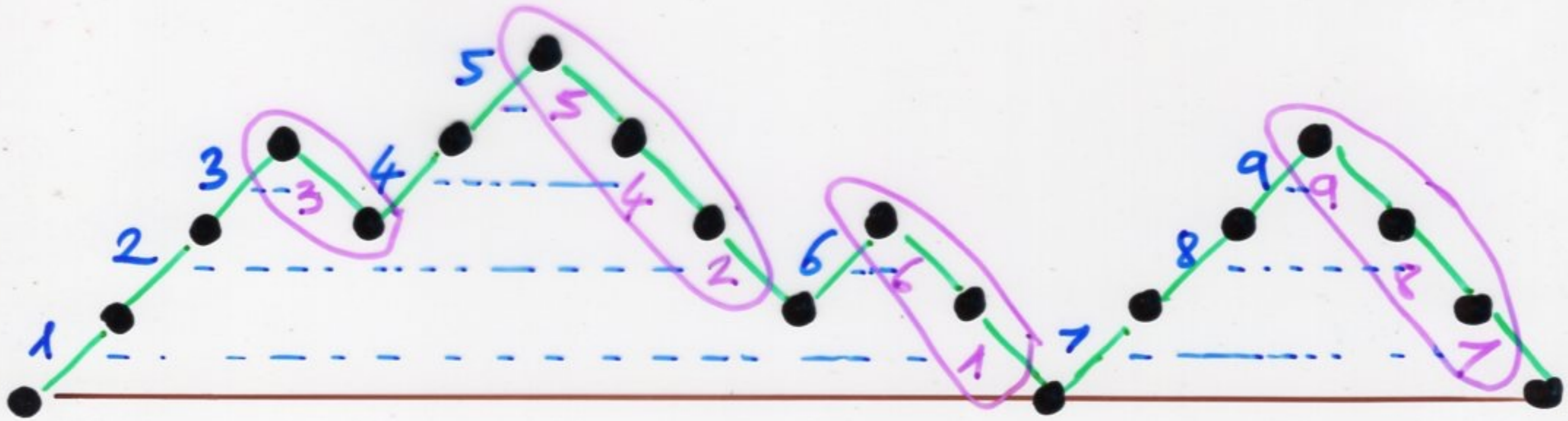

3 2 4 1 6 5

relation

Catalan permutations

non-crossing partition of $\{1,2,\ldots,n\}$



blocks (3) (5,4,2) (6,1) (9,8,7)

forbidden
pattern

$\sigma = \sigma_1 \ldots\ldots \sigma_n \in \mathfrak{S}_n$

$\tau = \sigma_{i_1} \ldots \sigma_{i_k}$

subsequence

$st(\tau) \in \mathfrak{S}_k$

(standardization)

<u>example</u>:

$\sigma = 5\,\underbrace{3}\,4\,\underbrace{1}\,6\,\underbrace{8}\,2\,\underbrace{7}$

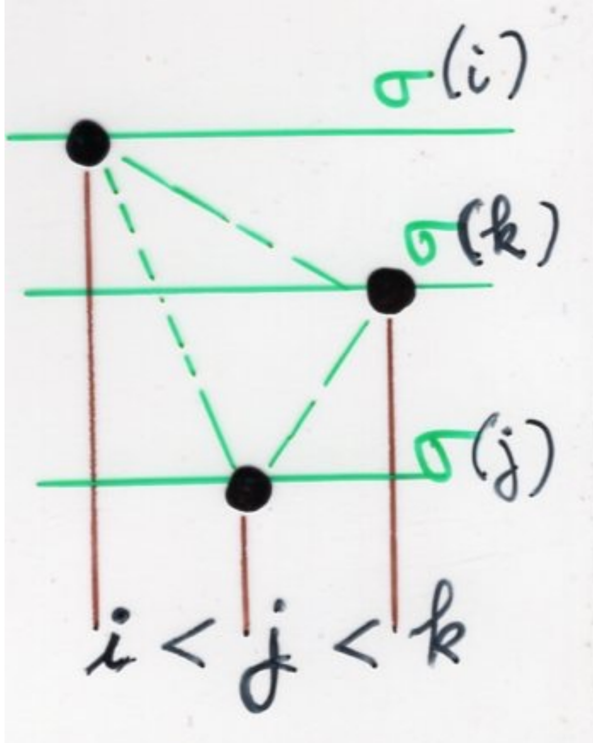
$\tau = 3\ 1\ 8\ 7$

$st(\tau) = 2\,1\,4\,3$

forbidden pattern

Permutations sortable with one stack

$\Longleftrightarrow$ (3 1 2)-avoiding



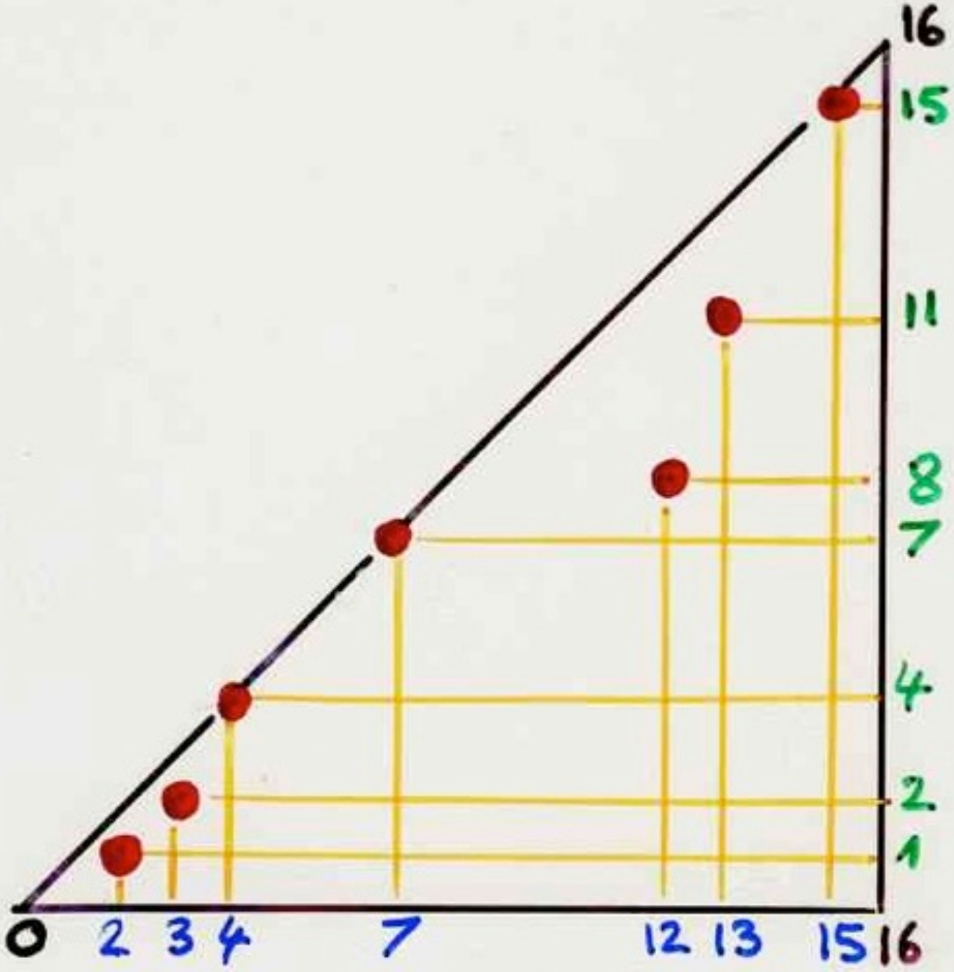
(321) - avoiding permutations

C_n Catalan numbers

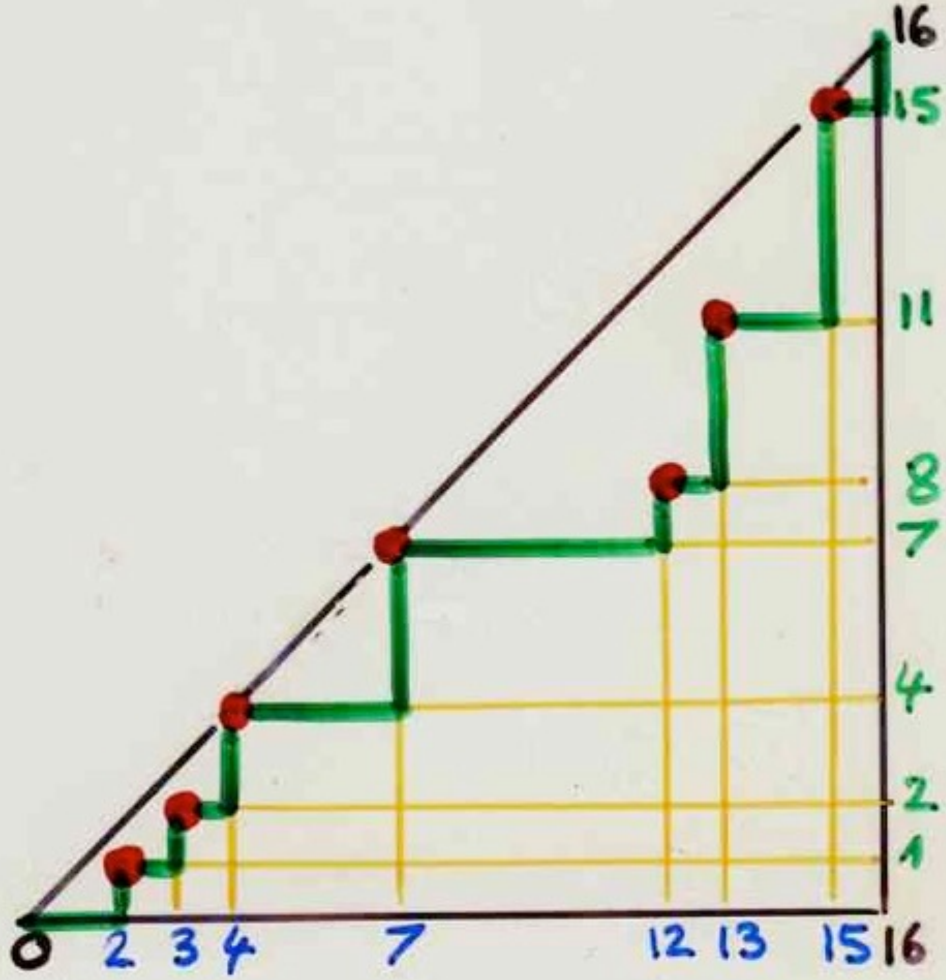
→ complements

basis of the Temperley - Lieb algebra with n generators

pair of sequences



$$\begin{array}{ccccccccccccccccc} & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\ & & \geq & & \geq & & \geq & & \geq & & \geq & & \geq & & \geq & & \\ 1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n \end{array}$$



$$\begin{array}{cccccccccccccccc} & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & \\ & & \geq & & \geq & & \geq & & \geq & & \geq & & \geq & & \geq & \\ 1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq n \end{array}$$

bijective proof for

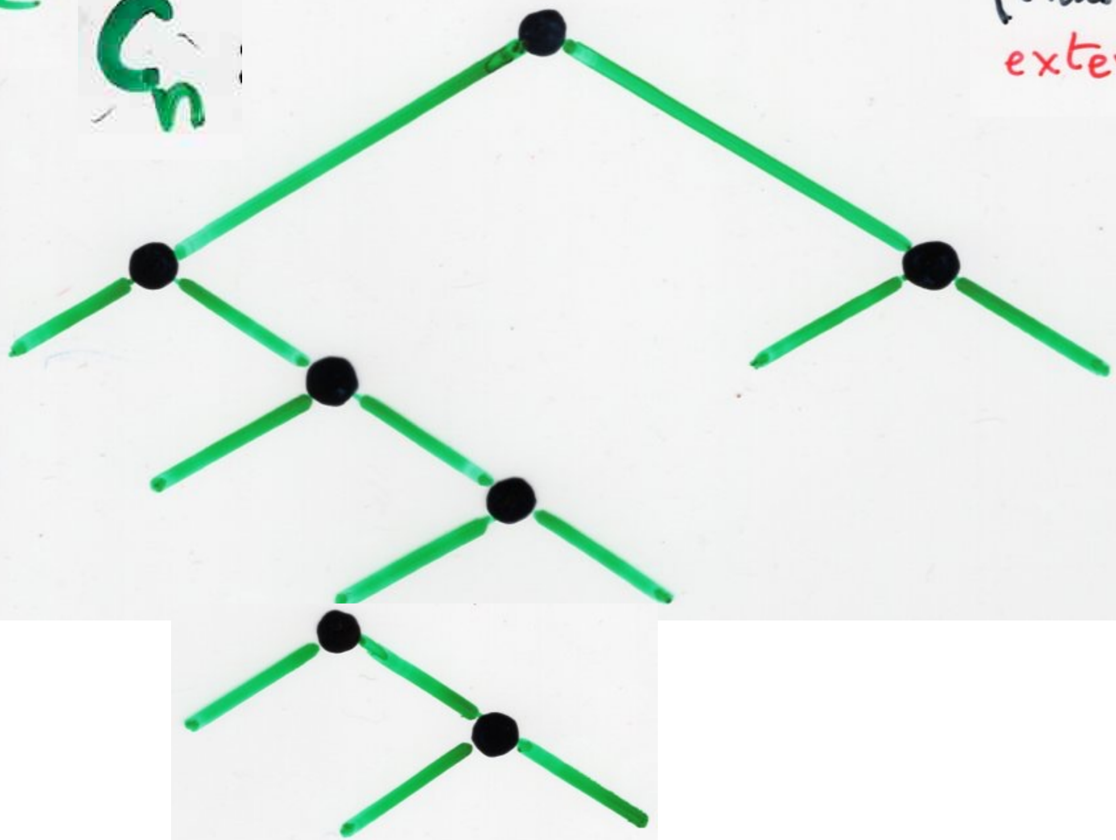
the multiplicative recurrence
of Catalan numbers

