# Introduction to Chapter 3 on continued fractions

[version 6, 29 January 2013]

by Xavier Viennot
LaBRI, CNRS, Université de Bordeaux

*[all along this paper, PF is used for Philippe Flajolet]*

PF fell in love with continued fractions around the year 1978 (paper [1]) when he discovered a general methodology for analyzing the cost of algorithms on a file, integrated on a sequence of primitive allowed operations (insertions, deletion, queries), once the average costs on single operations are known. The amazing connection resounds like a thunder in the sky of combinatorics and classical analysis, and a *"coup de foudre"* for PF. Analytic continued fractions theory was put at the combinatorial level with a beautiful interpretation in terms of certain weighted paths (the so-called Motzkin paths). This interpretation is described in the famous seminal paper [6]. Combining this interpretation with some combinatorial bijections between classical combinatorial objects and these weighted paths leads PF to a fireworks of combinatorial proofs of many classical developments of power series into continued fraction in connection with special functions (papers [6], [10], [12], [13]), together with the explicit computation of integrated costs for some classical data structures such as lists, priority queue, dictionary, each corresponding to classical orthogonal polynomials (Hermite, Laguerre, Charlier, ...). (papers [3], [4], [7], [8], [9], [11]).

In the last few years of his life, PF came back to the subject with deep results about continued fractions related to elliptic functions [16], [17]. The connection with combinatorics opens many deep and new questions and probably PF was feeling the same excitation about opening a new field of researches, as he was doing in the blessed year 1978. The astonishing and very deep new continued fractions appearing in his two Happy New Year cards for 2009 and 2010 underlie this promising and unachieved period.

Another loved area of PF is stochastic processes (urn models, birth-and-death, ...) and PF made fruitful connections between combinatorics and probability theory (papers [14], [16]), underlying the following quotation from [15]: *"Discrete and continuous mathematics willingly and harmoniously encounter and complement"*.

There are two main concepts through these 16 papers: *weighted Motzkin paths* and the concept of *histories*. This last concept is basic for the integrated cost analysis: an *history* is the exact sequence of primitive operations, together with some additional information about the "position" of the key which is added, deleted, or questioned, of the data structure. The same idea of histories underlies the bijections interpreting various expansions of power series into continued fractions. The bijections are combinatorial constructions, which are sequences of primitive operations, acting on some combinatorial objects, where the operations are performed on a certain position. The number of possible positions corresponds to the weight of the related elementary step in the path. Such sequences of operations are also described in urn models. In quantum mechanics, creation and annihilation operators, are very similar to the operations of add and delete a ball in the urn process, or add and delete in a priority queue. These two operators satisfy the well known quantum mechanics commutation relation $UD = DU + I$ (Weyl-Heisenberg algebra), which is the starting point of the paper [18].

## 1 Continued fractions before PF

A classical example of (arithmetic) continued fraction is the expansion of the golden ratio $\phi$

$$\phi - 1 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}$$

When truncating this continued fraction at level $k$, one gets a rational fraction as the quotient of two consecutive Fibonacci numbers $F_k/F_{k+1}$. This ratio is called the $k$-th *convergent* whose limit is $\phi - 1$.

Continued fraction expansions of numbers are implicit in the Euclidian algorithm and are important in giving rational approximation of real numbers. They are implicit since the beginning of science in many ancient civilizations, Greek, Indian and Chinese. For example, the approximation of $\pi$ by the rational $355/113$ appears in China in the $5$th century, and is in fact the third convergent of the expansion of $\pi$ into continued fraction.

Continued fractions have been used to prove the irrationality properties of number such as $\pi$ (Lambert in 1766 ) or more recently by Apéry for proving the irrationality of $\zeta(3)$ by using the following representation:

$$\zeta(3) = \cfrac{6}{\overline{\omega}(0) - \cfrac{1^6}{\overline{\omega}(1) - \cfrac{2^6}{\overline{\omega}(2) - \cfrac{3^6}{\dots}}}} \quad \text{with} \quad \overline{\omega}(n) = (2n+1)(17n(n+1)+5).$$

