An introduction to

enumerative
algebraic combinatorics
bijective

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IMSc
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Chapter 2

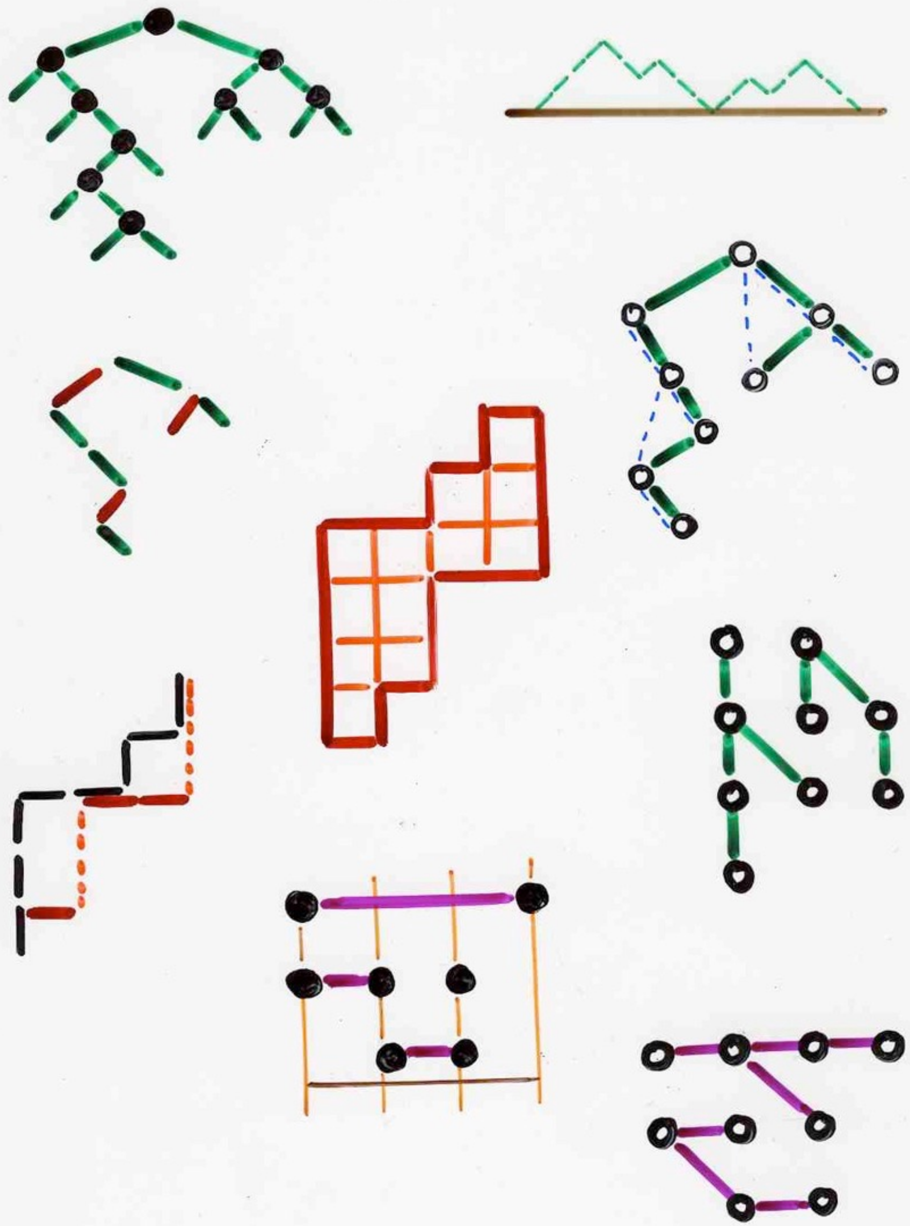
The Catalan garden

(1)

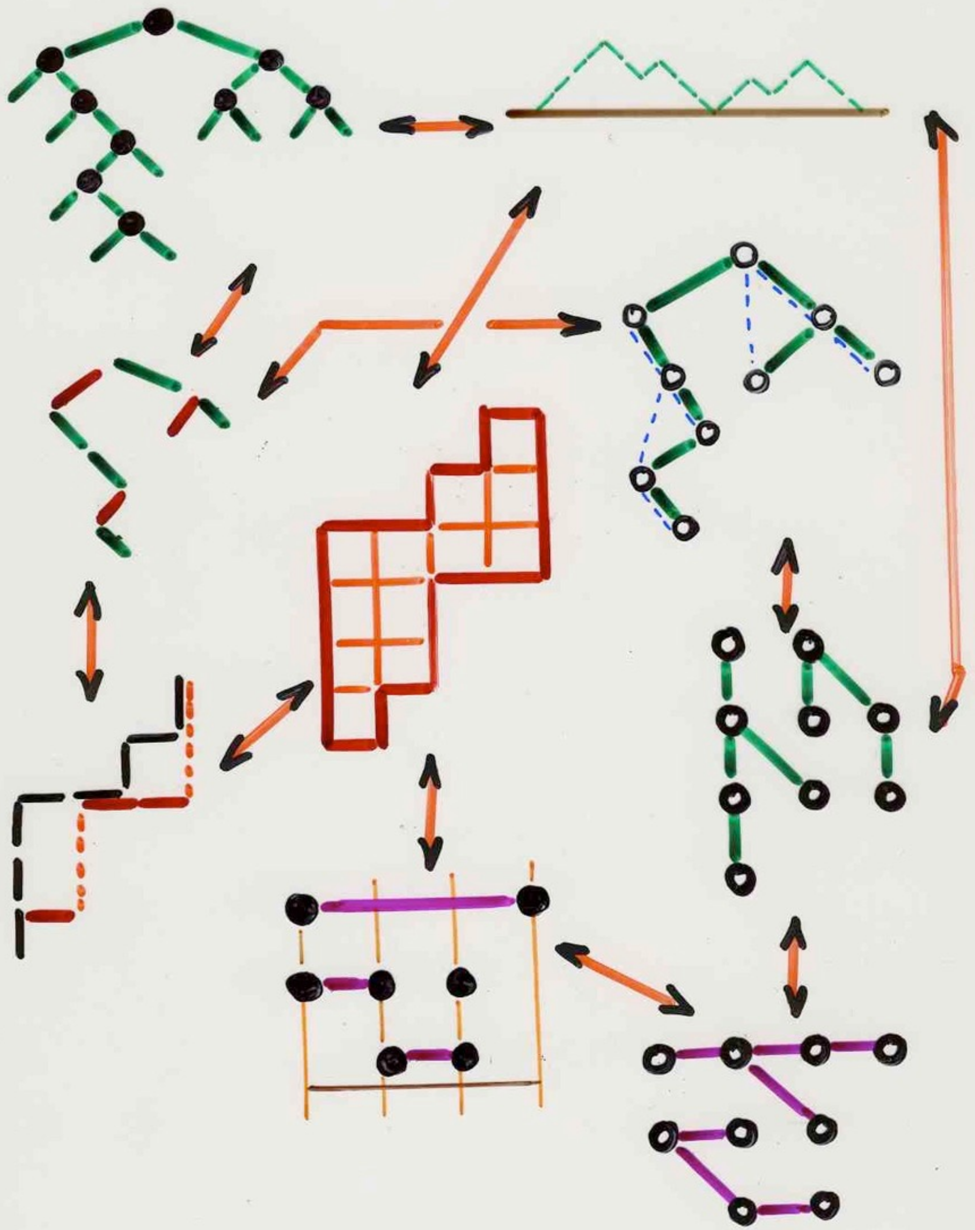
IMSc

21 January 2016

the Catalan garden



the Catalan garden



A selection
of the main

Combinatorial
interpretations
of Catalan numbers

and
Catalan bijections

- binary trees
- complete binary trees
- planar trees

- Dyck paths
- 2-colored Motzkin paths
- Lukasiewicz paths

- triangulations (of a convex polygon) L. Euler

- Staircase polygons

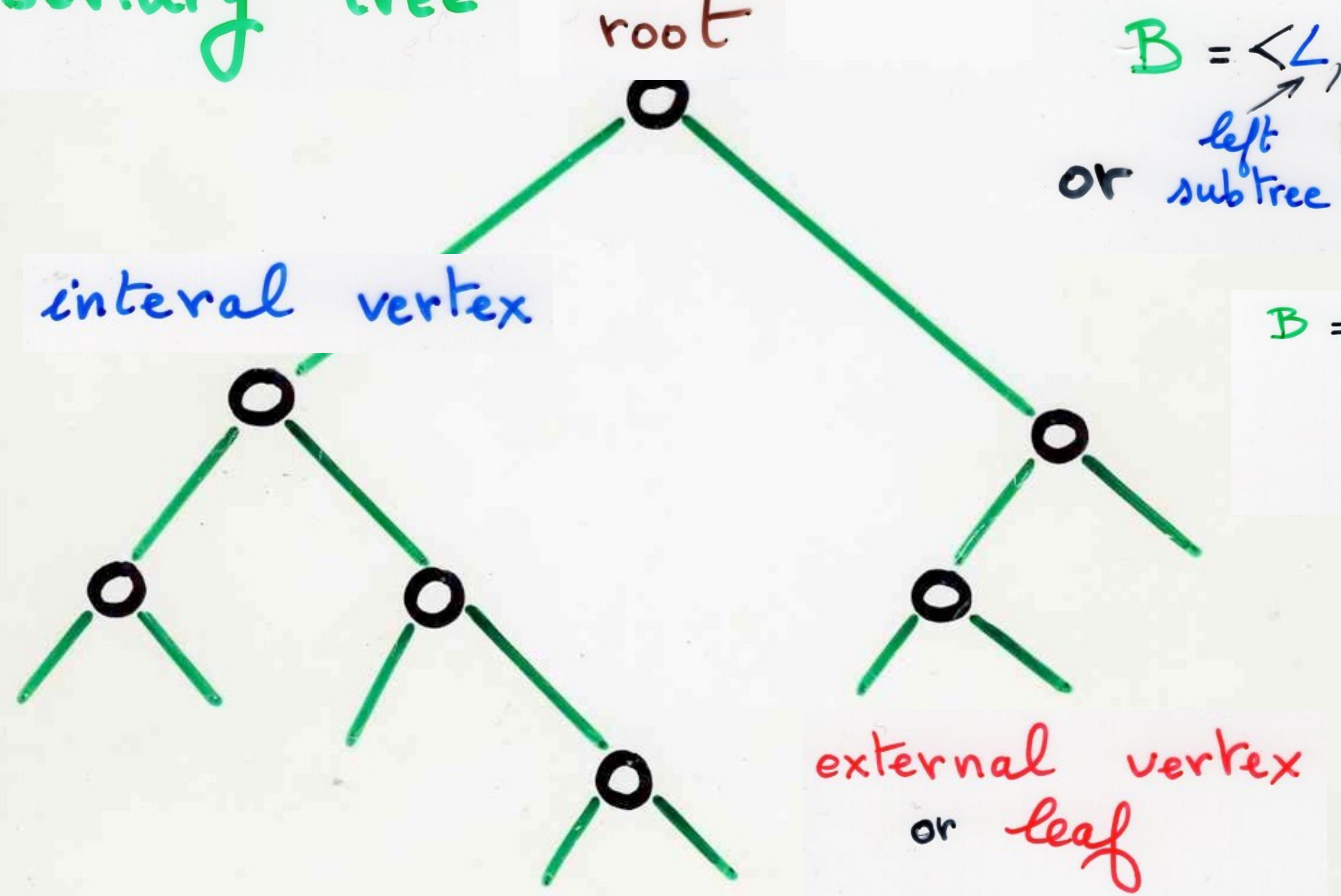
- non-crossing partitions

Binary trees

and

complete binary trees

binary tree



$B = \langle L, r, R \rangle$

left subtree, root, right subtree

or

$B = \langle v \rangle$

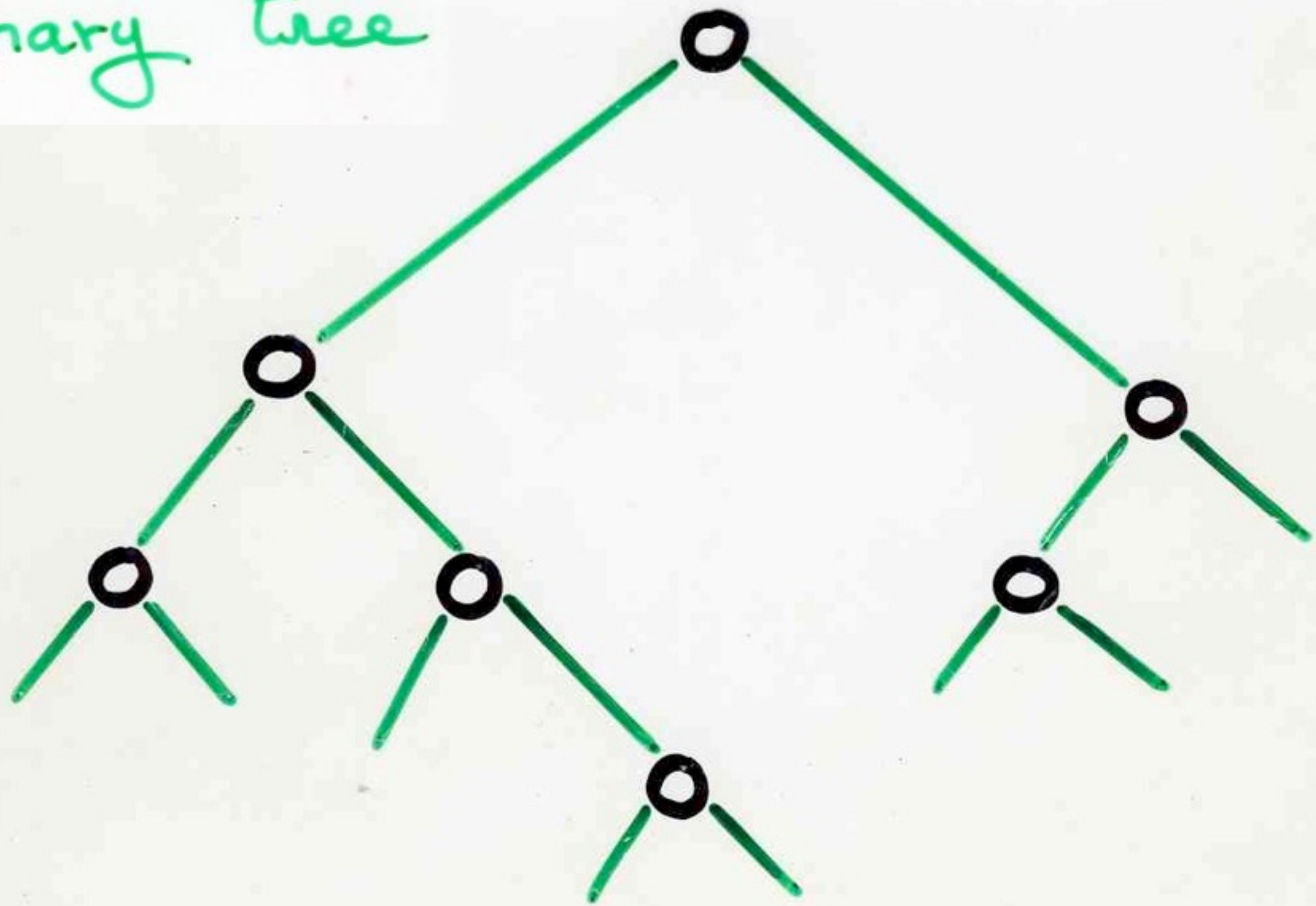
leaf or external vertex

C_n = number of binary trees having n internal vertices (or $n+1$ leaves = external vertices)

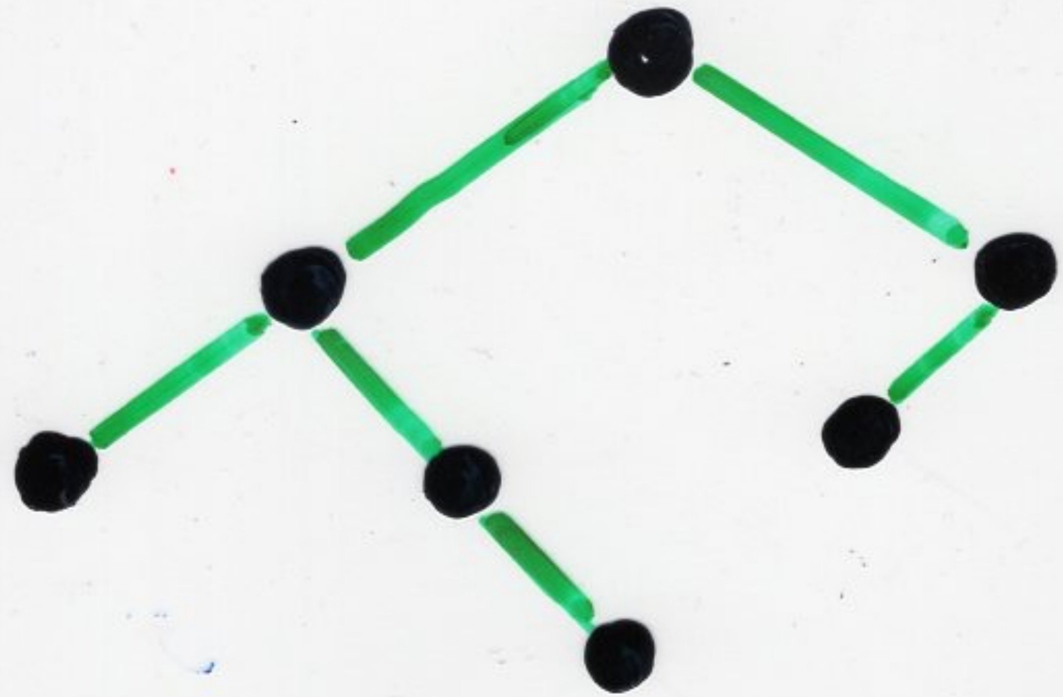
Catalan number

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}$$

previous binary trees (Ch 1) $\xrightarrow{\text{called}}$ complete binary trees complete binary tree



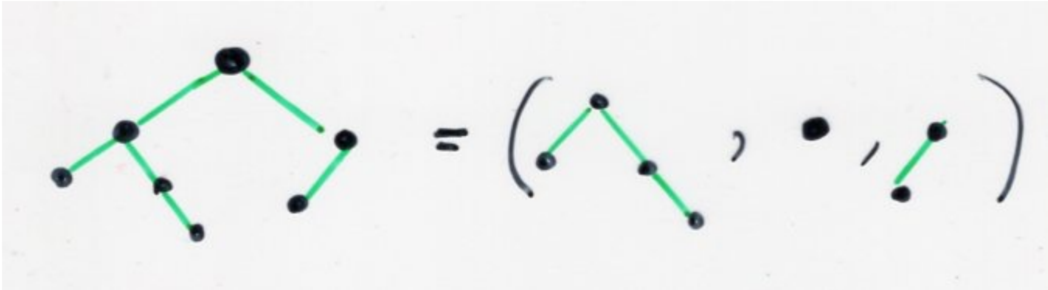
binary tree

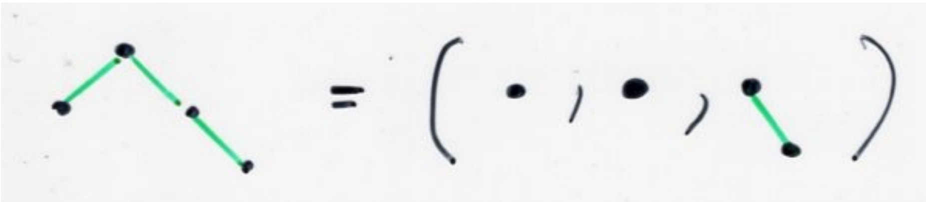


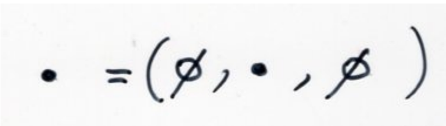
$$\begin{cases} B = (L, r, R) \\ \text{or} \\ B = \emptyset \end{cases}$$

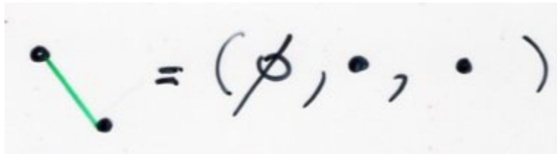
L, R binary trees

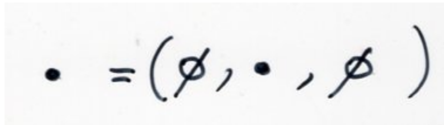
r root

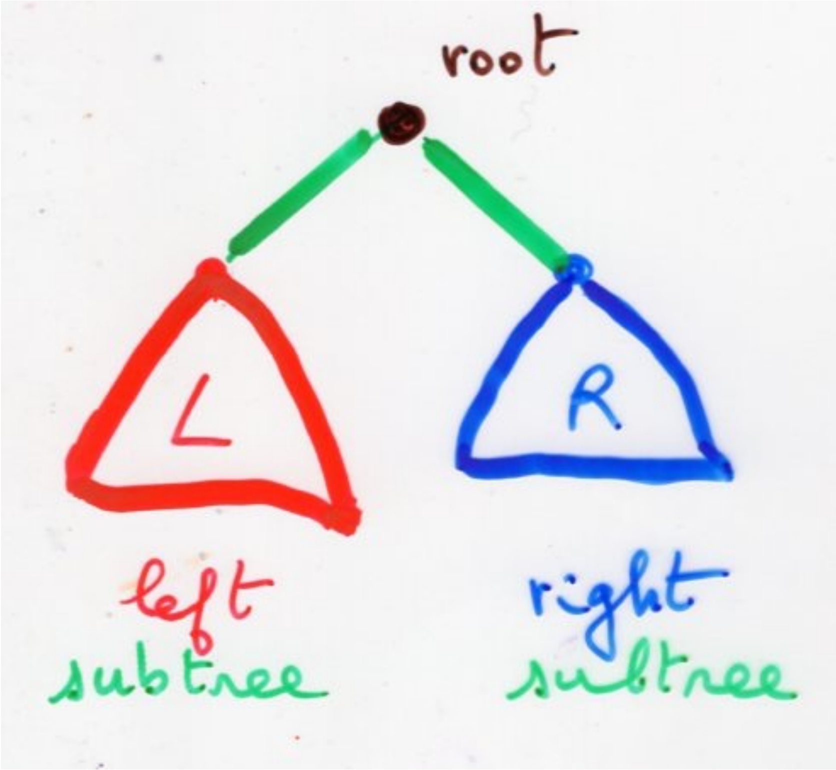


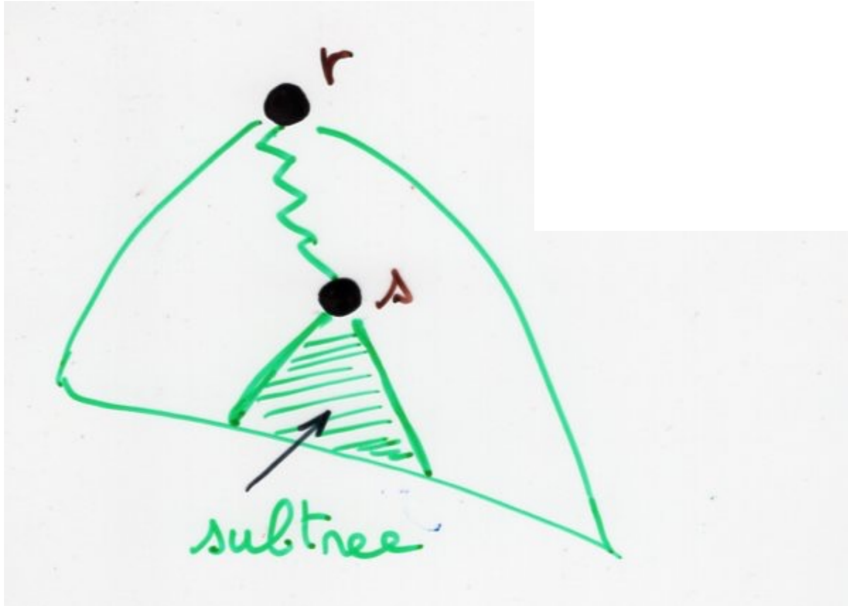






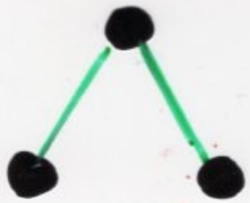



root
L
R
left
subtree
right
subtree


r
s
subtree


left
son
daughter


right
son
daughter

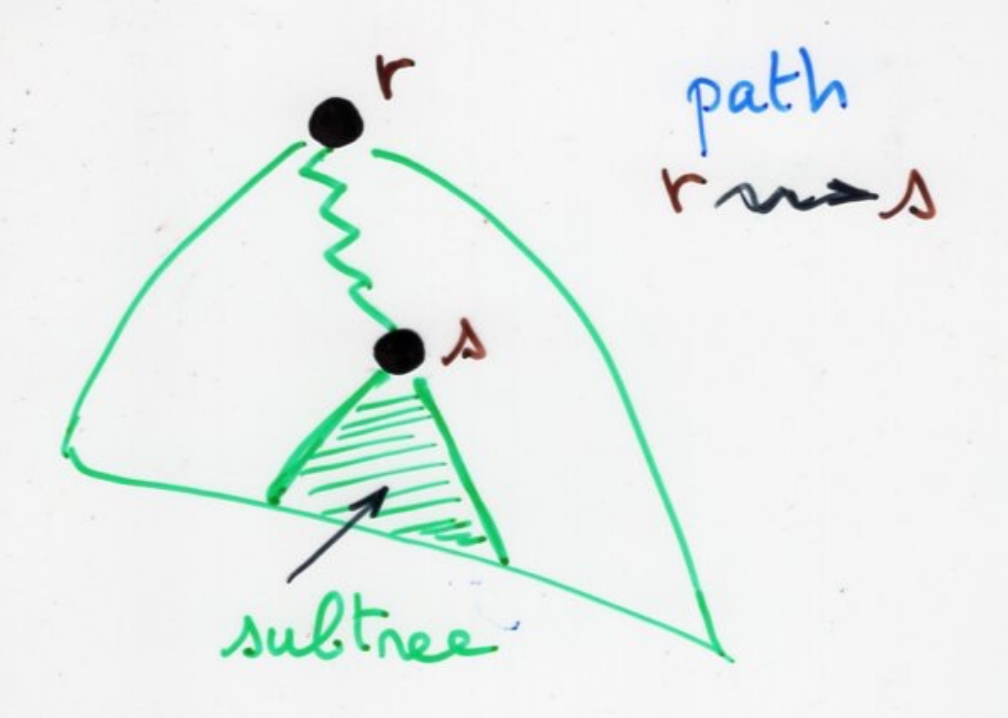


double vertex

right simple vertex

left simple vertex

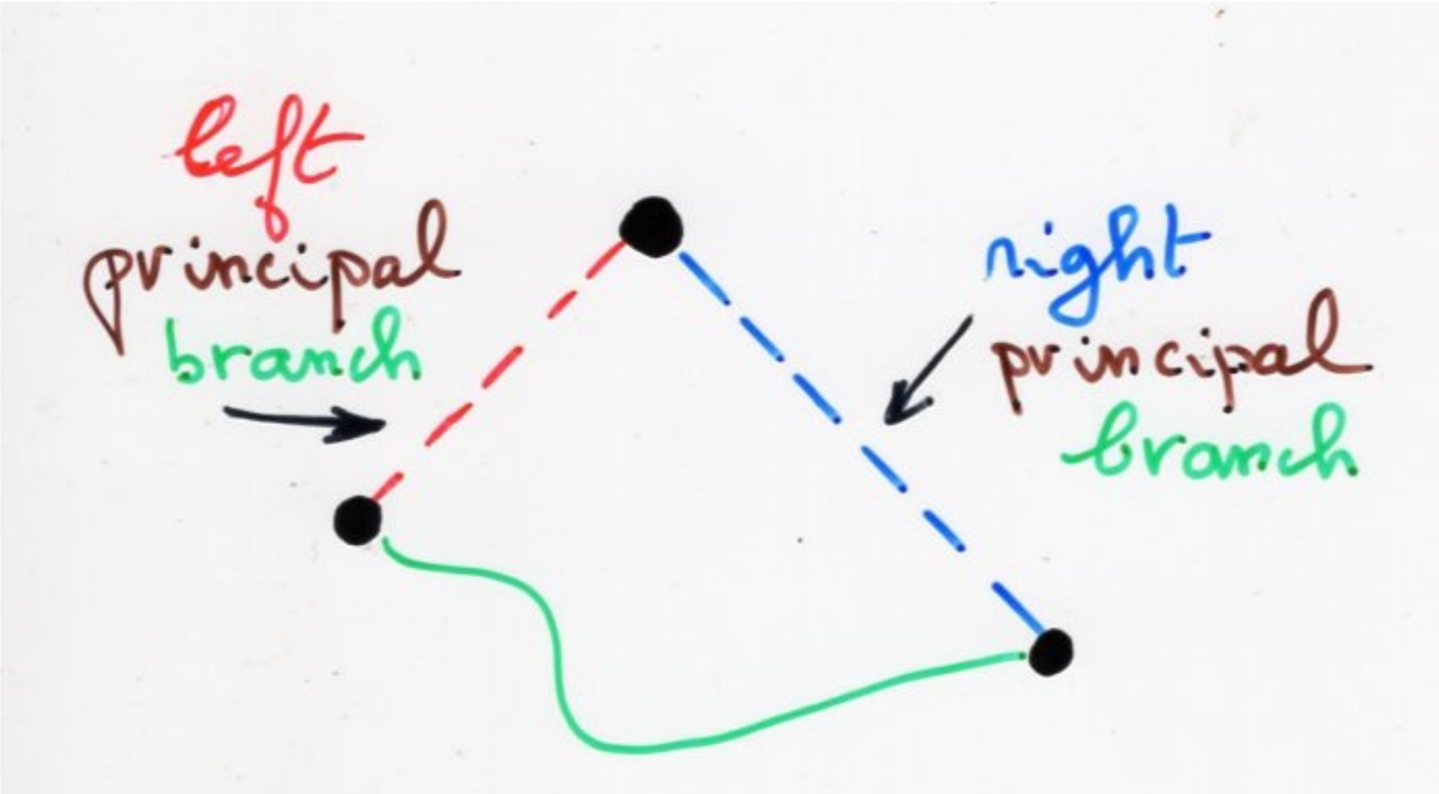
leaf



height $h(s)$
of the vertex s

left-height $hl(s)$
right-height $hr(s)$

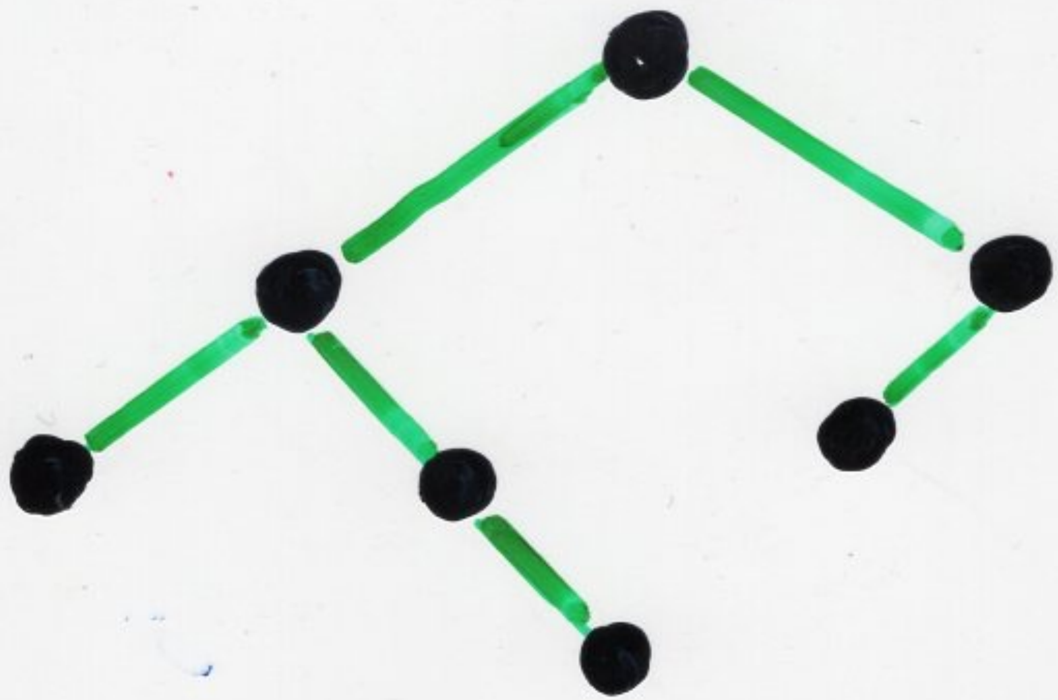
$$h(s) = hl(s) + hr(s)$$

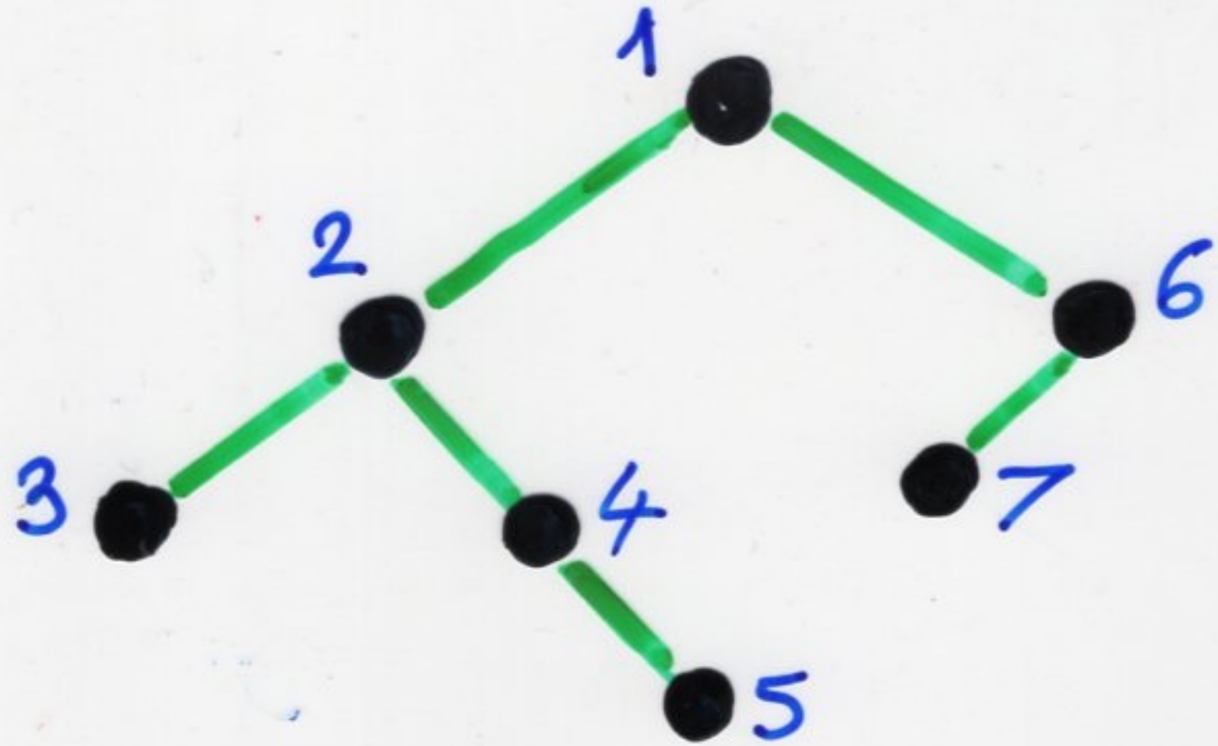

left
principal
branch
right
principal
branch

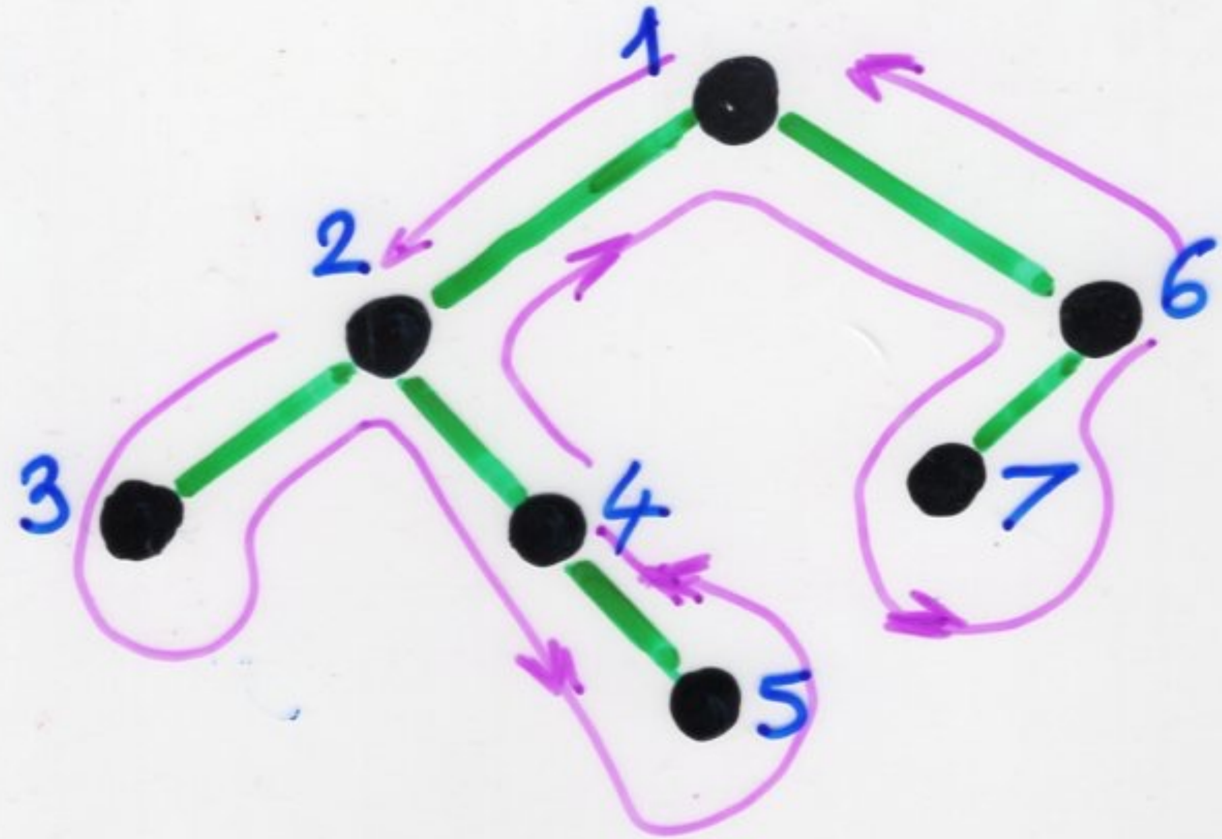
traversal of a binary tree

preorder

- visit the root (if $B \neq \emptyset$)
- then visit the left-subtree
- then visit the right subtree