$$2(2n+1)C_n = (n+2)C_{n+1}$$

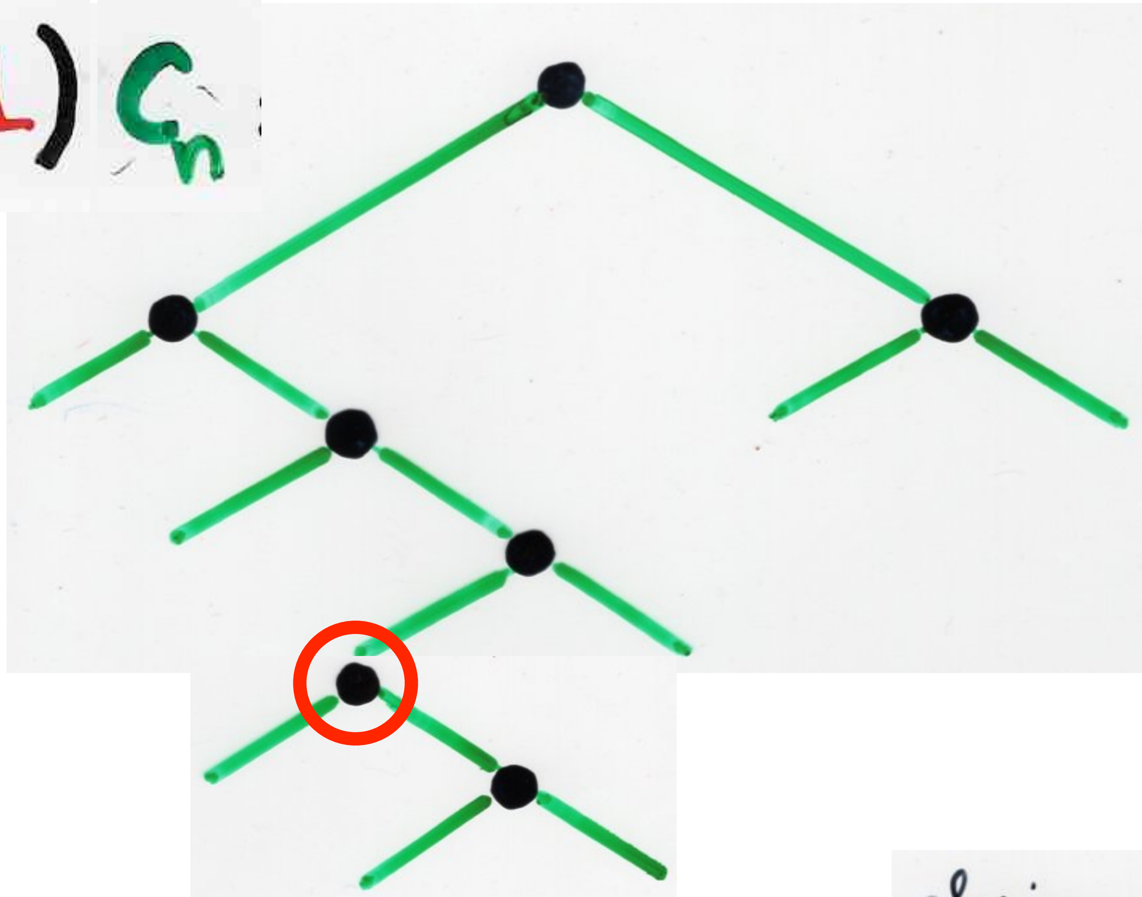
choice of a complete binary tree

C_n :

with n internal vertices (thus $(n+1)$ external vertices)

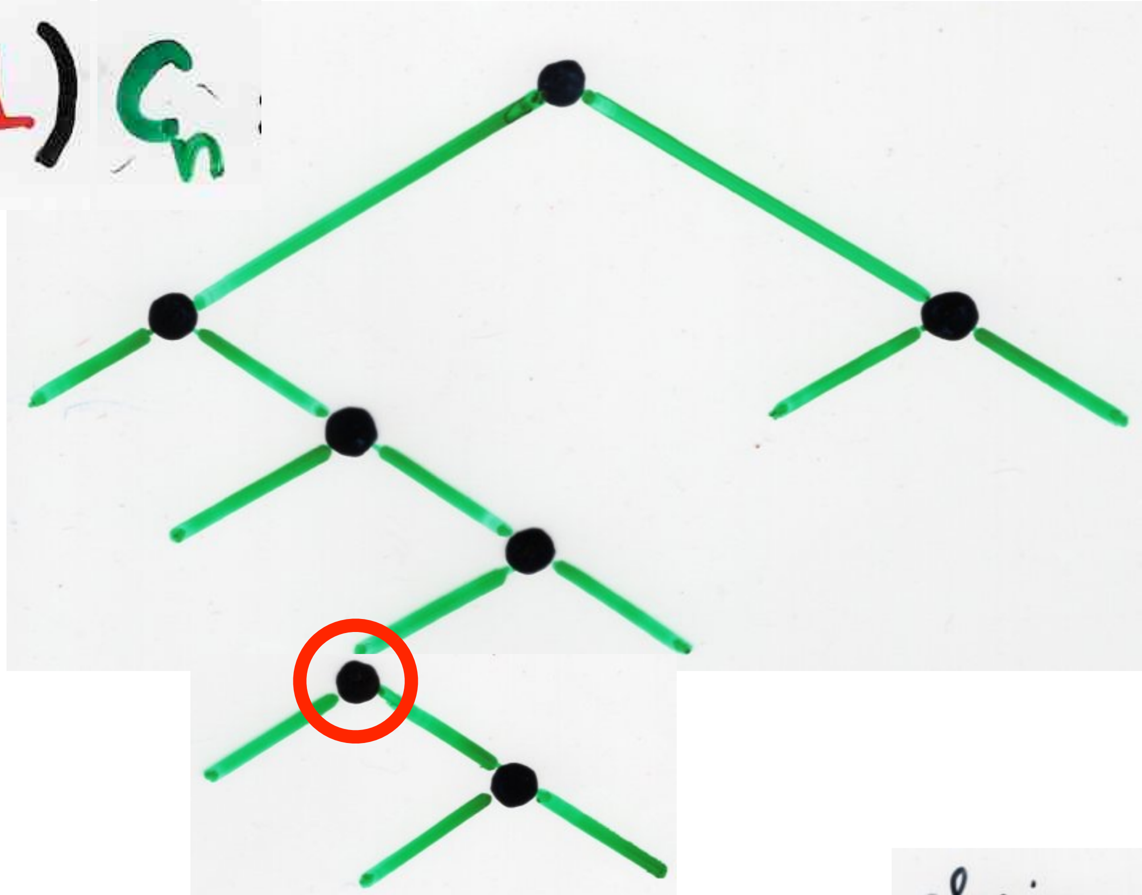


$(2n+1)\, C_n$:



choice of a vertex (internal or external)

$(2n+1)\, C_n$:



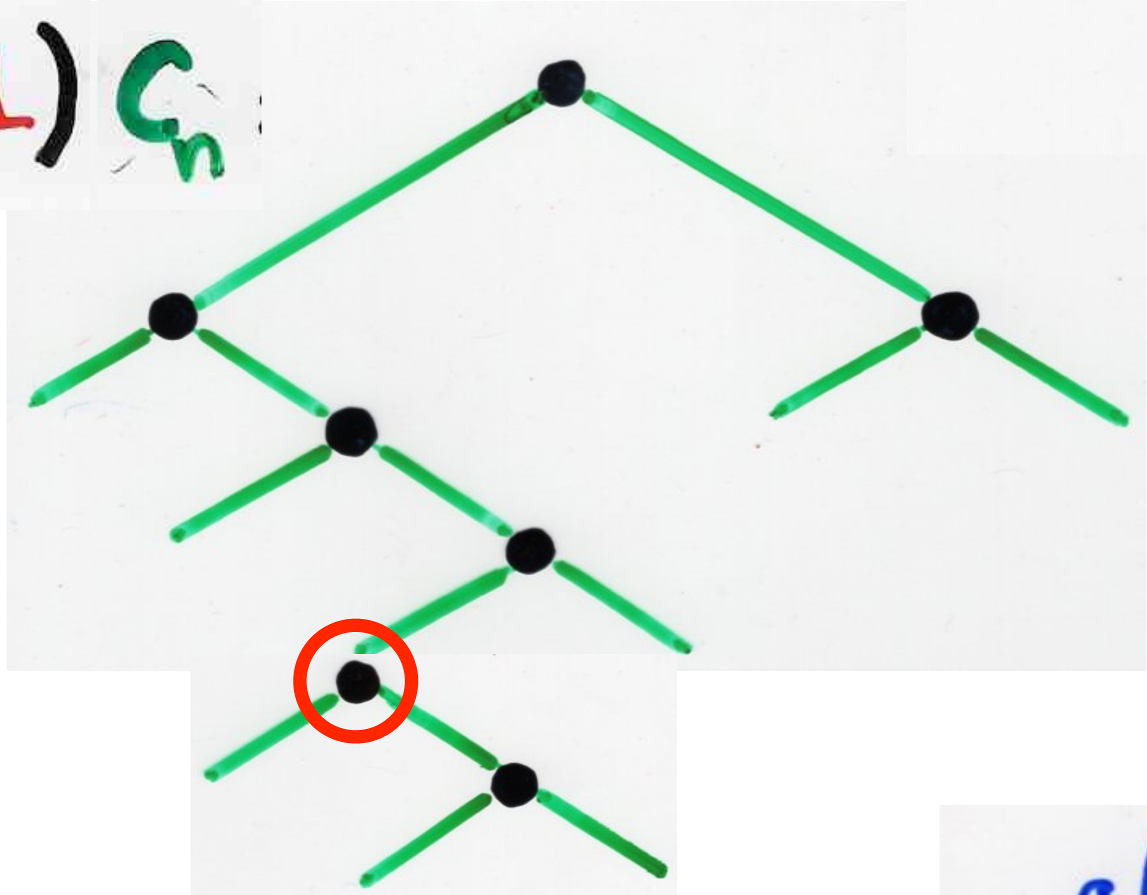
sub-tree

choice of a vertex (internal or external)

$2\,(2n+1)\,C_n$:

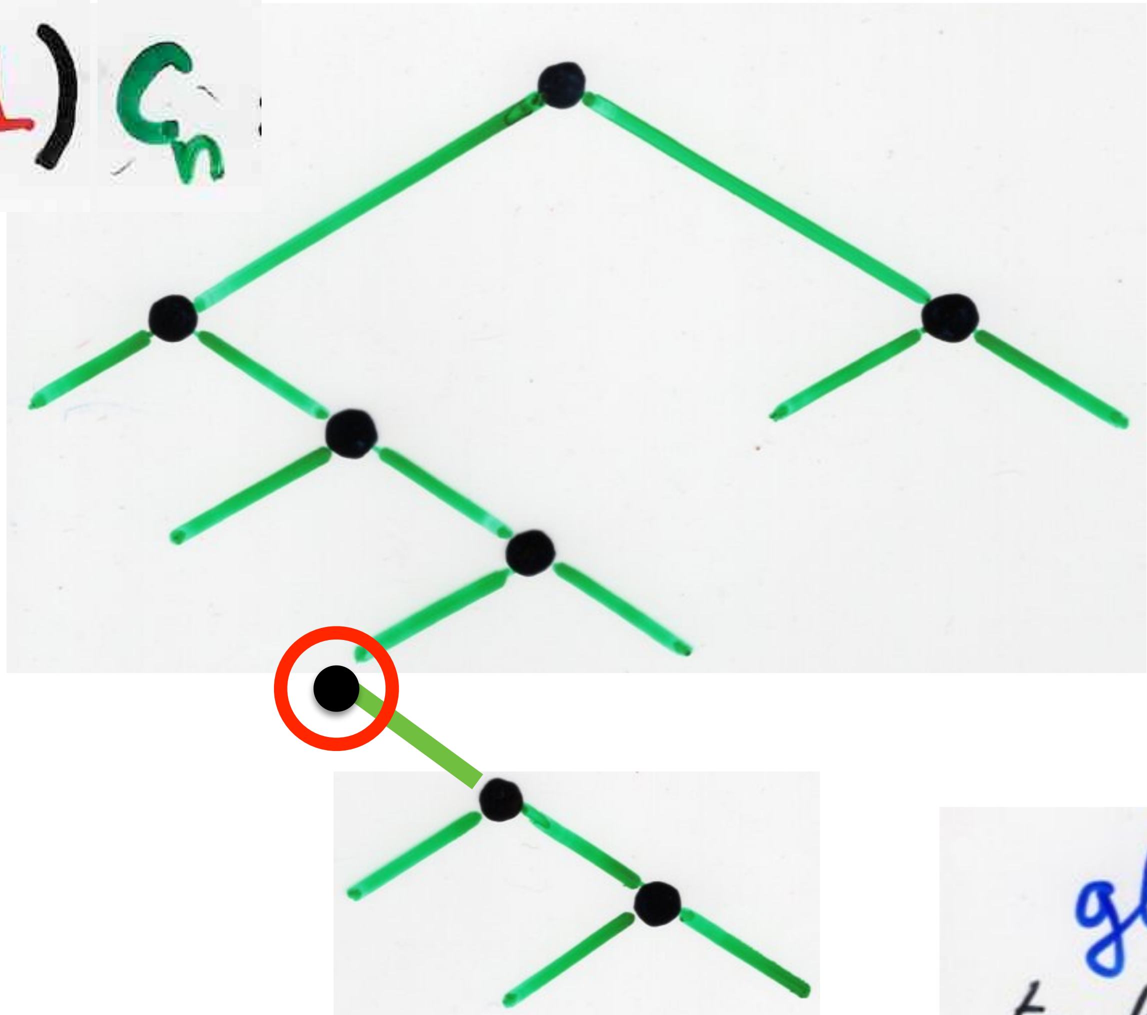
choice left

right



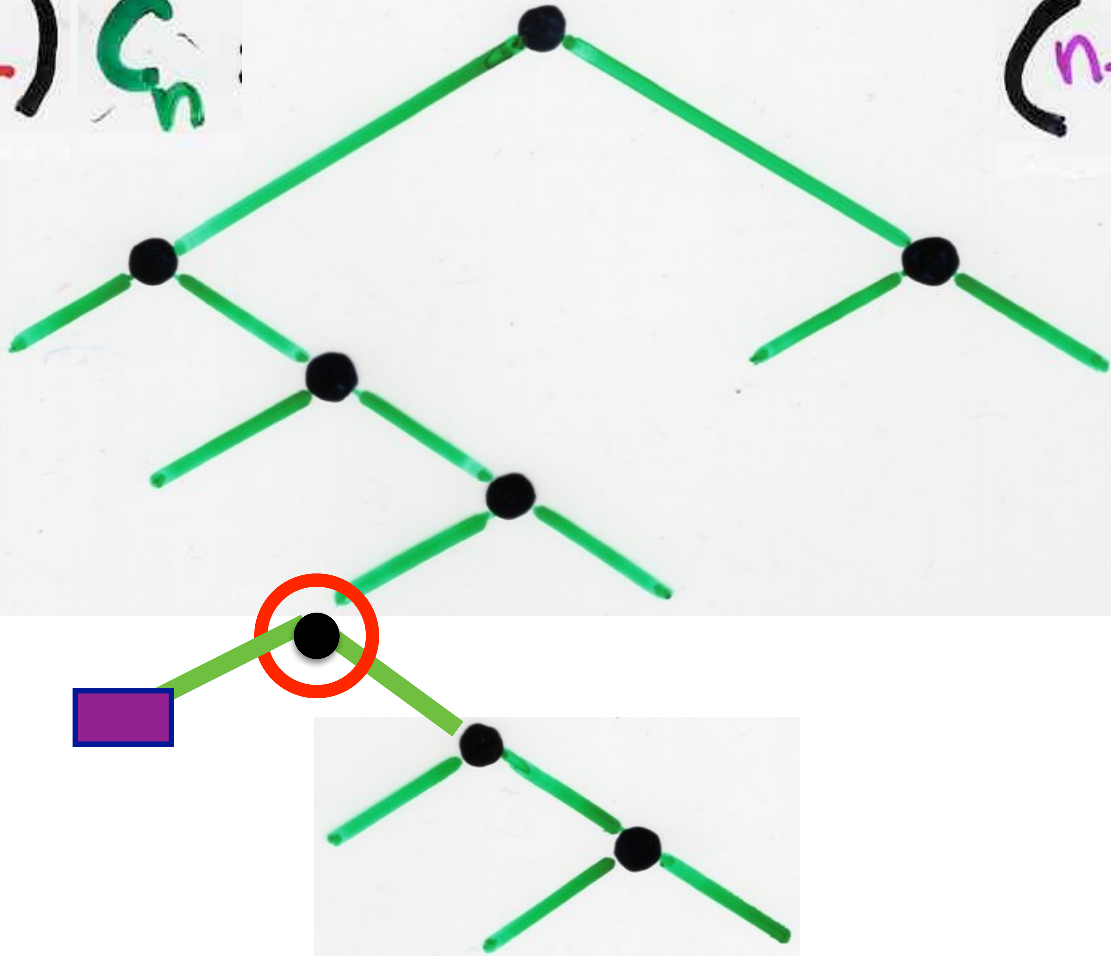
gliding to the right

$2\,(2n+1)\,C_n$:



gliding
to the right

$2\,(2n+1)\,C_n$ $\qquad$ $(n+2)\,C_{n+1}$



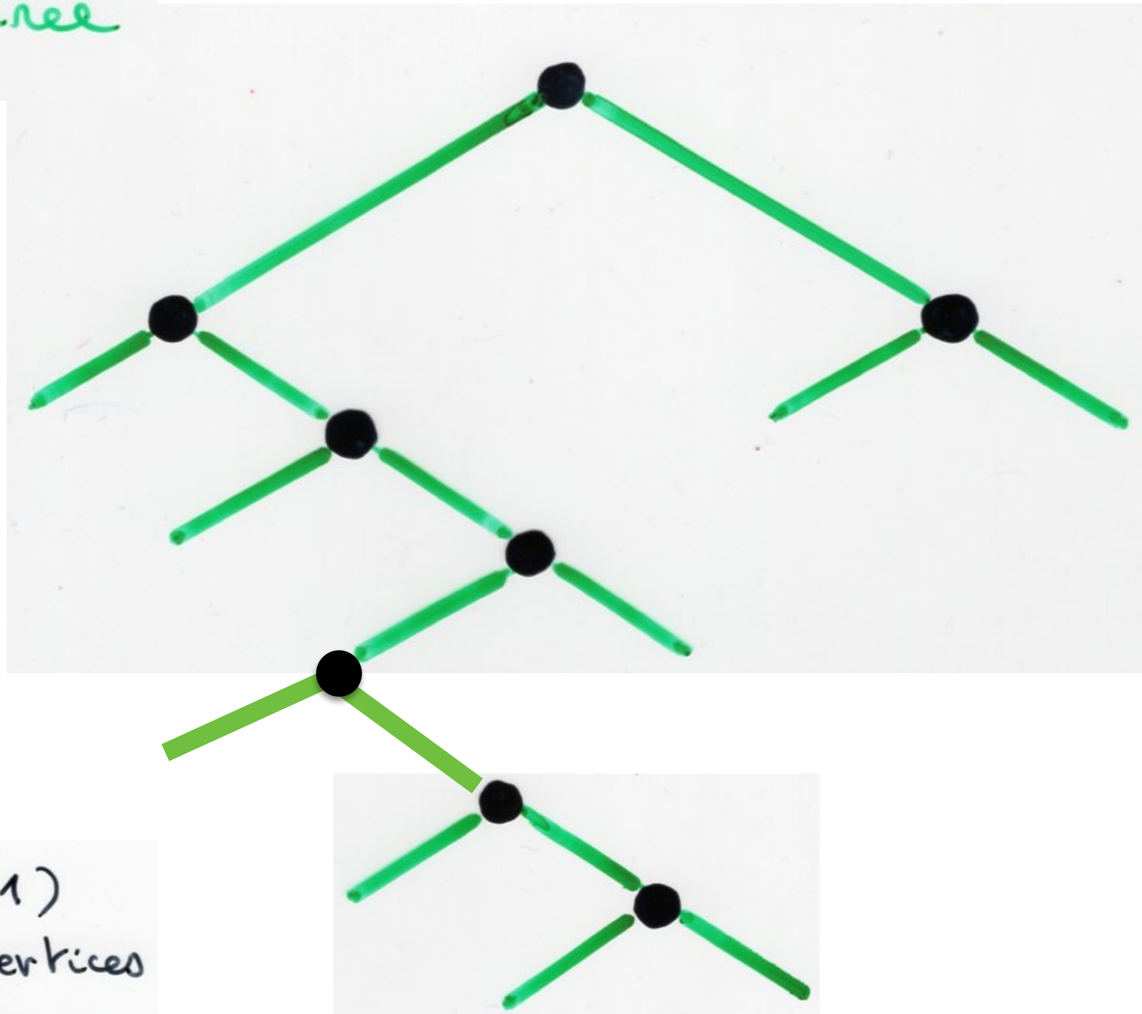
add a leaf
(= external vertex)

reciprocal
bijection

$$2(2n+1)C_n = (n+2)C_{n+1}$$

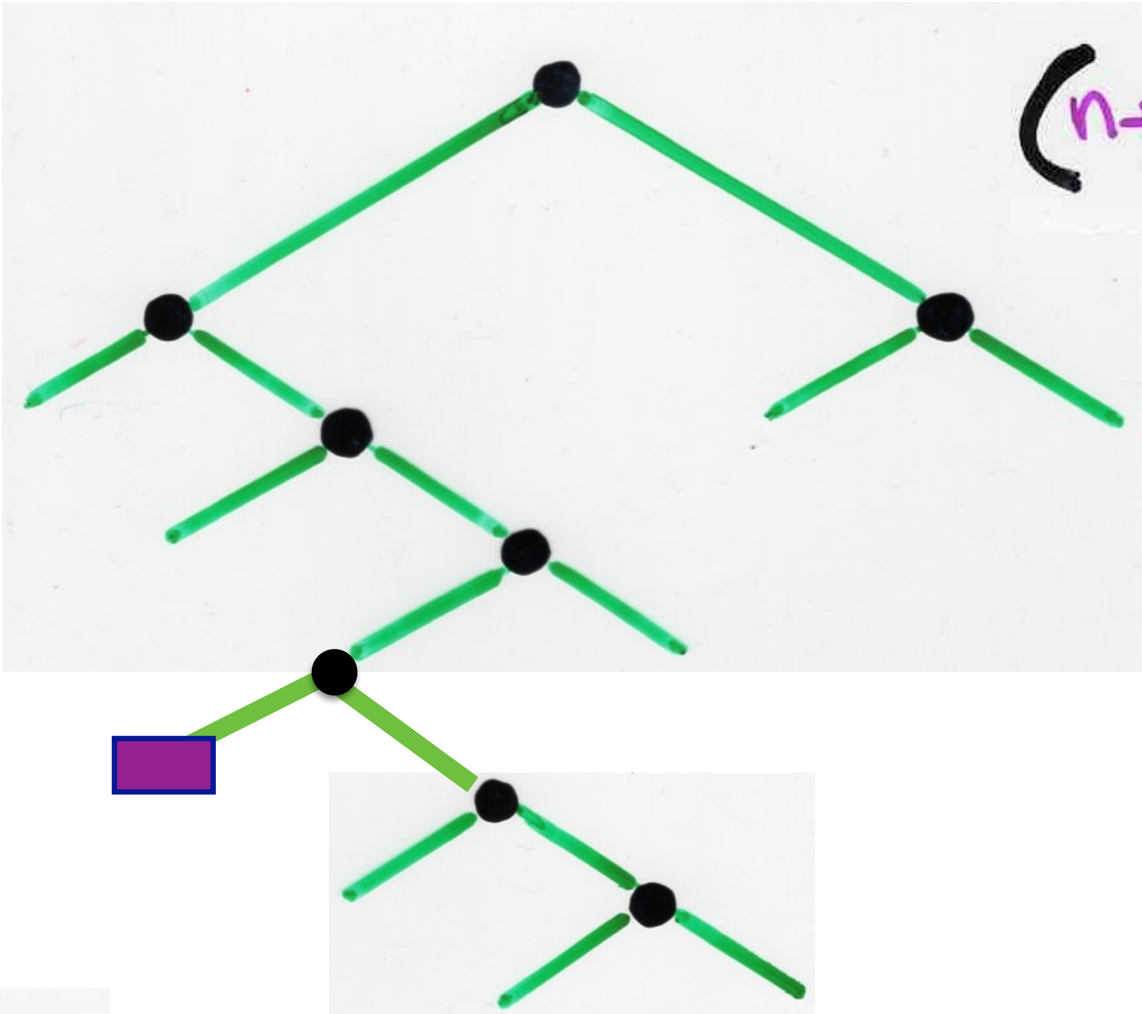
choice of a
binary tree

C_{n+1}



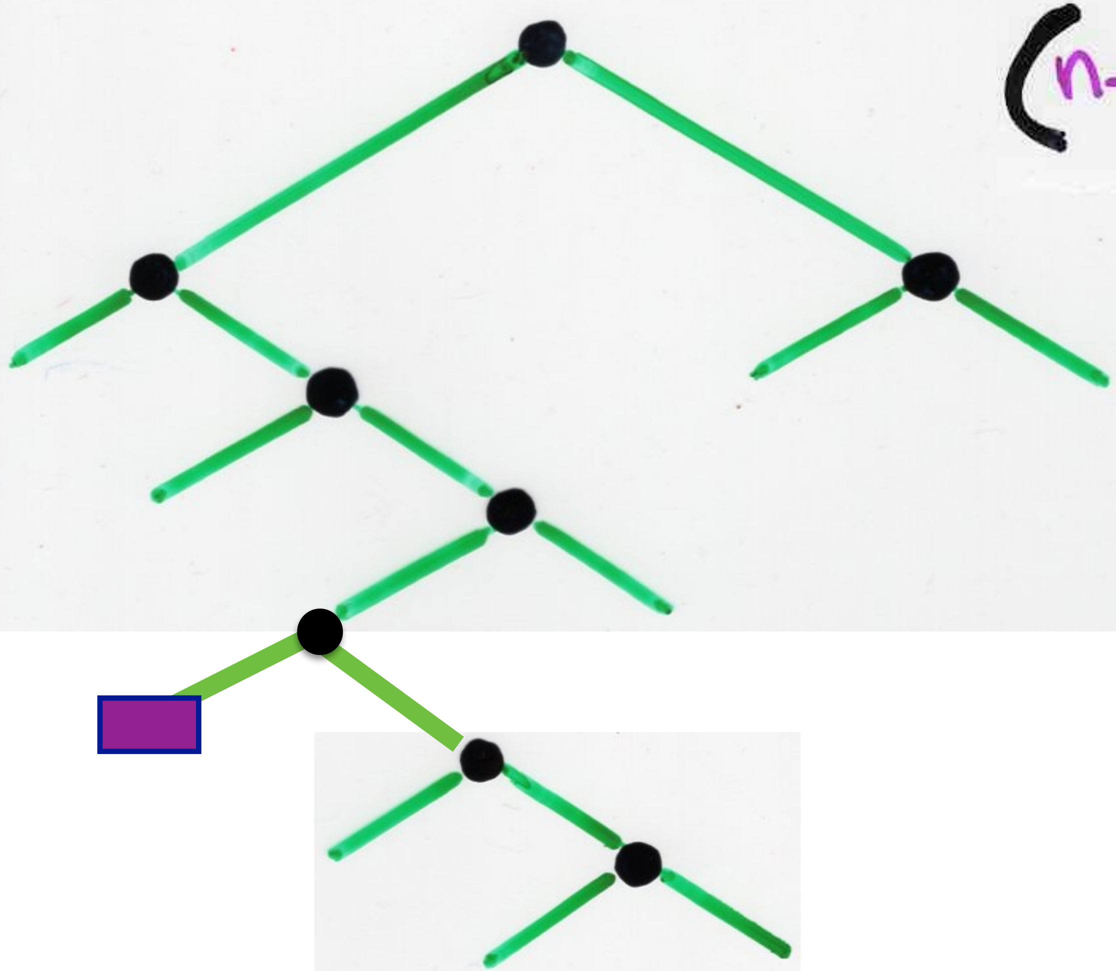
with $(n+1)$
internal vertices
(thus $(n+2)$
external vertices

$(n+2)\ C_{n+1}$

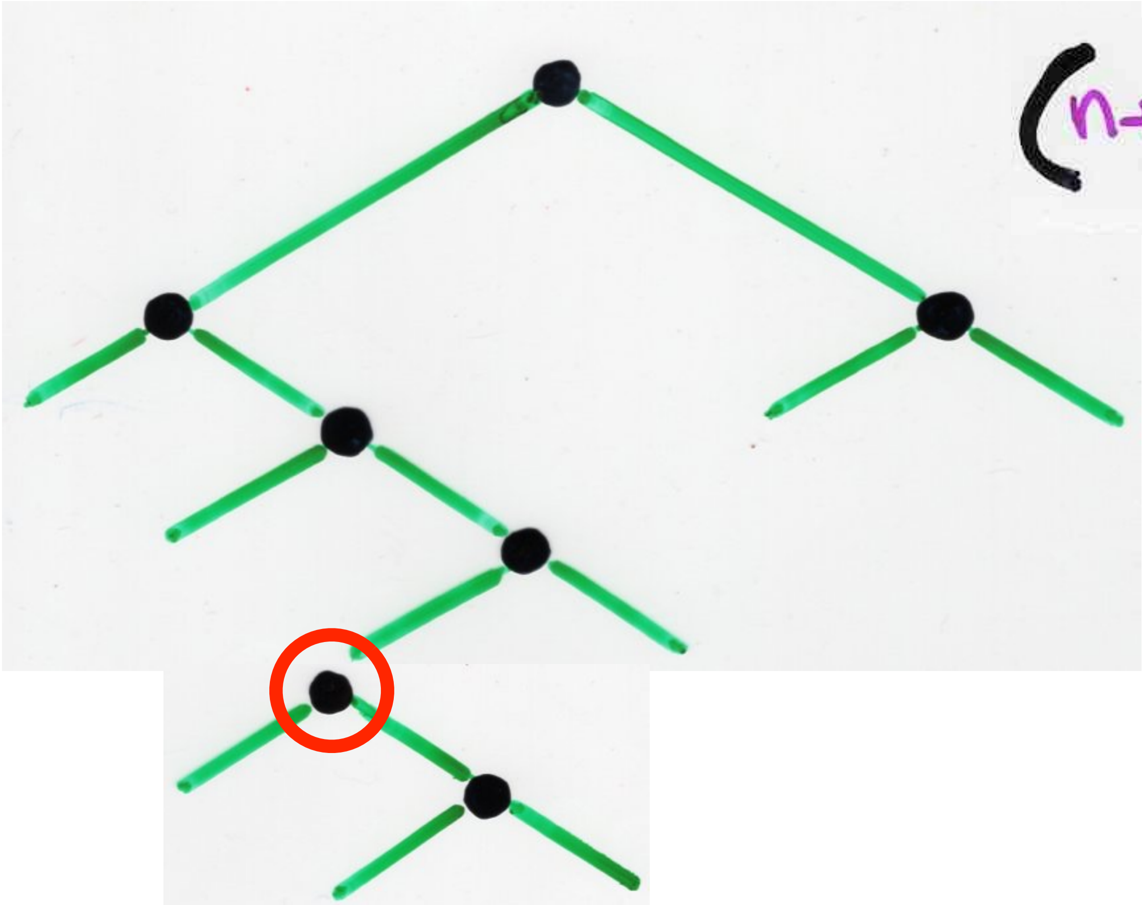


choice of a leaf (= external vertex)