Such continued fractions are also called *arithmetic* continued fraction, in contrast with the continued fractions making the subject of this chapter, that is *analytic* continued fractions. Such fractions contain a (real or complex) variable and give the expansion of a function in term of continued fraction. Here, with PF we consider the variable to be a formal variable, and the function is in fact a power series. The convergence is at the formal level. The pioneer work on analytic continued fractions is due to Leonhard Euler, who introduced several expansions of power series such as [41]:

$$\sum_{n\geq 0}(n+1)!z^n = \cfrac{1}{1 - 2z - \cfrac{1\cdot 2z^2}{1 - 4z - \cfrac{2\cdot 3z^2}{\cfrac{\dots}{1 - 2kz - \cfrac{k(k+1)z^2}{\dots}}}}}. \quad (1) \quad \text{and} \quad \sum_{n\geq 0} 1\cdot 3\cdot 5\cdots(2n-1)z^n = \cfrac{1}{1 - \cfrac{1z}{1 - \cfrac{2z}{\cfrac{\dots}{1 - \cfrac{kz}{\dots}}}}}. \quad (2)$$

The general theory of such continued fractions was mainly developed by Stieltjes in his memoir [55]. These continued fractions are particular case of the so-called Jacobi continued fraction or J-fractions, i.e., fractions of the following form

$$\sum_{n\geq 0} \mu_n z^n = \cfrac{1}{1 - c_0 z - \cfrac{\lambda_1 z^2}{1 - c_1 z - \cfrac{\lambda_2 z^2}{\cfrac{\cdots}{1 - c_k z - \cfrac{\lambda_{k+1} z^2}{\cdots}}}}} \tag{3}$$

where $\{c_k\}_{k\geq 0}$ and $\{\lambda_k\}_{k\geq 1}$ are two sequences of "coefficients", which can be integers, real or complex numbers or more generally "formal variables" of two infinite alphabets. In many examples, it will be polynomials with integer coefficients in some "formal variables" or "parameters" ($\alpha$, ... and $q$ for "$q$-analogs").

Most of PF 's work on continued fraction is concerned with analytic continued fractions (this chapter), but also with some considerations with arithmetic continued fraction, see papers [26], [27]. Nevertheless, the two apparently distinct domains are related. For example the Apéry continued fraction for $\zeta(3)$ with cubic and sextic terms appears in the paper [16] and has some intriguing similarity with the expansion of Dixmioninan functions into continued fractions with cubic denominator and sextic numerators.

## 2 The fundamental Lemma of PF on continued fractions

The fundamental combinatorial (or geometric) interpretation of J-continued fraction is the main theorem of the seminal paper [6], which we propose to call "The fundamental Lemma of PF".

We define a Motzkin path as a sequence of vertices $(s_0, \ldots, s_n)$ of the square lattice $\mathbb{N} \times \mathbb{N}$ , satisfying the following conditions

- $s_0 = (0, 0), \quad s_n = (0, n)$

- each elementary step $(s_i, s_{i+1})_{0\leq i<n}$ is of 3 possible types: North-East (NE), South-East (SE), East (E) corresponding respectively to $s_{i+1} = s_i + (x, y)$ with $(x, y) = (1, 1), (1, -1), (1, 0)$.

Three sequences of "coefficients" or "letters" from 3 alphabets are given: $(a_k)_{k\geq 0}, (b_k)_{k\geq 1}, (c_k)_{k\geq 0}$. We define the weight $v(\omega)$ of a Motzkin path $\omega$ as to be the product of the weight of its elementary steps, where the weight of a NE (resp. SE, resp. E step) starting at height $k$ is $a_k$ (resp. $b_k$, resp. $c_k$ ).

An example is displayed on Figure 1, $v(\omega) = a_0^2 a_1 b_1^2 b_2 c_0 c_1^2$.

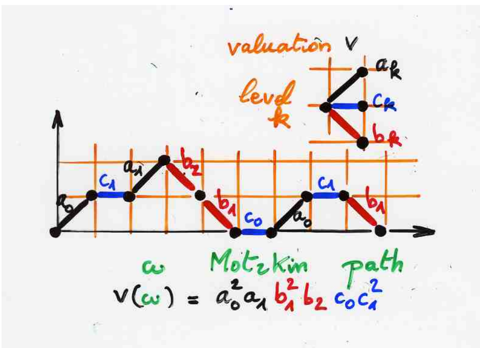


Figure 1: Weighted Motzkin path

Denote by $|\omega|$ the length of the path $\omega$, i.e., the number of elementary steps. Then the PF fundamental *Lemma* (paper [6]) gives an interpretation of the general J-fraction as the generating function of weighted Motzkin paths:

PF Fundamental Lemma on continued fractions

$$\sum_{\omega:\text{Motzkin paths}} \nu(\omega) z^{|\omega|} = \cfrac{1}{1 - c_0 z - \cfrac{a_0 b_1 z^2}{1 - c_1 z - \cfrac{a_1 b_2 z^2}{\cfrac{\cdots}{1 - c_k z - \cfrac{a_k c_{k+1} z^2}{\cdots}}}}}. \tag{4}$$

In different talks and papers such as [16], PF calls his main theorem with some ironic adjective, referring to the way a colleague welcomes this proposition. I propose here in this introduction to lift it at the level of a "Lemma" in the sense used in chapter 29 the beautiful book of Aigner and Ziegler "Proof from the BOOK" [31]:

*"The essence of mathematics is proving theorems - and so, that is what mathematicians do: They prove theorems. But to tell the truth, what they really want to prove, once in their lifetime, is a Lemma, like the one by Fatou in analysis, the Lemma of Gauss in number theory, or the Burnside- Frobenius Lemma in combinatorics.*

*Now what makes a mathematical statement a true Lemma? First, it should be applicable to a wide variety of instances, even seemingly unrelated problems. Secondly, the statement should, once you have seen it, be completely obvious. The reaction of the reader might well be one of faint envy: Why haven't I noticed this before? And thirdly, on an esthetic level, the Lemma - including its proof - should be beautiful!"*

It is immediate from the fundamental Lemma that the $k$-th convergent $J_k(t)$ of the Jacobi continued fraction $(4)$ is the generating functions of weighted Motzkin paths bounded by height $k$

$$\sum_{\substack{\omega:\text{Motzkin paths}\\ \text{height}\le k}} \nu(\omega) z^{|\omega|} = \cfrac{1}{1 - c_0 z - \cfrac{a_0 b_1 z^2}{1 - c_1 z - \cfrac{a_1 b_2 z^2}{\cfrac{\cdots}{1 - c_k z}}}}. \tag{5}$$

In the case $c_k = 0$ for all $k \geq 1$, i.e., the paths are Dyck paths, the continued fraction $(4)$ is called Stieltjes continued fraction or S-fraction.

Many known expansions of functions or power series into continued fractions can be proved combining PF fundamental Lemma with some combinatorial constructions. The coefficients of the power series are interpreted by some weighted combinatorial objects. Some *weighted histories* (see sections $4$, $5$) are related to the continued fraction, and a weight preserving bijection between these objects and these *histories* will give a combinatorial proof to the continued fraction expansion using PF fundamental Lemma. Such combinatorial proofs are also related to the combinatorial theory of orthogonal polynomials.

# 3 The classical correspondence between analytic continued fractions and orthogonal polynomials

First let us recall the classical definition of (formal) orthogonal polynomials. We consider the algebra $K[x]$ of polynomials in one variable $x$ and with coefficients in a commutative ring $K$. In the classical theory of orthogonal polynomials, $K$ is the field $\mathbb{R}$ of real numbers, with sometimes extension to complex numbers $\mathbb{C}$. Here everything is done in a formal way, and for combinatorial purposes, very often $K$ is the ring $\mathbb{Z}$, $\mathbb{Q}$ or the ring of polynomials in some formal variables $\alpha, \beta, \ldots$ (the "parameters") and $q$ (for $q$-analogues).

A sequence $\{P_n(x)\}_{n\geq 0}$ of polynomials of $K[x]$, each $P_n(x)$ being of degree $n$ , is said to be a sequence of orthogonal polynomials if and only if there exist a linear functional $f : K[x] \to K$ such that for every $k, l \geq 0, k \neq l$, $f(P_k P_l) = 0$ , and for every $k \geq 0$ , $f(P_k^2) \neq 0$.

In general, for a linear functional there exist, up to a multiplicative factor, a unique sequence of orthogonal polynomials, and conversely, the linear functional $f$ is uniquely determined by the sequence $\{P_n(x)\}_{n\geq 0}$, up to a multiplicative factor. The linear functional $f$ is uniquely determined by its value on the monomial basis: $f(x^n) = \mu_n$, $n \geq 0$. The $\mu_n$ are called the moments of the orthogonal polynomials.