1
2
6
3
4
7
5



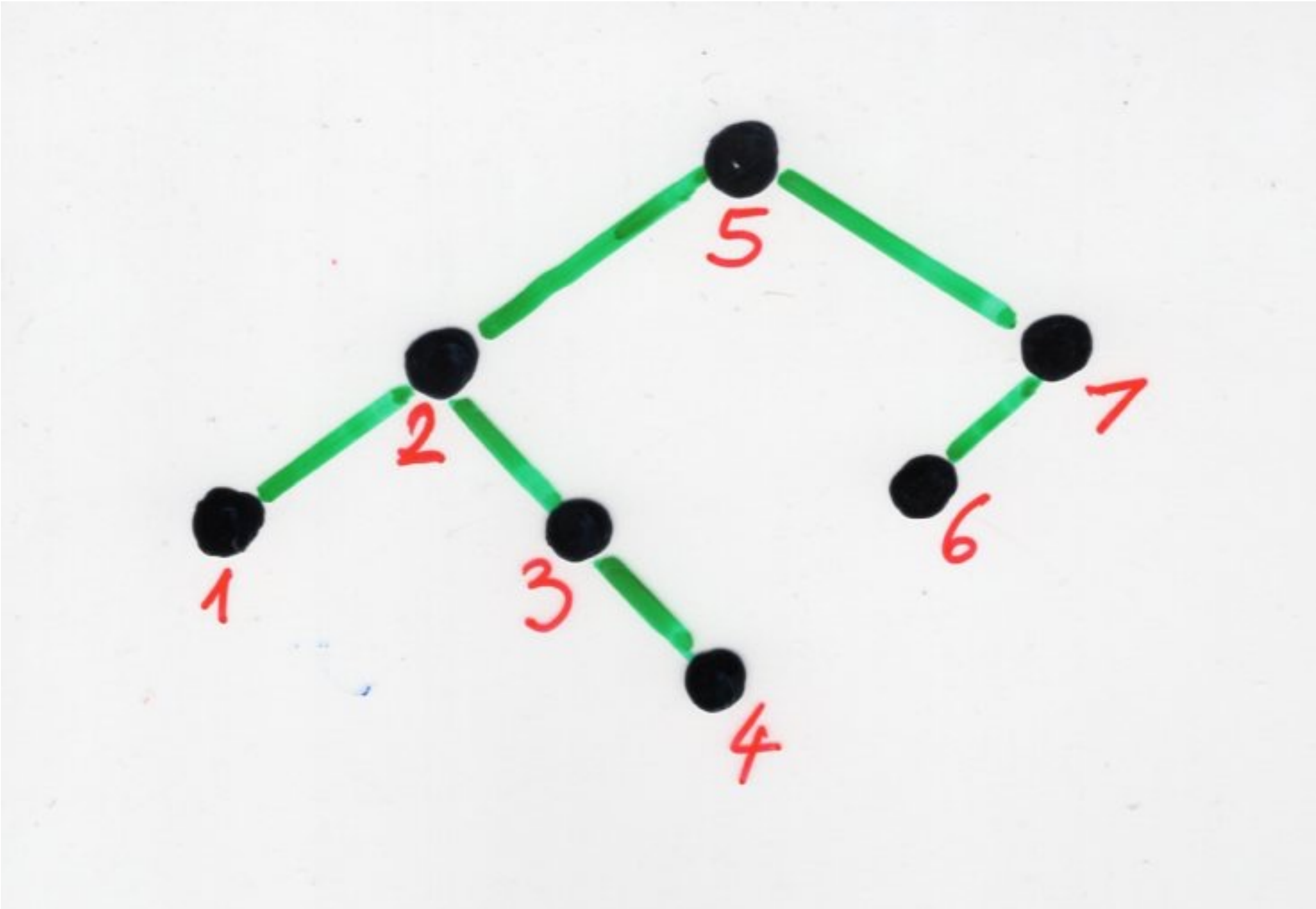
1
2
6
3
4
7
5

inorder
(symmetric order)

visit the left subtree

visit the root

visit the right subtree

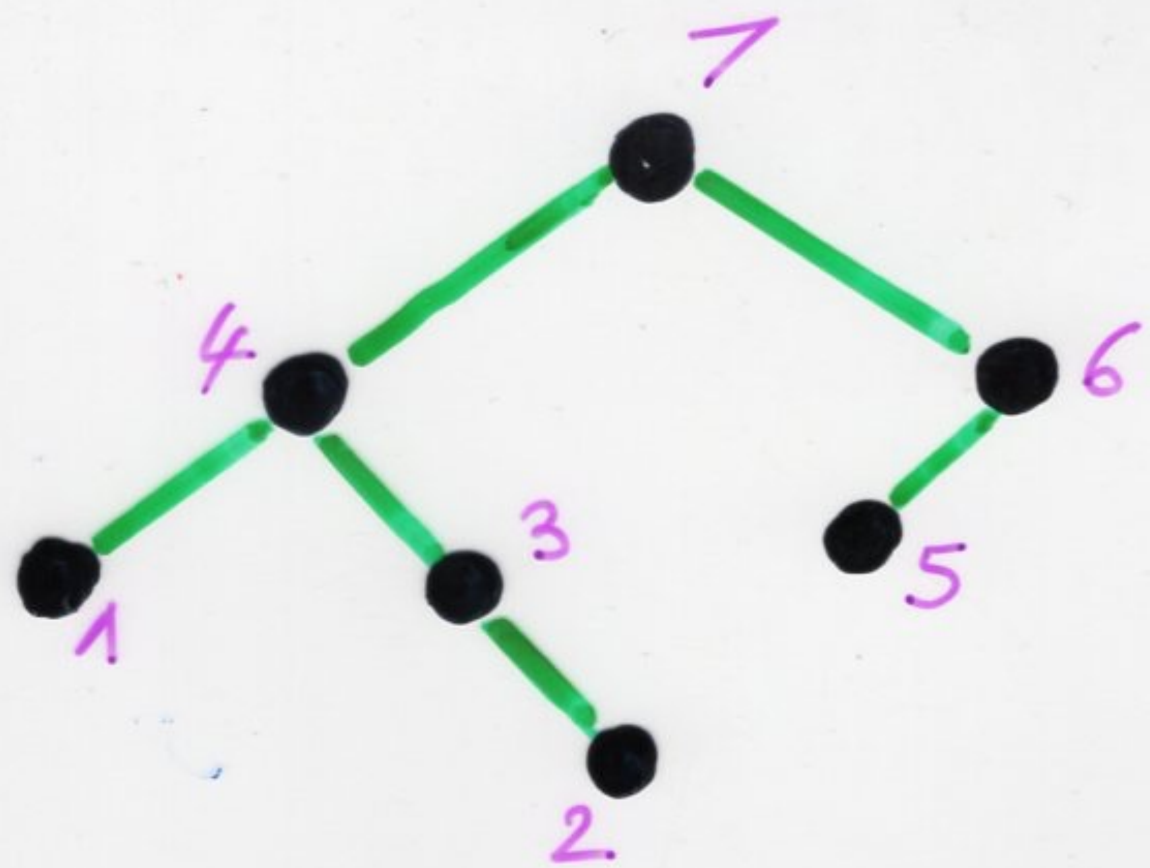

5
2
7
1
3
6
4

postorder

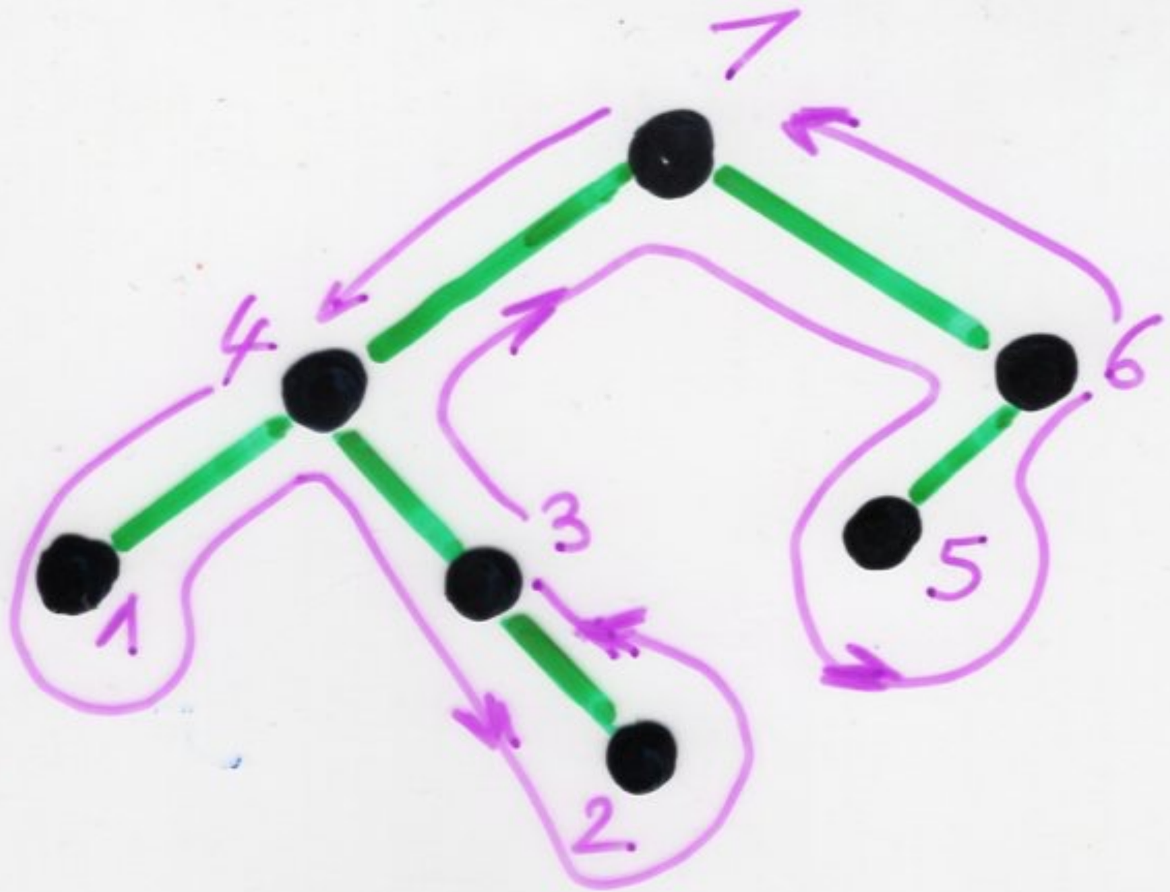
visit the left subtree

visit the right subtree

visit the root



7
4
6
3
1
5
2


7
4
6
3
5
1
2

binary trees

n vertices

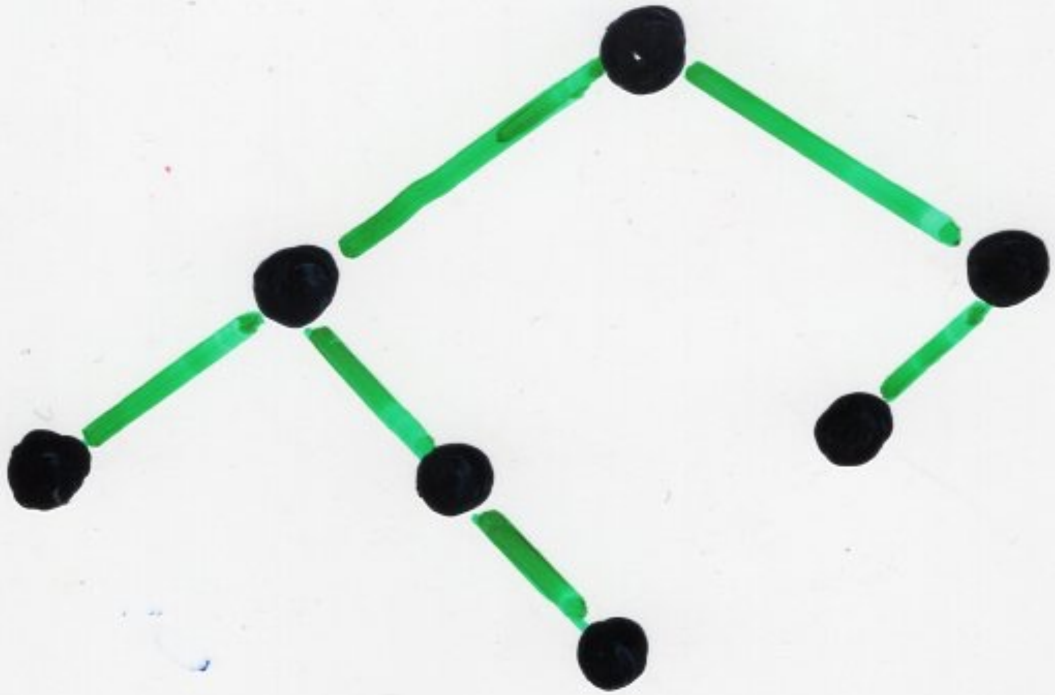
bijection

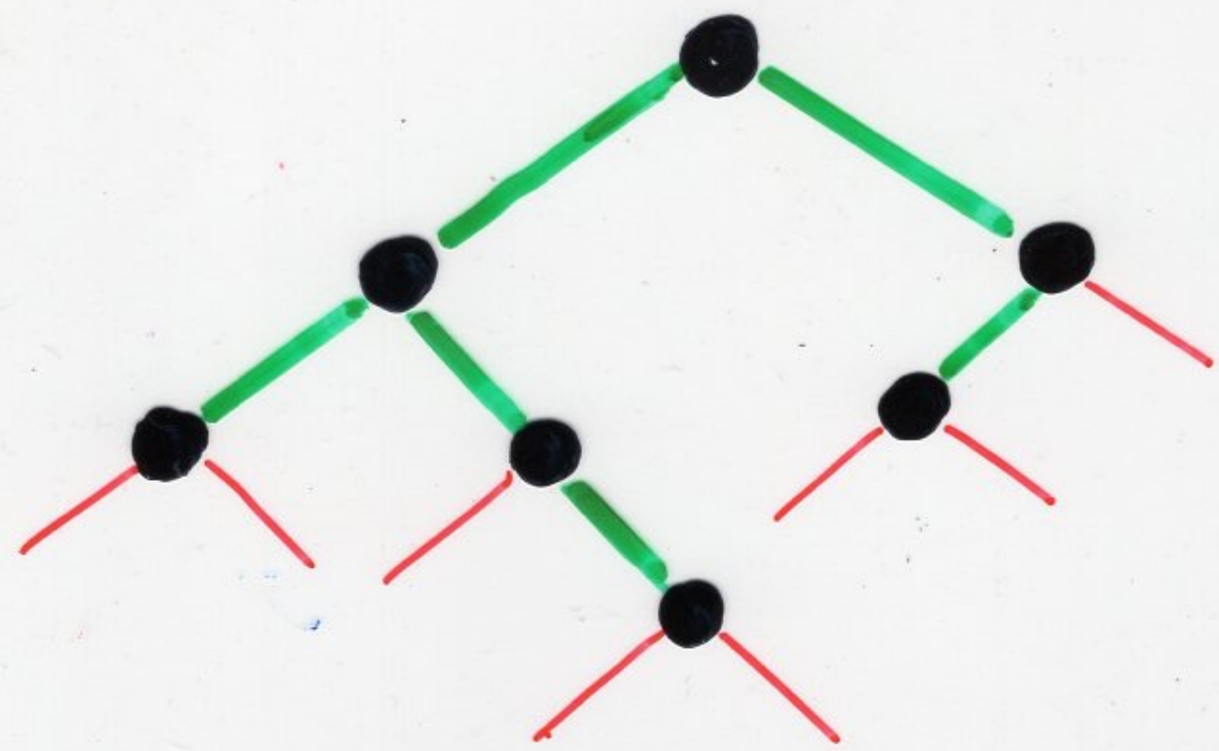
complete

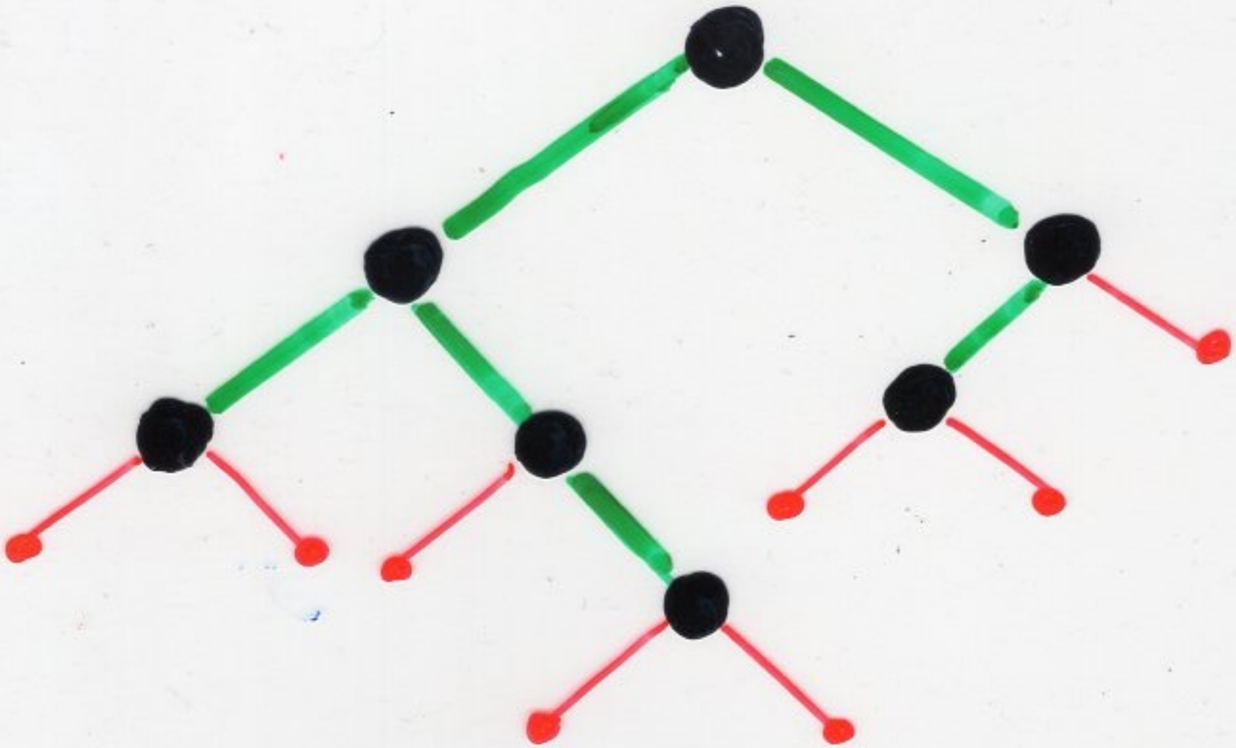
binary trees

(2n+1) vertices

$\begin{cases} n & \text{internal} \\ n+1 & \text{external} \end{cases}$ vertices







exercise

bijective proof of

$$2(2n+1)C_n = (n+2)C_{n+1}$$

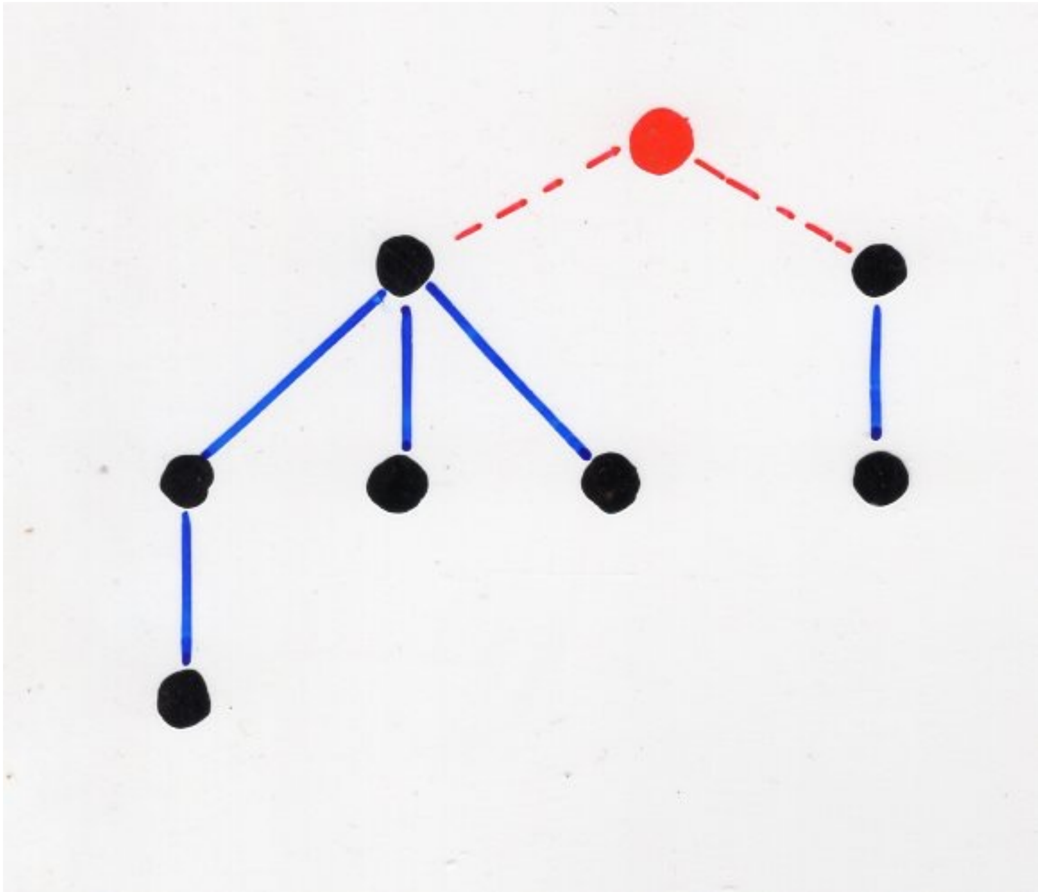
planar trees

planar (ordered) tree

$T = \{ r \}$ root

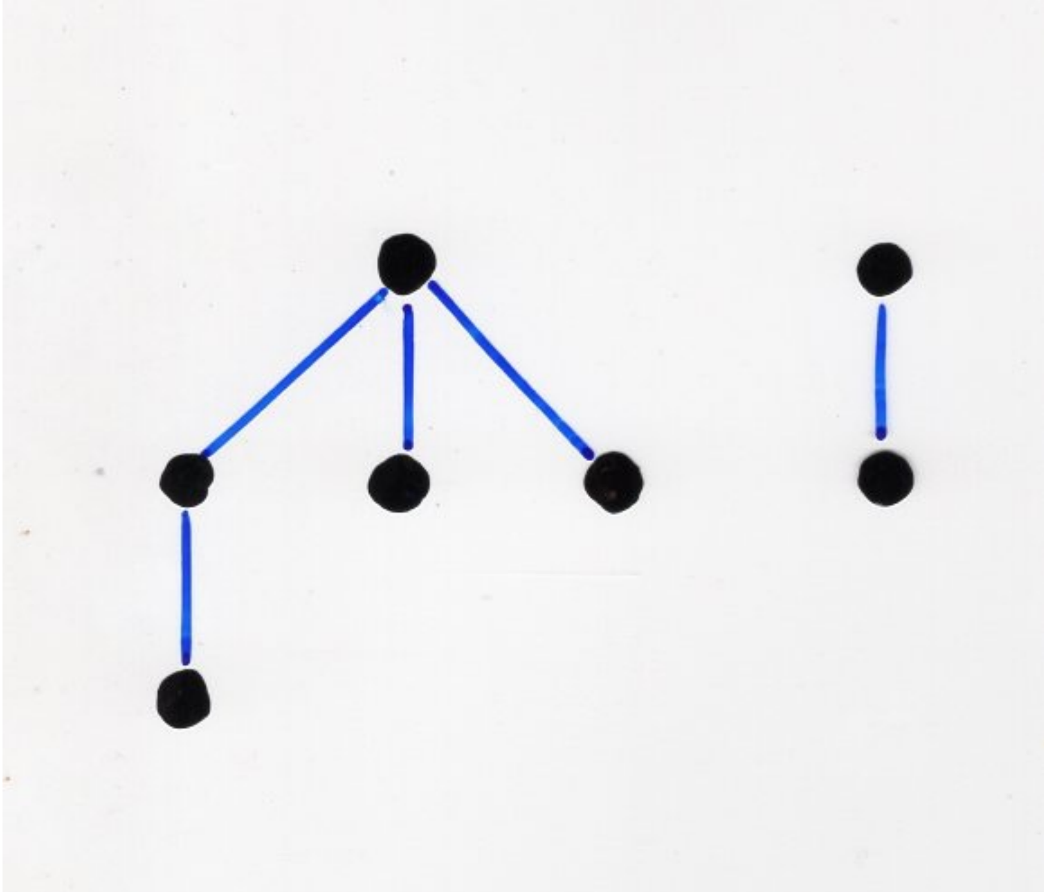
$T = \{ r ; \underbrace{T_1, T_2 \dots T_k}_{\text{sequence of planar trees}} \}$

root



forest of planar tree

sequence
$\{T_1, \ldots, T_k\}$
of
planar trees



The number of planar trees with $(n+1)$ vertices is the Catalan number C_n

check !

$T = \{r\}$ root $\bullet$

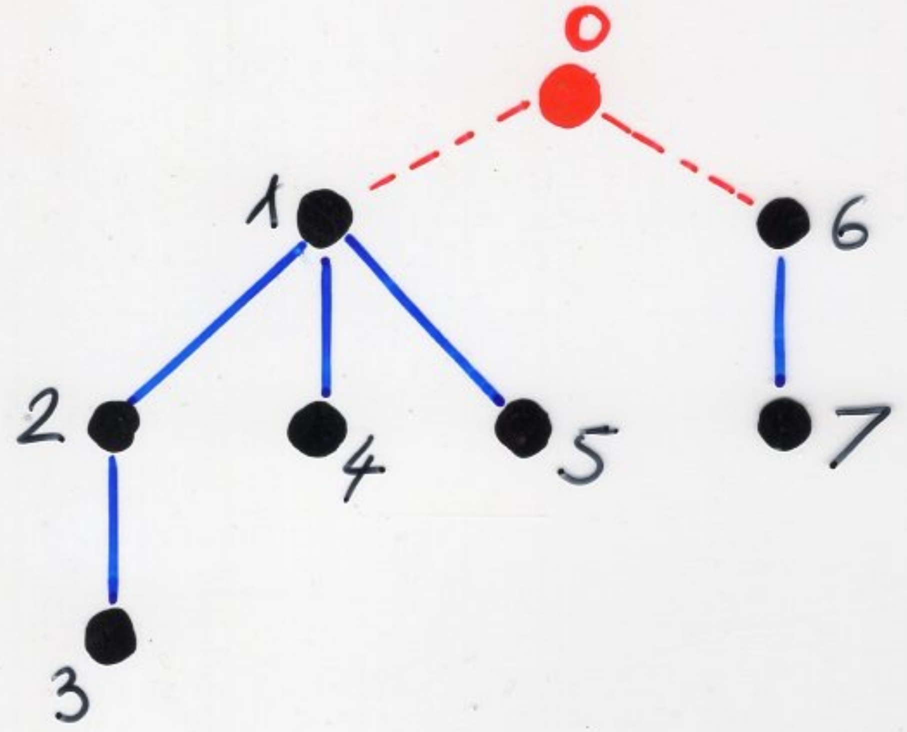
$$T = \{\underset{\text{root}}{r} \,; \underbrace{T_1, T_2 \ldots T_k}_{\text{sequence of planar trees}}\}$$

$$y = t + \frac{t\,y}{1-y}$$

preorder
- visit the root
- then visit T_1,

  .....

- visit T_k.



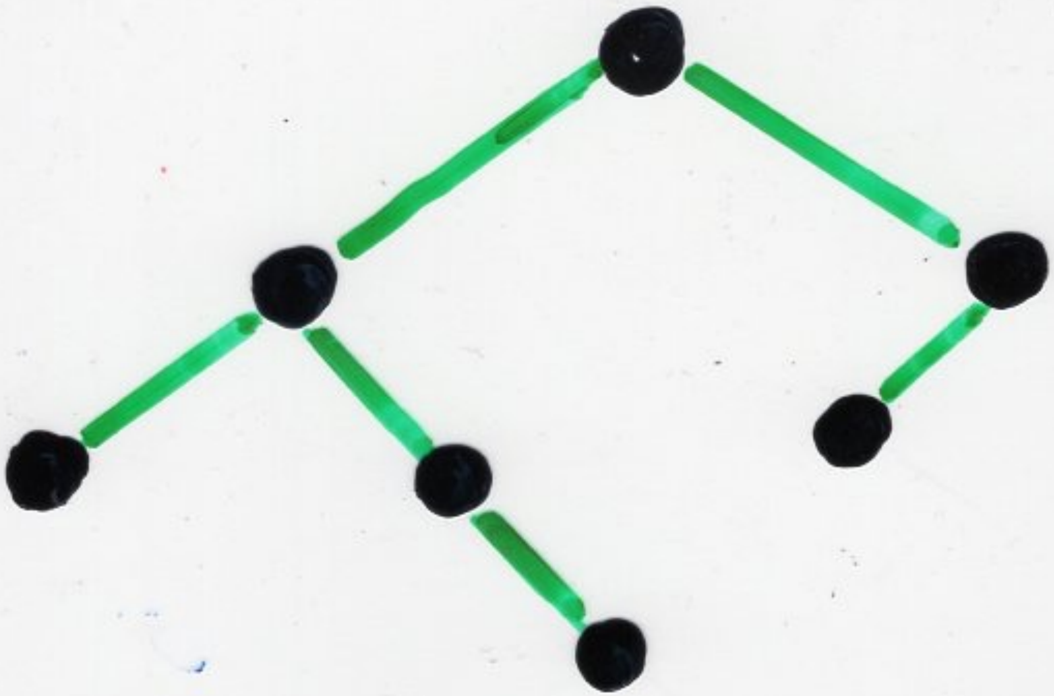
depth-first search algorithm (for a tree)

Charles Pierre Trémaux
(1859-1882)