$(n+2)\ C_{n+1}$



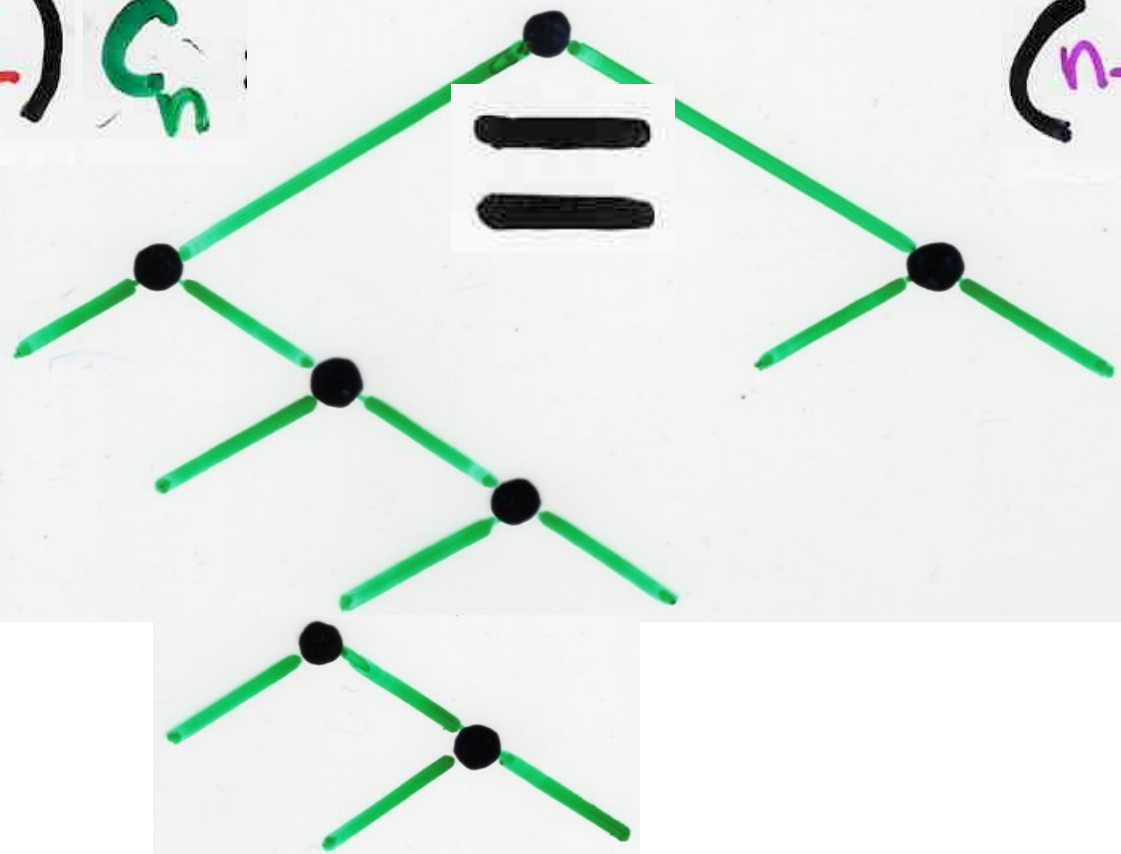
contraction



$(n+2)\ C_{n+1}$

contraction

$$2\,(2n+1)\,C_n = (n+2)\,C_{n+1}$$



Touchard's identity

$$C_{n+1} = \sum_{0 \leqslant i \leqslant \lfloor n/2 \rfloor} \binom{n}{2i} C_i \, 2^{n-2i}$$

C_i

choice of a Dyck path

length $2i$ $(=6)$

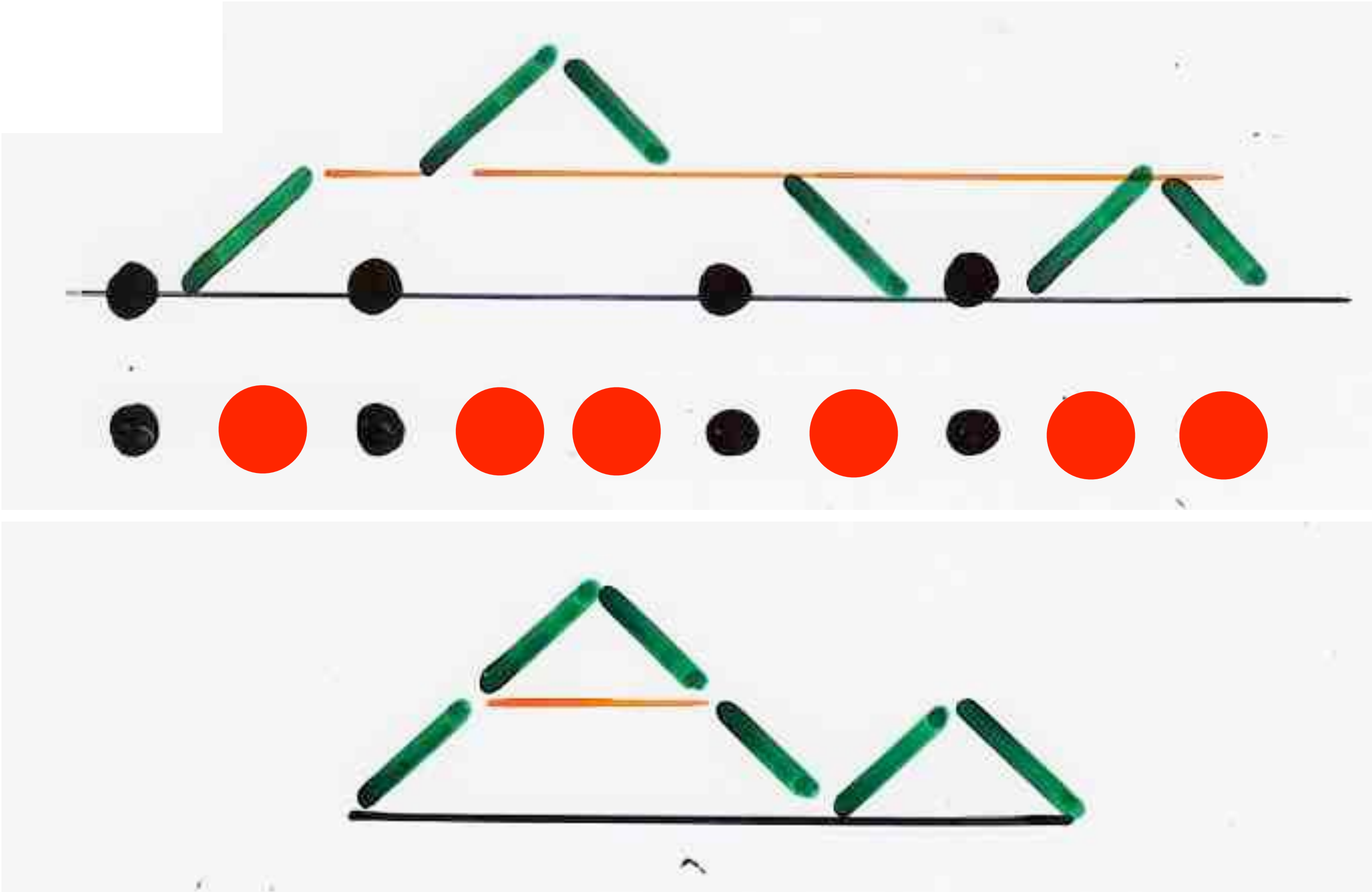
$$\binom{n}{2i}$$

choice of a subset with $2i$ elements among n $(=10)$ elements

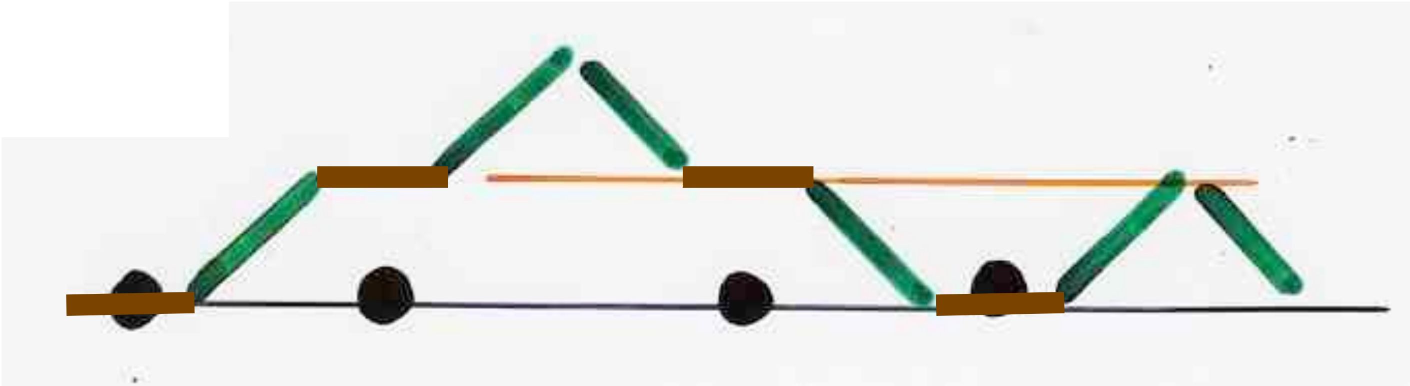
$$\binom{n}{2i}$$

choice of a subset with $2i$ elements among n $(=10)$ elements

$$\binom{n}{2i} \; C_i$$



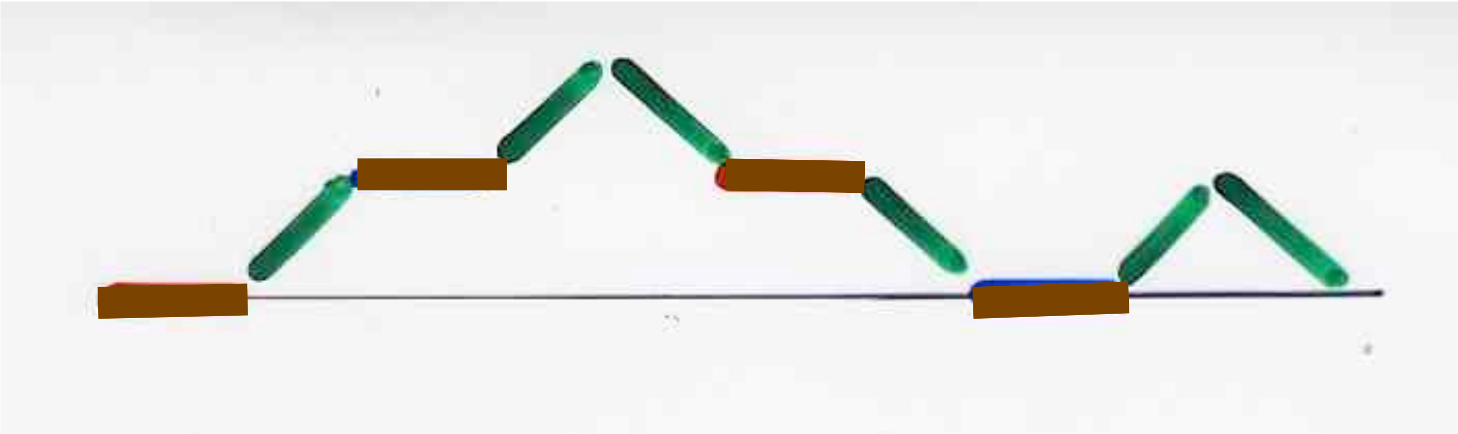
placement of the Dyck path
on the subset