In classical analysis, the orthogonality is defined by a certain (Stieltjes) integral involving a certain measure $\int_a^b P_k(x)P_l(x)d\psi$. But in this chapter the essential point of view is in a formal way, in a similar way to the book [33] of Chihara. But for applications, moments can be put into integral form and parameters and variable can be real numbers.

For a given sequence of moments $\{\mu_n\}_{n\geq 0}$, there exist orthogonal polynomials related to such moments if and only if the so-called Hankel determinants $\Delta_n$ are non zero for every $n \geq 0$. These determinants are defined by the following: the term $(i, j)$ of the determinant $\Delta_n$ is $\mu_{i+j}$.

A main classical theorem on orthogonal polynomials theory is Favard theorem, which we state in its formal way. A polynomial of degree $n$ is monic if the coefficient of $x^n$ is 1.

(formal) Favard theorem

Let $\{P_n(x)\}_{n\geq 0}$ be a sequence of monic polynomials of degree $n$ with coefficient in $K$. This sequence is a sequence of orthogonal polynomials if and only if there exist two sequences $\{c_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ of coefficients in $K$ such that the polynomials satisfy the following 3-term linear recurrence

$$P_{k+1}(x) = (x - c_k)P_k(x) - \lambda_k P_{k-1}(x), \quad k \geq 1. \tag{6}$$

In the classical theory, there are two relations between continued fraction and orthogonal polynomials.

The generating function of the Jacobi continued fraction $J(t)$ defined by the coefficients $\{c_k\}_{k\geq 0}$ and $\{\lambda_k\}_{k\geq 1}$ as in (3) is exactly the power series $\sum_{n\geq 0} \mu_n z^n$, generating function of the moments $\mu_n$ associated with the orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ defined by the 3-term linear recurrence (6). Moreover, the convergents of the Jacobi continued fraction $J(t)$ are the following rational fractions $\delta P_k^*(z)/P_{k+1}^*(z)$, where $P_k^*(z) = z^k P_k(1/z)$ denotes the reciprocal of $P_k(x)$ and

$\delta P_k(x)_{k\geq 0}$ denotes the "shift" of the sequence $\{P_k(x)\}_{k\geq 0}$, i.e., orthogonal polynomials obtained from the 3-term recurrence relation $(6)$ by replacing the sequences $\{c_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ respectively by $\{c_{k+1}\}_{k\geq 0}$, $\{\lambda_{k+1}\}_{k\geq 1}$.

Thus, combining the classical analytic theory of orthogonal polynomials and continued fractions with PF fundamental Lemma, we deduce that the $n^{\text{th}}$ moment of the most general sequence of orthogonal polynomials defined by the 3-terms linear recurrence relation $(6)$ is the sum of the weights of Motzkin paths of length $n$:

$$\mu_n = \sum_{\substack{|\omega|=n \\ \text{Motzkin paths}}} \nu(\omega), \tag{7}$$

where the weight of the Motzkin paths $\omega$ is defined (with the notations of section PF Lemma) by the 3 sequences: for NE steps $\{a_k = 1\}_{k\geq 0}$, for SE step $\{b_k = \lambda_k\}_{k\geq 1}$ and for E step $\{c_k\}_{k\geq 0}$.

# 4 Histories and continued fraction: the toy example with Hermite histories

Applying PF's fundamental Lemma, a combinatorial or "bijective" proof of Euler's identity $(2)$ is obtained by establishing a bijection between chord diagrams and the so-called Hermite histories of length $n$ defined below.

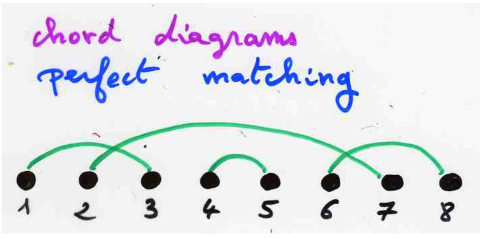


Figure 2:

A chord diagram is a set of $n$ arcs joining two by two $2n$ points (or perfect matchings of the complete graph, or involutions without fixed points). They are enumerated by the product $1.3.\cdots(2n-1)$ interpreting the left hand side of the identity $(2)$. The right hand side is interpreted by a "Hermite history", i.e., a pair $h = (\omega, f)$ where $\omega = (\omega_1, \ldots, \omega_{2n})$ is a Dyck path with elementary steps $\omega_1, \ldots, \omega_{2n}$ and $f = (p_1, \ldots, p_{2n})$ (the possibilities function, also called choice function) is such that $p_i = 1$ when $\omega_i$ is a NE step, else $1 \leq p_i \leq \nu(\omega_i) = b_{k_i}$ where $k_i$ is the height of the starting point of the step $\omega_i$.