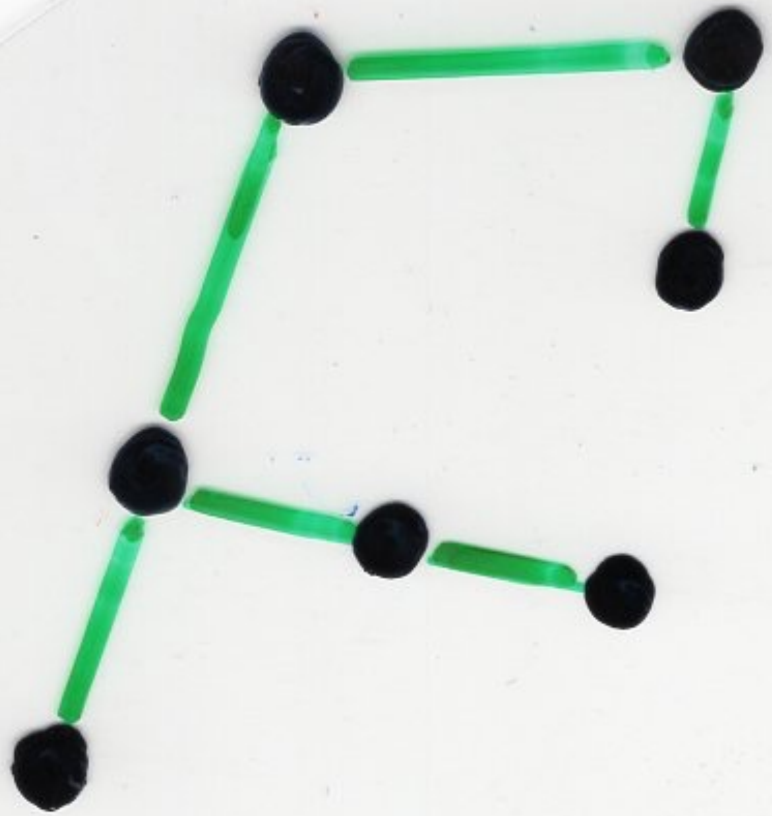
bijection

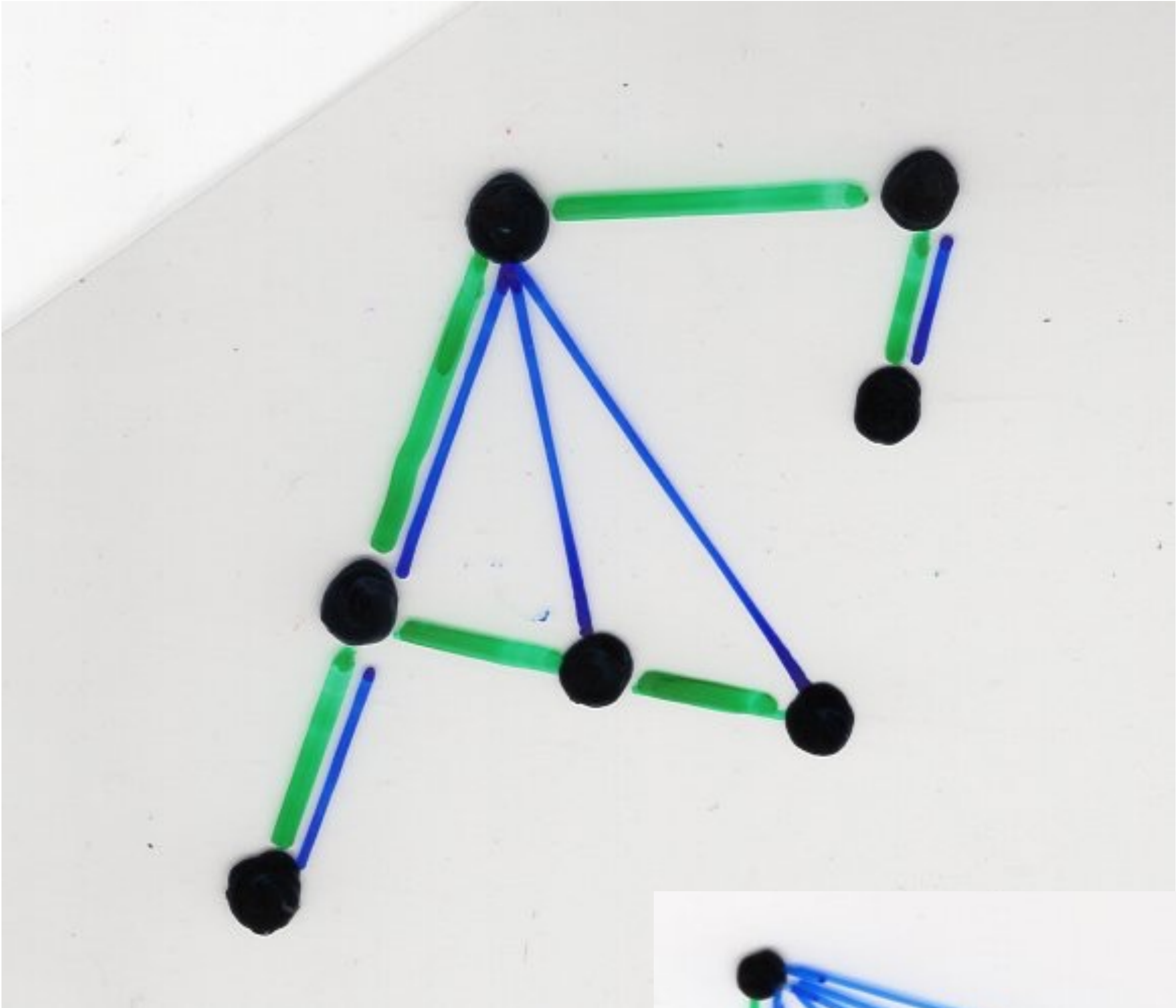
binary trees

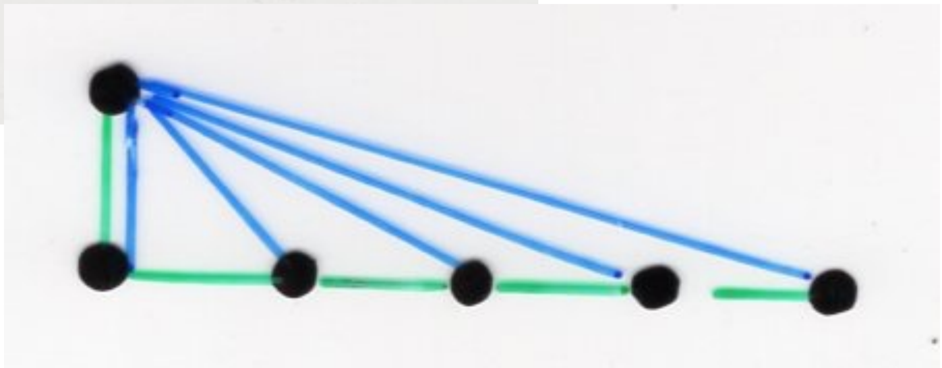
(forest of) planar trees

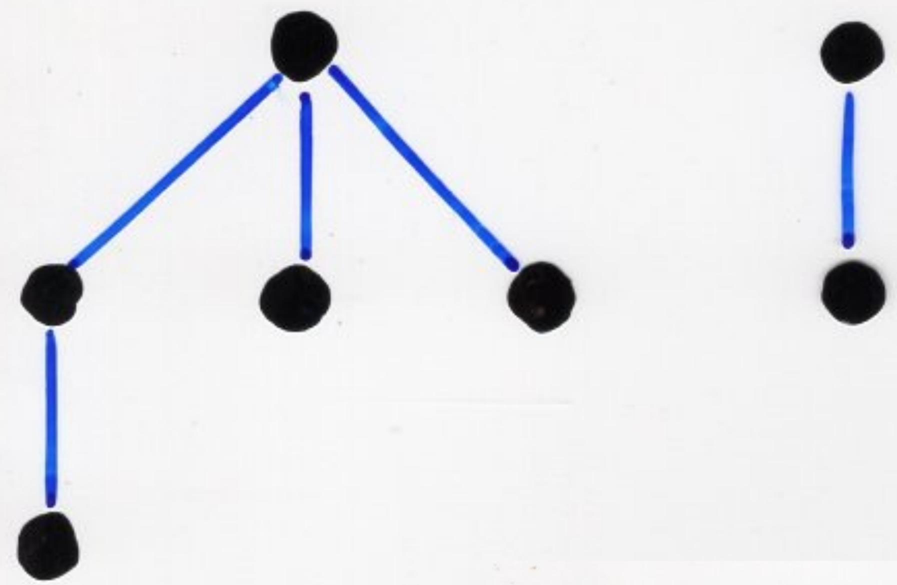


right rotation

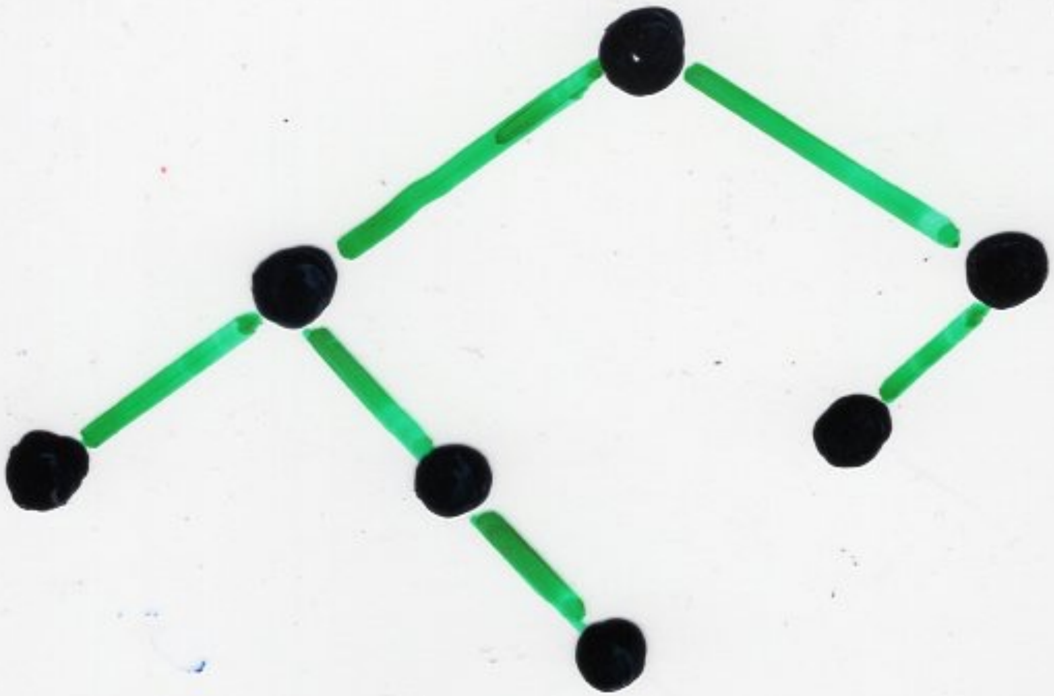




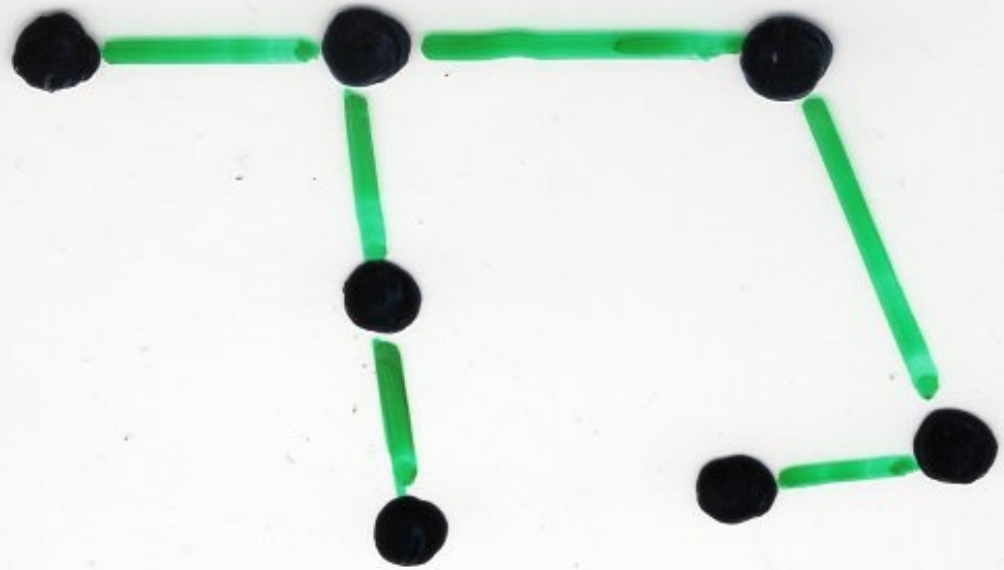


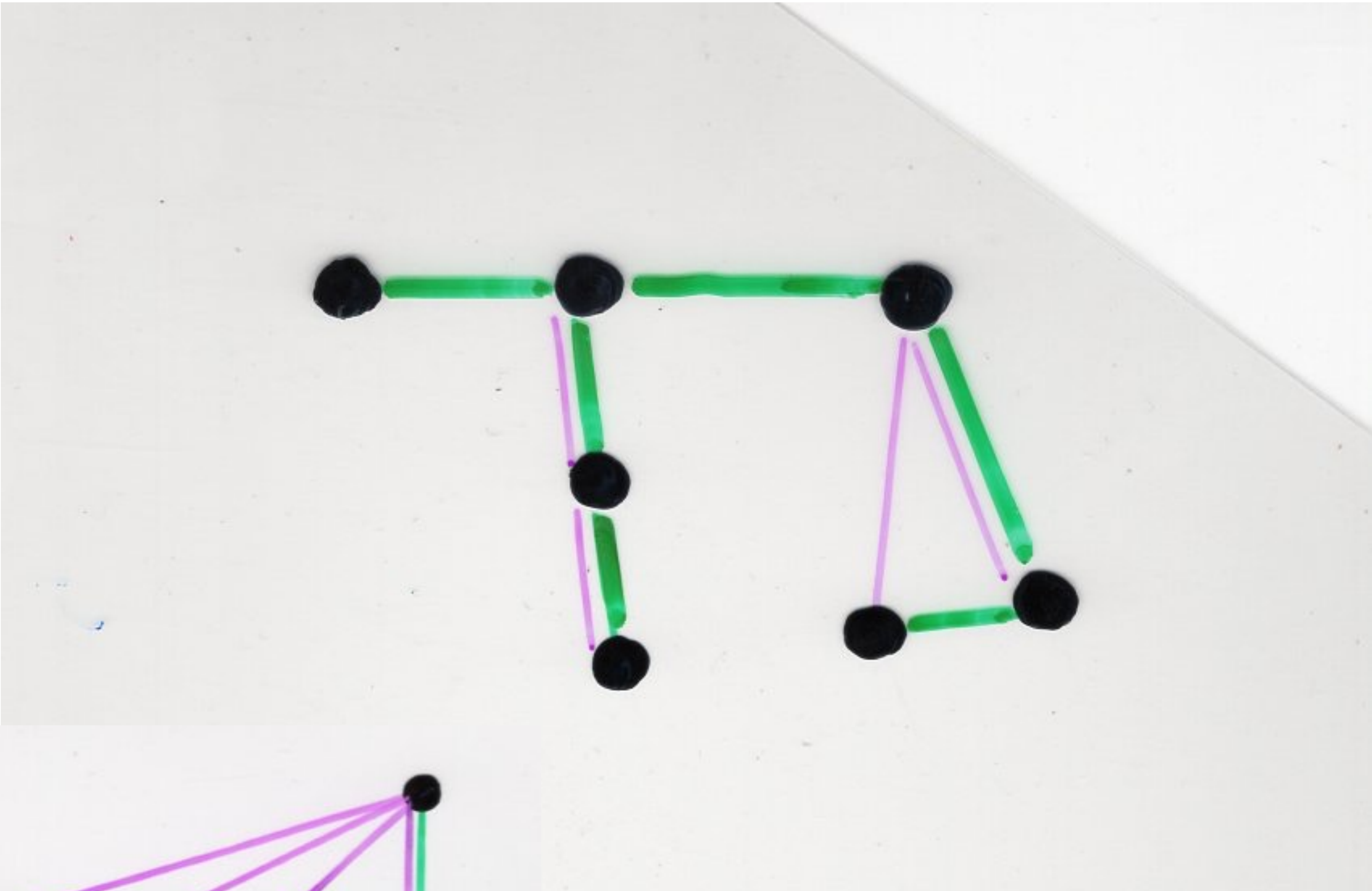


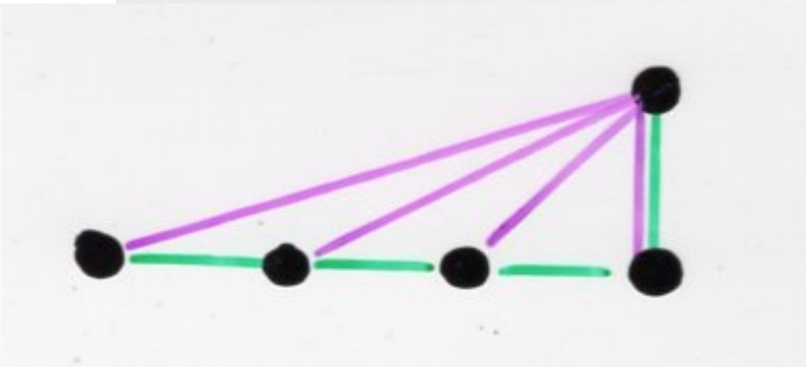
"fundamental transform"
in Knuth,
The Art of Computer Programming
Vol 1

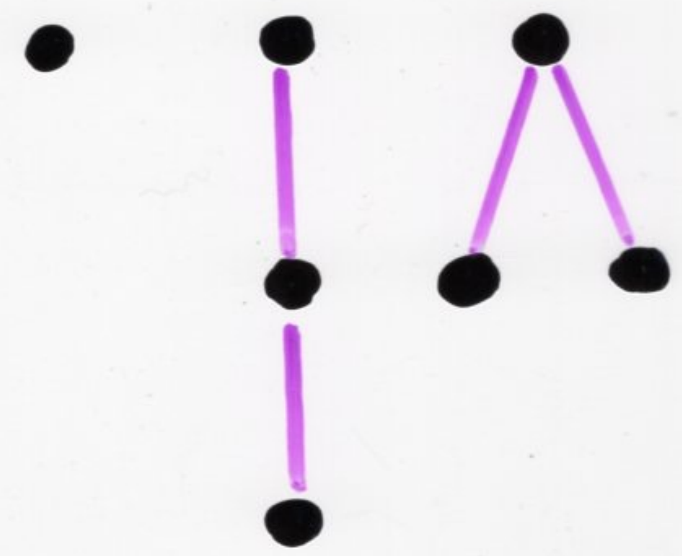


left rotation







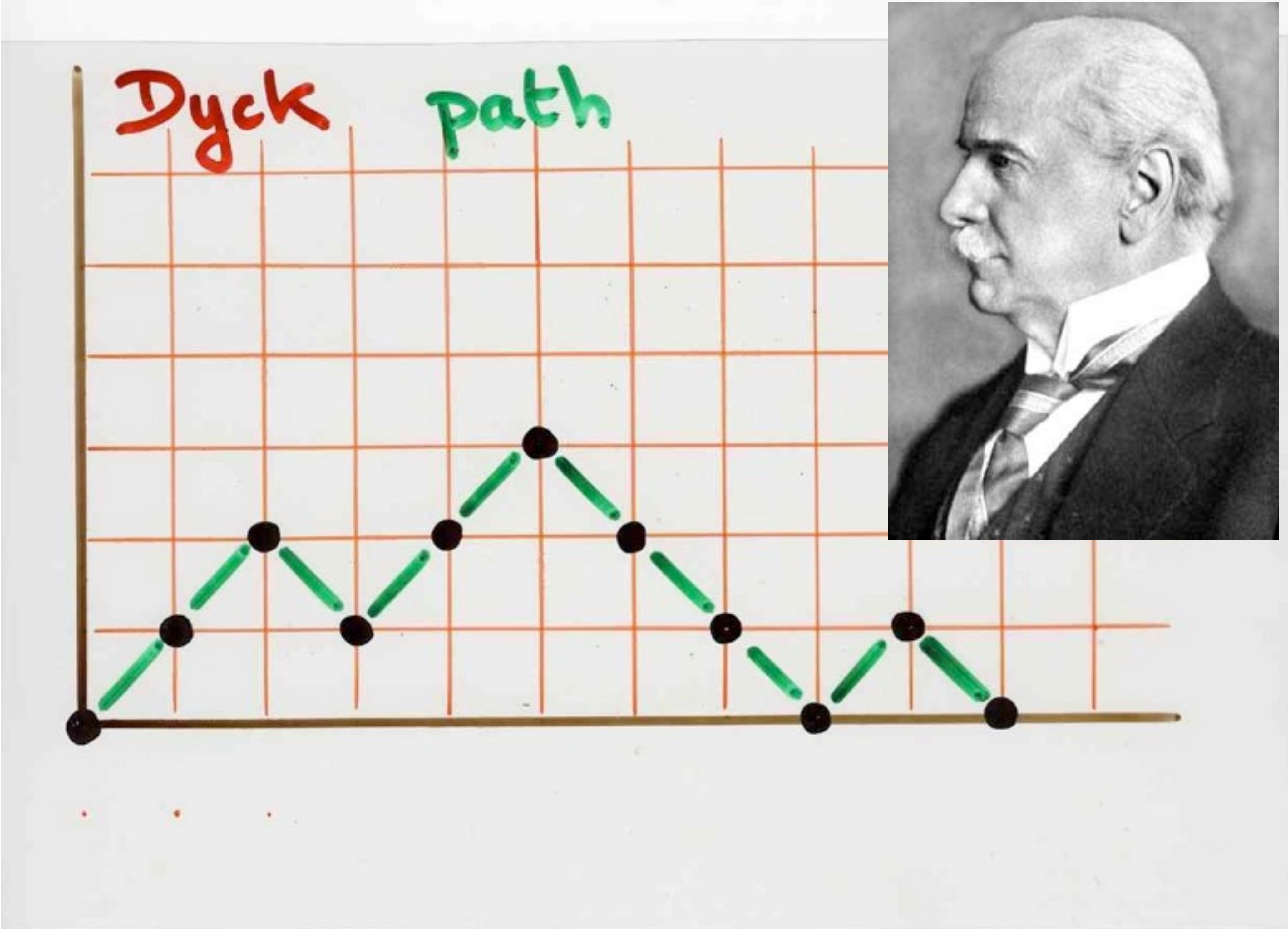


paths

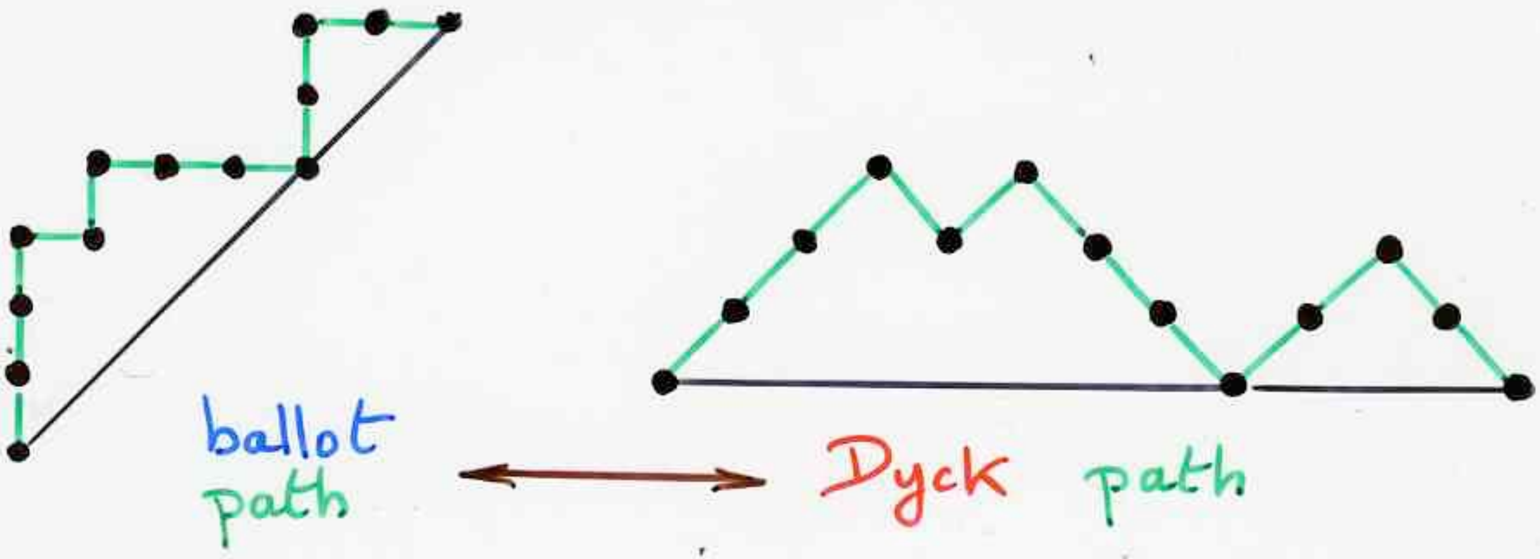
Dyck paths

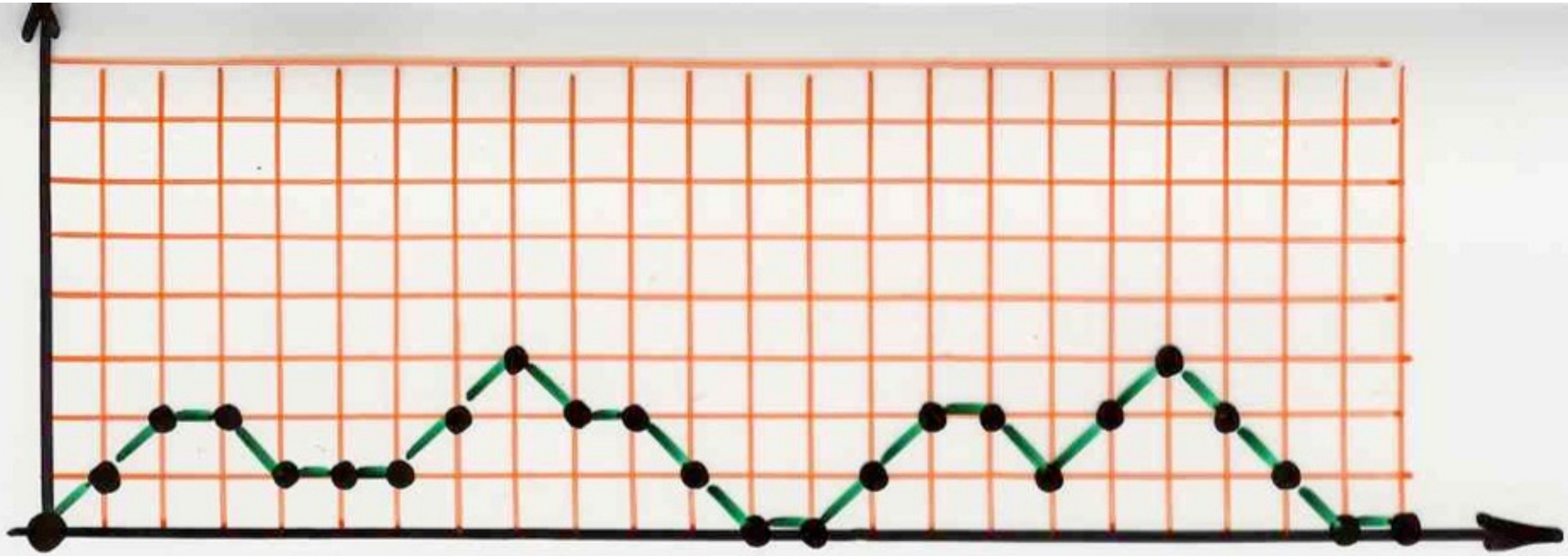
2-colored Motzkin paths

Lukasiewicz paths


Dyck path

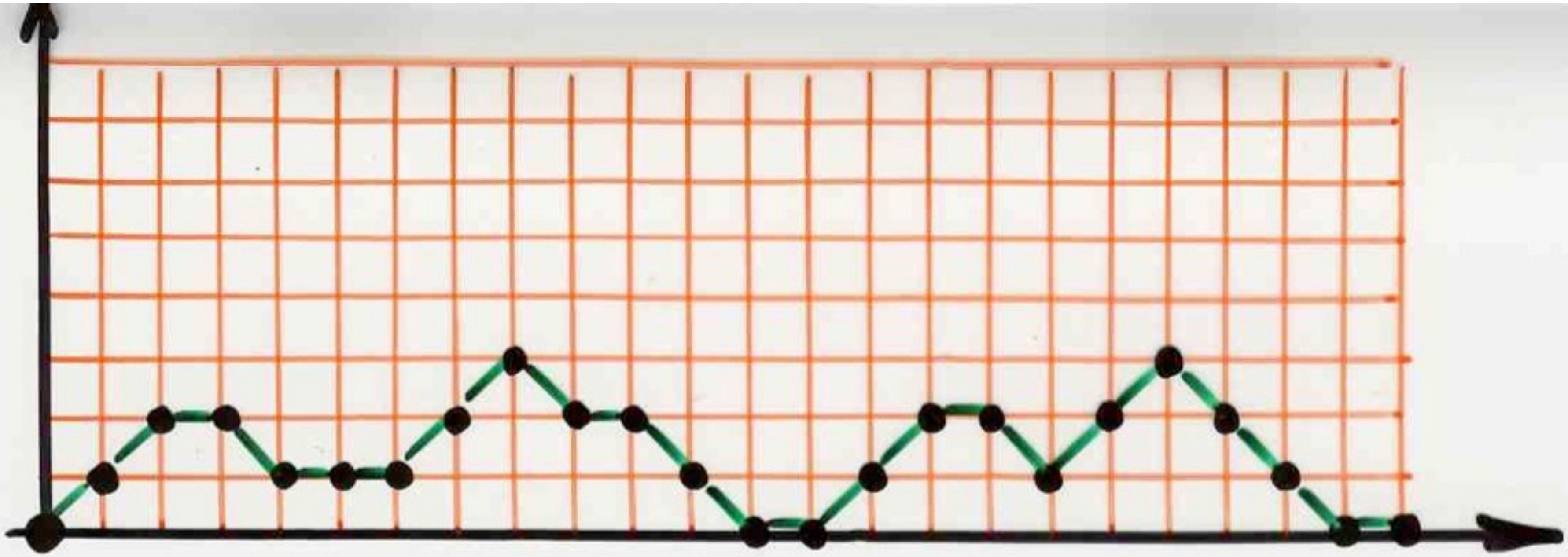
vocabulary: ballot path
Dyck path





- Motzkin path $=$
  - $\emptyset$
  - $(\bullet\!-\!\bullet) \times (\text{Motzkin path})$
  - $(\nearrow) \times (\text{Motzkin path}) \times (\searrow) \times (\text{Motzkin path})$

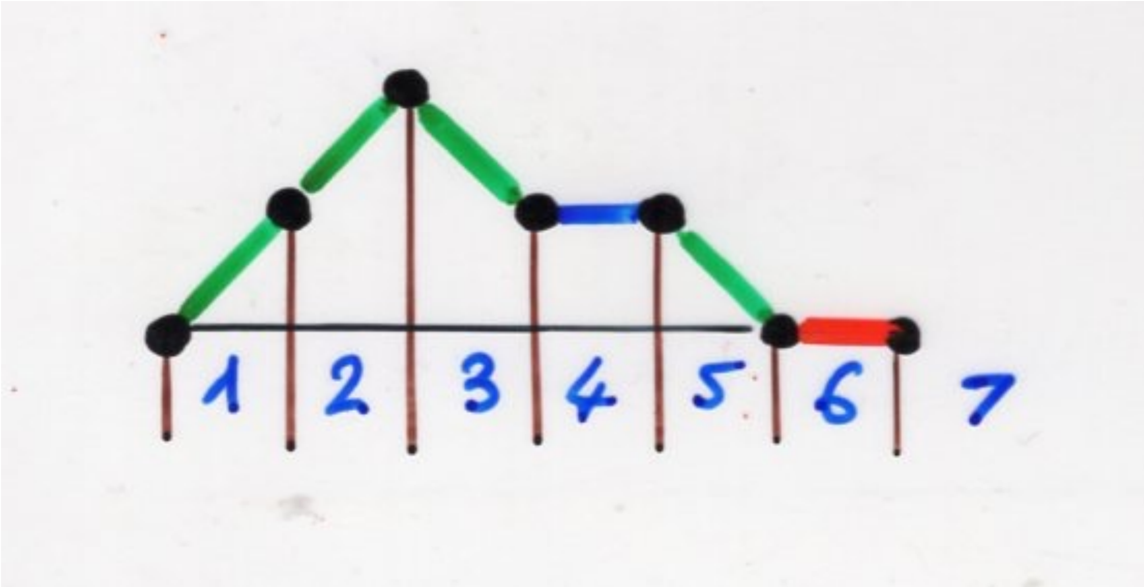
$$m = 1 + t\,m + t^2 m^2$$



- $\emptyset$
- $(\bullet\!-\!\bullet) \times$ (Motzkin path)
- $(\nearrow) \times$ (Motzkin path) $\times (\searrow) \times$ (Motzkin path)

$$1 + t\,m + t^2 m^2$$

2-colored
Motzkin
path



$a_n = C_{n+1}$

number of such paths of length n

$$z = 1 + 2tz + t^2 z^2$$
$$y = 1 + tz$$

$$y = 1 + ty^2$$

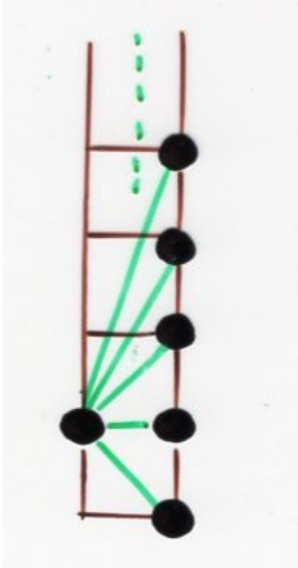
Lukasiewicz path

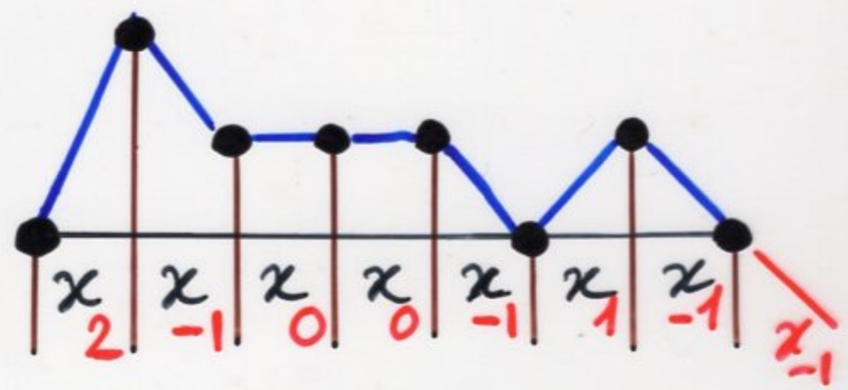
$\omega = (s_0, \ldots, s_n)$

$s_0 = (0,0)\,, \quad s_n = (n,0)$

elementary step $\quad s_i = (x_i, y_i) \quad s_{i+1} = (x_{i+1}, y_{i+1})$

$x_{i+1} = 1 + x_i \quad$ with $\quad y_{i+1} \geqslant y_i - 1$





Łukasiewicz language $L \subseteq X^*$

$X = \{x_{-1}, x_0, \ldots, x_p, \ldots$

$\delta : X^* \longrightarrow \mathbb{Z}$ monoid morphism

$\delta(x_i) = i$ $\delta(uv) = \delta(u) + \delta(v)$

$w \in L$

iff (i) $\delta(w) = -1$

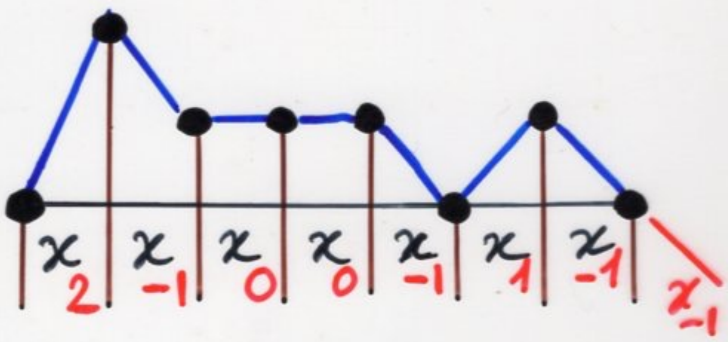
(ii) $\delta(u) \geq 0$ for every u left factor of w