placement of the East steps

Two colors on the East steps

$$2^{n-2i}$$



Two colors on the East steps

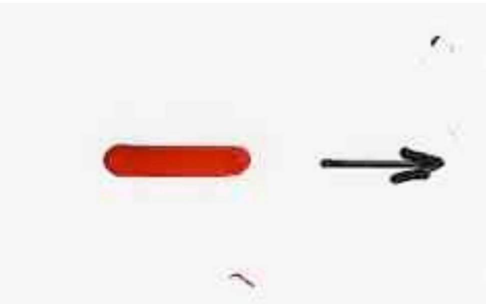
/ →

substitution

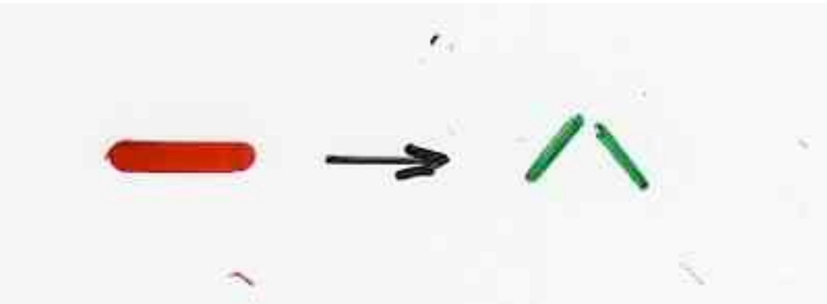
substitution

substitution

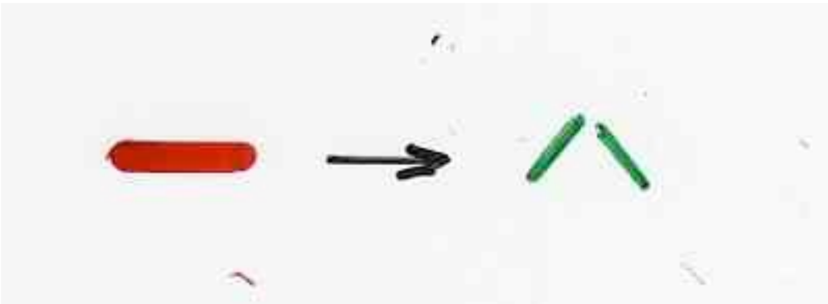
substitution

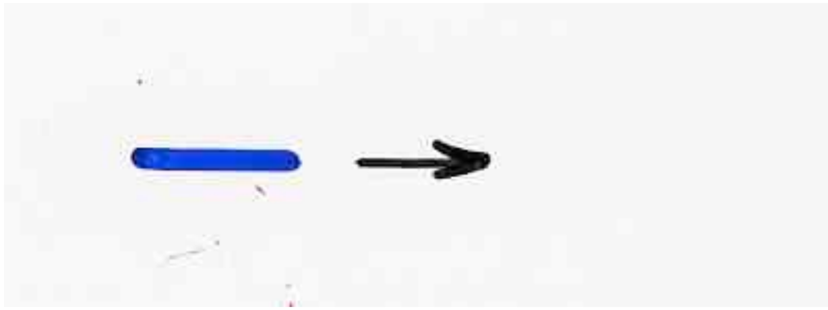


substitution

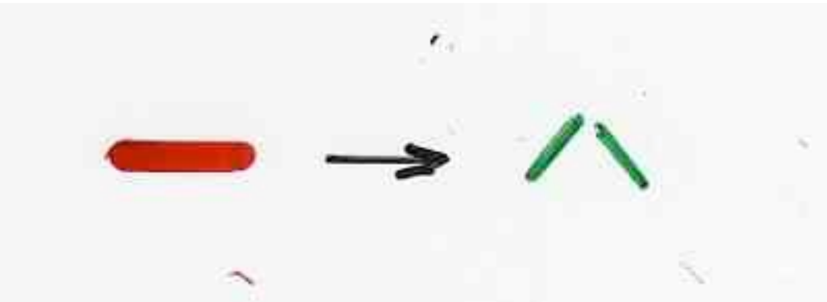


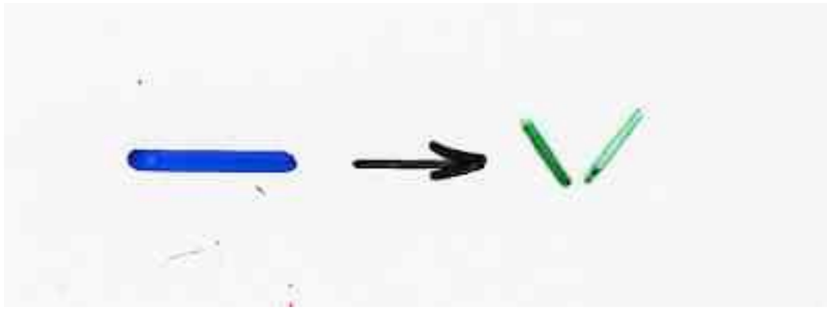
substitution



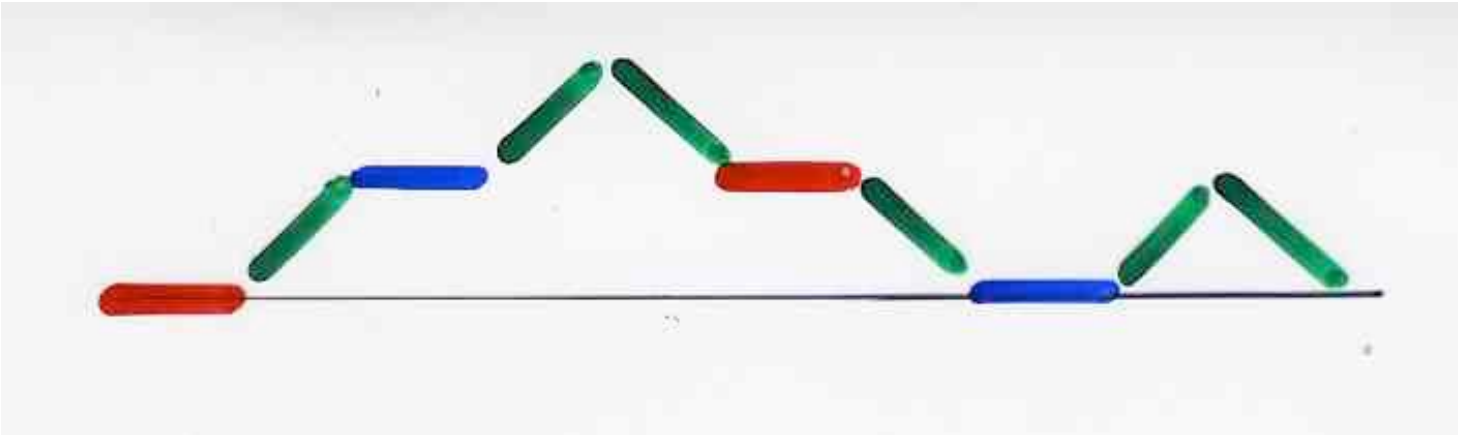


substitution

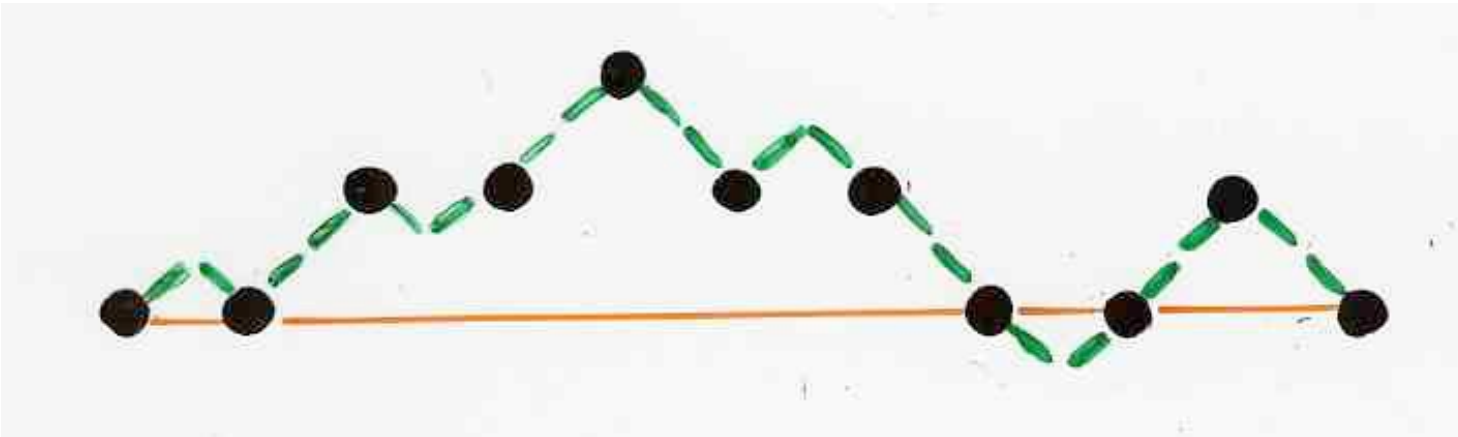


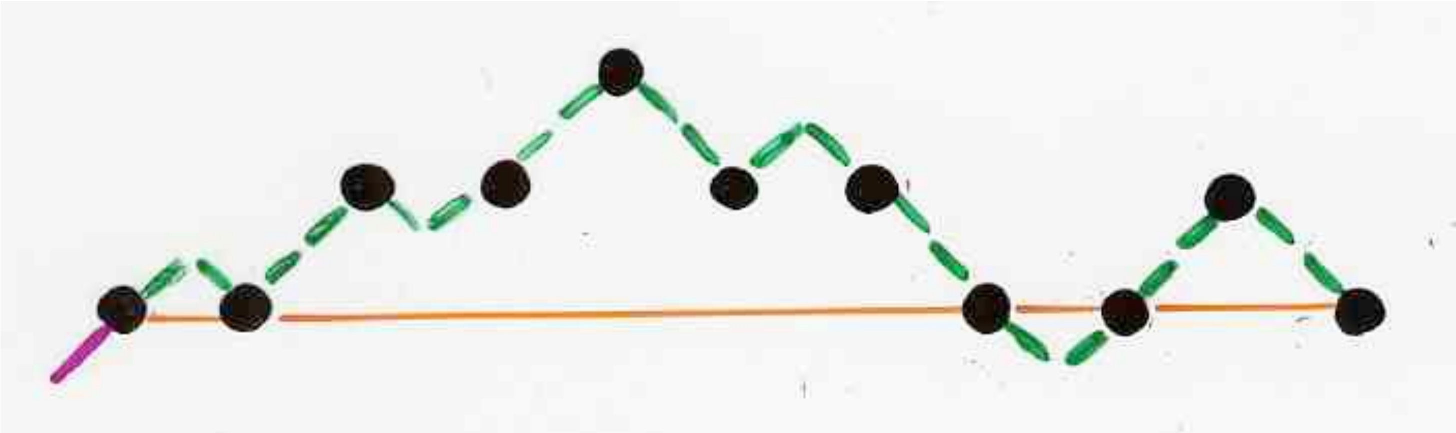


substitution

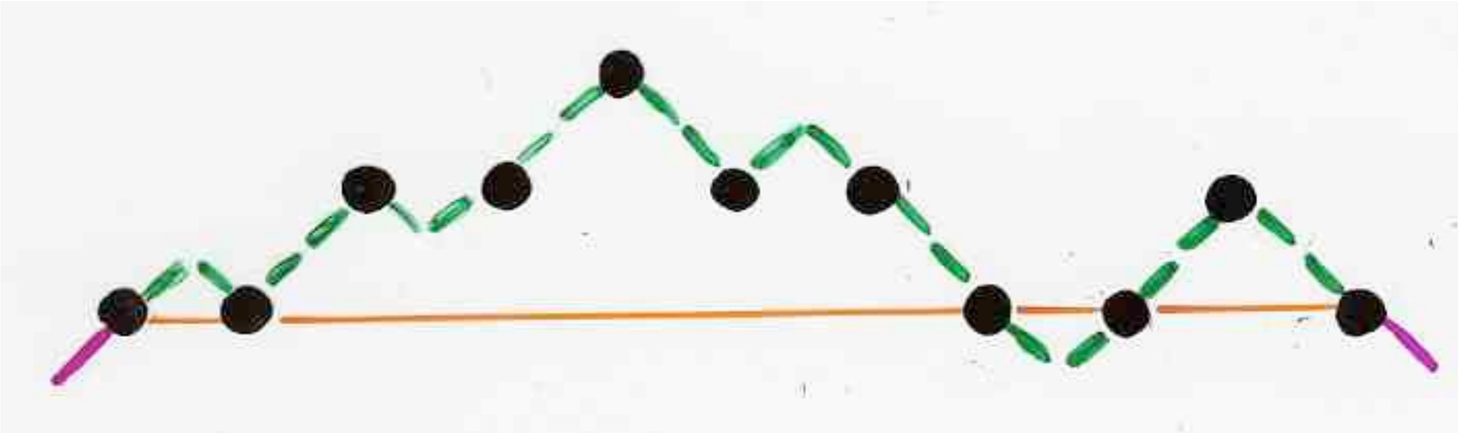


substitution



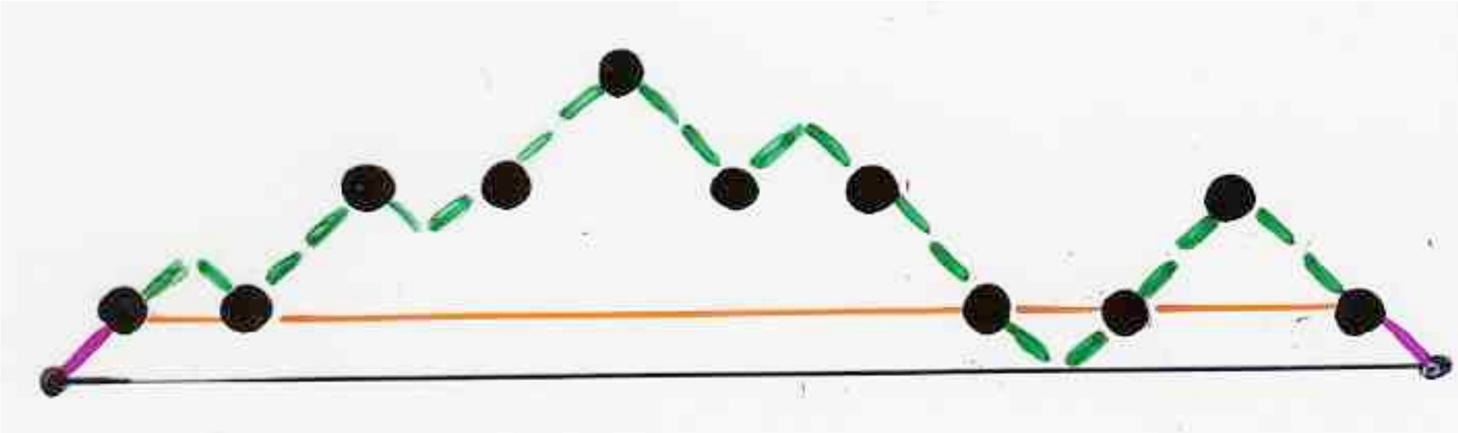


adding at the
begining of the path



adding at the end of the path

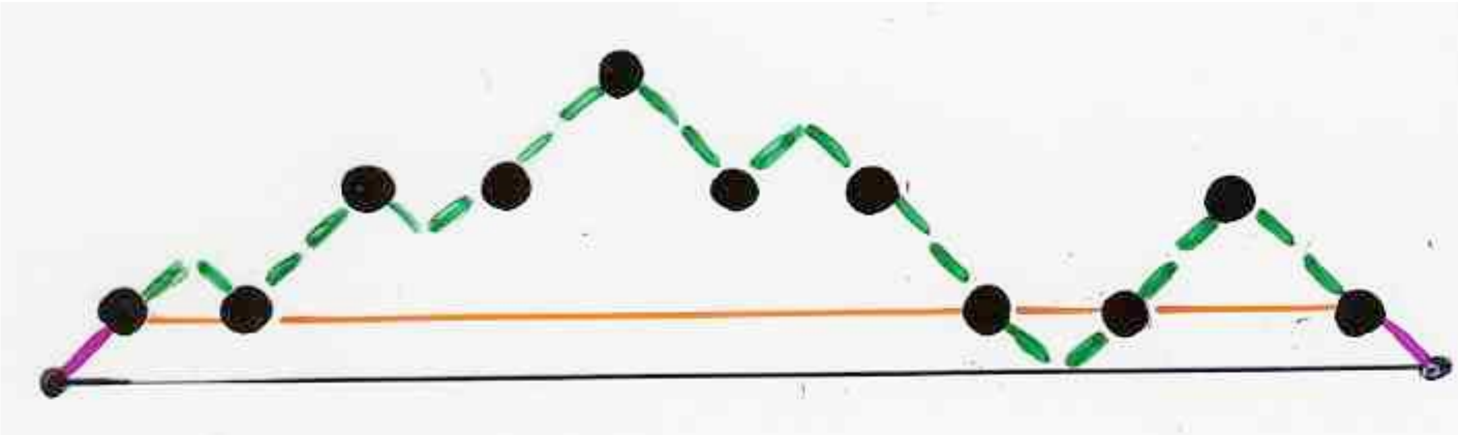
C_{n+1}



Dyck path
length $2i+2$

C_{n+1}

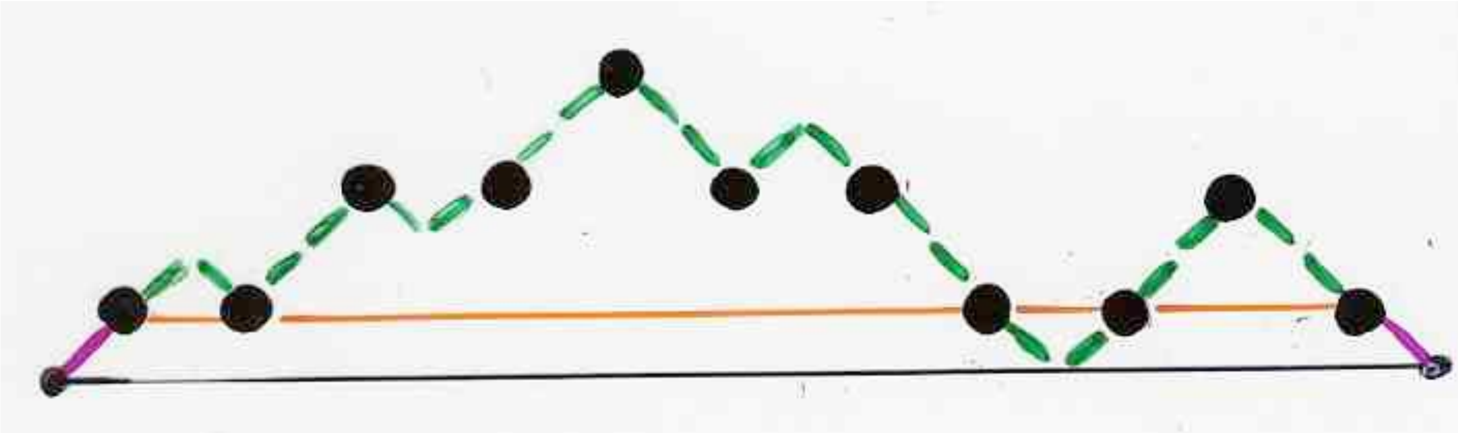
$\binom{n}{2i}$ C_i 2^{n-2i}



Dyck path

length $2i+2$

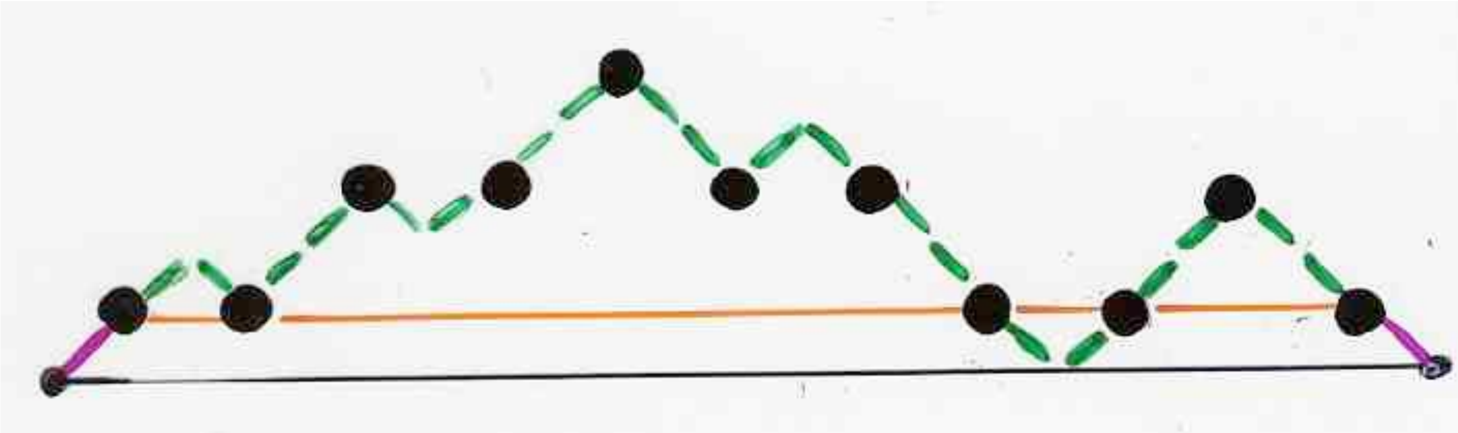
$$C_{n+1} = \sum_{0 \leq i \leq \lfloor n/2 \rfloor} \binom{n}{2i} C_i \, 2^{n-2i}$$