As explained in the introduction, the name "histories" refers here to the idea of a sequence of operations corresponding to the sequence of NE or SE steps of the Dyck path, each operation being performed on certain combinatorial objects with a certain number of possible choices. Here a NE step ($\nearrow$) corresponds to "open" a new chord, a SE step ($\searrow$) corresponds to "close" a chord with the possible "open" chords waiting to be closed.

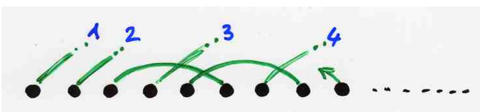


Figure 3:

Figure 3 is a visualization of the beginning of a Hermite history $h = (\omega, f)$ with $\omega = (\nearrow, \nearrow, \nearrow, \nearrow, \nearrow, \searrow, \nearrow, \searrow, \searrow, \dots)$ and $f = (1, 1, 1, 1, 1, 3, 1, 4, p_9, \dots)$. The $k_i$ possible choices for closing a chord are here numbered from left to right. Here, at this $9th$ step, there are $4$ possible values for $p_9$.

In the context of the PF's work on Polya urns, exactly the same toy example of Hermite histories appears in relation with the following elementary rule: one can add a ball (NE step), or take by random a ball from an urn containing $k$ balls, with $k$ possible choices (SE step). This corresponds exactly in the context of PF's work on integrated cost of data structure to the structure of priority queue: adding a key anywhere in a $k$-elements file (with $k + 1$ possibilities) and deleting the minimum (with $1$ possibility). In fact the sequence of such primitive operations in a priority queue, with its possibilities functions, would correspond to the mirror image of a Hermite history.

Such add and delete primitive operations satisfy the commutation relation $UD = DU + I$, developed in the paper [18].

The name Hermite histories refers to the Hermite orthogonal polynomials, whose relation with continued fractions was explained in section 3. From that section, the moments of the Hermite polynomials become the number of Hermite histories. PF used the term "system of paths diagram". In subsequent papers, many other "systems of paths diagram" have been introduced, corresponding to different valuation of the Motzkin paths. The name "Hermite histories", together with other terms corresponding to other orthogonal polynomials, "Laguerre histories", "Charlier histories", seem to be used to distinguish between all these systems of paths diagram.

Also, in many papers (see for example [37], [58]), once an integer valuation of the paths is given and gives rise to histories related to this valuation, one needs to consider weighted histories with usually two kind of weights: the $q$-analog, and the "alpha valuation".

## 5 Weighted histories and $q$-analogs

The simplest q-analog is done by giving a weight $q^{p_i - 1}$ when the possibility function is $p_i$. The weight of the Hermite history is the product of the weight of each $p_i$ corresponding to each elementary step. Through the above bijection between chord diagrams and Hermite histories, the qweight of the history corresponds to the classical parameter "nesting" on chord diagrams. If one reverse the convention of labeling the possible choices (open chord) for closing a chord from right to left, instead from left to right as described above with figure 3, then one gets as $q$-parameter on chord diagrams the classical parameter "number of crossings". Thus we get the well known fact that the parameters "nesting" and "crossing" have the same distribution on the set of chord diagrams. This is an illustration of the power of this concept of "histories" underlying most of the papers of this chapter. Such a parameter "number of crossings" for chord diagrams was already considered in a pioneer work by Touchard [56] where he established the following continued fraction for its generating function, $q$-analogue of the Euler continued fraction $(2)$

$$\cfrac{1}{1 - \cfrac{[1]_q z}{1 - \cfrac{[2]_q z}{\cfrac{\cdots}{1 - \cfrac{[k]_q z}{\cdots}}}}}, \tag{8}$$

where $[k]_q = 1 + q + \cdots + q^{k-1}$.

In the correspondence between continued fractions and orthogonal polynomials, these q-valuations give a certain q-analog of Hermite polynomials.

In a very similar way, some $q$-analog of Euler continued fraction $(1)$ can be defined, leading to some q-analog of Laguerre polynomials defined in the paper [11] about integrated cost of simple list structures. Here, the weight $q^{p_i-1}$ for the choice $p_i$ of the history is exactly the cost the primitive operation.