(i.e. $w = uv$, $u, v \in X^*$)

exercise

Prove that the number of Lukasiewicz paths of length n is the Catalan number C_n

(establishing an equation for the generating function)



bijections paths to paths

Dyck paths

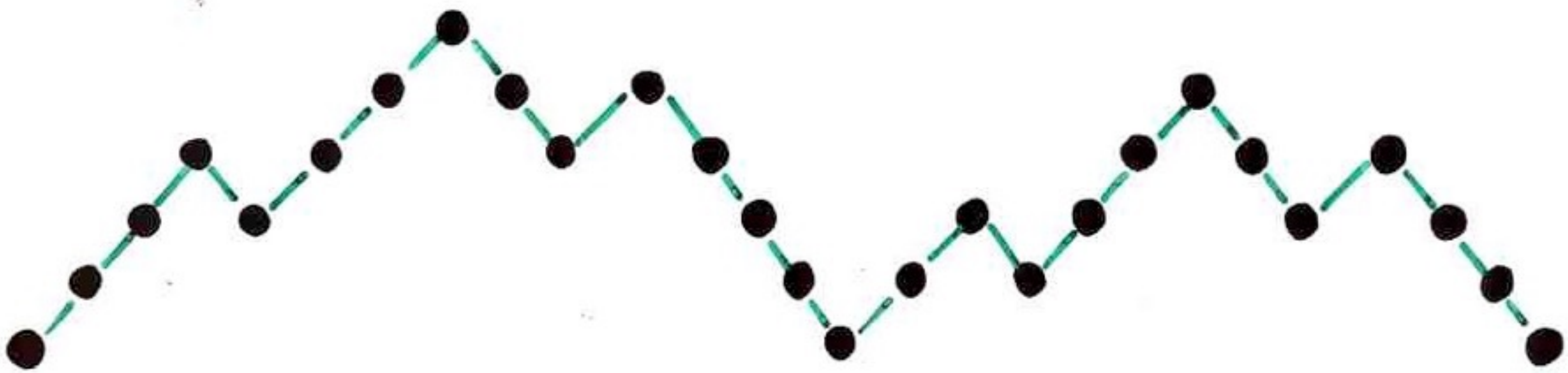
2-colored Motzkin paths

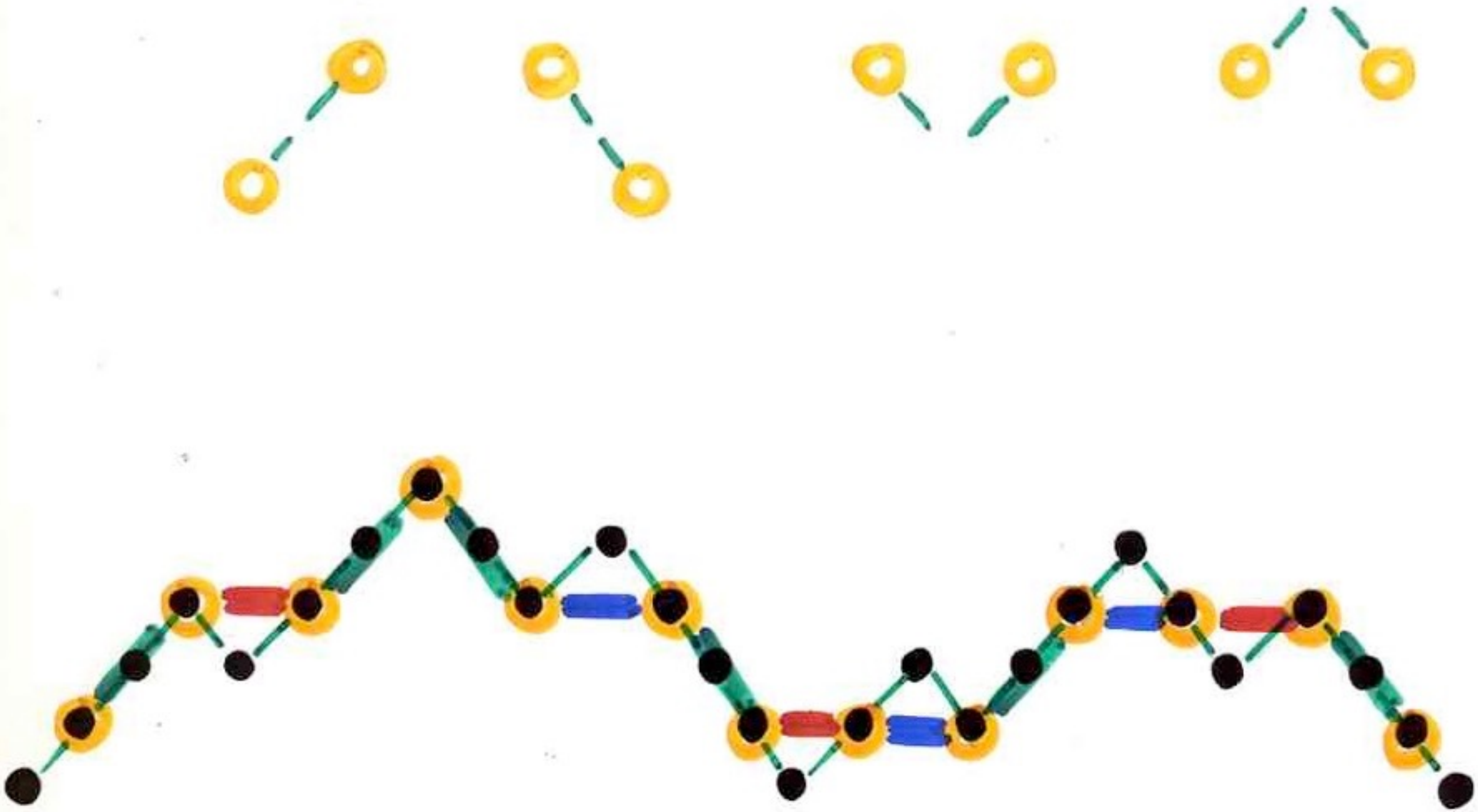
Lukasiewicz paths

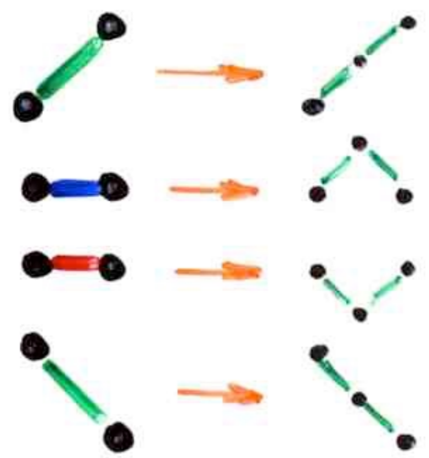
bijection

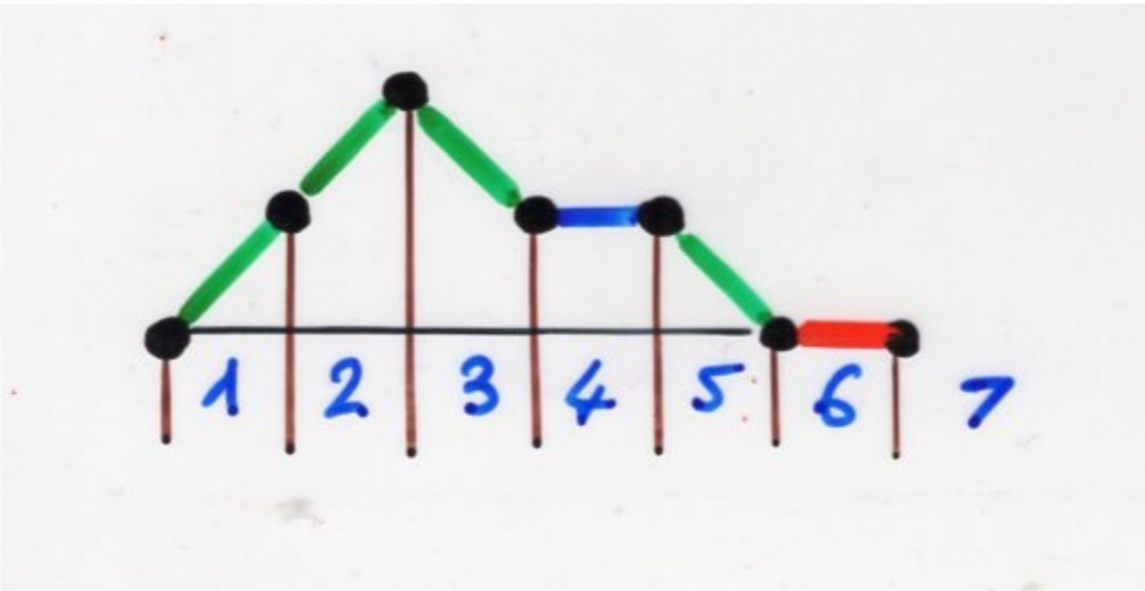
2-colored Motzkin paths

Dyck paths








1
2
3
4
5
6
7

exercise

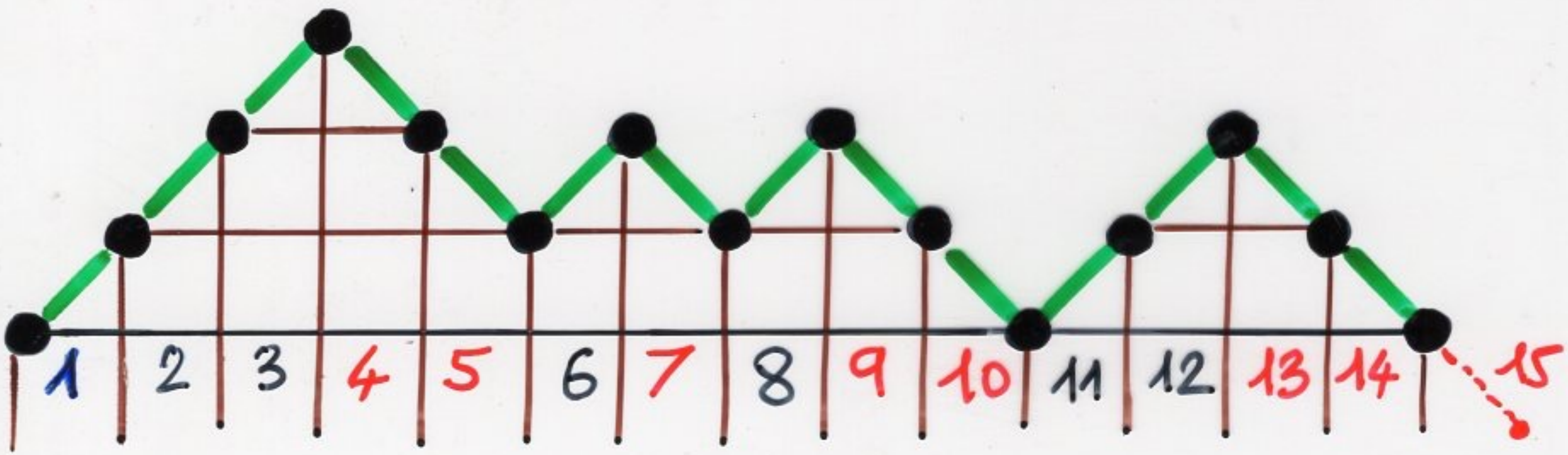
Touchard identity

$$C_{n+1} = \sum_{0 \leq i \leq \lfloor n/2 \rfloor} \binom{n}{2i} C_i \, 2^{2n-i}$$

bijection

Dyck paths

Lukasiewicz paths



1
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7
8
9
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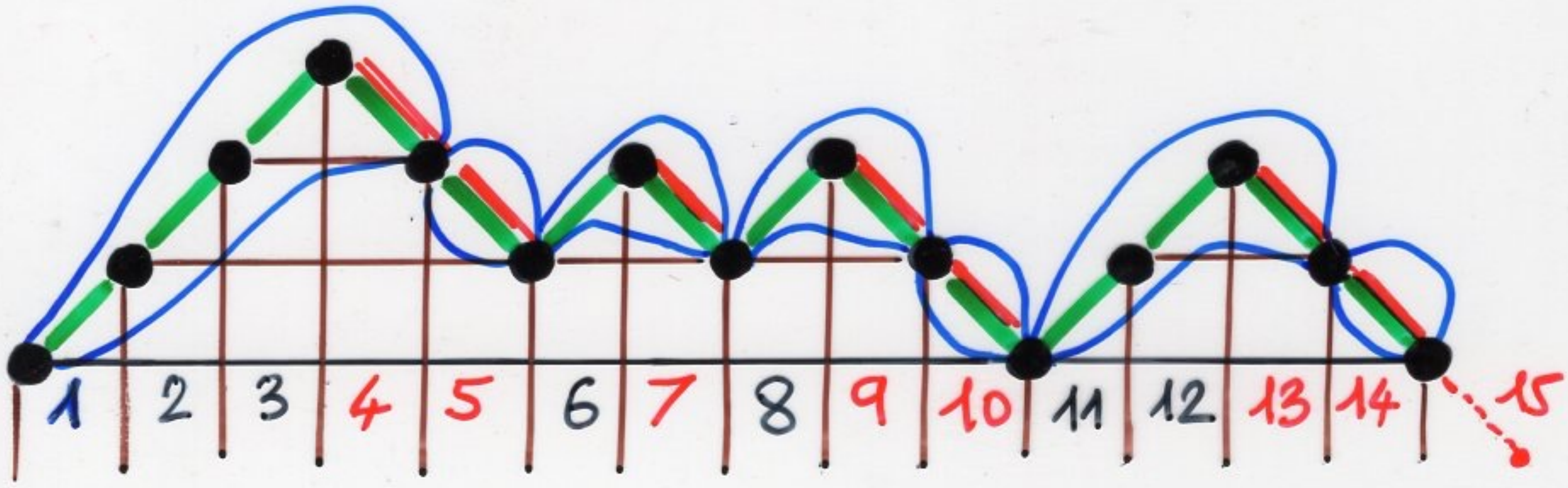
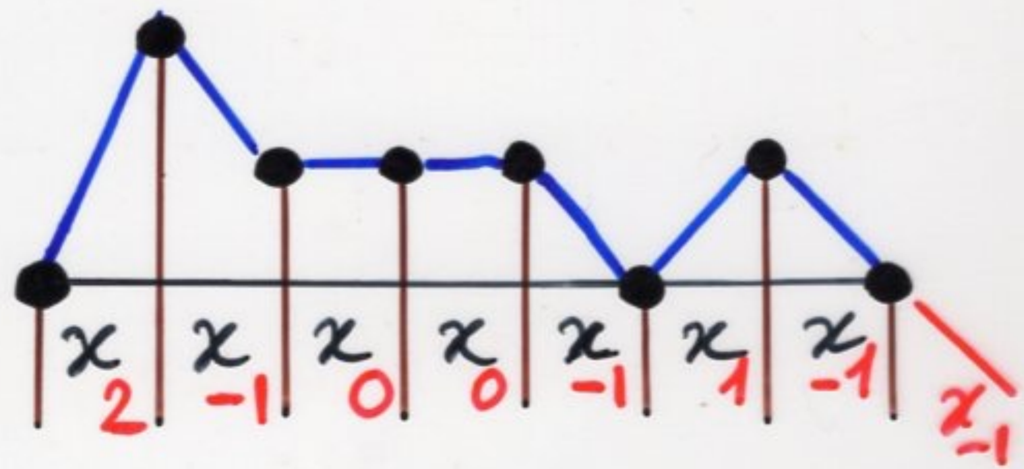
11
12
13
14
15


1 2 3 4 5 6 7 8 9 10 11 12 13 14 15



x_2
x_{-1}
x_0
x_0
x_{-1}
x_1
x_{-1}
x_{-1}

bijections trees to paths

complete binary trees —— Dyck paths

binary trees —— 2-colored Motzkin paths

planar trees — Dyck paths

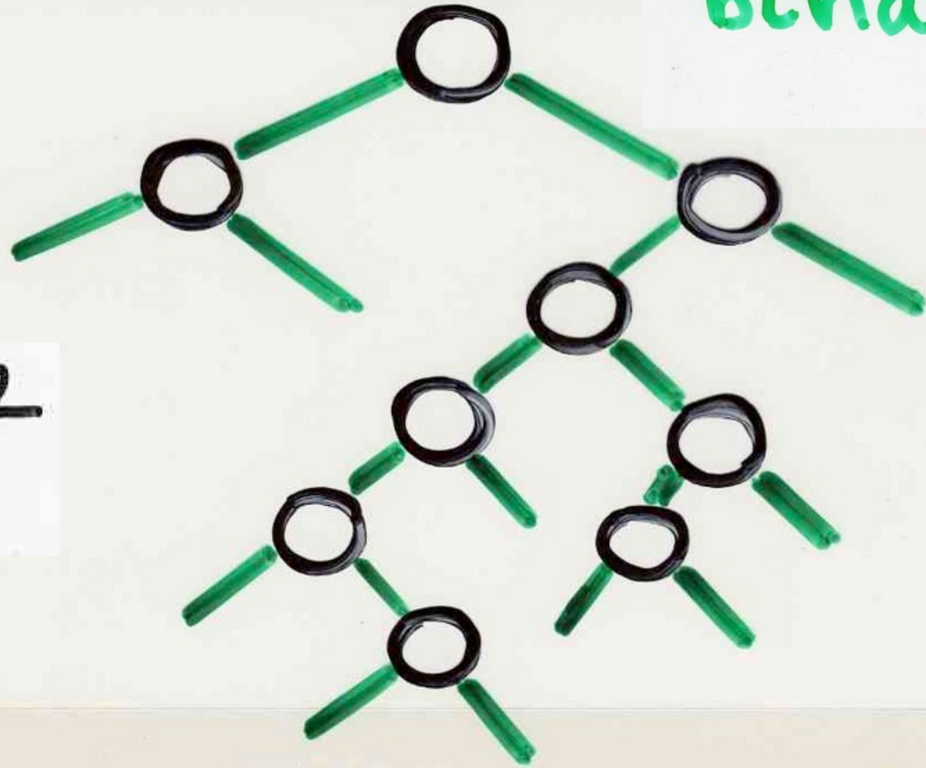
planar trees — Lukasiewicz paths

bijection

(complete) binary trees

Dyck paths

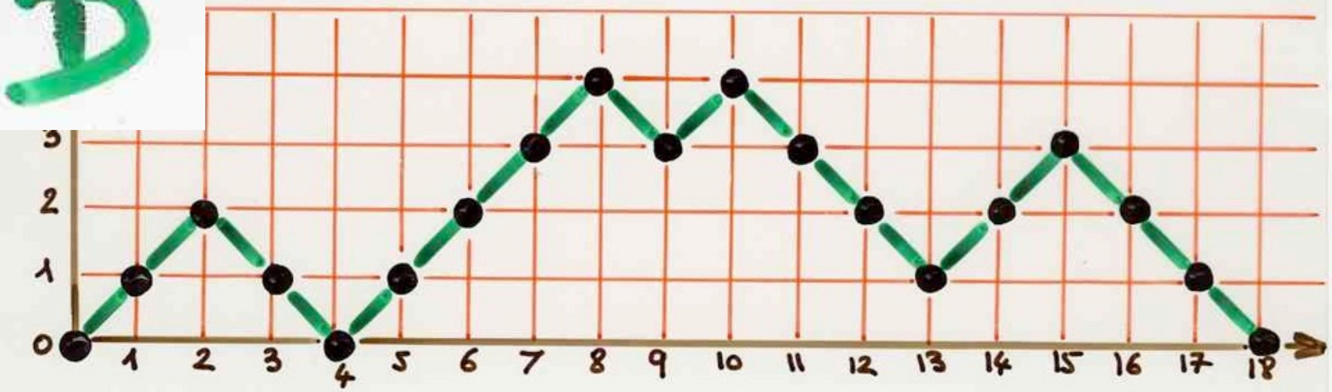
binary tree

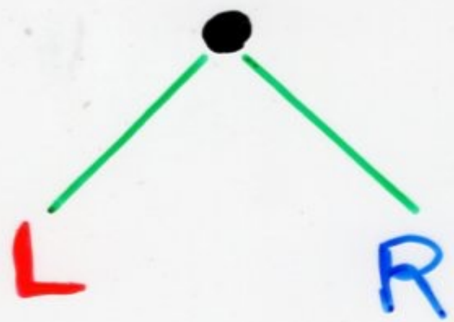


$$A = 1 + tA^2$$

Dyck path

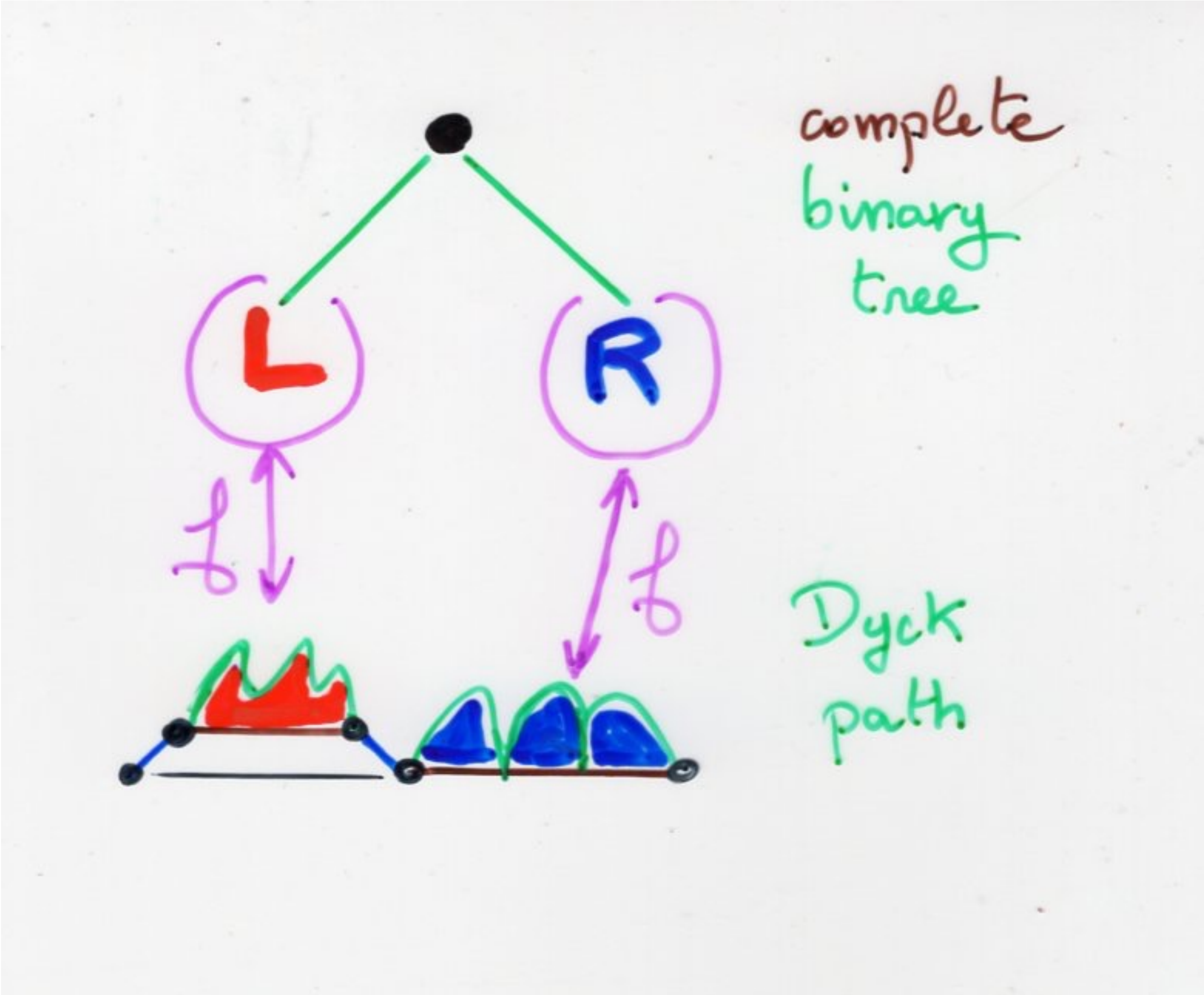
$$D = 1 + tD^2$$

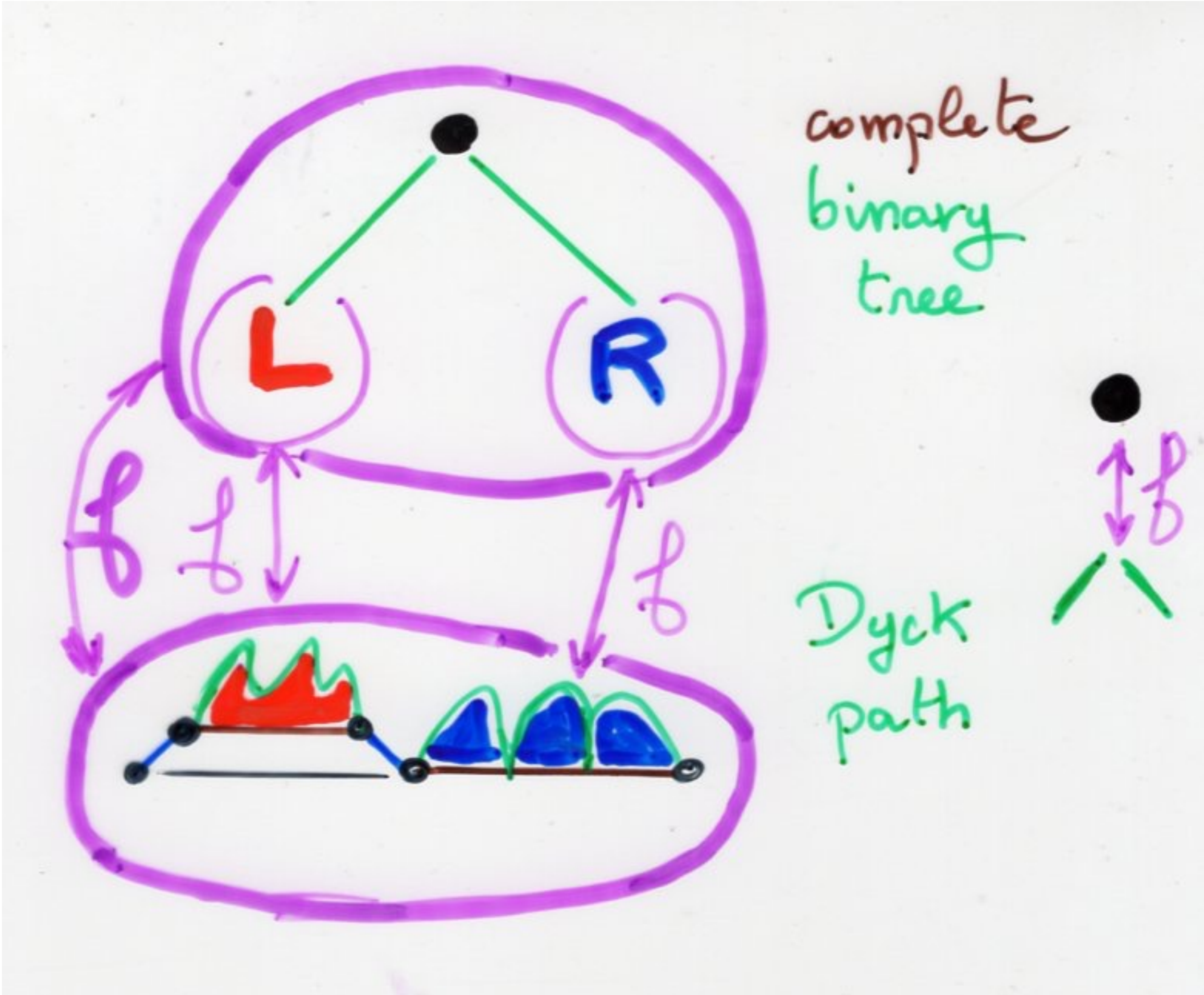




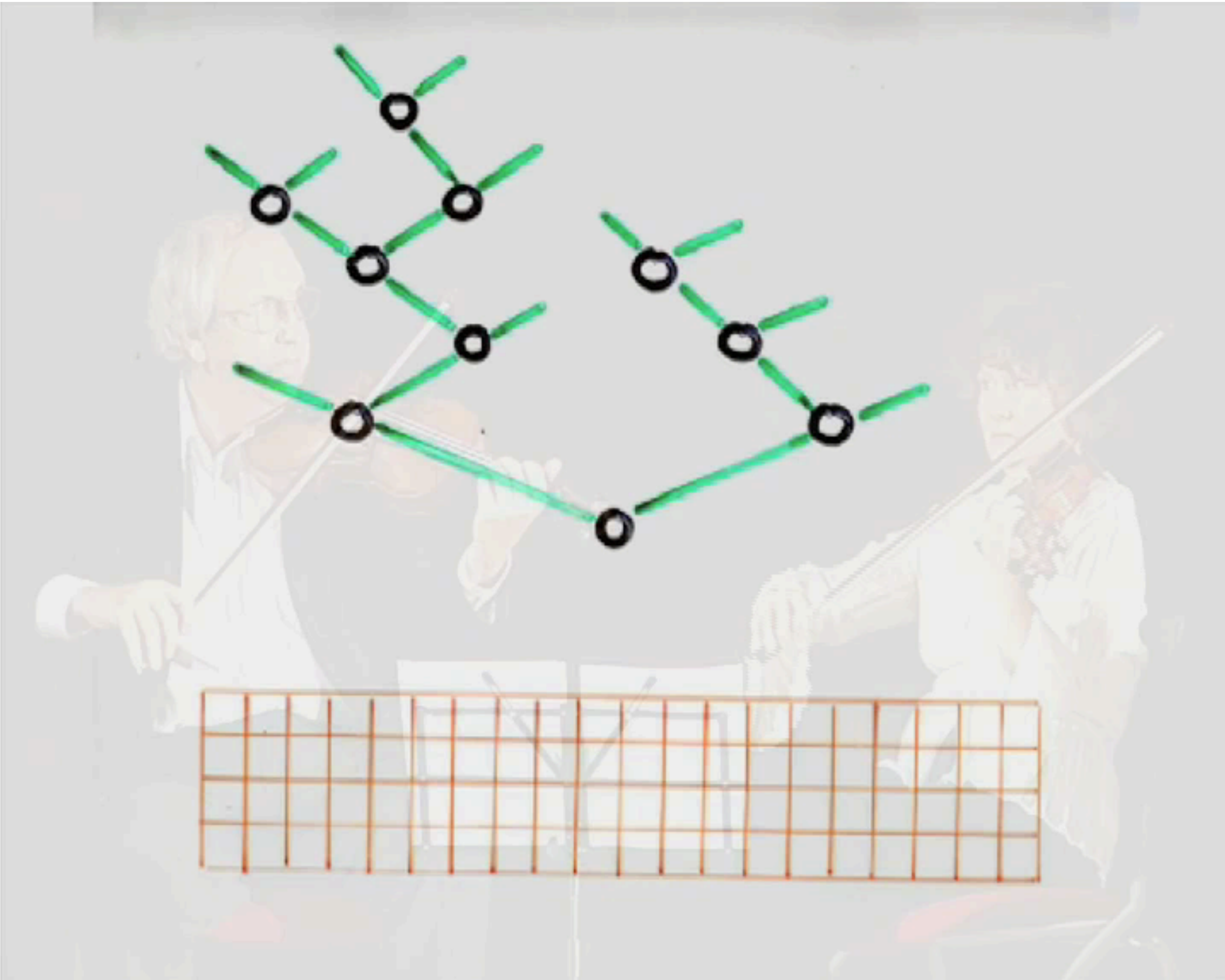
complete binary tree

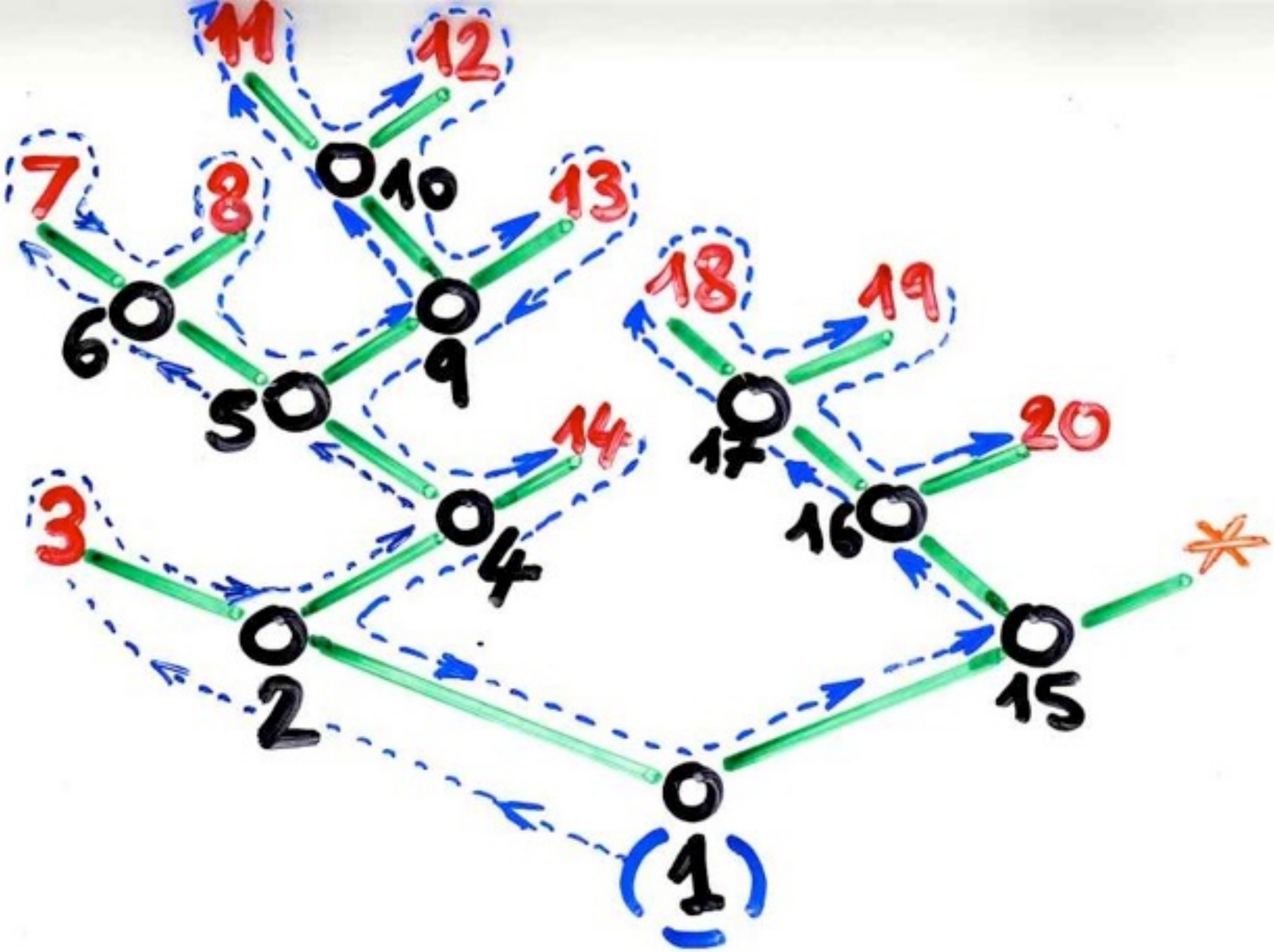
Dyck path


complete
binary
tree
L
R
f
f
Dyck
path



L
R
f
complete
binary
tree
Dyck
path





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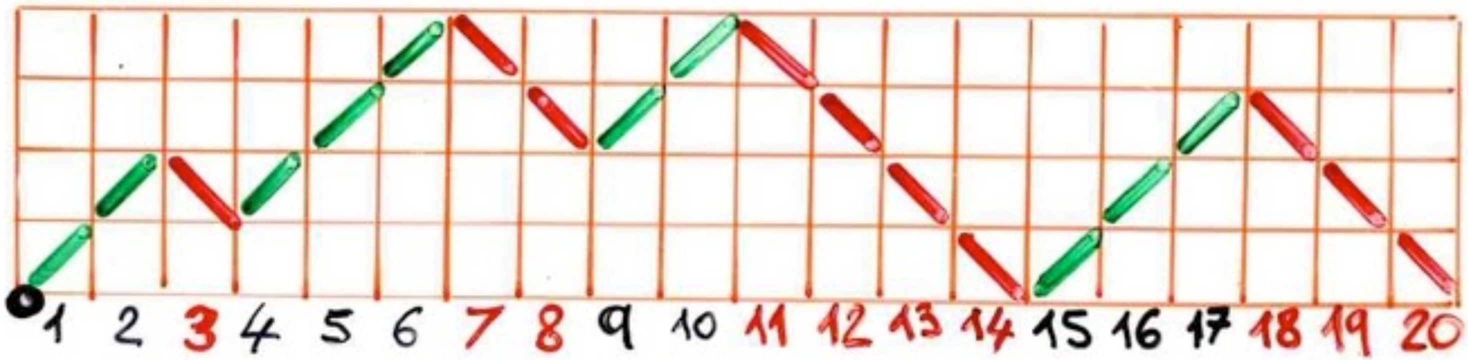
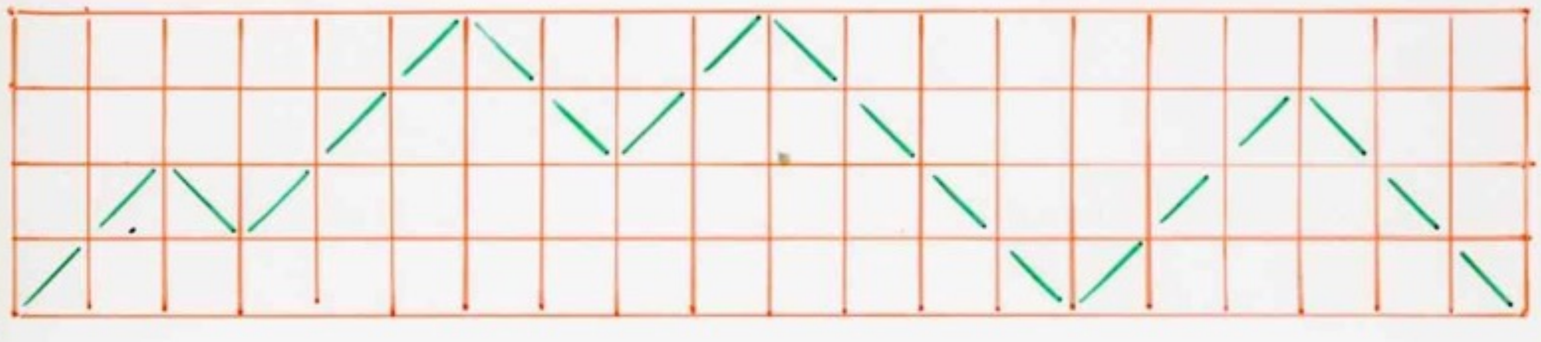


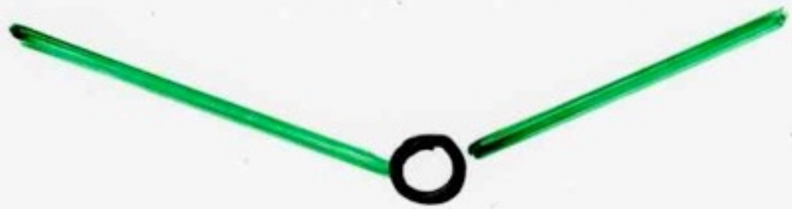
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