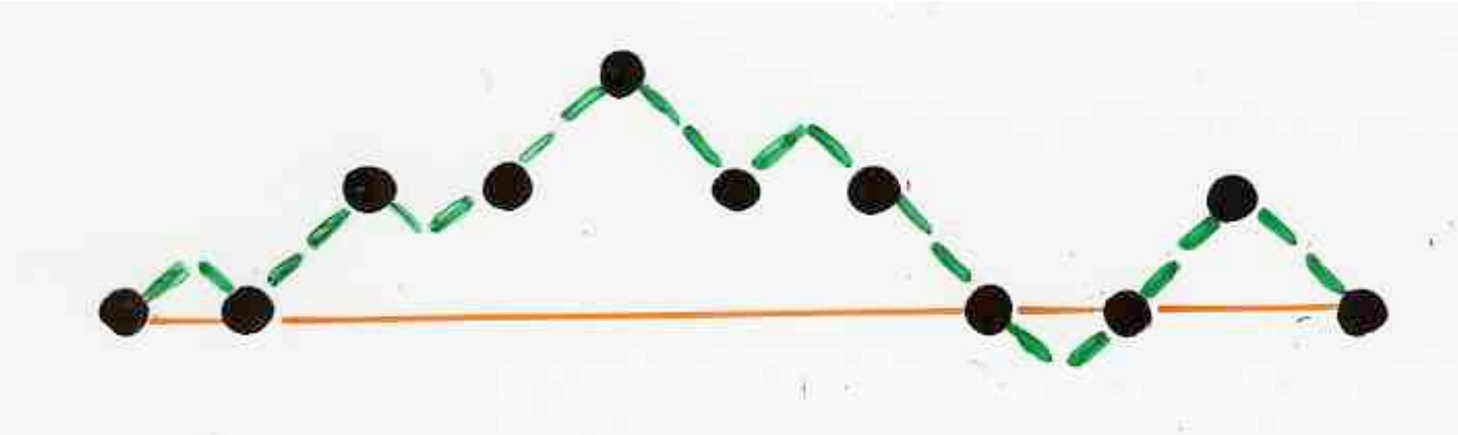


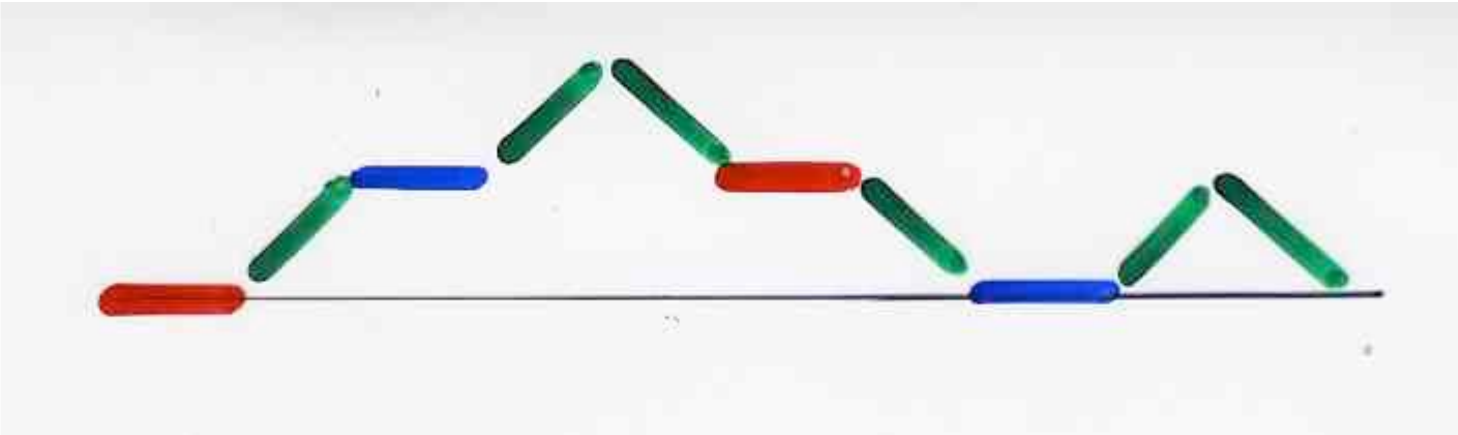
Dyck path
length $(2n+2)$

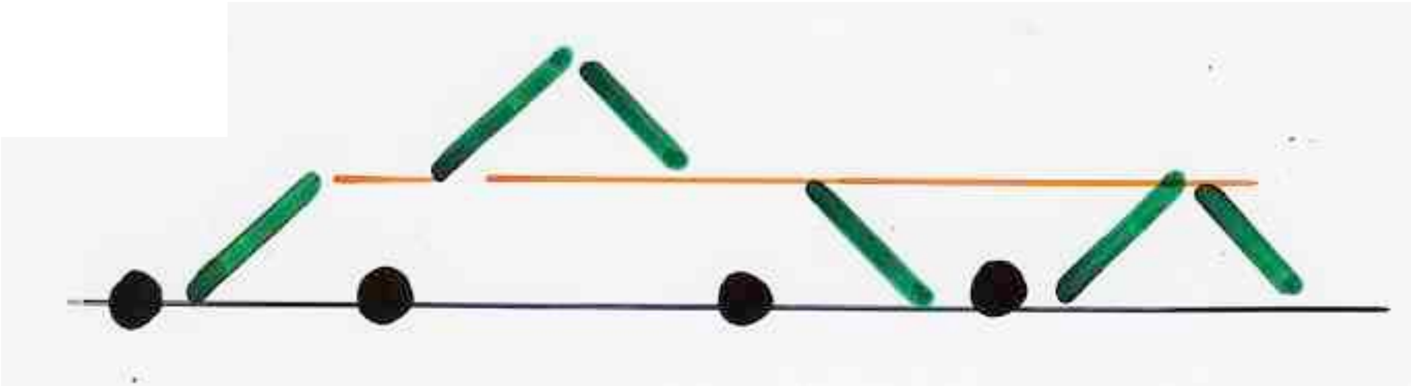
reciprocal

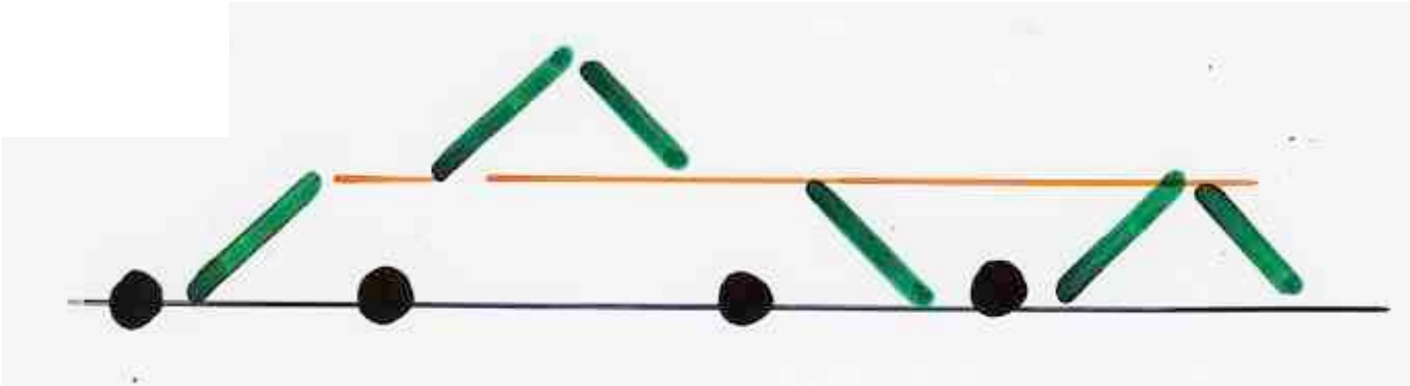
bijection

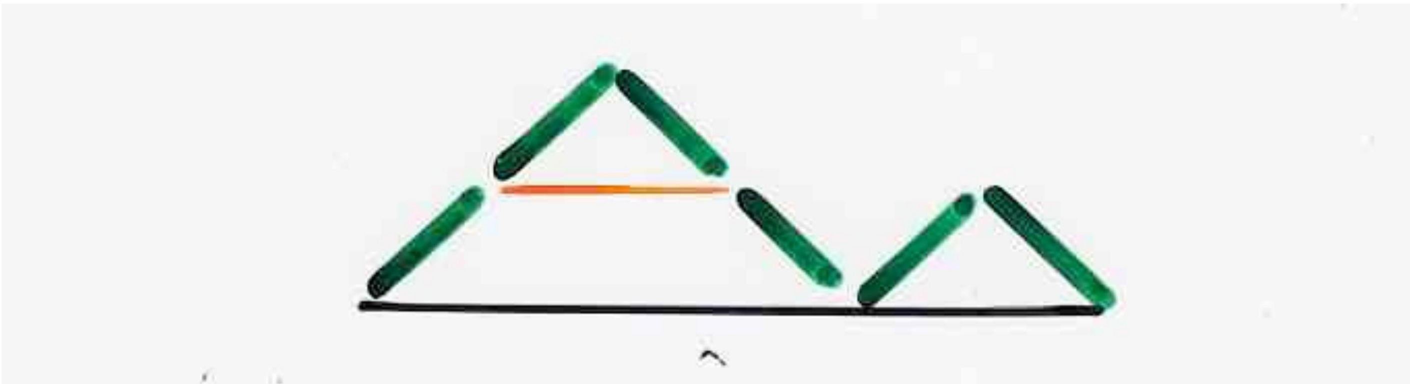








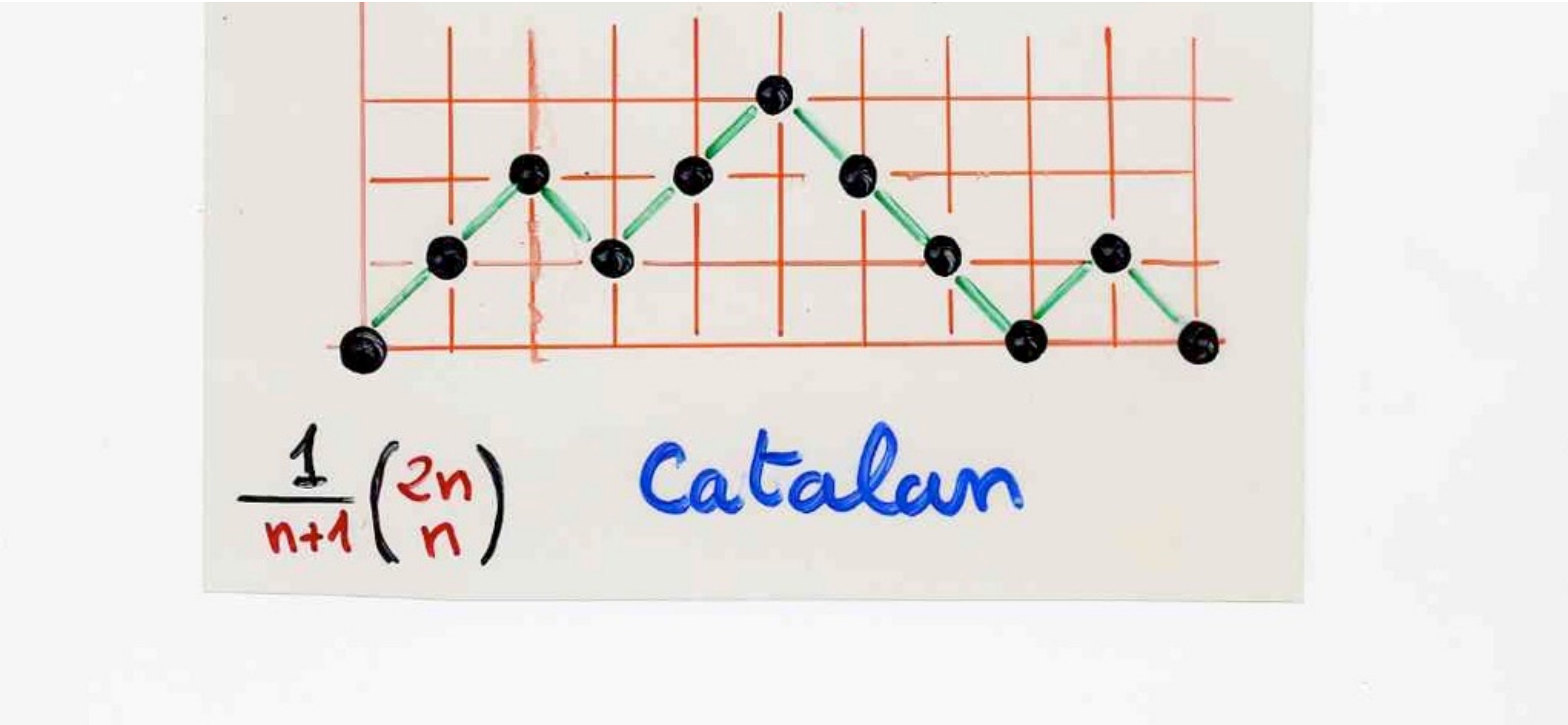




logarithmic height

of a Dyck path

Dyck path



Dyck path w

Height $h(w)$

logarithmic height $lh(w)$

$$= \lfloor \log_2 (1+h(w)) \rfloor$$

$$lh(w) = k$$

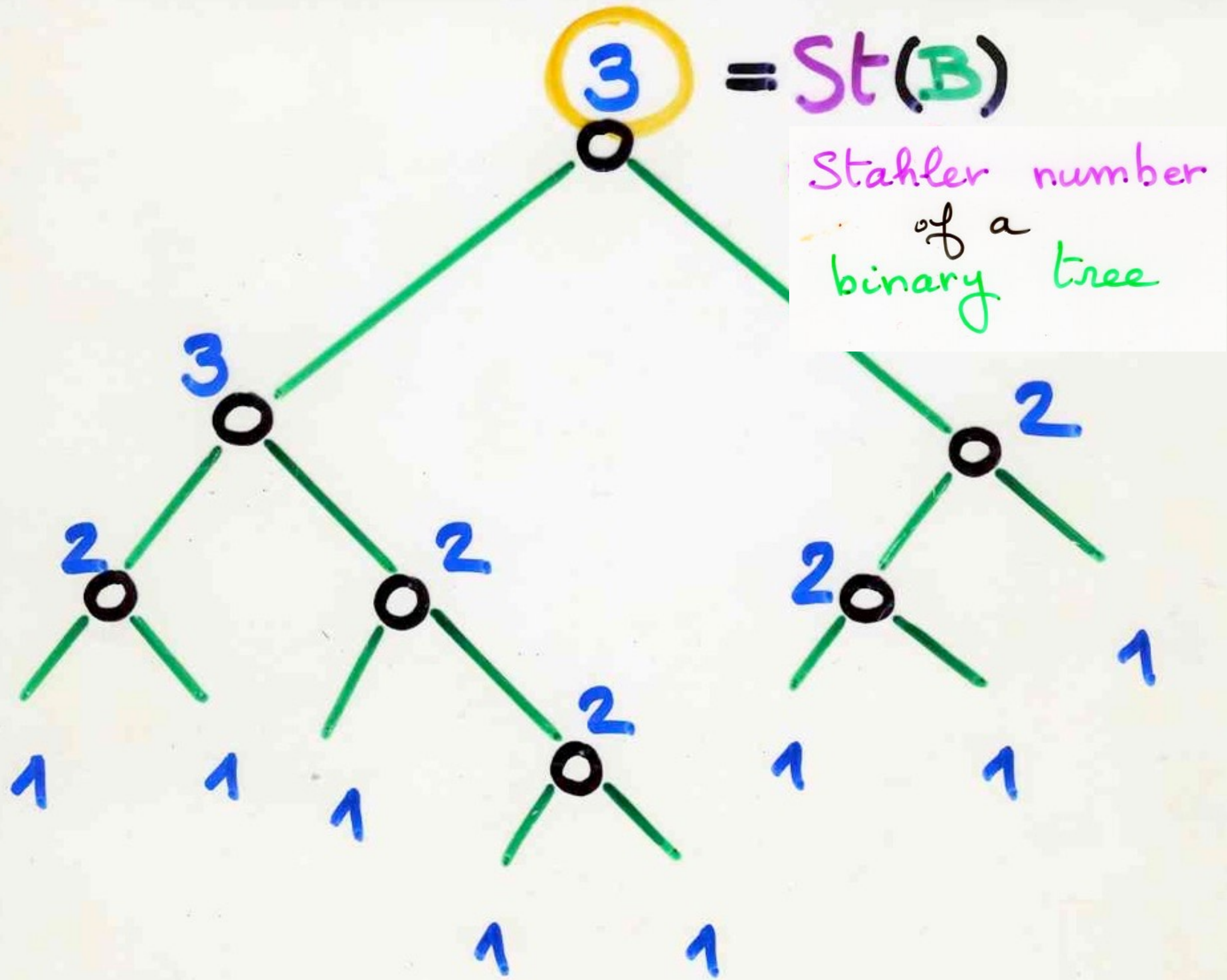
$$\Leftrightarrow \quad 2^k - 1 \leq h(w) < 2^{k+1} - 1$$

$$D(t,x) = \sum_{n,k} D_{n,k}\, x^k t^n$$

number of Dyck paths ω
with length $|\omega| = 2n$
and logarithmic height
$lh(\omega) = k$

$D(t,x)$ satisfies the same functional equation than $S(t,x)$

enumerating binary trees according to the number of internal vertices (t) and Strahler number (x)


3 = St(B)
Stahler number of a binary tree
3
3
2
2
2
2
2
1
1
1
1
1
1
1
1

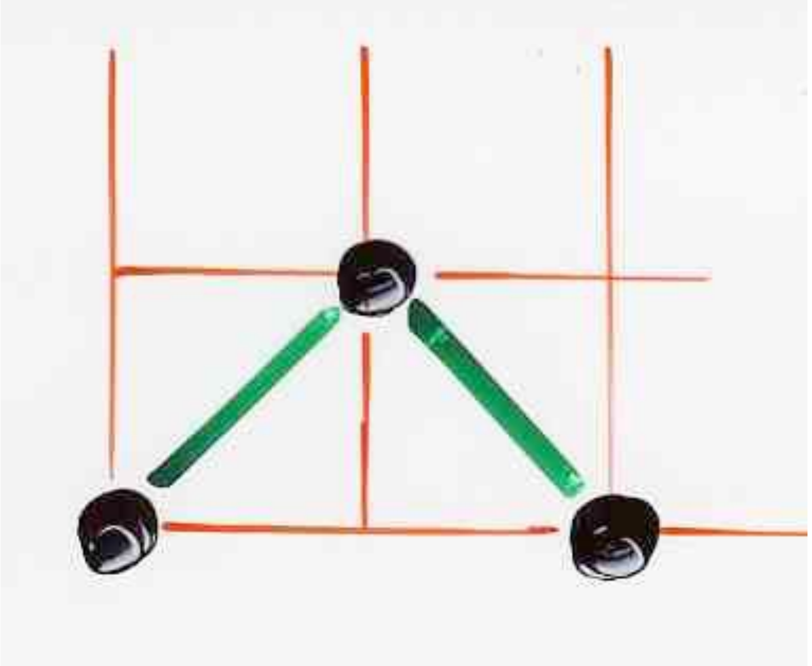
$$S(t,x) = \sum_{k\geq 0} S_k(t)\, x^k$$

$$= \sum_{n,k} S_{n,k}\, x^k t^n$$

$$S(t,x) = 1 + \frac{xt}{(1-2t)}\, S\left(\left(\frac{t}{1-2t}\right)^2, x\right)$$

Françon (1984)

Knuth (2005)