A second elementary idea to put weight on Hermite histories is to put a weight $\alpha$ each time in the above bijection it is the first position (or the last) which is chosen when closing the chord. This corresponds to replace the integer valuation $k$ by $(\alpha + k - 1)$. Again this is related to some classical one-parameter orthogonal polynomials. In the case of the continued fraction $(1)$ related to permutations and Laguerre histories (defined below in section 6), the similar weight $\alpha$ (taking the first choice in the history) corresponds to cycles in permutation and classical Laguerre orthogonal polynomial $L_n^{(\alpha)}(x)$ with parameter $\alpha$.

More generally, as mentioned at the end of section 2, many continued fractions expansions can be proved combinatorially by combining PF fundamental Lemma with some construction of weight preserving bijections between weighted objects and weighted histories. The "weight $\alpha$" plays a key role in many such combinatorial proofs. Other possible weights on histories are very useful, in particular for the case of the Sheffer class of orthogonal polynomials (see section 6 and [58]).

In this introduction, I have distinguished between the weighted paths, the histories related to the weighted path, and then the possibility to put some weight on the history. This distinction does not appear clearly in PF papers, where often the single term "system of path diagram" is used. But nowadays, in many papers the term "history" appears, in distinction of the term "weighted path" or "system of path diagram".

# 6 More combinatorial interpretations of continued fractions

In his seminal paper [18], PF gave many combinatorial proofs for some classical expansions of functions and power series into analytic continued fractions, using the philosophy of bijection related to the idea of "histories", combined with his fundamental Lemma. In particular he extended the above toy example with Hermite histories and chord diagrams to partitions and "Charlier histories", corresponding to Charlier polynomials. More generally, he made an intensive use of the FV bijection [43] between permutations and some histories related to the following path valuation

$$a_k = k + 1, b_k = k + 1, c_k = 2k + 2. \tag{9}$$

This bijection is constructed with the same philosophy than for Hermite histories. The $(n + 1)!$ permutations are constructed by using $4$ kinds of operators (one for NE step, one for SE step and two for E step), acting on words (or binary trees) with some empty positions (or "open" edges), each operator is done on a certain "open" position, and according to the fact that for an elementary step is NE (resp. E, resp. SE) the number of open positions is increased by one (resp. is invariant, resp. decreases by one). The histories related to these valuations are often called Laguerre histories, and give a combinatorial proof of Euler continued fraction $(1)$.

Playing with these histories, or putting some restrictions, or putting various kinds of weights (for example similar $q$ or $\alpha$ weight as described in section 5), PF got in the seminal paper [6] many combinatorial proofs of analytic continued fractions of the literature, without any calling for analytic method as was done for more than two centuries before his seminal paper. The above Hermite and Charlier histories are particular cases where some restrictions are made on the possible choices. By making more restriction for the possible choices for the Charlier histories, PF and R.Schott get in [13] a complete combinatorial study of continued fractions expansions related to some partitions called “non-overlapping” and to Bessel functions. This class of partitions introduced by PF and his co-author lies somewhere between unconstrained partitions and the classical noncrossing partitions enumerated by the Catalan numbers.

Another correspondence between analytic continued fractions and combinatorial objects is made in a reverse way using bijections based on histories. The expansion of a special function or a power series into continued fraction is supposed to be known, and the bijection with histories gives a combinatorial interpretation of the special function or the particular power series.

See for example the paper [12] giving, after the combinatorial interpretation of Dumont [40] and Viennot [57], another interpretation of the Jacobi elliptic functions in terms of alternating permutations. The problem remains open to give a relation between this interpretation and the others. Also in the paper [16], an interpretation is given for other elliptic functions, the Dixmionian function sm, cm. Here too, the problem remains open to go in the reverse way, and give a combinatorial proof for the expansion into continued fractions of these Dixmionian elliptic continued fractions from the system of differential equations defining these functions.

Other combinatorial consequences of the path interpretation of continued fractions are developed in [10] about congruence properties.