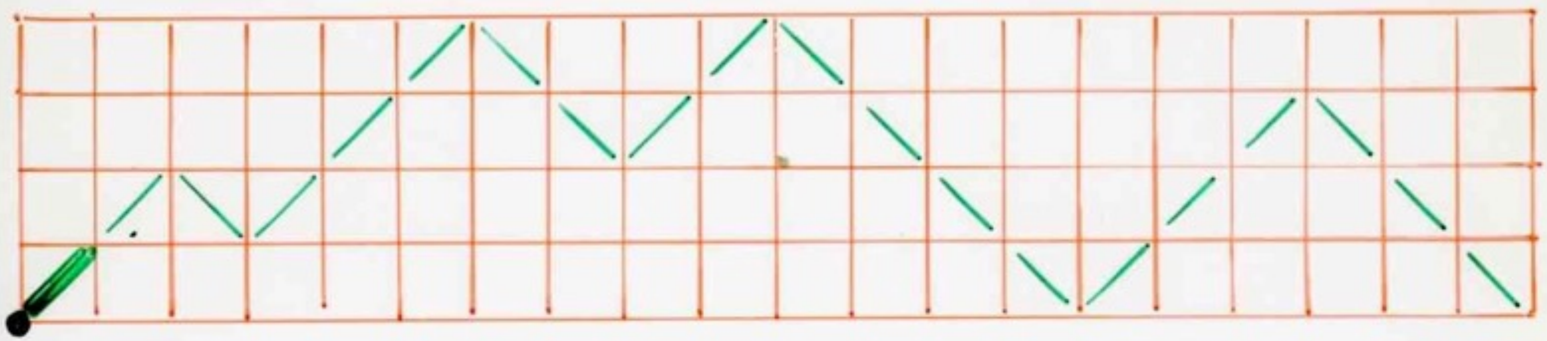
reciprocal bijection

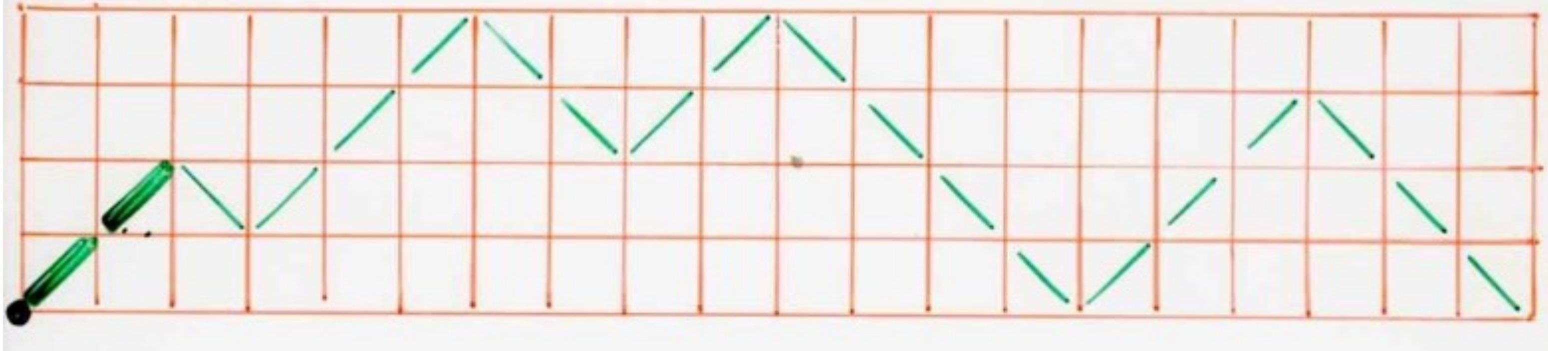
(complete) binary trees

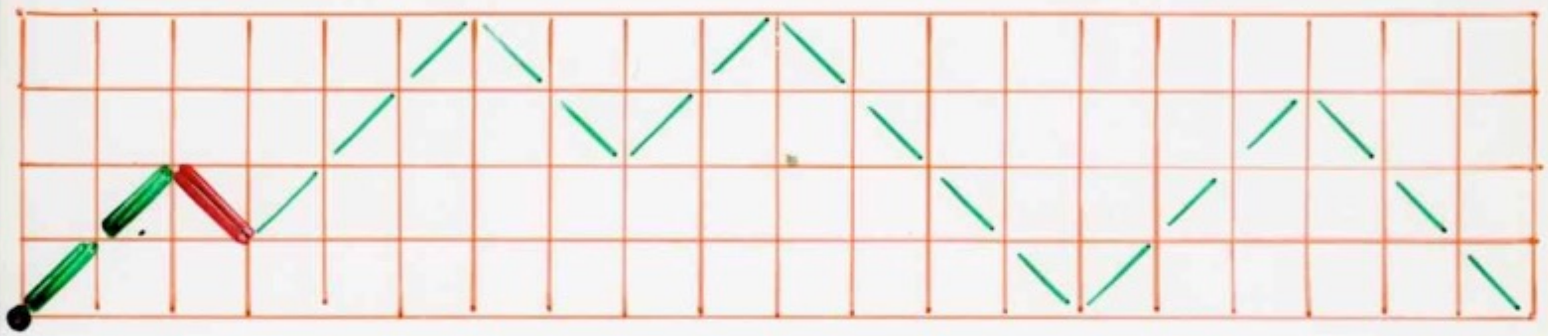
Dyck paths

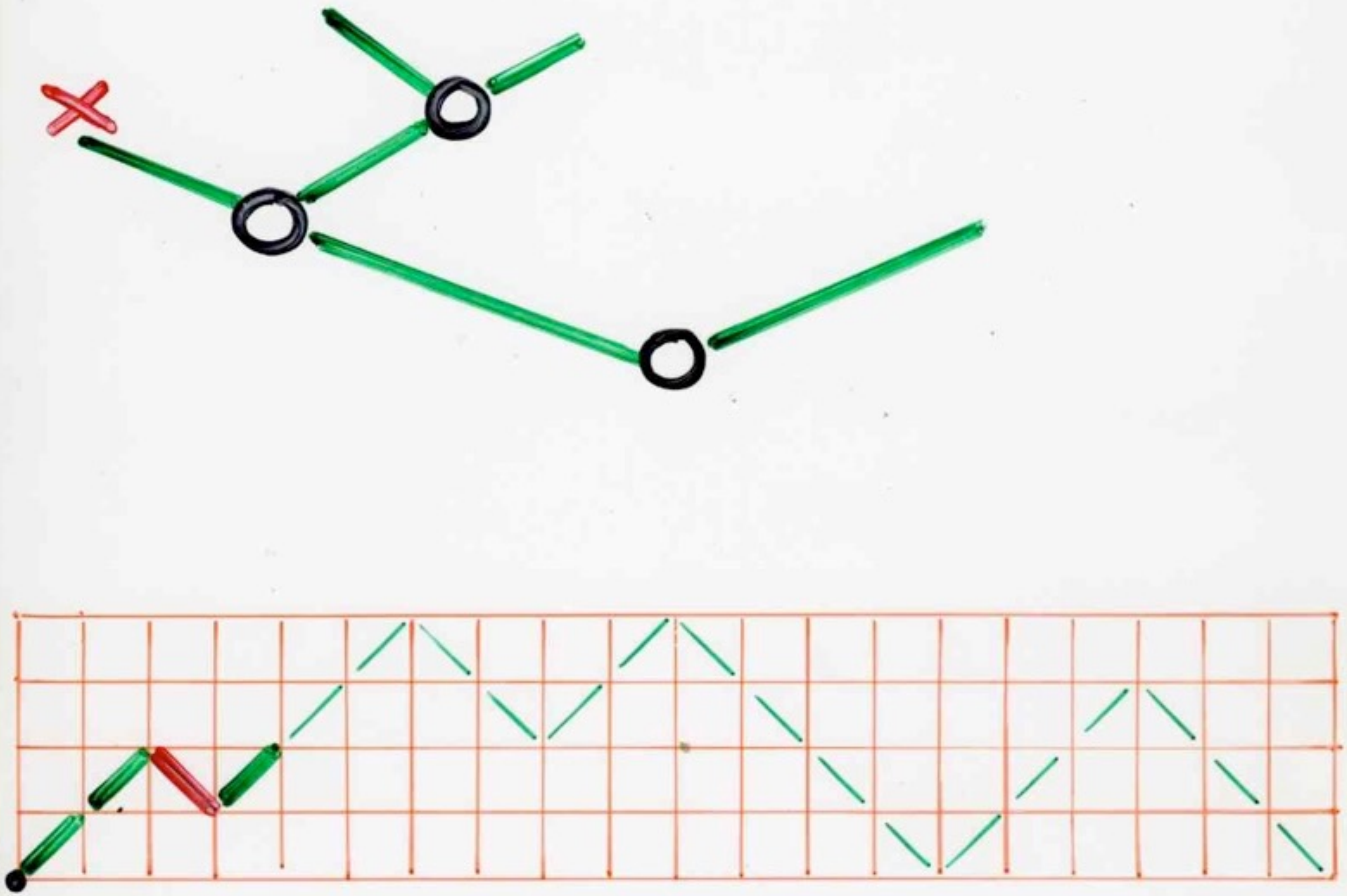


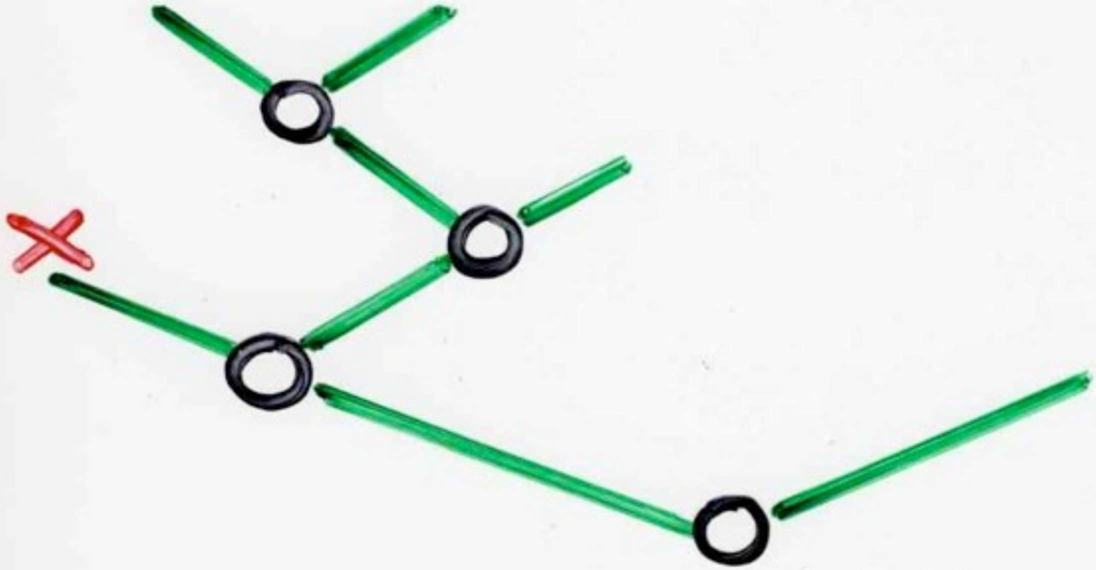


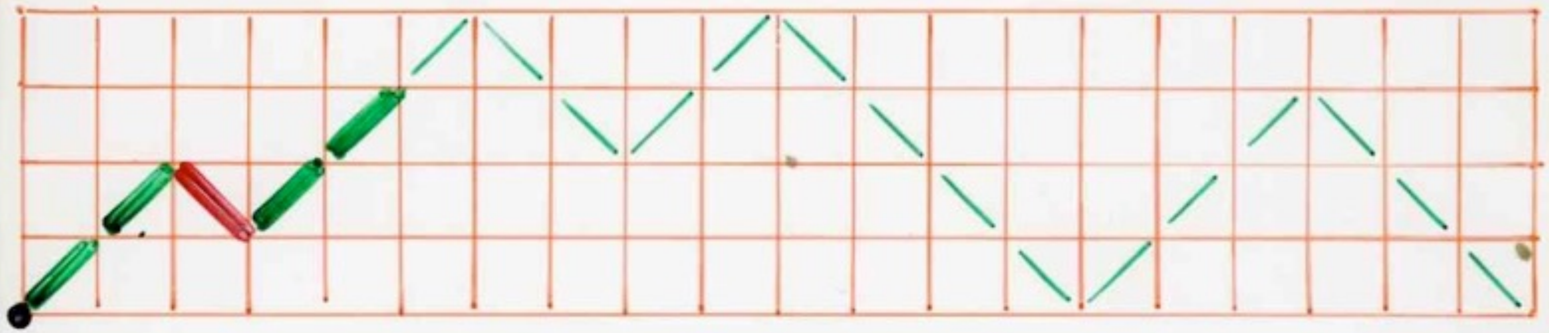


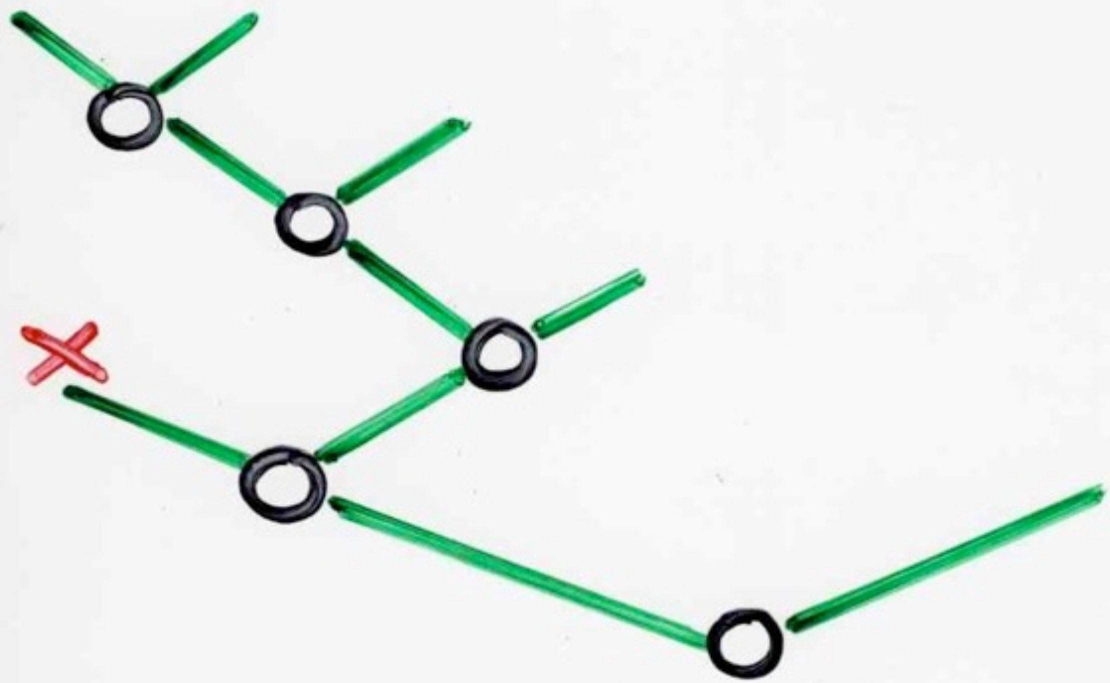


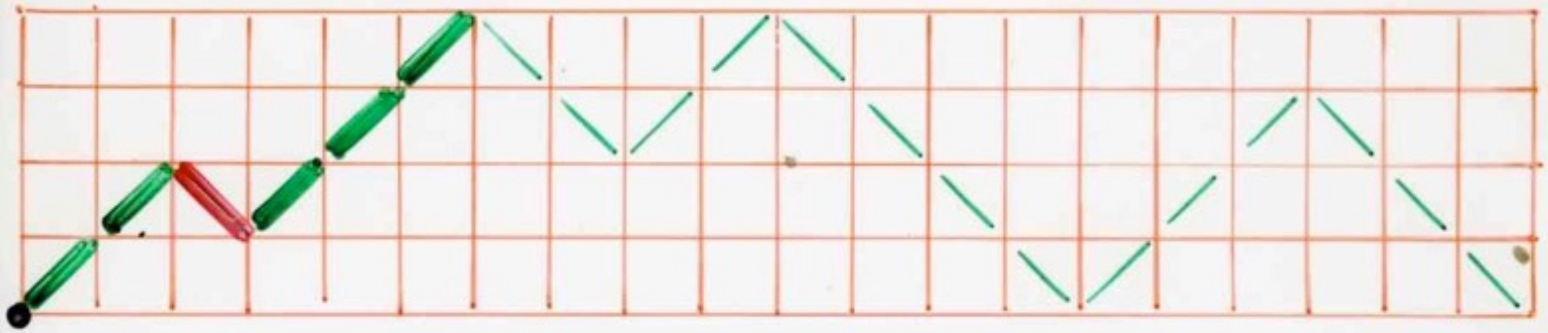


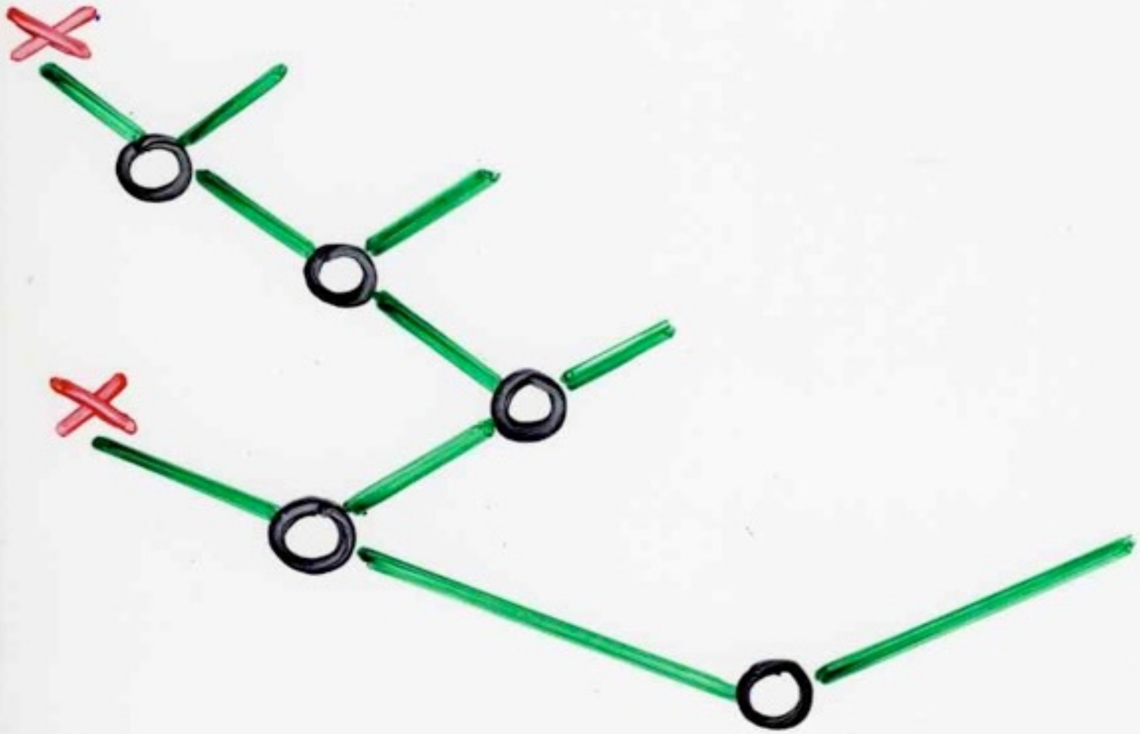


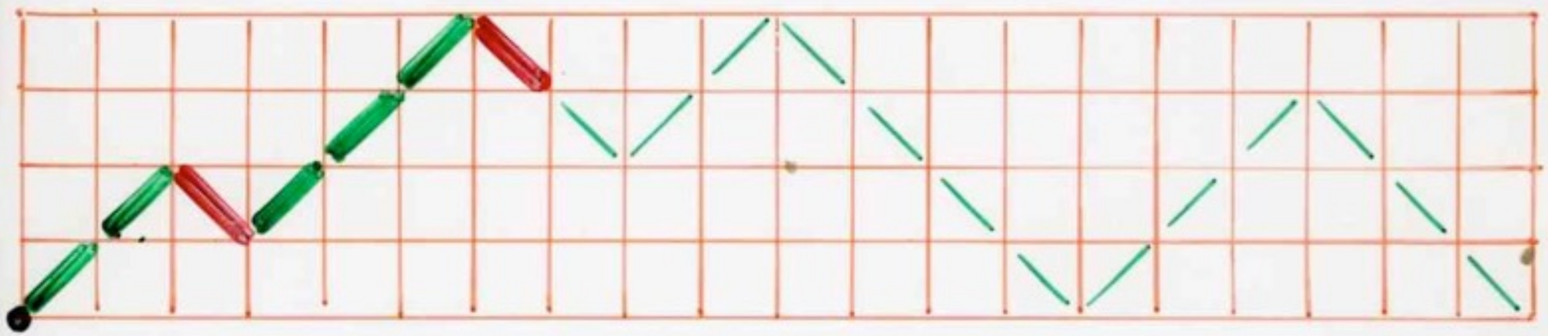


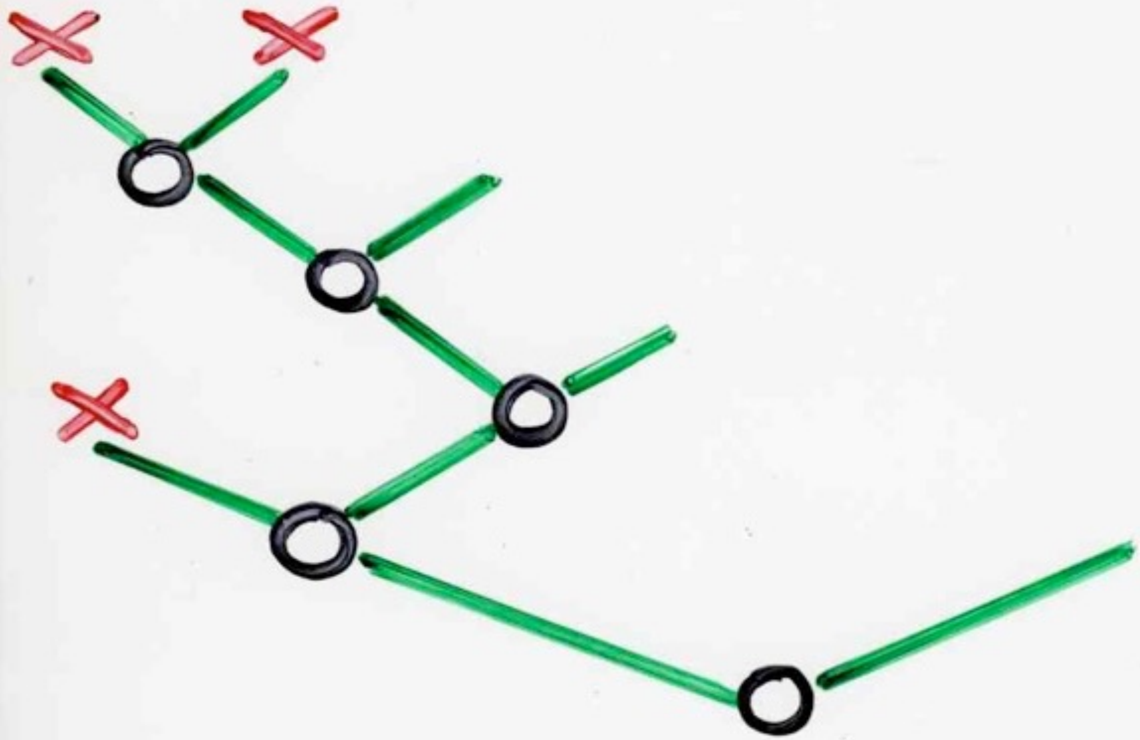


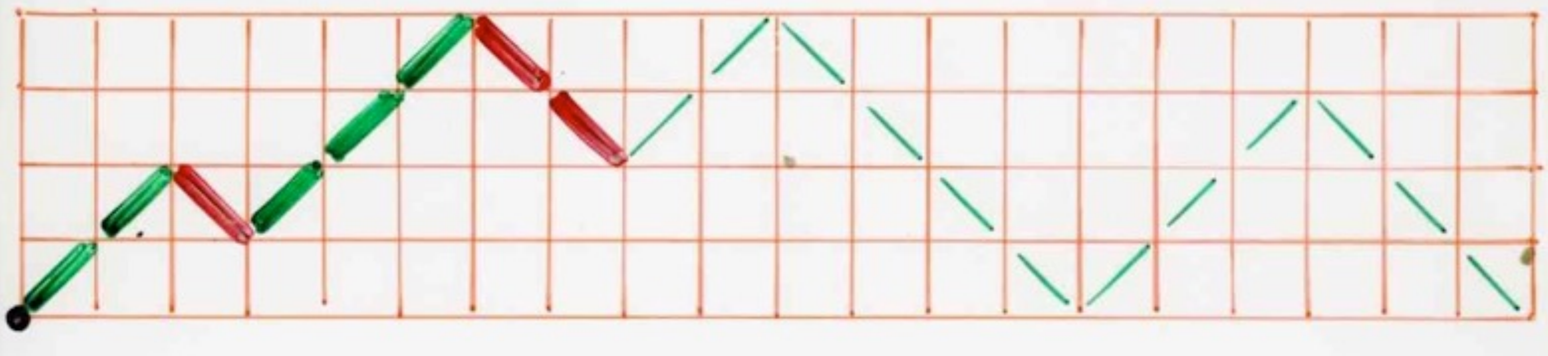


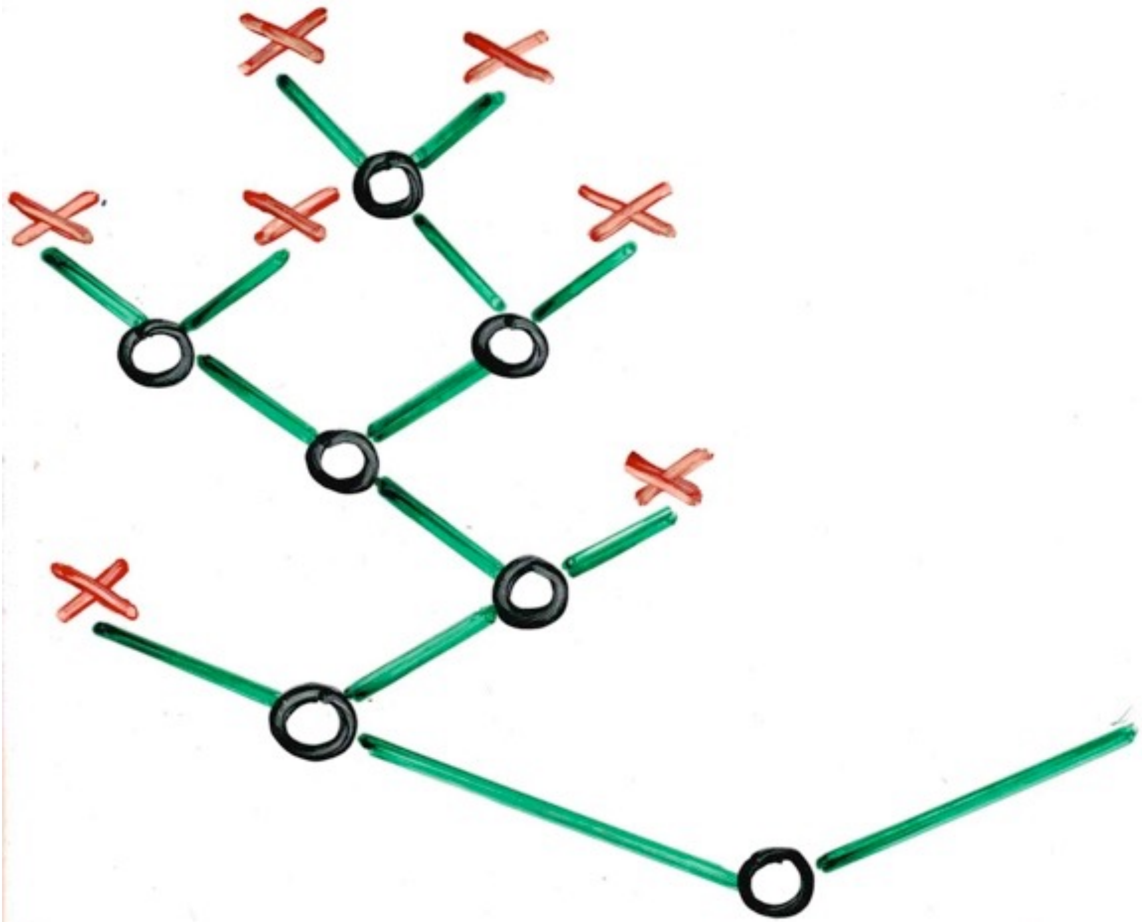


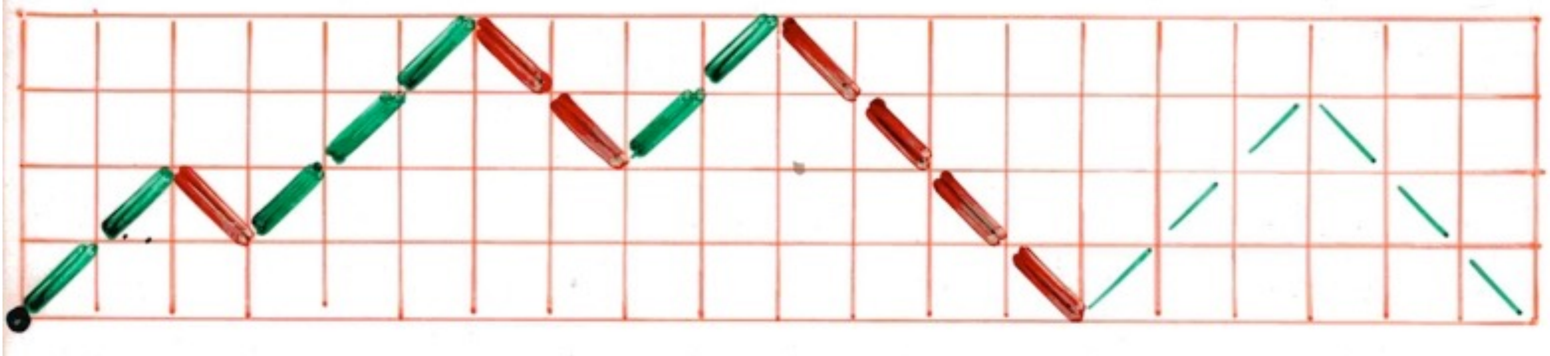


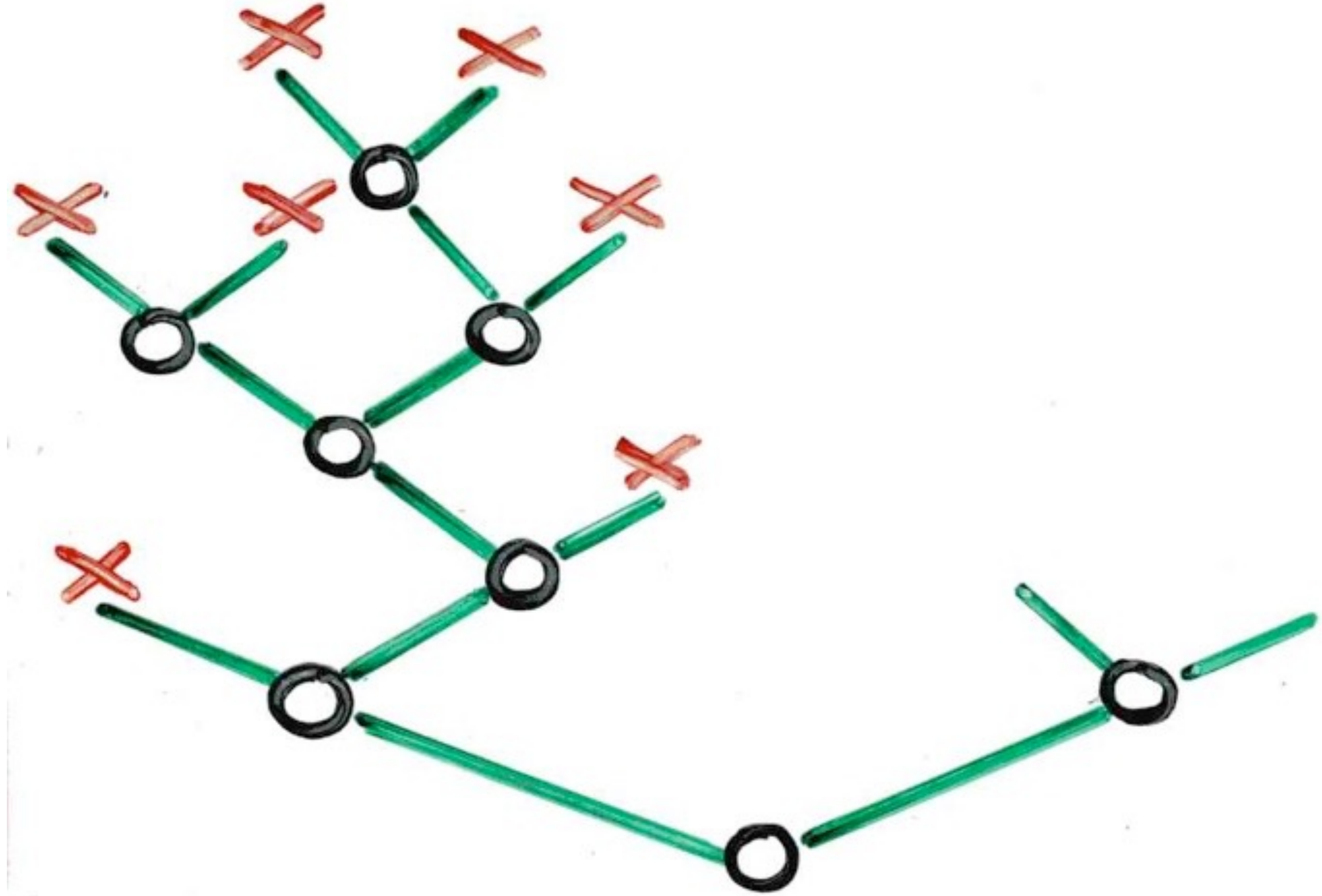