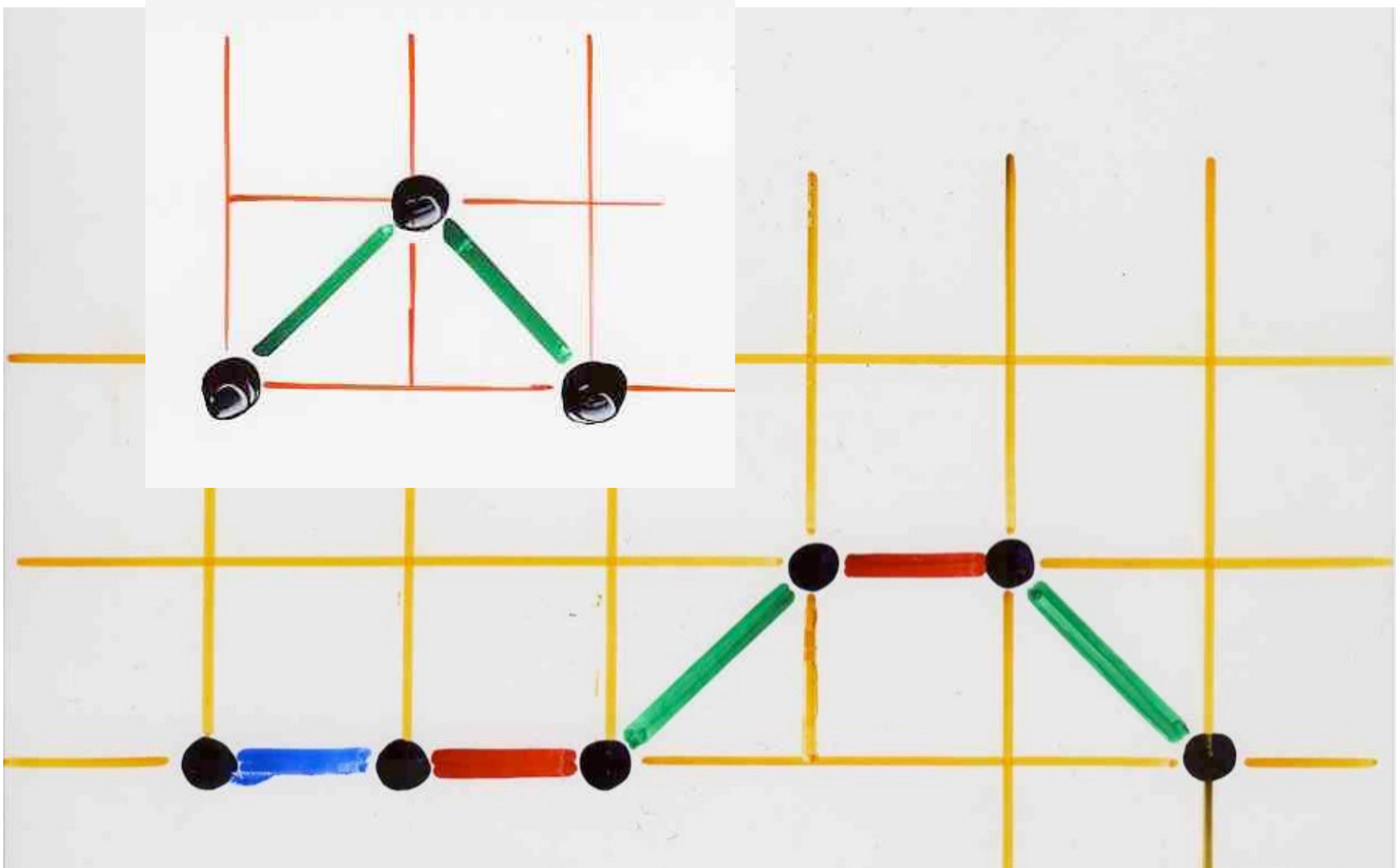


Dyck path
length $2n$ $\rightarrow$ $(2n+1)$ vertices
t^{n} $\quad$ t^{2n+1}

$$S(u,x) \to u\,S(u^2,x)$$

$$u \leftarrow \frac{t}{1-2t}$$

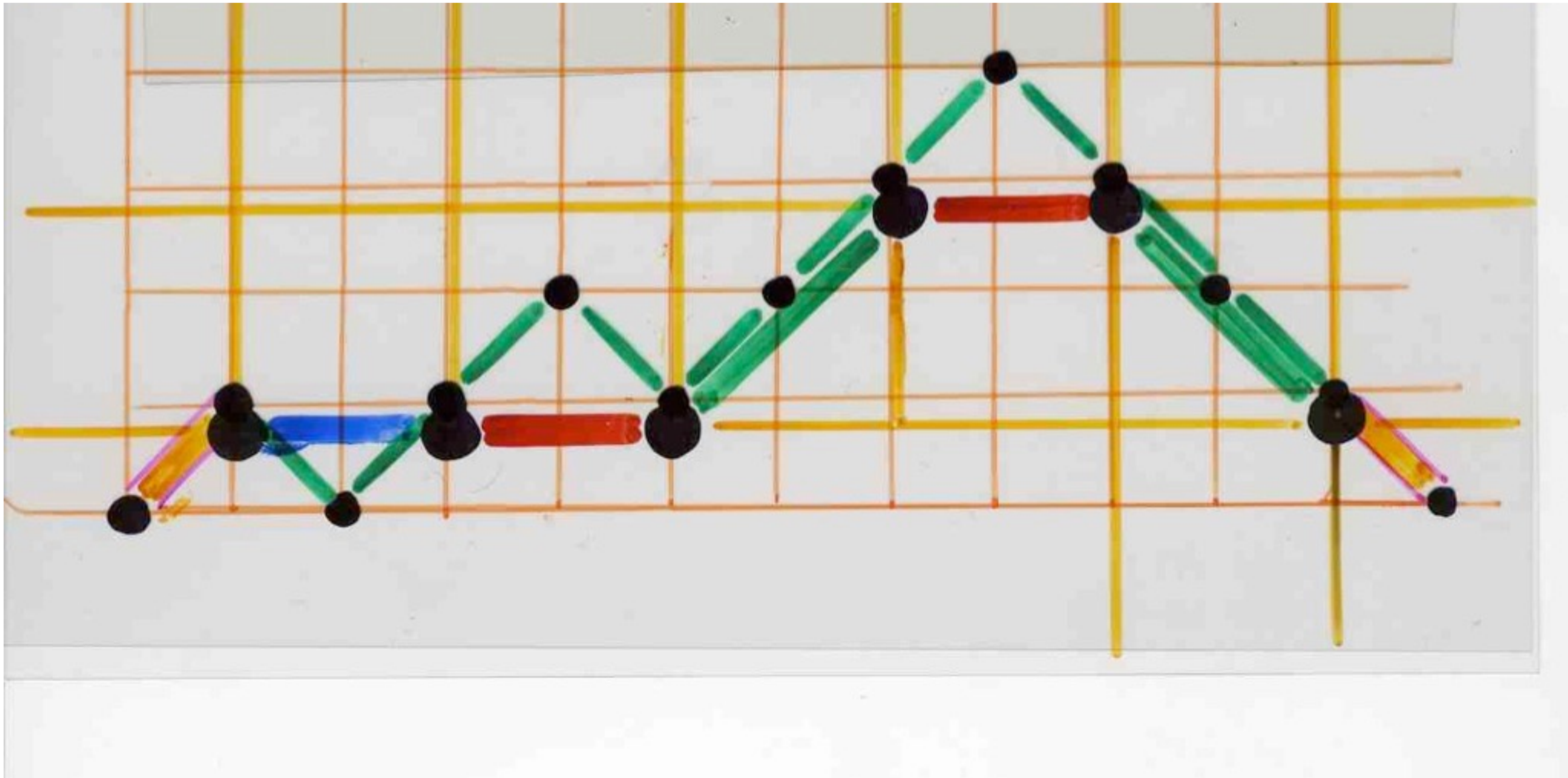
substitution

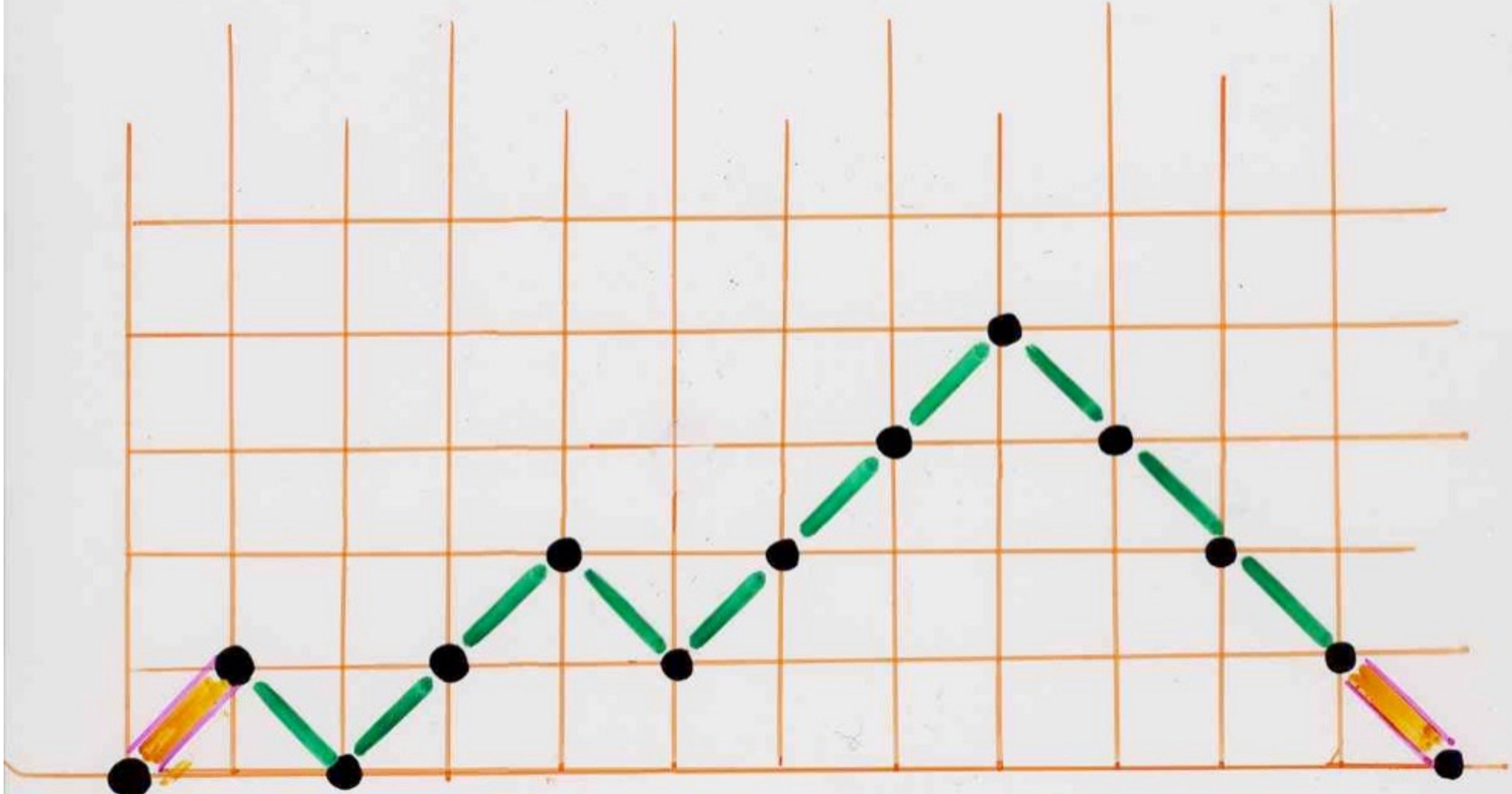


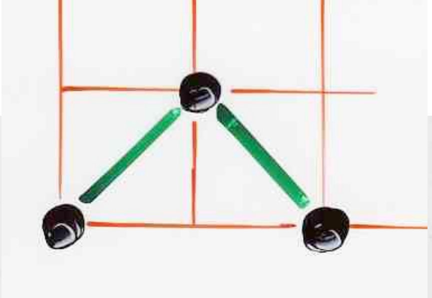
Dyck path

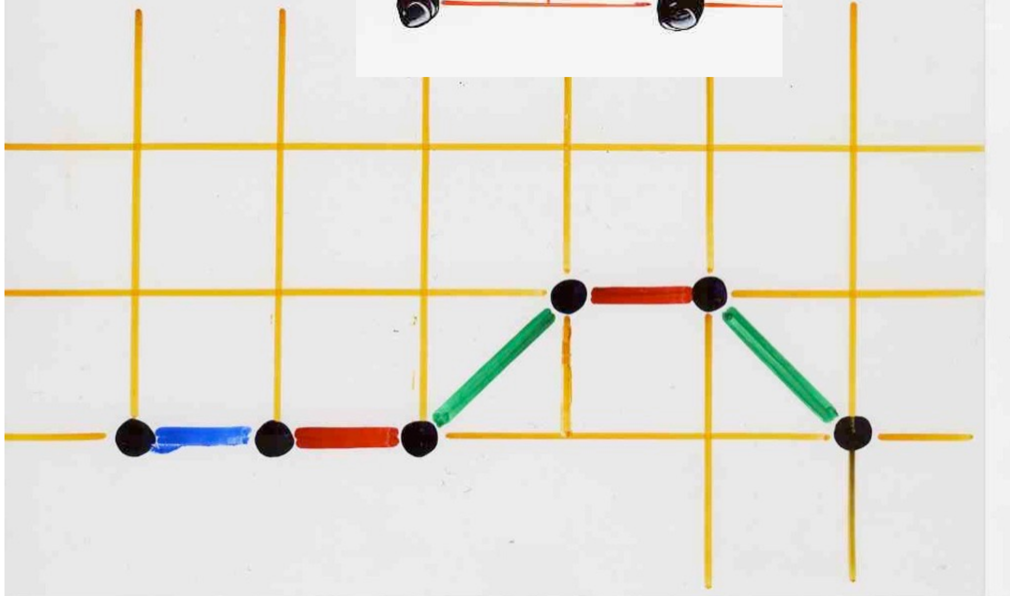
length $2n$ $\rightarrow$ $(2n+1)$ vertices

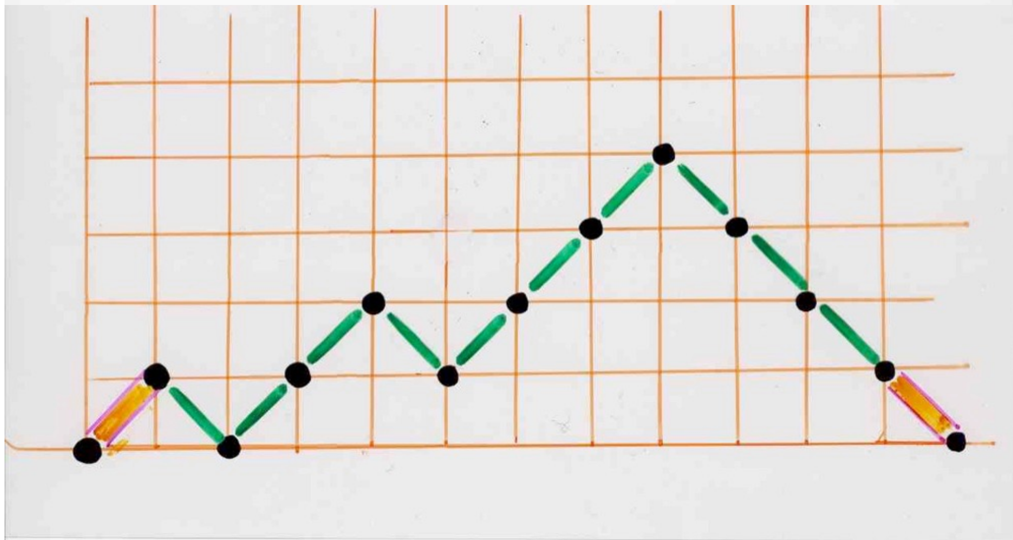
t^n $\qquad$ t^{2n+1}

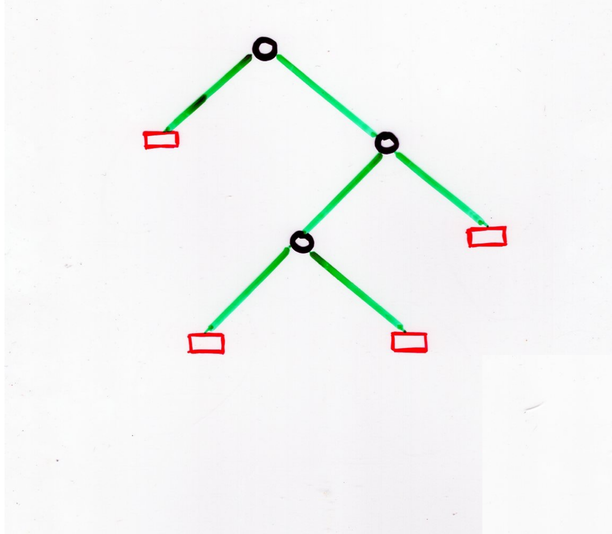


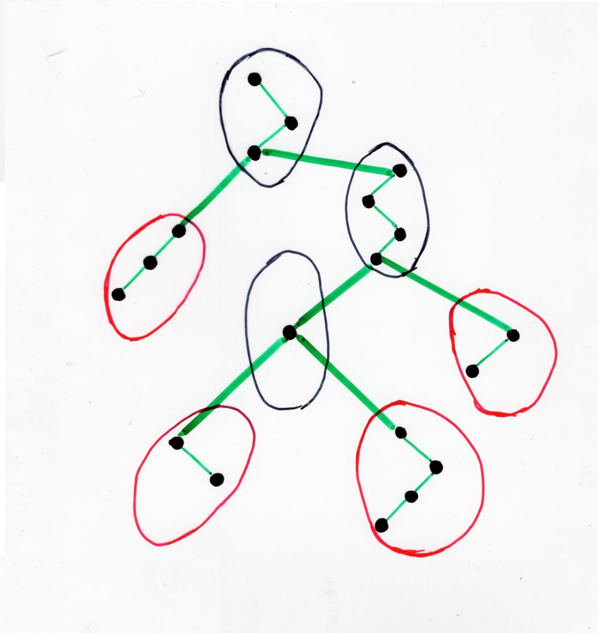


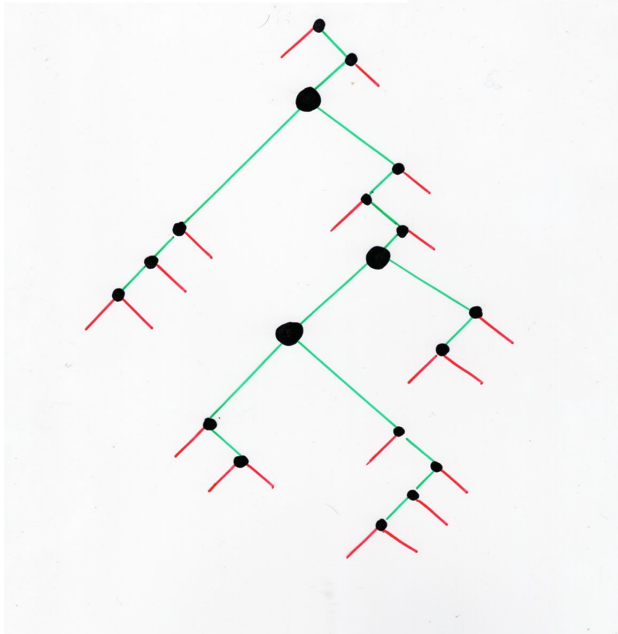












$$S(t,x) = \sum_{k\geq 0} S_k(t)\, x^k$$

$$= \sum_{n,k} S_{n,k}\, x^k t^n$$

$$S(t,x) = 1 + \frac{xt}{(1-2t)}\, S\left(\left(\frac{t}{1-2t}\right)^2, x\right)$$