From the equivalence between continued fractions and orthogonal polynomials described in section 3, the combinatorial proofs of expansion into continued fractions of functions or power series given above can be reinterpreted as combinatorial interpretation (and proof) of the moments of the associated orthogonal polynomials. In particular, for the following classical orthogonal polynomials:

| orthogonal polynomials | moments | 3-terms recurrence relation |
|---|---|---|
| Hermite | $1 \cdot 3 \cdots (2n-1)$ chord diagrams | $c_k = 0, \lambda_k = k$ |
| Charlier | $B_n$ set partitions | $c_k = k+1, \lambda_k = k$ |
| Laguerre | $n!$ permutations | $c_k = 2k+1, \lambda_k = k^2$ |

Table 1

Sheffer polynomials are defined by polynomials having exponential generating function of the following form:

$$\sum_{n \geq 0} P_n(x) \frac{t^n}{n!} = g(t) \exp(x f(t)). \tag{10}$$

A classical theorem says that there are only five classes of Sheffer orthogonal polynomials, called Hermite, Charlier and Laguerre (with one parameter $\alpha$), Meixner I and Meixner II (with two parameters $c, \delta$). These polynomials play a key role in the series of papers of PF and his coauthors about analysis of integrated cost of data structures. Roughly speaking, the “Sheffer” property for orthogonal polynomials is equivalent to say that the 3 valuations $a_k, b_k, c_k$ of Motzkin paths are linear expression in $k$.

# 7 File histories and integrated cost in data structures

As said in the introduction, the pioneer paper [1] in 1978 about the analysis of integrated cost in data structures was the starting point of a beautiful theory in computer science for data structures, and at the same time the combinatorial theory of analytic continued fractions.

In [42] J. Françon introduced the concept of file history (in French *histoire de fichiers*). Primitive operations are of 3 kinds: adding, deleting a key, or asking a question (inspired from the dictionary data structure represented by a binary search tree). The idea of data history is to keep in memory the exact sequence of operations, up to an isomorphism of relative values of the different keys. If the $k$ keys are totally ordered, the only pertinent information is to remember the "position" where the operation is performed, which can be a key itself in the case of deletion or question about a key (positive query) (with $k$ different possibilities), or an interval (with $k+1$ possibilities) in the case of adding a key or a negative query. Starting and ending with an empty data structure, the total number of such file histories is the sum of weighted Motzkin paths of length $n$, according to the valuation $a_k = k+1, b_k = k, c_k = 2k+1$. Using the FV bijection, with a restriction in the construction, gives the number of such histories as $n!$, moments of the Laguerre polynomials $L_n^{(0)}(x)$, (instead of $(n+1)!$, moments of $L_n^{(1)}(x)$ for the general case described in section 5).

The idea of file histories is fruitful in the case the algorithms performing the allowed primitive operations use only comparisons between keys. PF considers the analysis of the average cost of a data structures, integrated over the set of all possible histories. Under some additional hypothesis called "stationarity" of the data structure, PF shows that it is possible to give explicit expression for this average integrated cost, as a function of the average cost of each primitive operation, and of some coefficients given by the combinatorics of weighted Motzkin paths. These coefficients involve the number of times a primitive operation (add, delete or query) is performed on a file of size $k$, over all possible histories. Such coefficients can be computed from the continued fraction interpretation, using the generating function for weighted paths starting from level 0 and ending at the level $k$. Classical data structures are just defined by the possibility function to each primitive operations, which corresponds to a certain continued fraction and thus a certain family of orthogonal polynomials. Main examples are given on table 2.

| data structure | possibility function | orthogonal polynomials |
|---|---|---|
| stack | $a_k = 1, b_k = 1, c_k = 0$ | Tchebycheff (second kind) |
| priority queue | $a_k = k+1, b_k = 1, c_k = 0$ | Hermite |
| symbol table | $a_k = k+1, b_k = 1, c_k = k+1$ | Charlier |
| linear list | $a_k = k+1, b_k = k, c_k = 0$ | Meixner |
| dictionary | $a_k = k+1, b_k = k, c_k = 2k+1$ | Laguerre |

Table 2

The remarkable fact is that for each classical data structure, such as the five listed on table 1, corresponds a family of classical orthogonal polynomials. Except for the most simple case of stack, where the moment of the corresponding orthogonal polynomials are the Catalan numbers themselves, the 4 other data structures involve 4 of the 5 classes of Sheffer orthogonal polynomials.