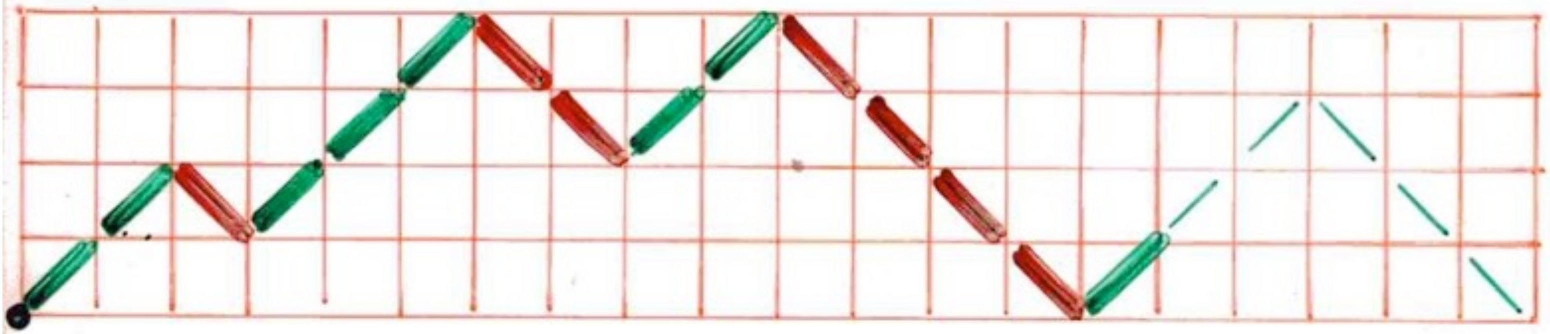


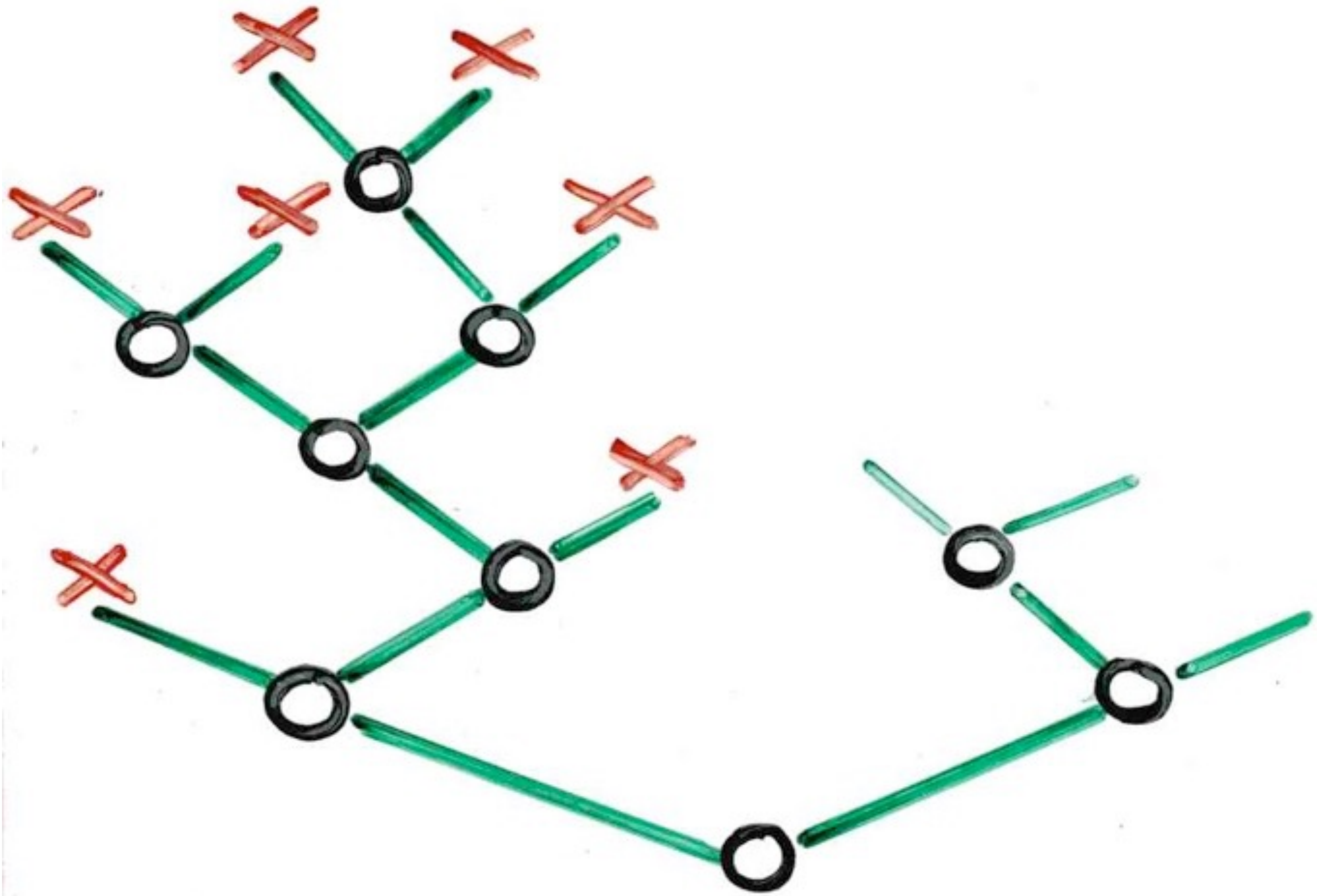


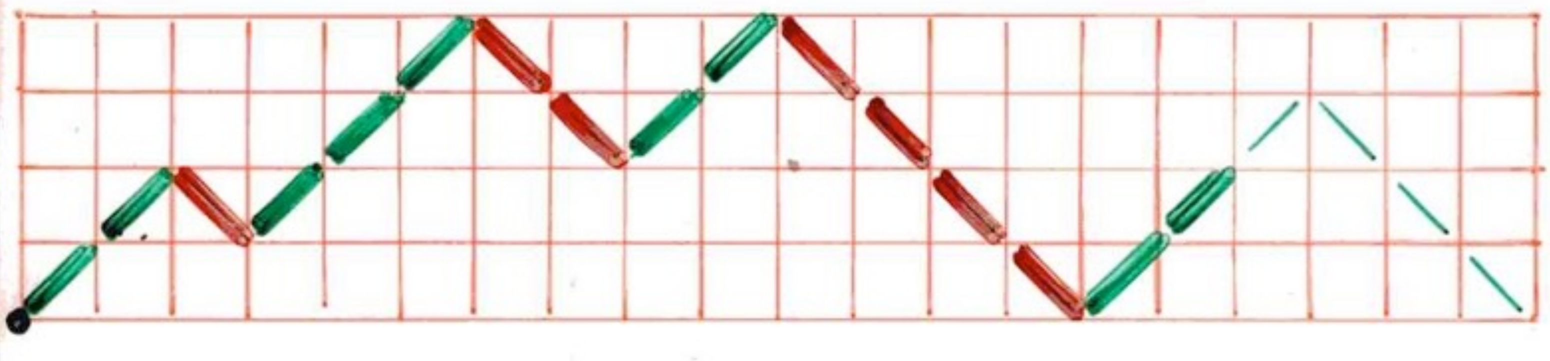


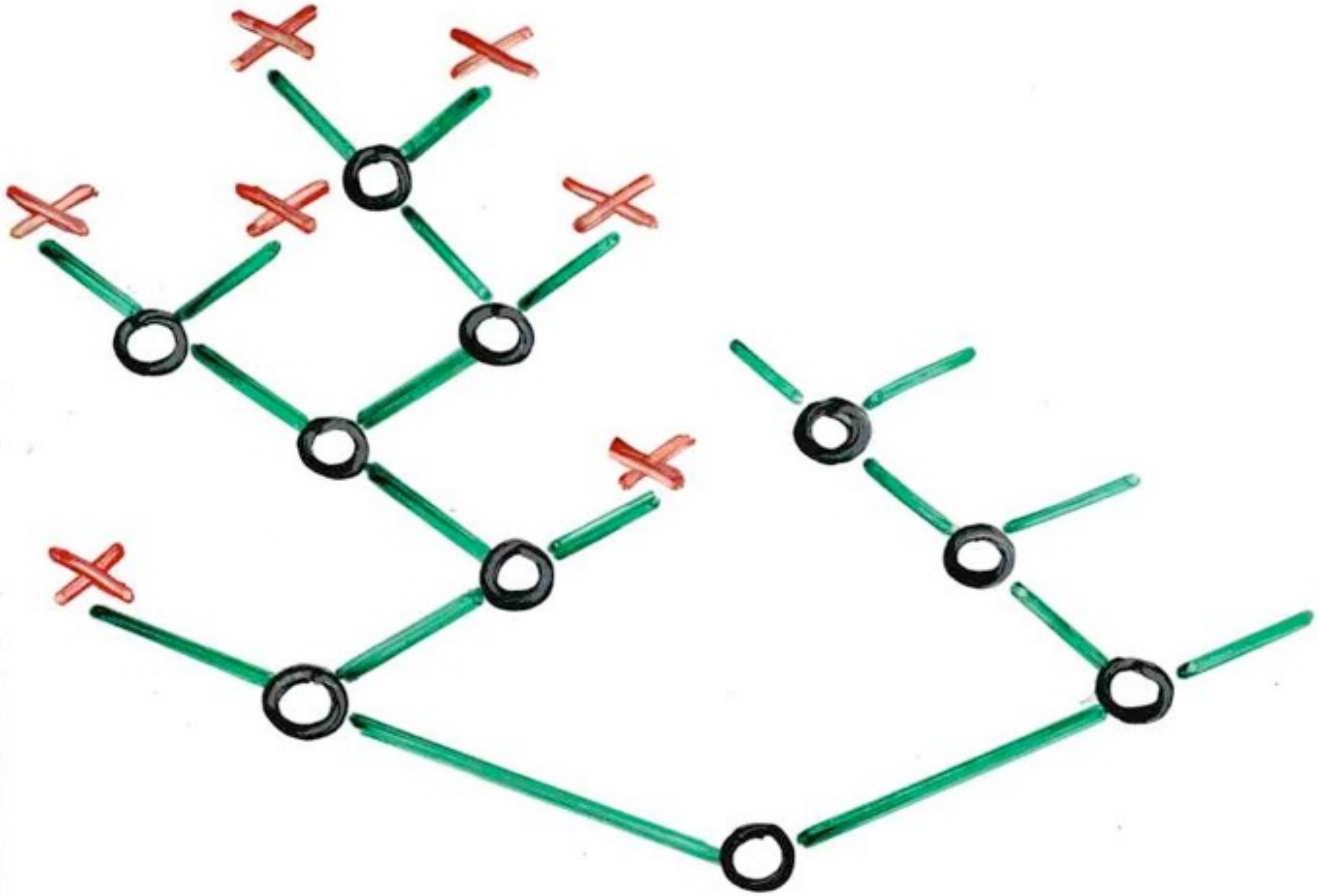


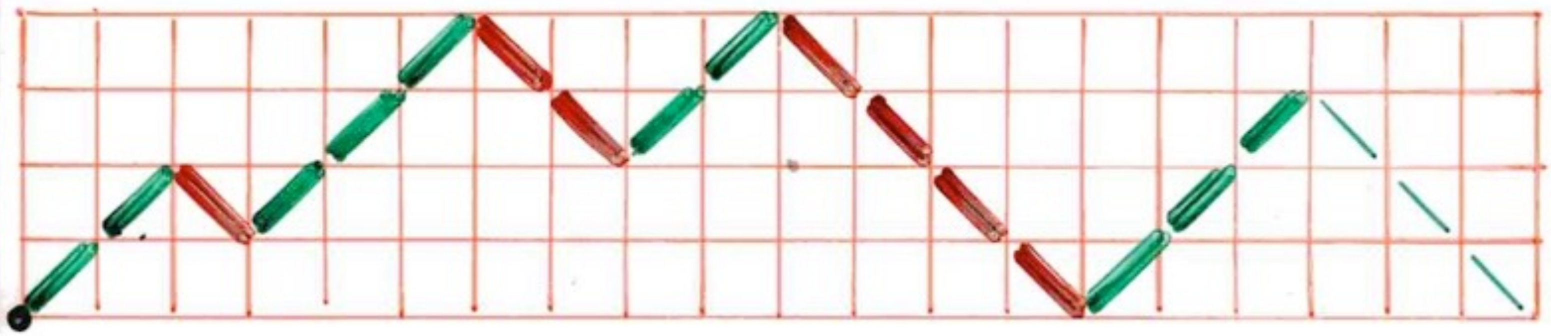


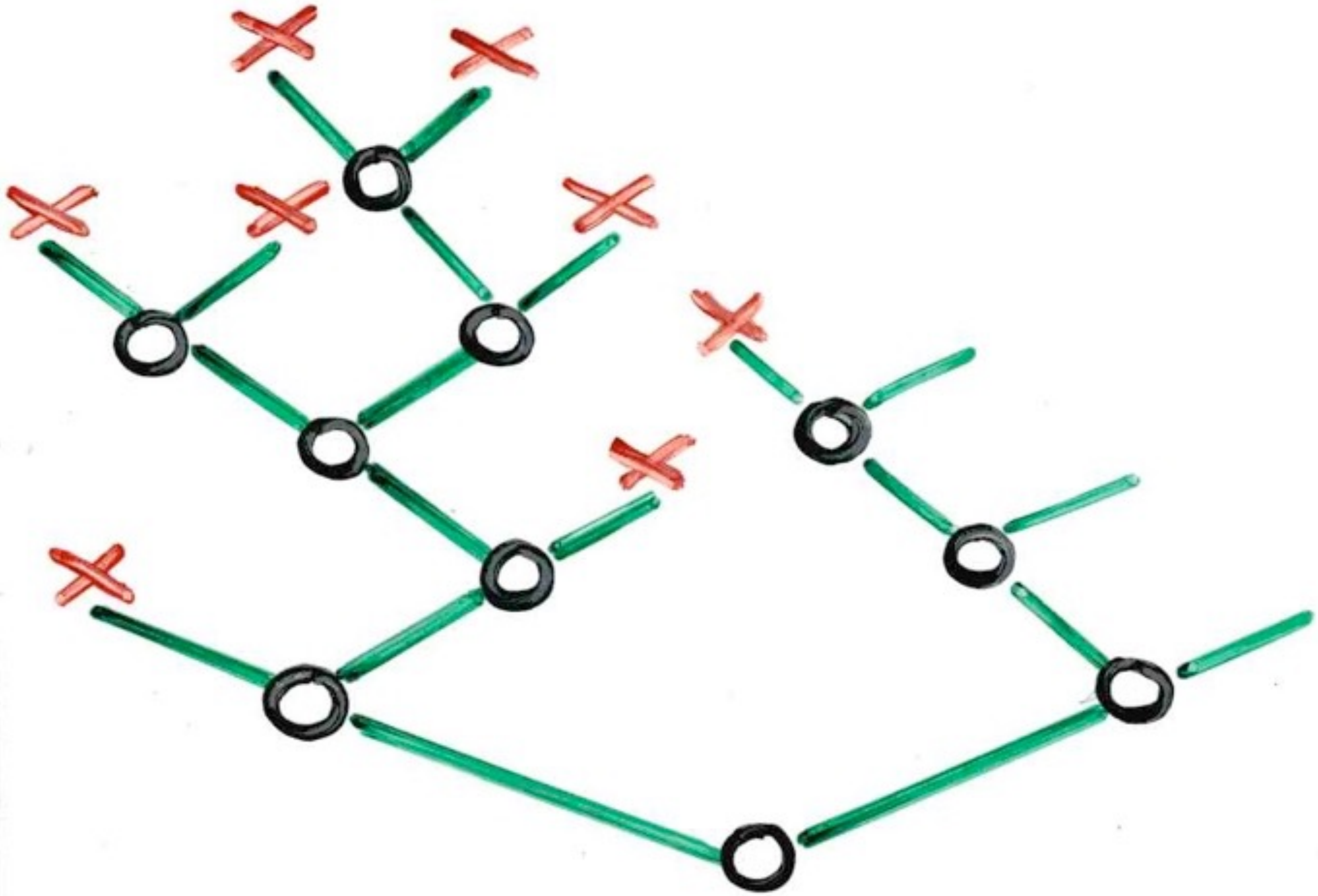


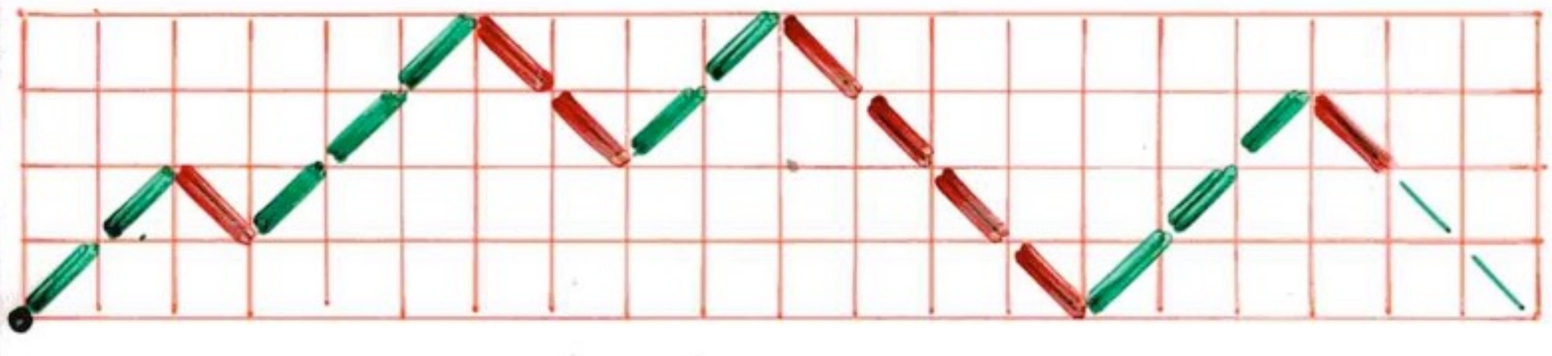


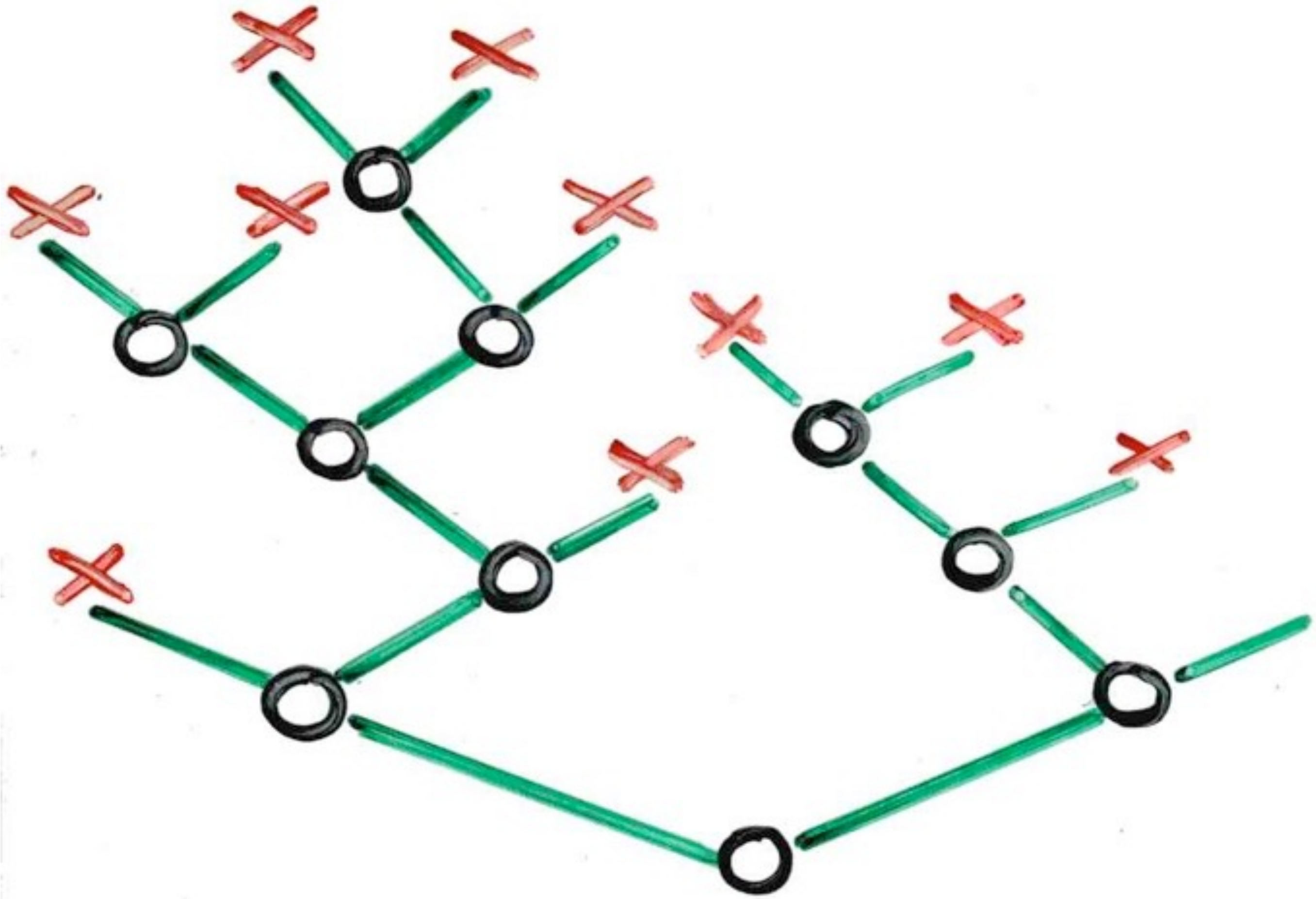


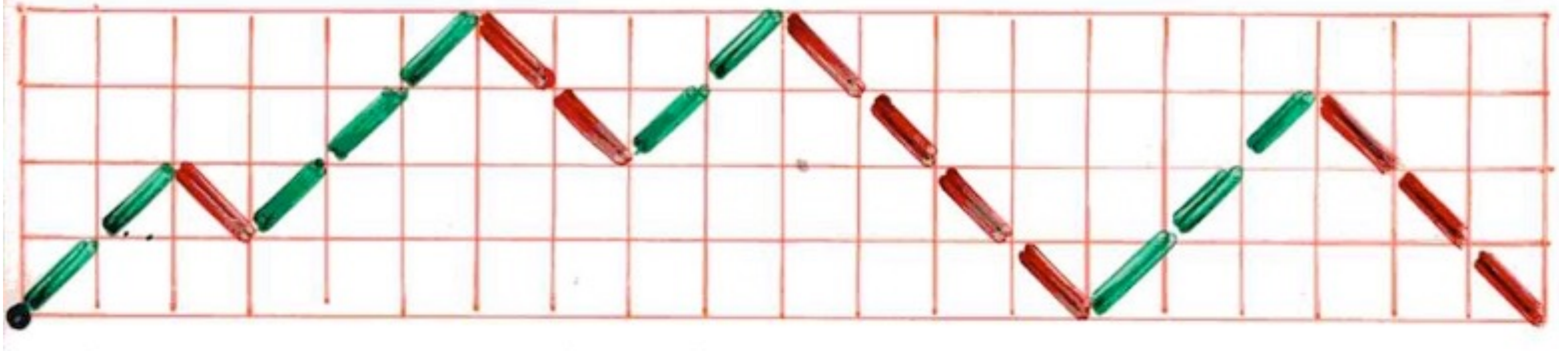


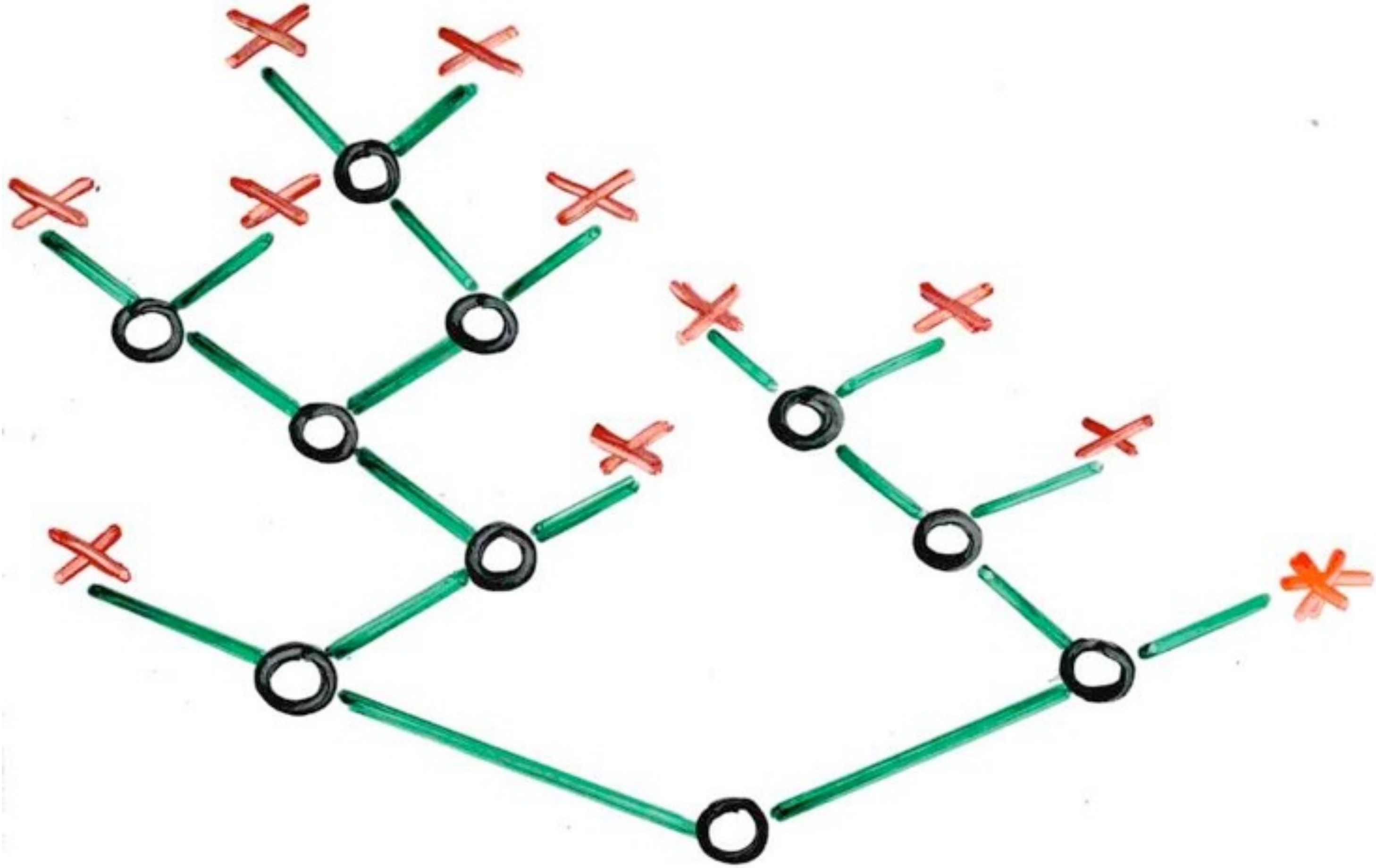


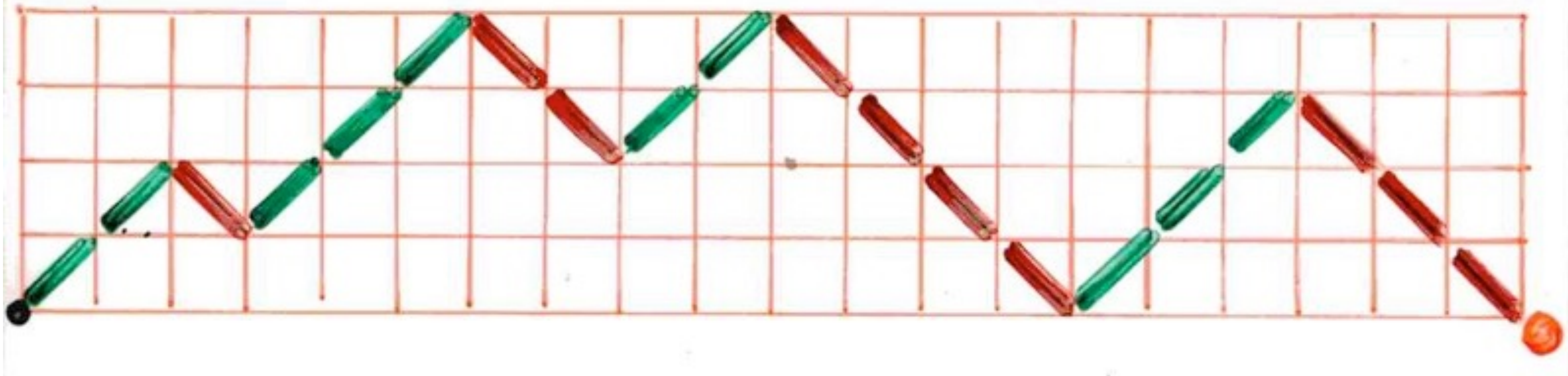


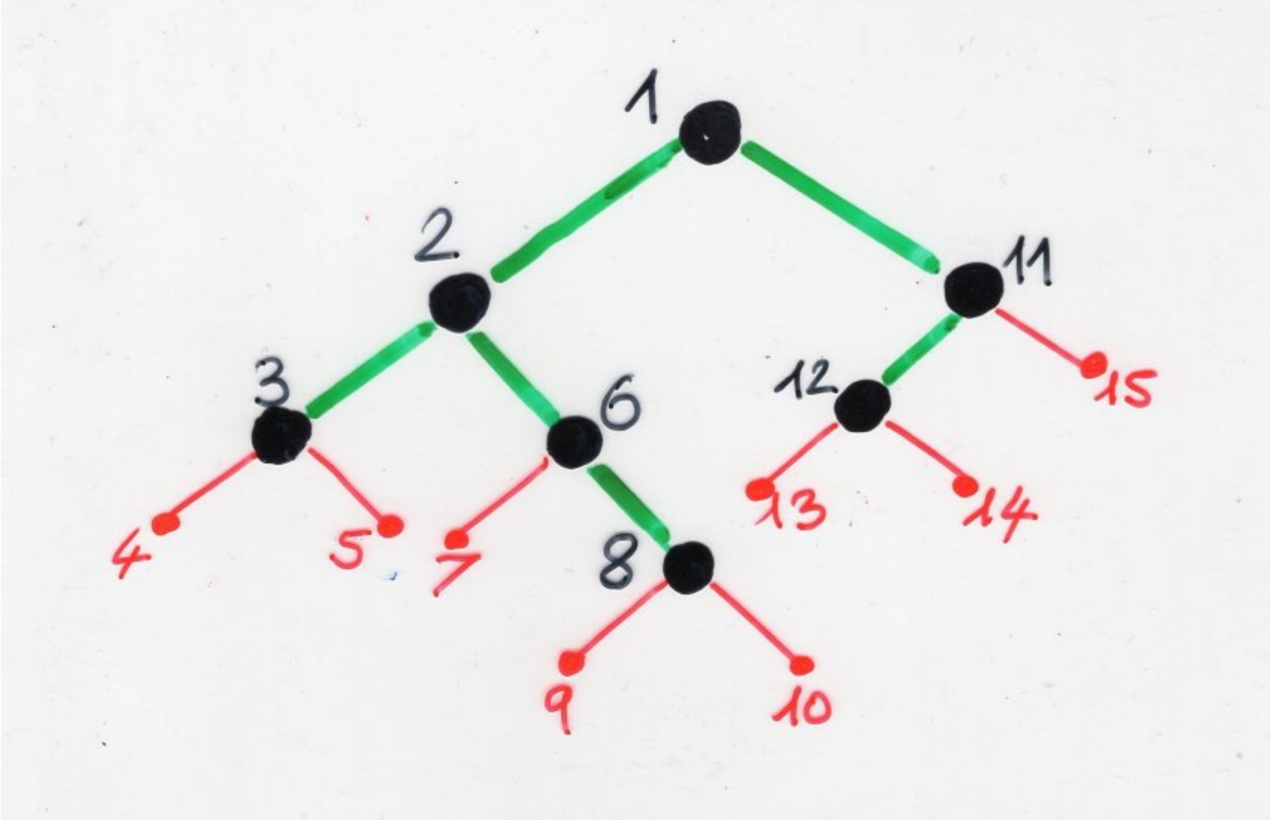








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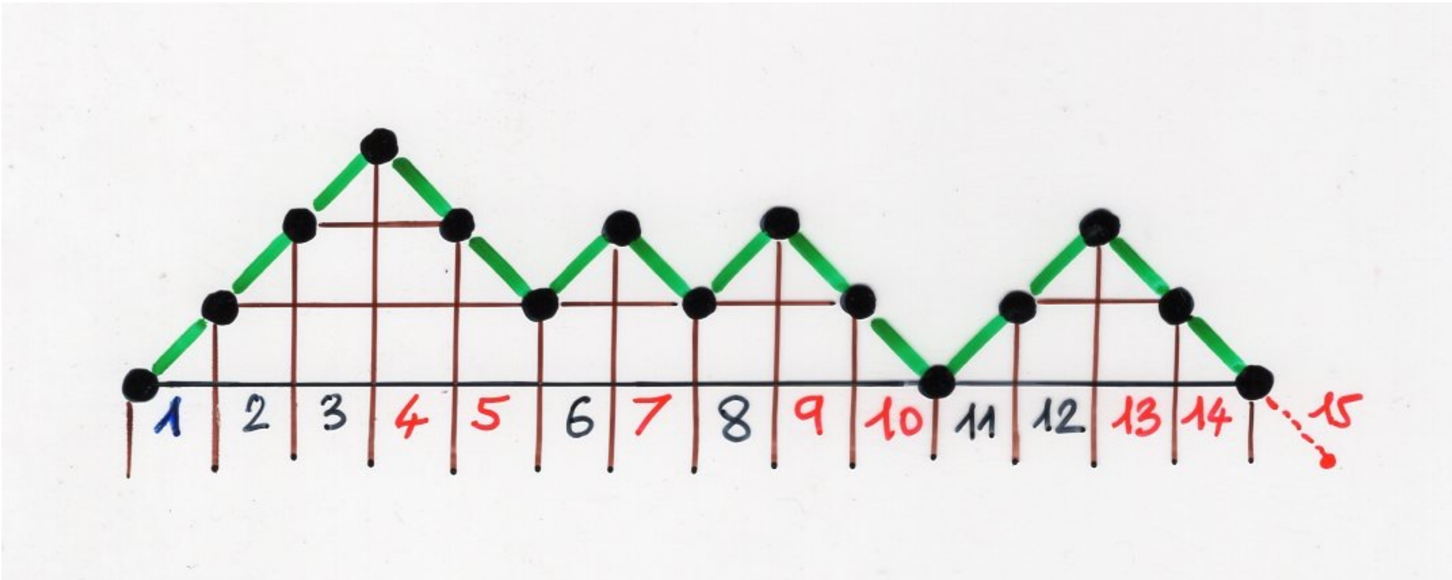
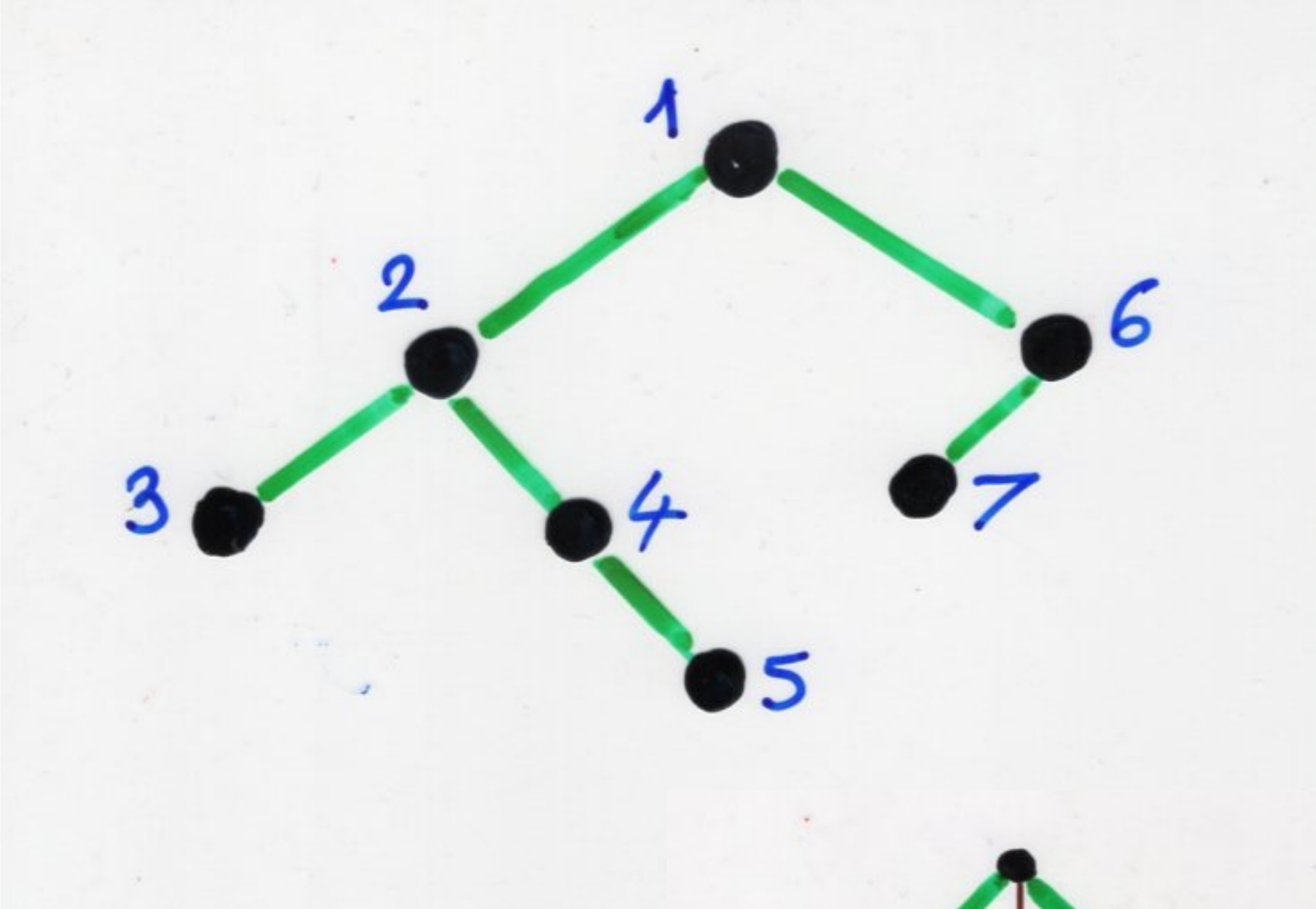