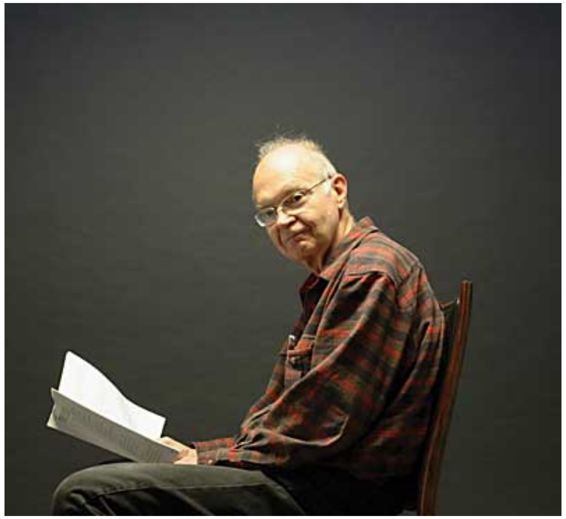
Françon (1984)
Knuth (2005)

(complete)

binary trees $\longleftarrow$ Françon (1984) $\longrightarrow$ Dyck paths

n (internal) vertices — length $2n$

Strahler nb $= k$ — log. height $lh(w) = k$



$\longleftarrow$ Knuth (2005) $\longrightarrow$

$$S_{\leq k}(t) = \frac{F_{2^k-2}(t)}{F_{2^k-1}(t)} \qquad (k \geq 2)$$

generating function
for binary trees
with Strahler number $\leq k$ $\quad = k$

$$S_{\leq k}(t) = \frac{F_{2^k-2}(t)}{F_{2^k-1}(t)} \qquad (k \geqslant 2)$$

generating function for binary trees with Strahler number $\leq k$ $= k$

$$S_k(t) = S_{\leq k}(t) - S_{\leq (k-1)}(t) \qquad (k \geqslant 2)$$

$$= \frac{t^{(2^{k-1}-1)}}{F_{2^k-1}(t)}$$