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

bijection

binary trees

2-colored Motzkin paths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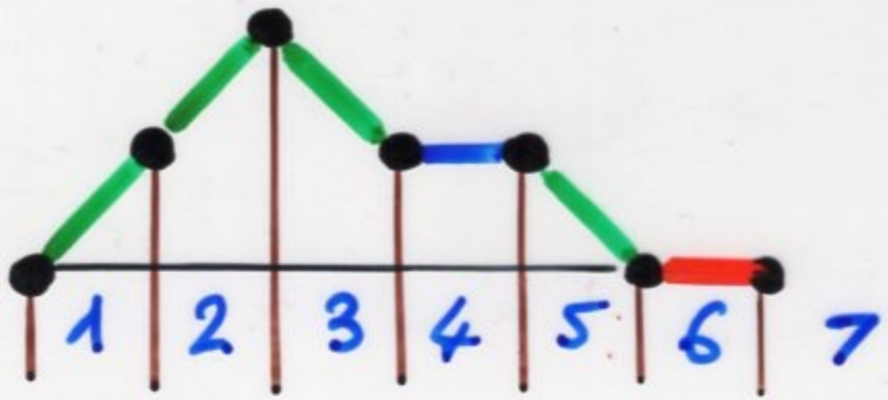
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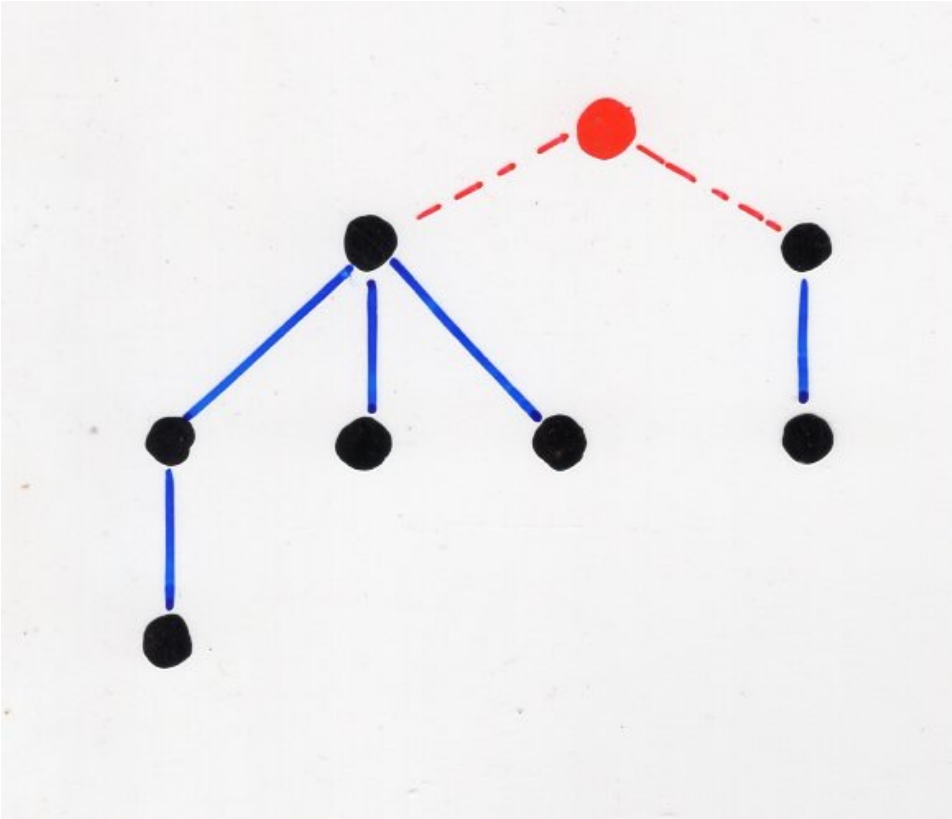
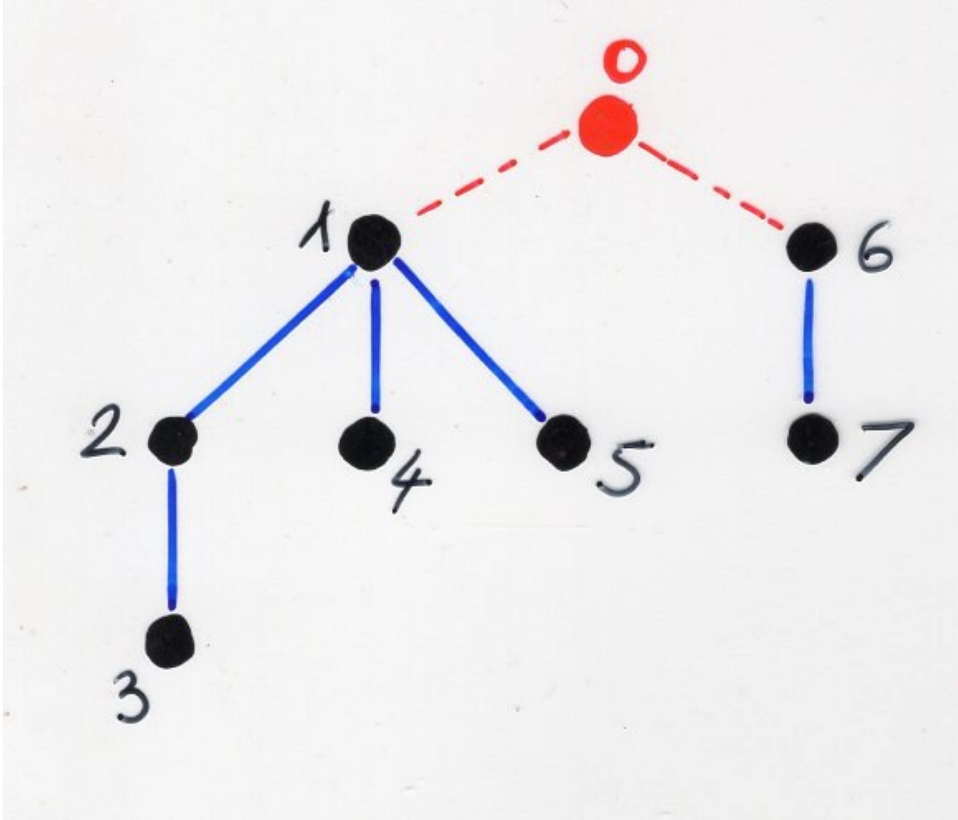
bijection

planar trees

Dyck paths

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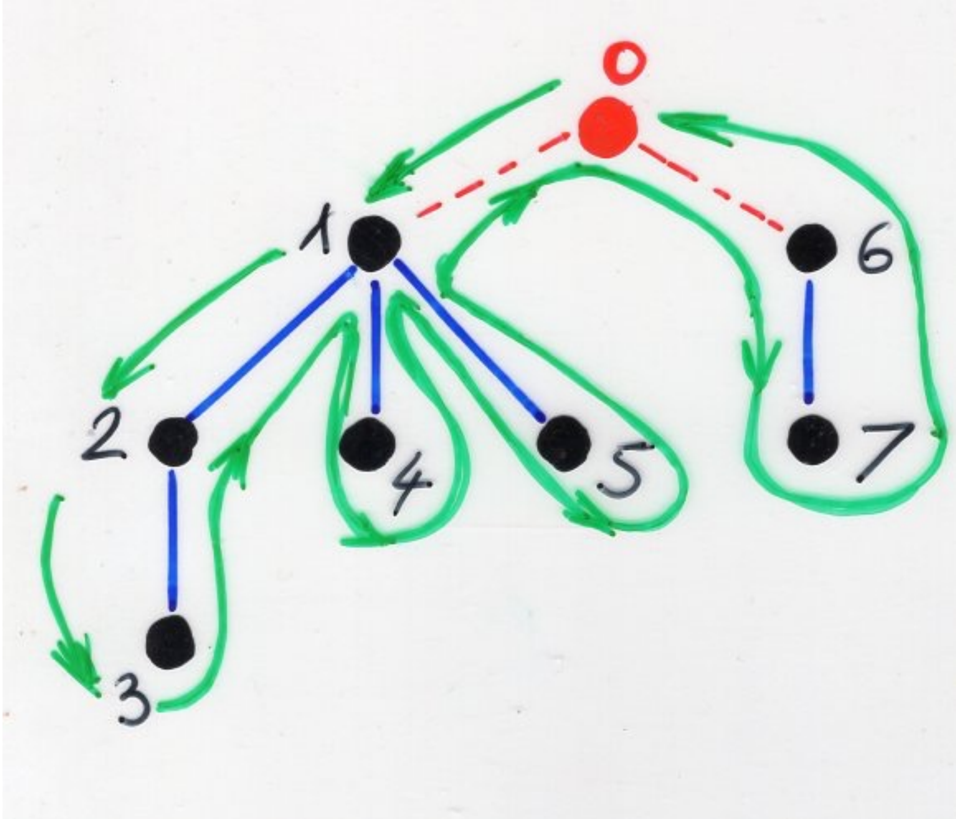

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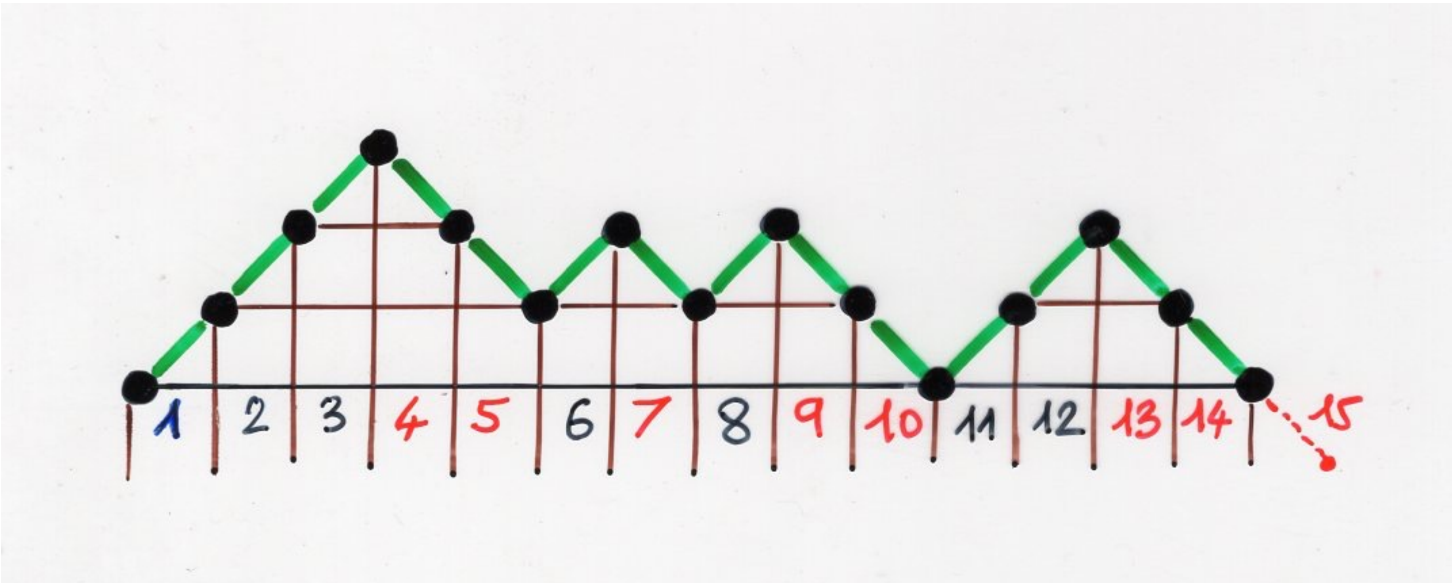

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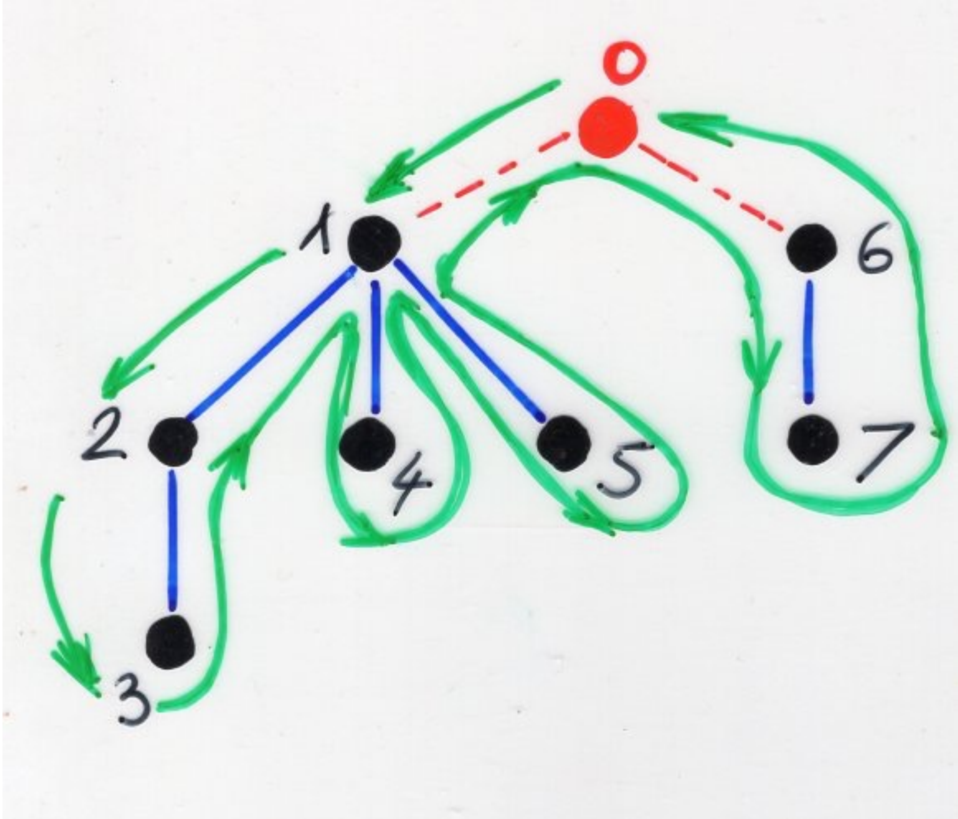
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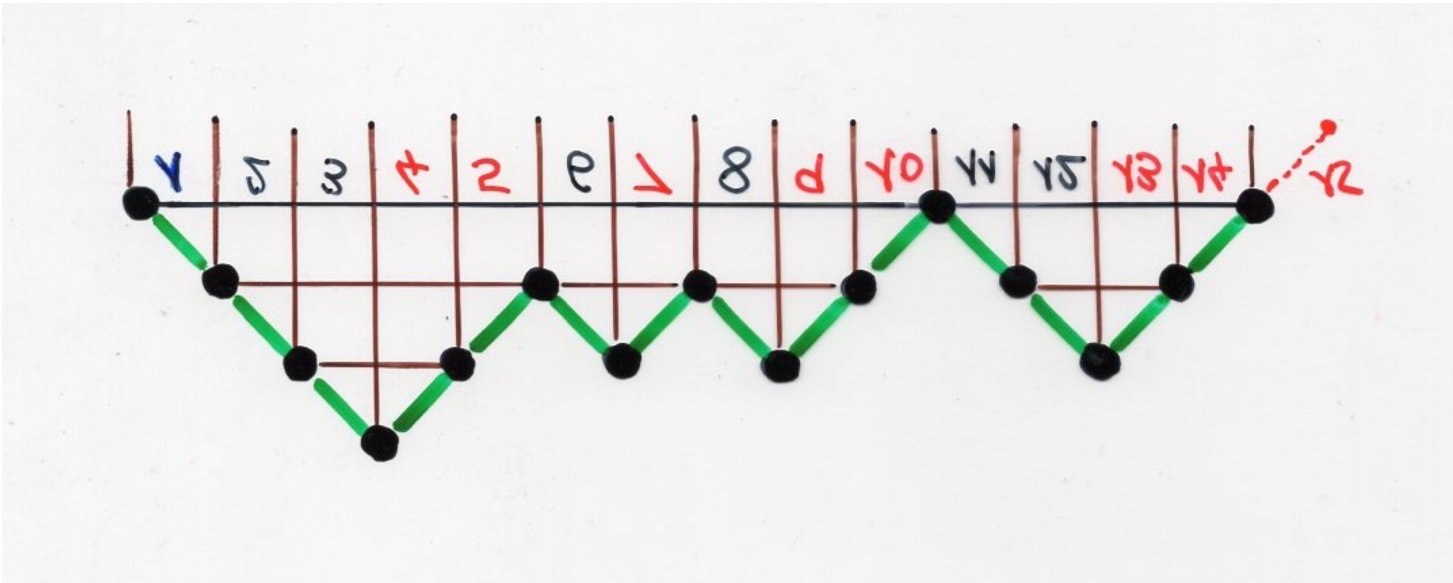
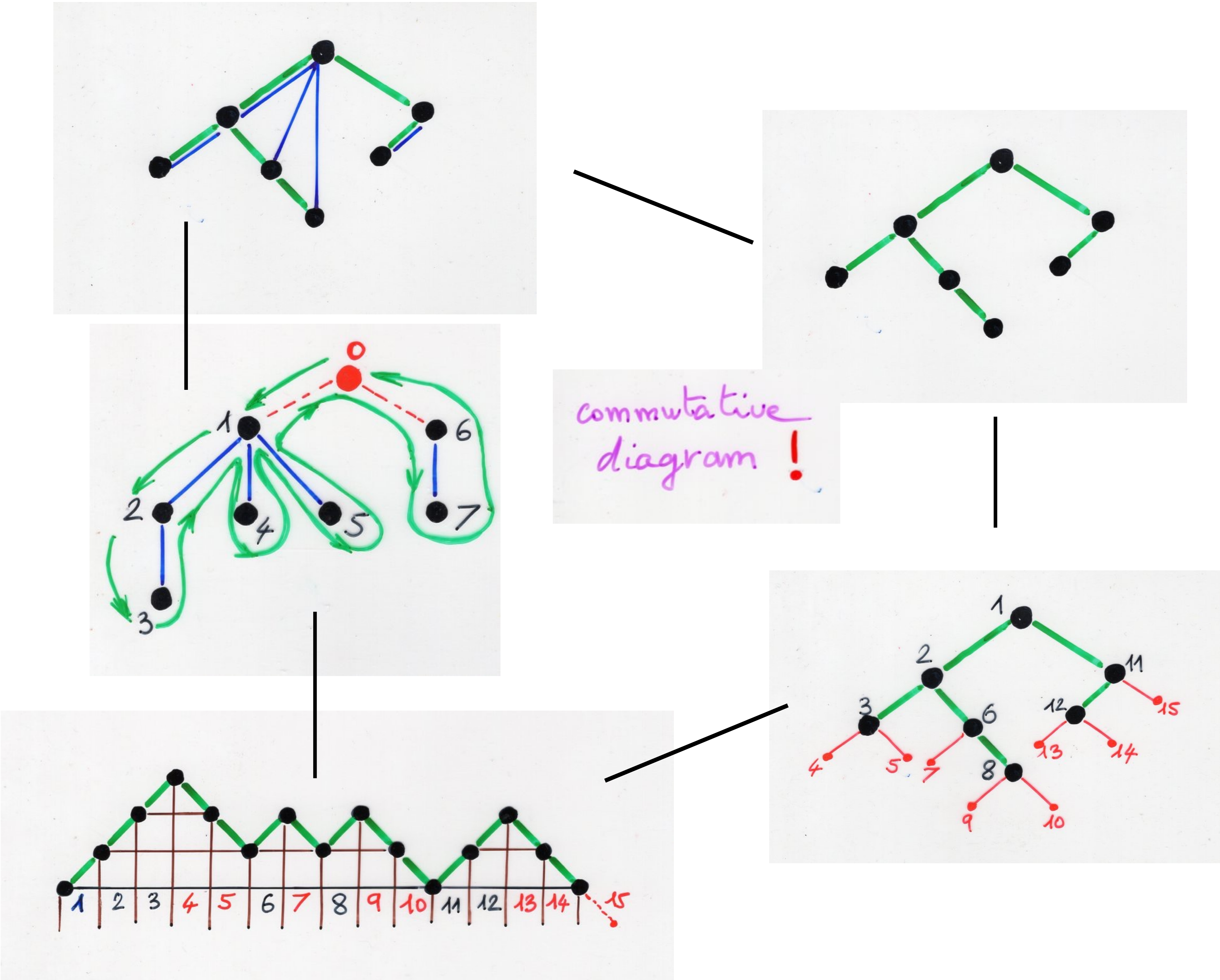
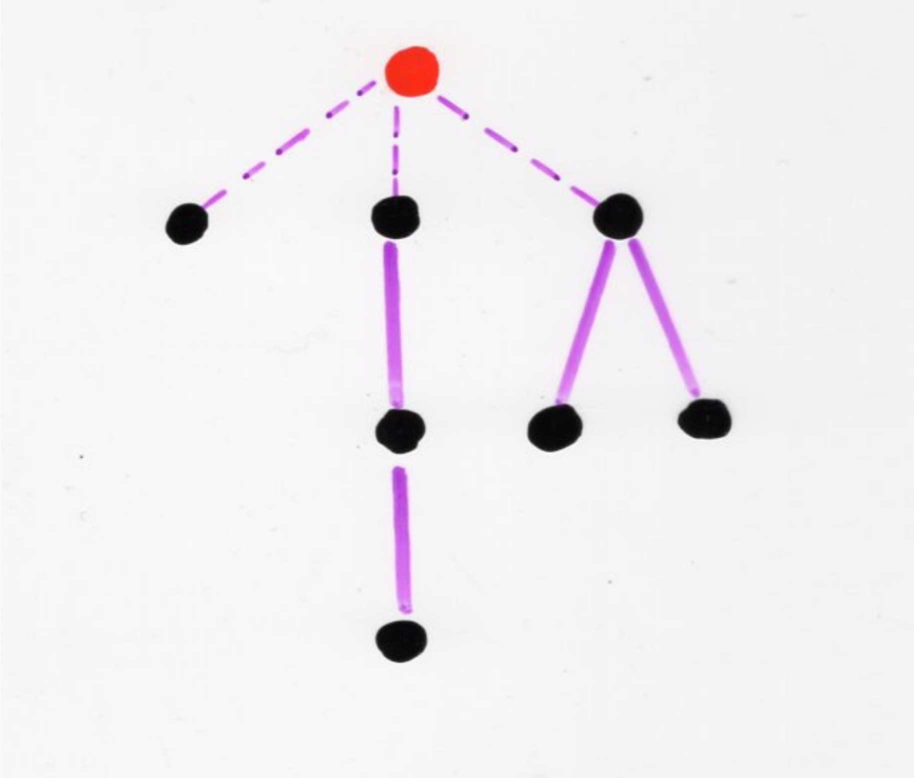



commutative diagram !

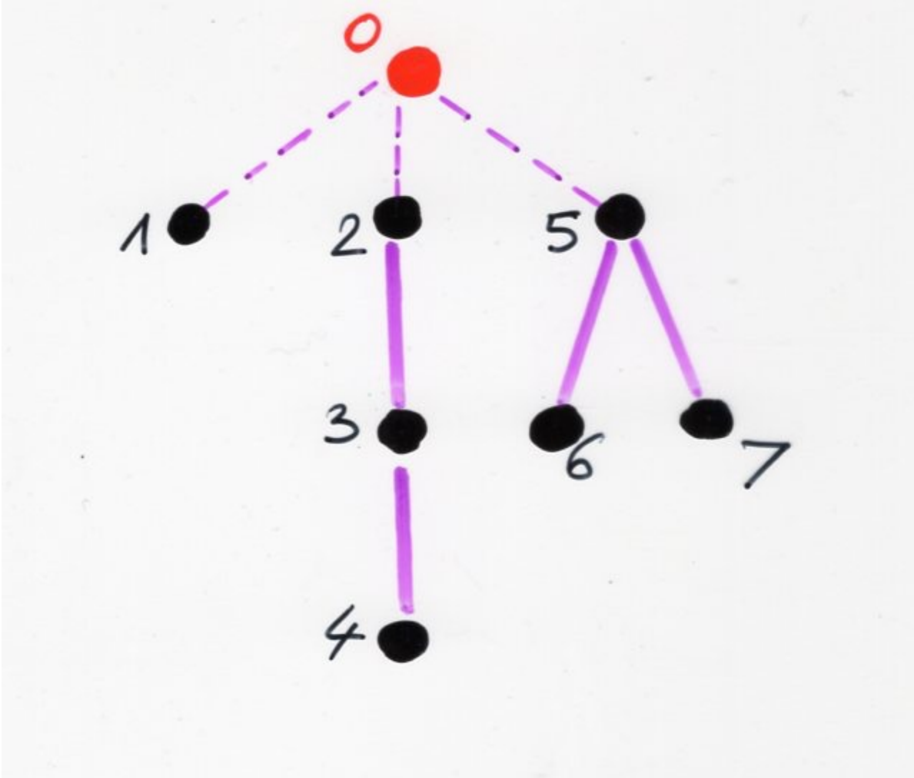
bijection

planar trees

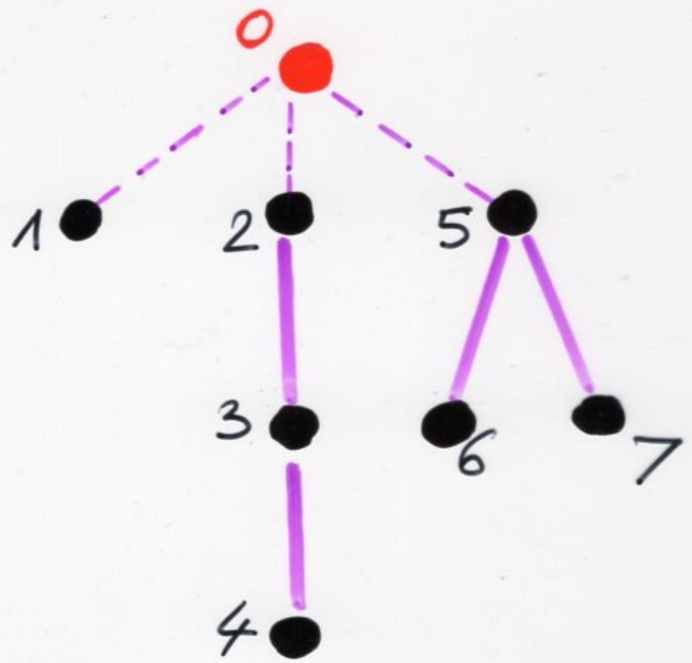
Lukasiewicz paths



prefix order for T planar tree



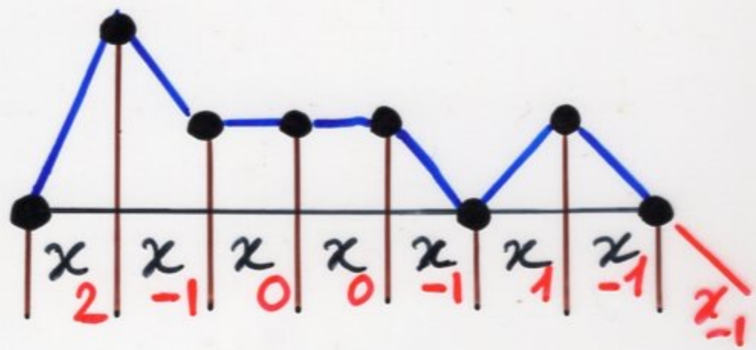
prefix order for T planar tree

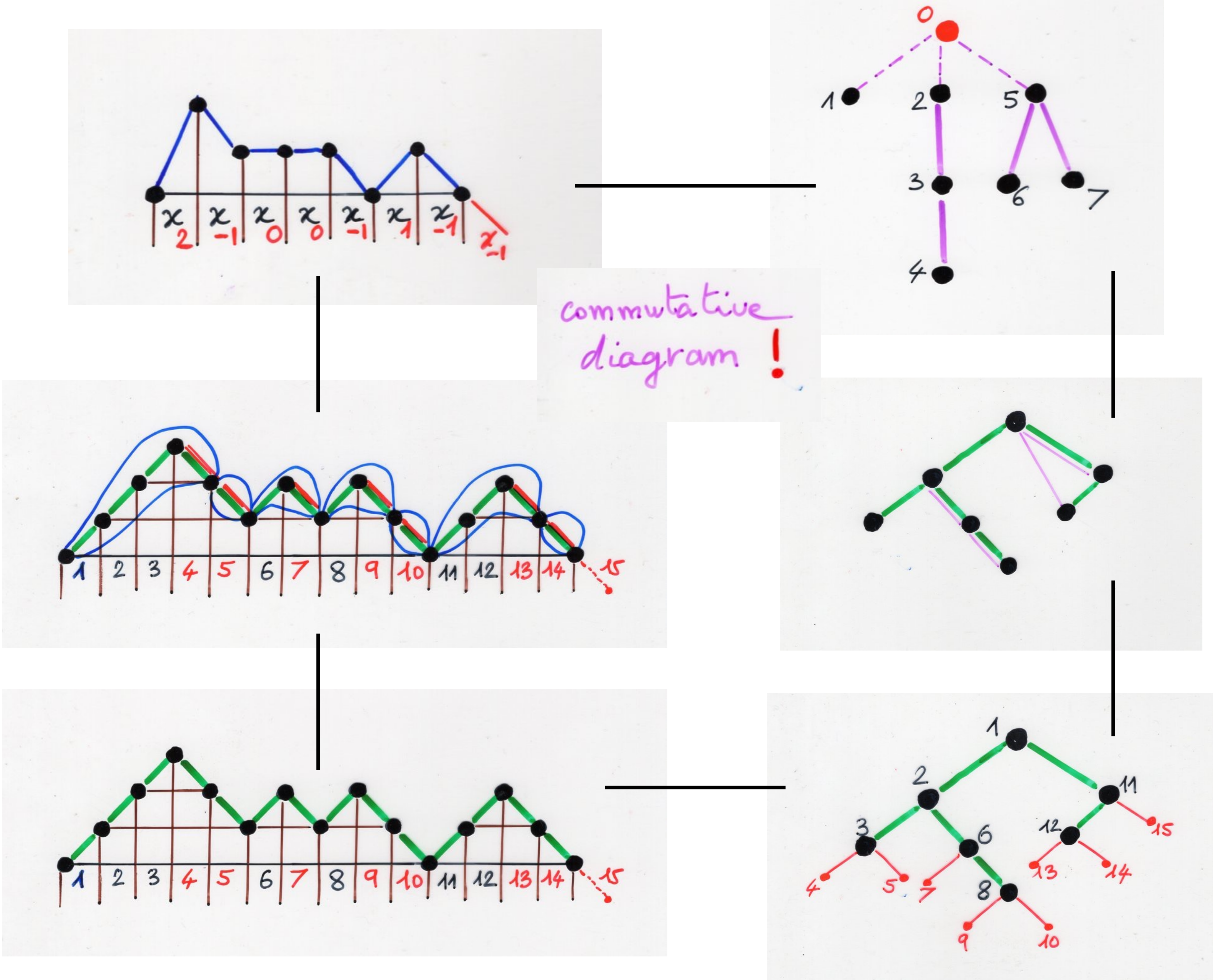


i^{th} vertex s_i of T has j_i sons

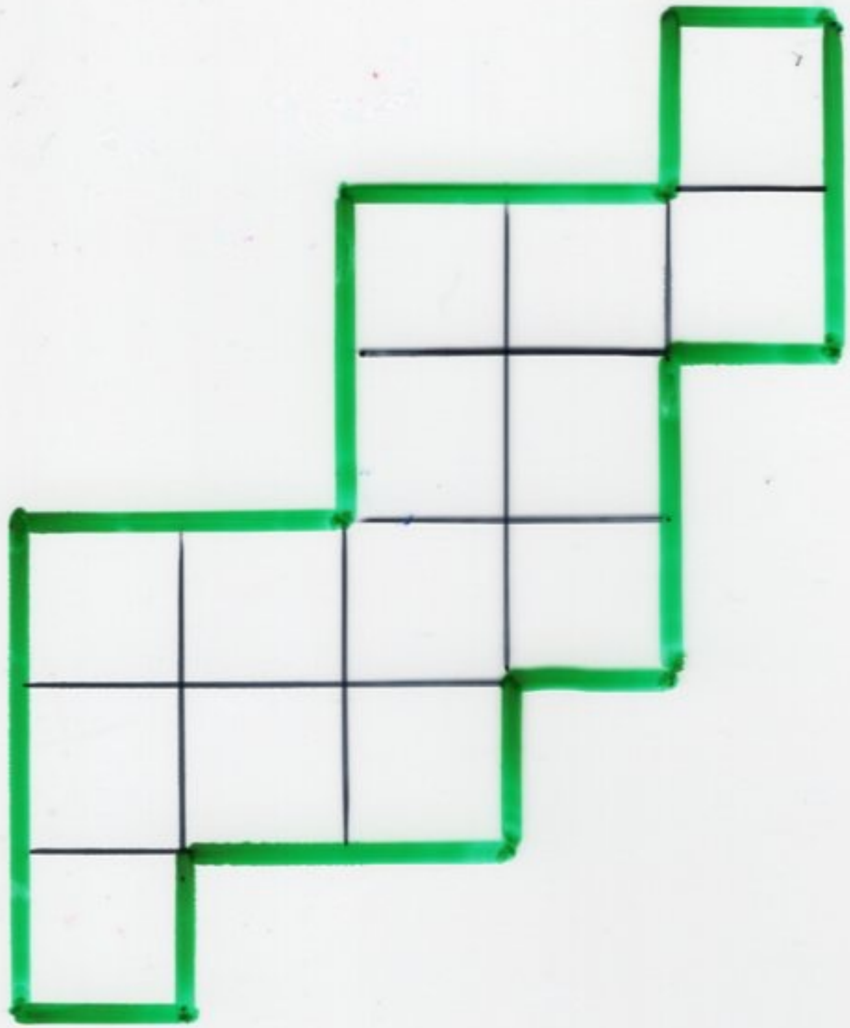
then $a_i = x_{j_i - 1}$

$w = a_1 \cdots a_{n+1} \in L$




commutative diagram !

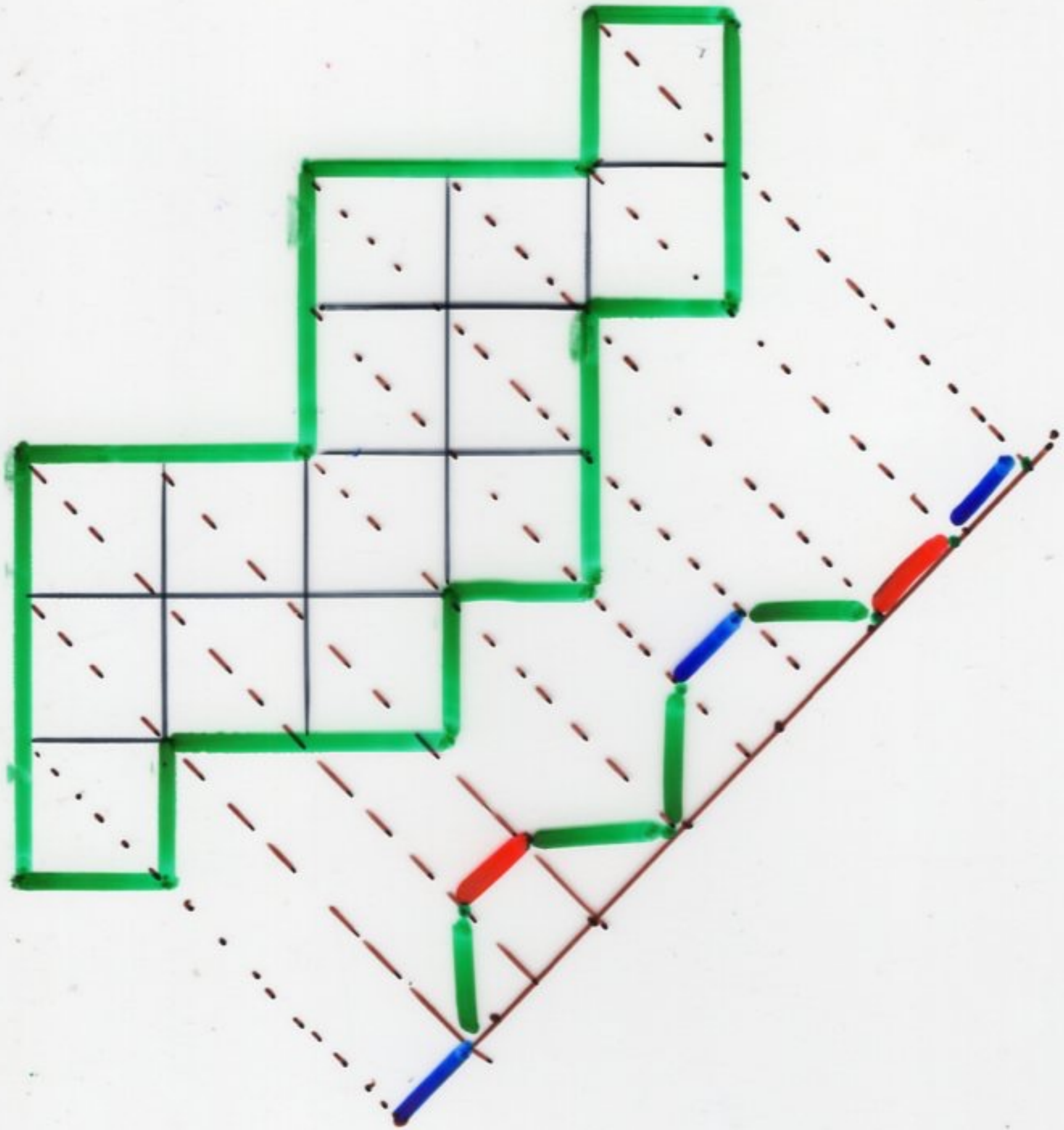
staircase polygons

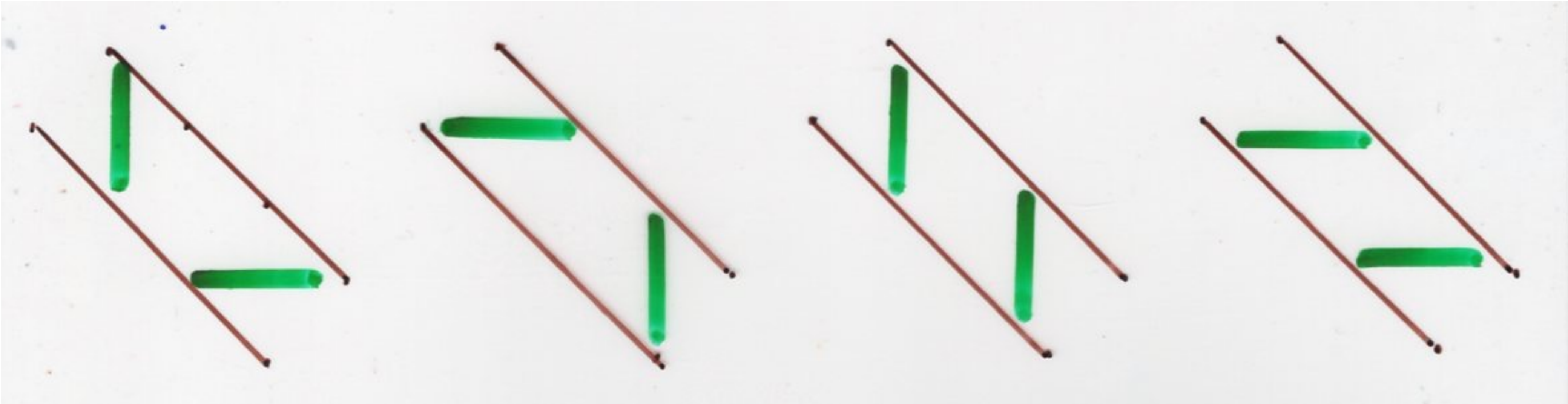


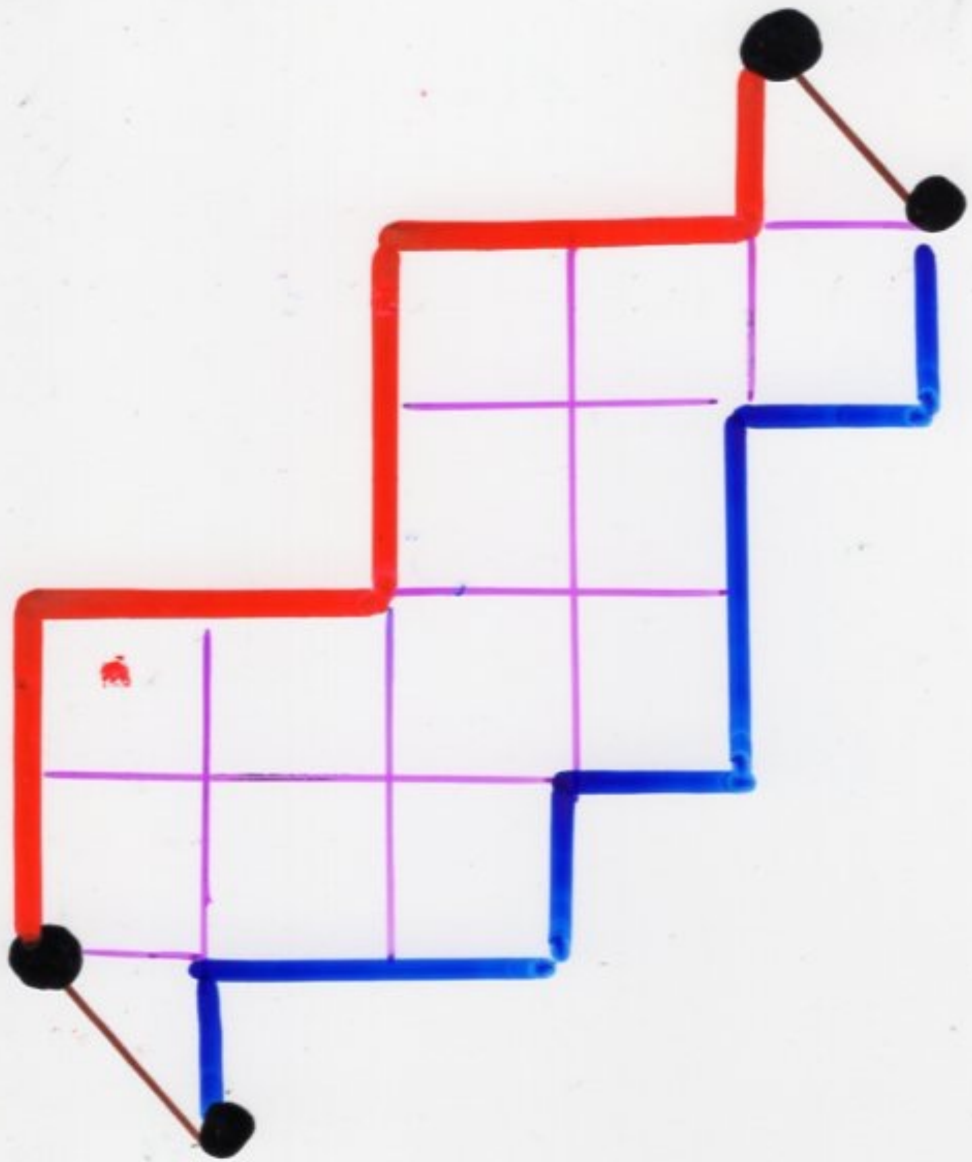
bijection

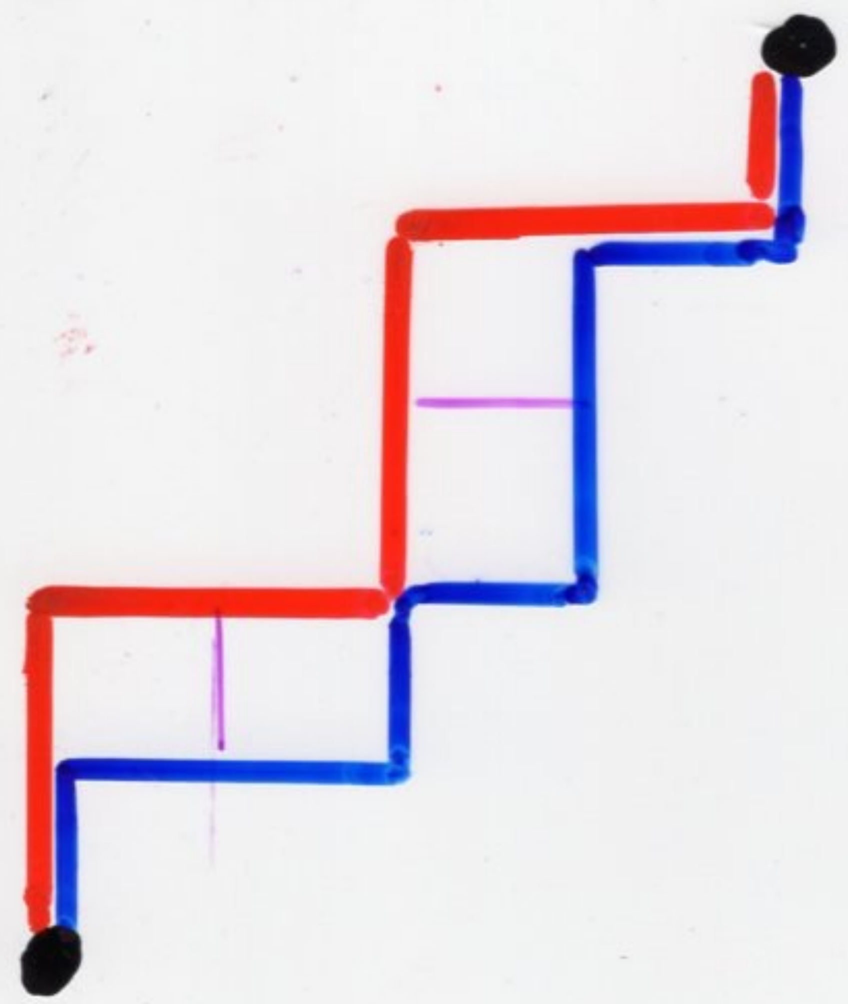
staircase polygons

2-colored Motzkin paths





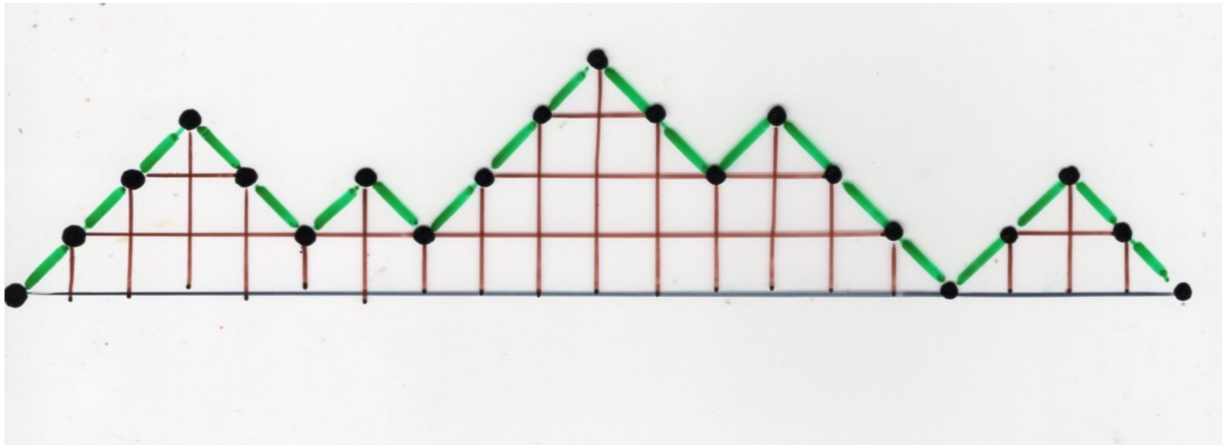


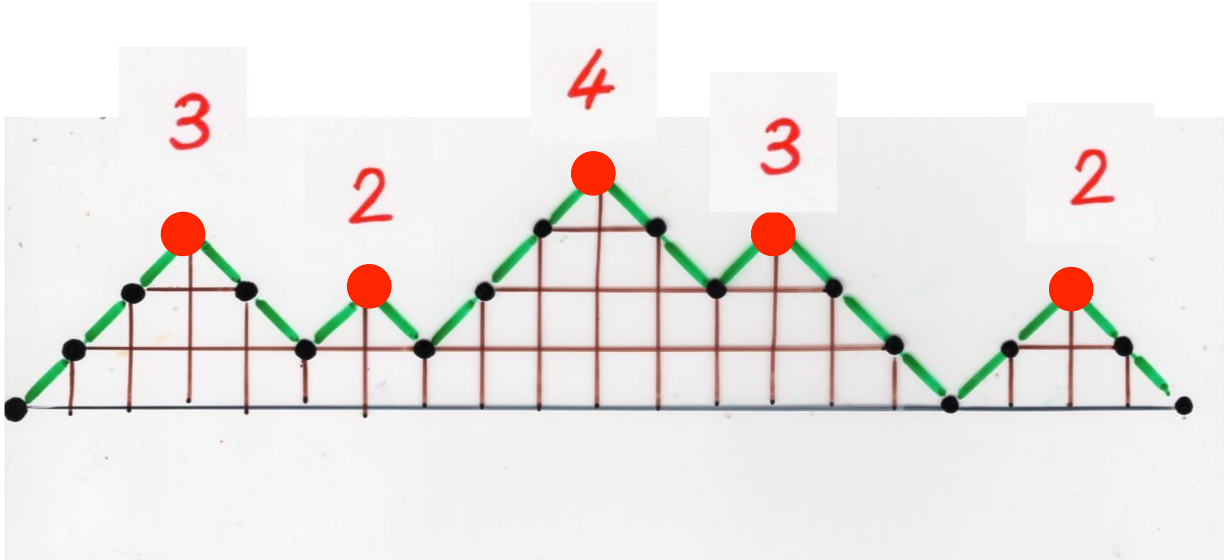


bijection

staircase polygons

Dyck paths




3
2
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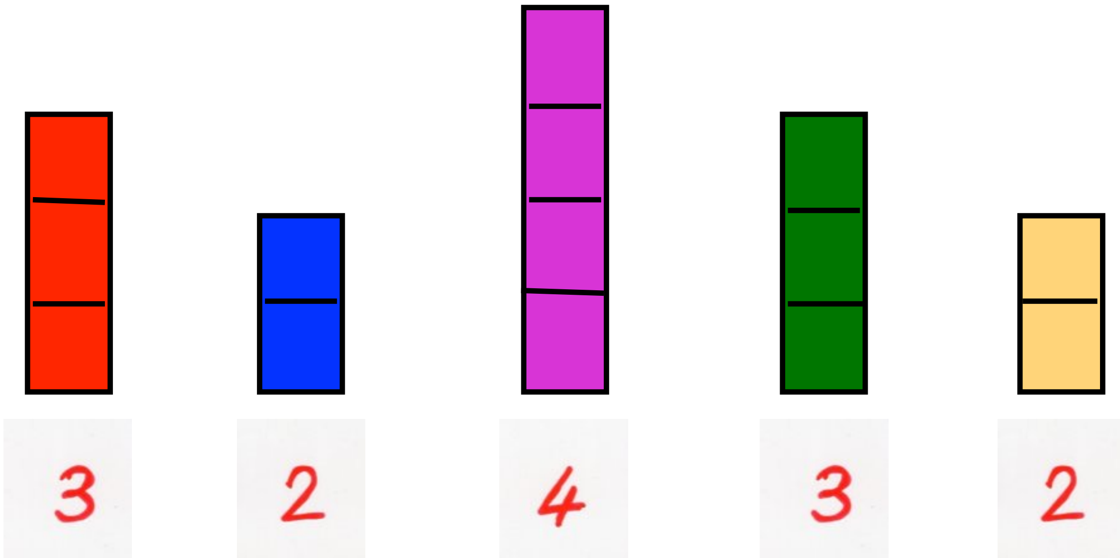
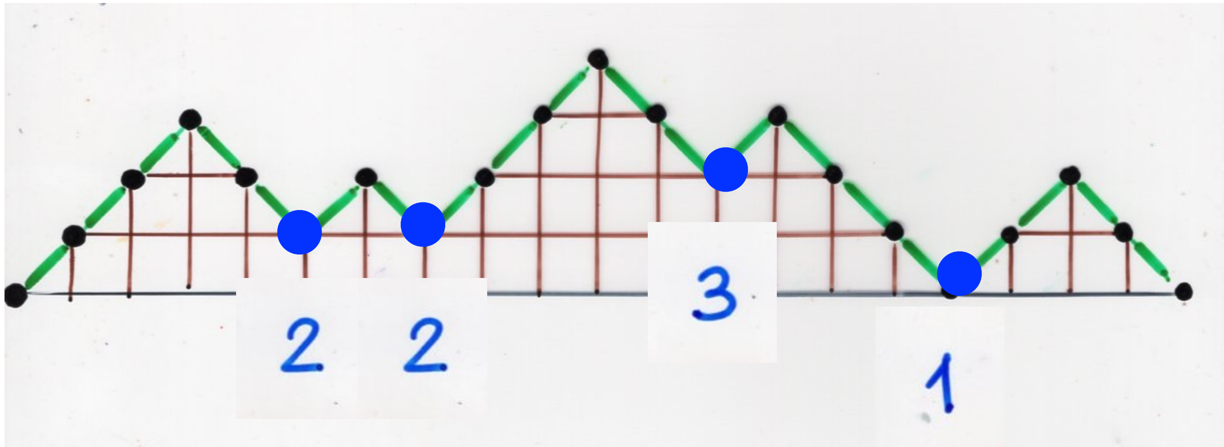
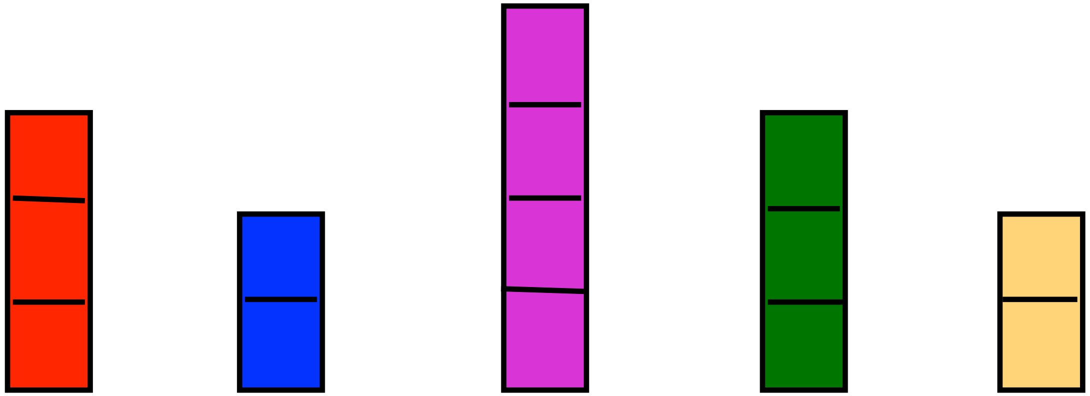
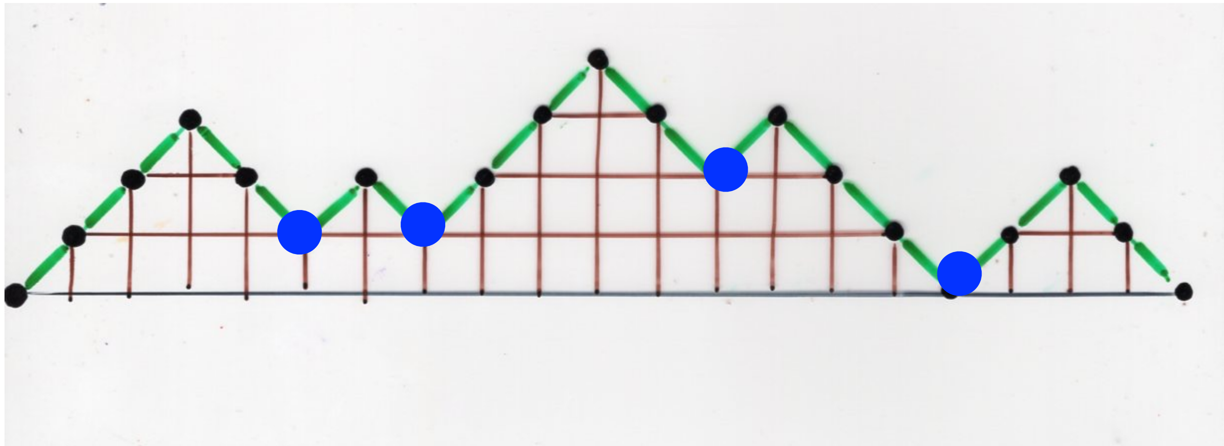
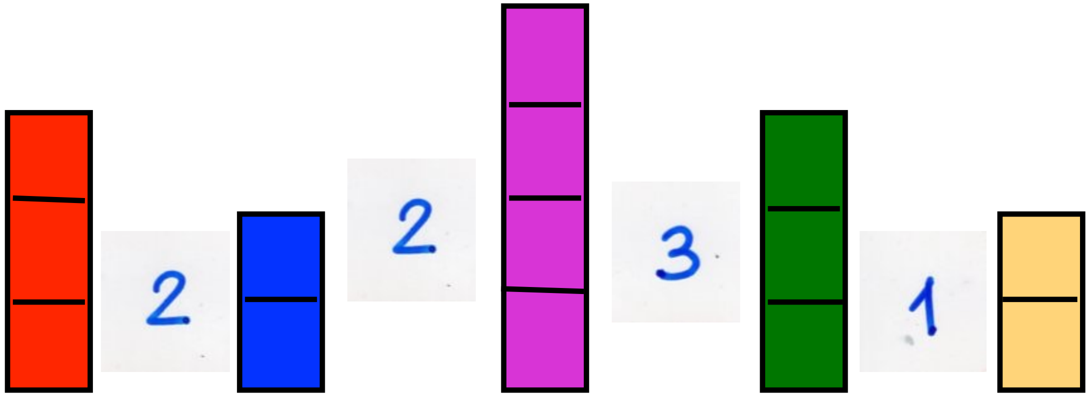
3
2


3
2
4
3
2


2
2
3
1






2
2
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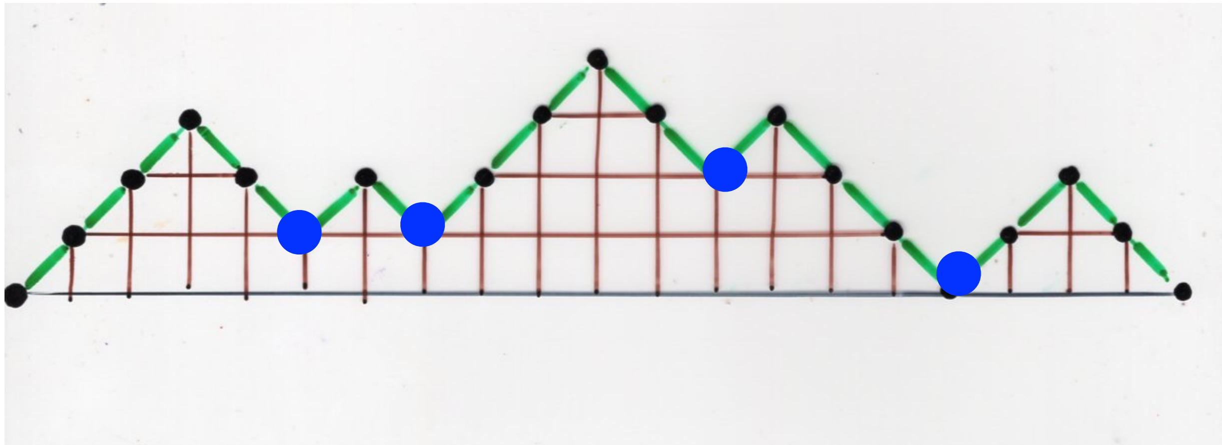
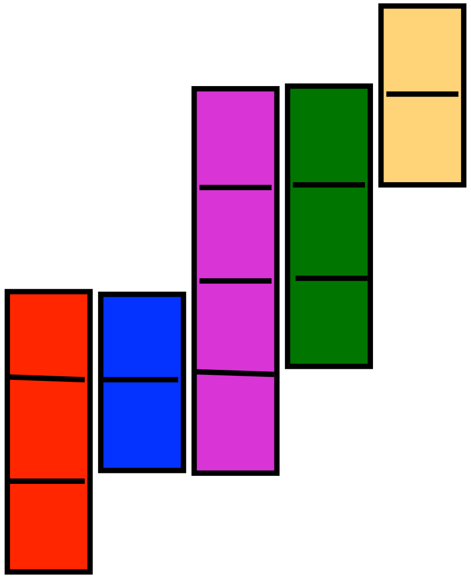
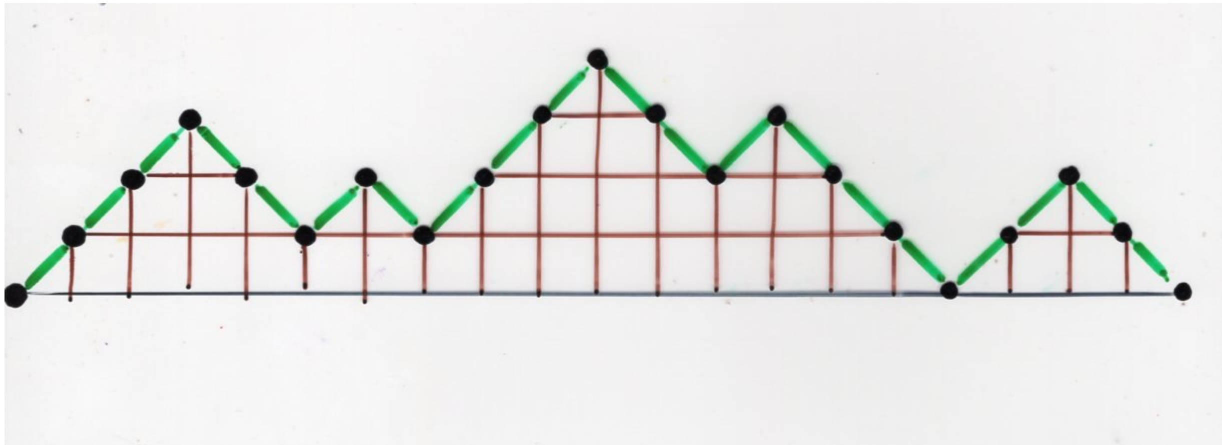
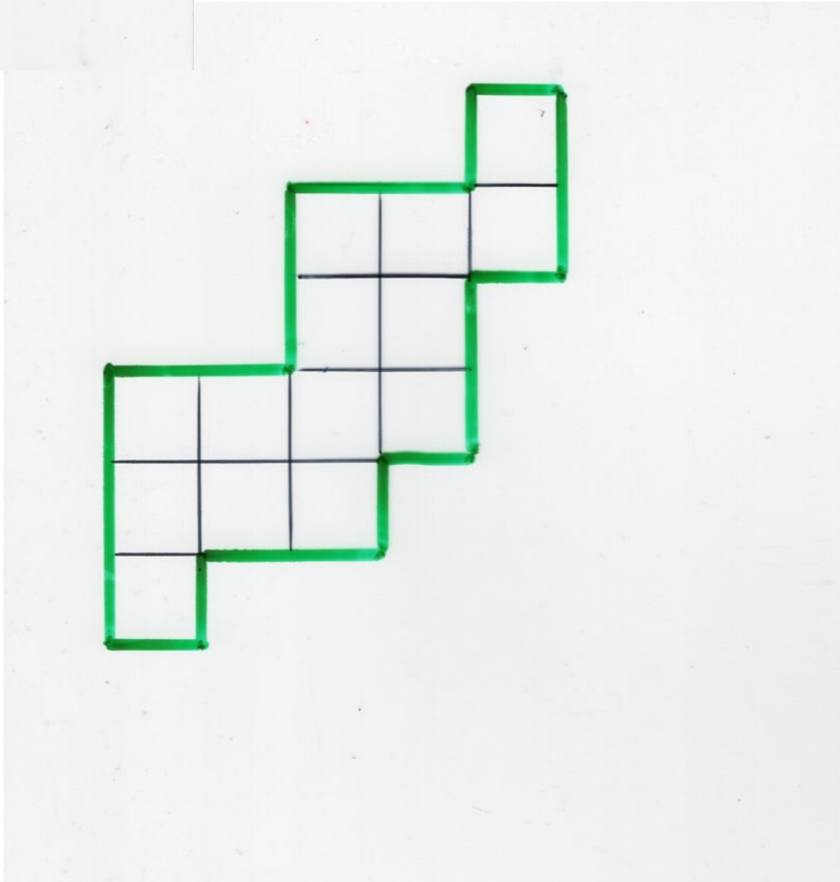
1

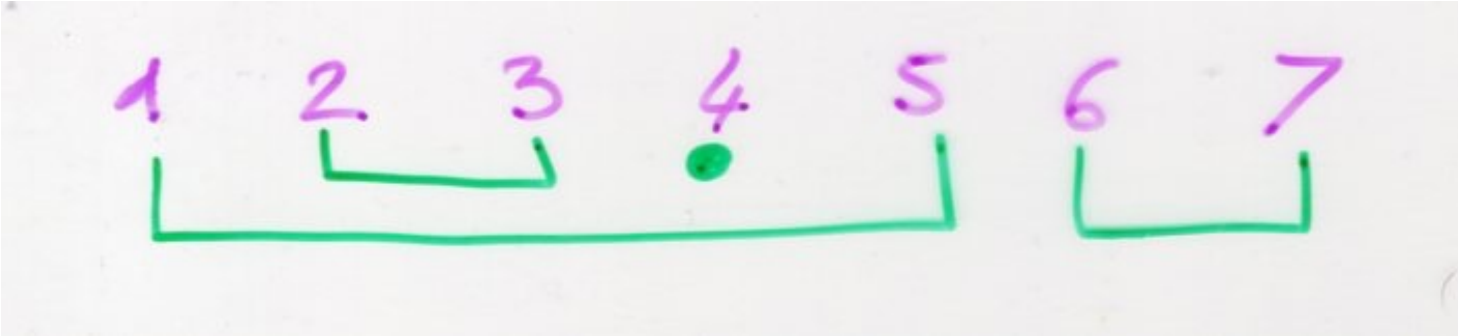








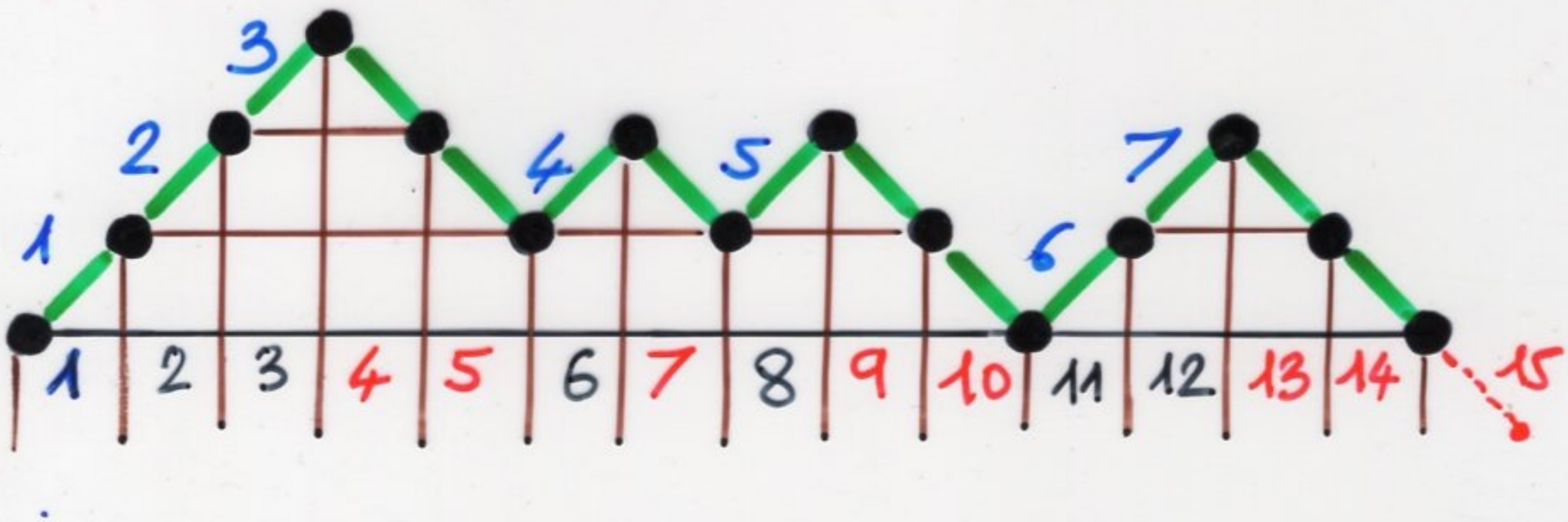
non-crossing partitions


1 2 3 4 5 6 7

bijection

non-crossing partitions

Dyck paths



1
2
3
4
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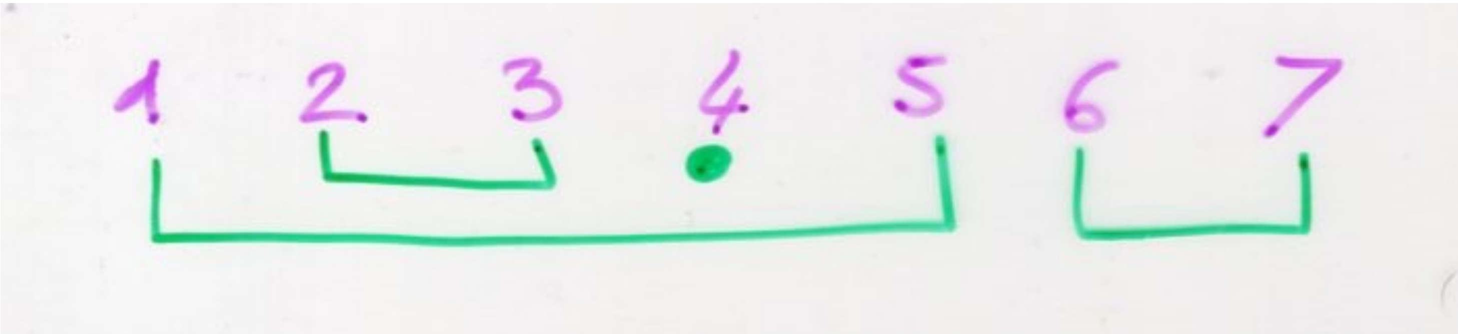
6
7
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

$\{3, 2\}$ $\{4\}$ $\{5, 1\}$ $\{7, 6\}$

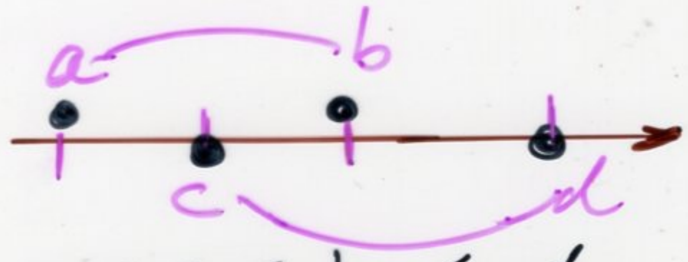


Definition

non-crossing partition of $\{1, 2, \ldots, n\}$

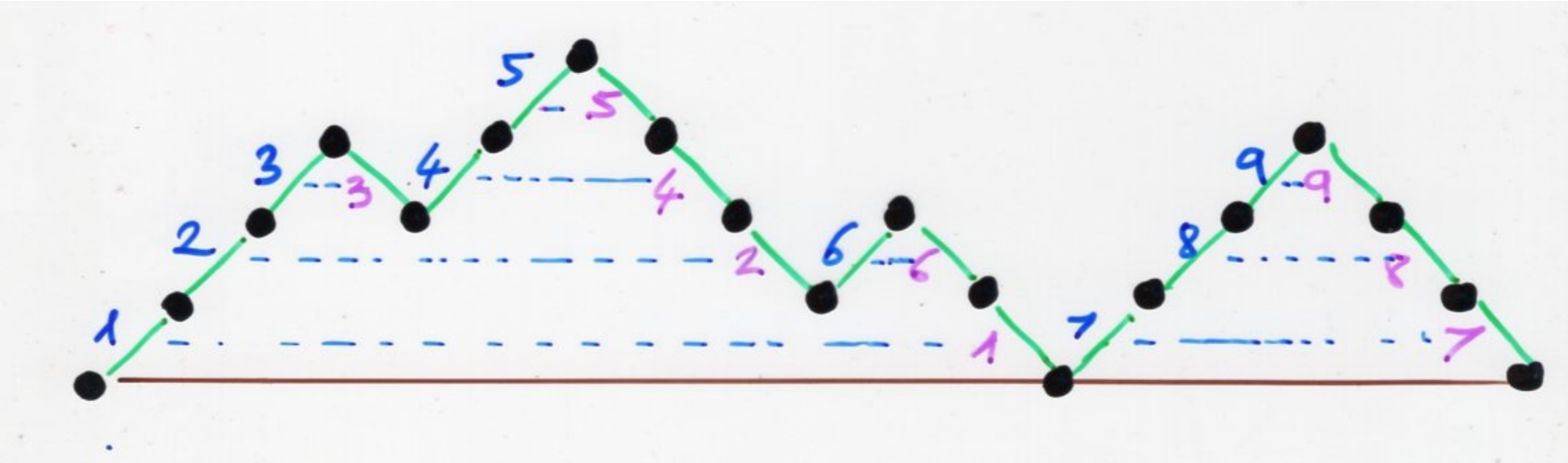
$\{B_1, \ldots, B_k\}$
k blocks

if $a, b \in B_i$
$c, d \in B_j$ $(i \neq j)$



$a < c < b < d$
impossible

(3 slides added after the class)



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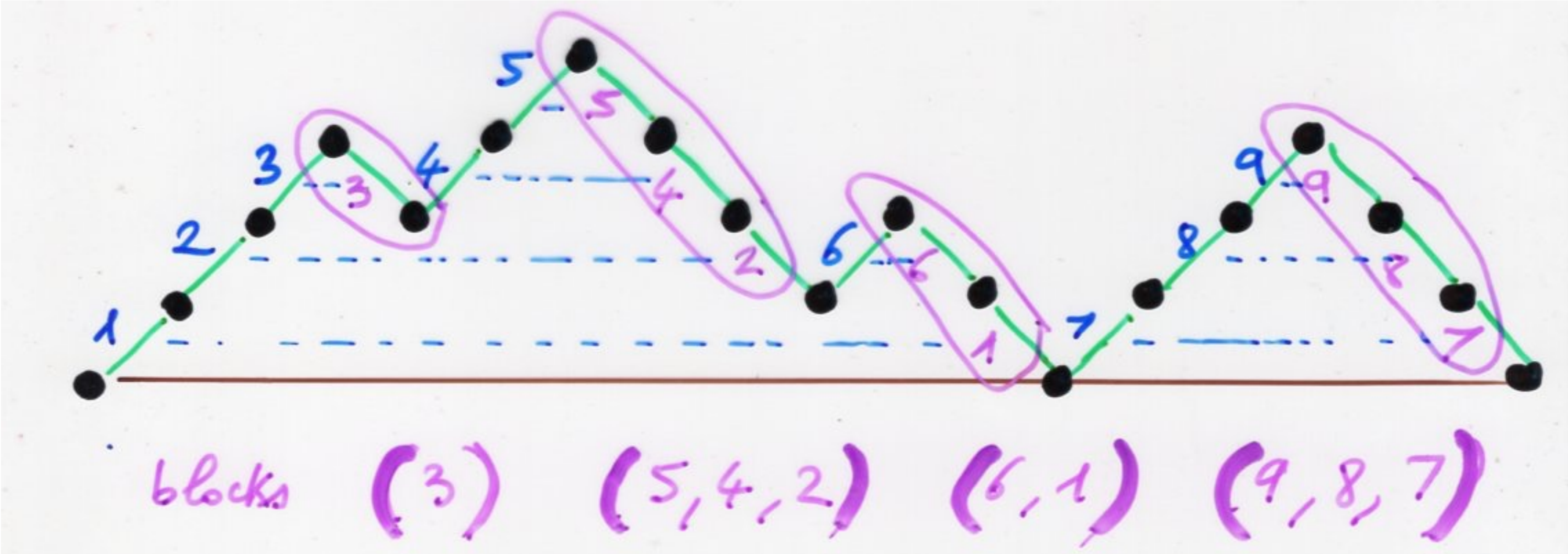
1
2
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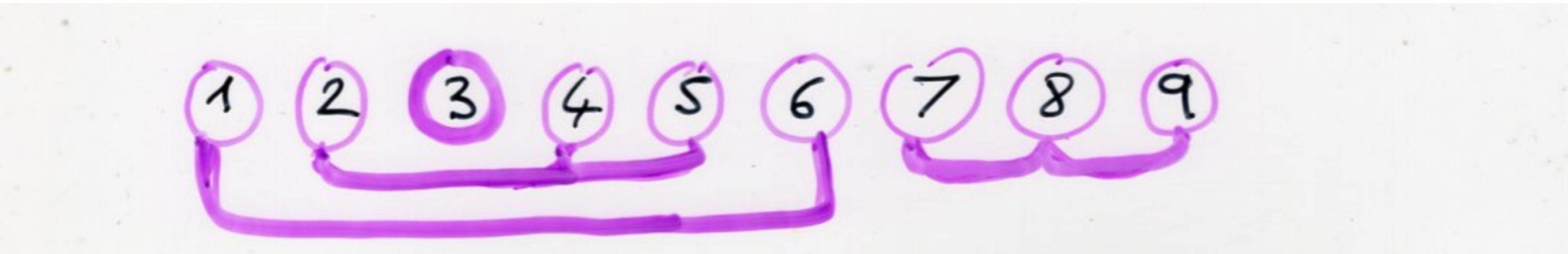
9
blocks (3) (5,4,2) (6,1) (9,8,7)


1 2 3 4 5 6 7 8 9