In the sequel of papers [4], [8], [9], [11], PF and his coauthors make a complete computation of the analysis of such integrated costs for various data structures. The principal ingredient explained in the fundamental paper [8], is to give an expression of the (exponential) generating function of the integrated cost in term of the generating function for the average cost of a single primitive operation. The path interpretation leads to a certain expression with an integral and a convolution product.

The orthogonality of the related polynomials appearing as numerators and denominators of the convergents (which are the same as the polynomials whose moments are the number of histories) plays a key role. For particular example, explicit expression for the exponential generating function of each Sheffer orthogonal polynomials is used and leads to a complete analysis. What a beautiful interplay between computer science, classical analysis and combinatorics !

Similar continued fraction techniques can also be used for the analysis of the integrated cost of data structures in the case of bounded number $N$ of keys. The continued fraction becomes finite, with valuation depending on the size $N$ and the level in the Motzkin path (paper [7]).

In the paper [11], with a simple cost in linear lists analysis reduced to $q^{p_i-1}$ for the choice $p_i$ of the corresponding data history, leads PF and his coauthors to introduce a non-classical $q$-analog of Laguerre polynomials, as explained in section 4, and get an explicit computation of the variance of the integrated cost. Curiously, this same $q$-analog of Laguerre polynomials plays a fundamental role in the recent study of the PASEP model in physics of dynamical systems (see section 11).

# 8 From combinatorics to probability theory

In a series of papers (starting with [48]), Karlin and McGregor studies probabilistic birth-and-death process and showed the strong relationship with orthogonal polynomial theory. A population is increasing or decreasing according to some rate depending of the size of the population. In discrete time, this process reminds some Motzkin paths, with some weight on the edges depending of the height elementary step (NE, SE or E). Using methods such as backward or forward differential equation, Karlin and McGregor gave explicit expression for some random behavior in terms of integral involving related orthogonal polynomials. Certain ideas underlying are similar in their spirit to some computations of PF and coauthors related to the analysis of integrated cost in data structures explained in the previous section. In particular, a general expression for computing the generating function of weighted paths starting from a certain level $k$ and ending in a level $l$. It relies on the expression:

$$\sum_{\substack{|\omega|=n \\ \text{Motzkin paths}}} \nu(\omega) = a_0 \cdots a_{k-1} b_1 \cdots b_l f(x^n P_k P_l), \tag{11}$$

where the summation is over all paths with steps NE, SE and E going from level $k$ to level $l$, $\{P_n(x)\}_{n\geq 0}$ is the sequence of orthogonal polynomials defined by the 3-term recurrence $(8)$ related to the valuation with $\lambda_k = a_{k-1}b_k$, $k \geq 1$, and $f$ is the linear functional associated to the orthogonal polynomials. For computation, $f$ is written in form of an integral with a certain measure $d\psi$. (see section $6$).

Following his philosophy, *"Discrete and continuous mathematics willingly and harmoniously encounter and complement"*, PF in the paper [14] with F.Guillemin, returns on this birth-and-death classical subject, but having in mind his discrete (or combinatorial) theory of continued fractions with weighted paths. He shows the power of discrete thinking by getting new results on this well known and classical topic by introducing a fruitful methodology with some morphisms, which make possible to get probabilities results as a consequence of discrete analysis. This methodology is summarized in the abstract [21] of the talk of his co-author.

The same philosophy is used in the paper [16], where some urn processes analysis are made using combinatorial interpretations (permutations, obtained from certain "histories" related to some Dixmonian elliptic functions) and from which some analysis of a kind of Yule process is deduced (some particles called *foaton* and *viennon* disintegrate with a certain probability decay law, each particle of one kind giving rise to two particles of the other kind).

# 9 Continued fractions for elliptic functions

We just mentioned above the paper [14]. The starting point is the amazing expansion of Dixmioninan functions into continued fractions with cubic denominators and sextic numerators obtained by Conrad in his thesis [34]. Such functions are related to the cubic Fermat curve $x^3 + y^3 = 1$ and, as we say in the section about arithmetic continued fractions, the amazing Apéry continued fraction for $\zeta(3)$ has intriguing similarity with Conrad analytic continued fraction. PF gives two combinatorial interpretations in terms of permutations of such Dixmionian functions sm and cm, starting from the system of differential equations for sm and cm, and for the second interpretation, starting from the continued fraction. This second interpretation uses a similar idea as in the paper [12] interpreting Jacobian elliptic functions in term of the classical alternating permutations going back to D.André in the years 1880s. The idea is to glue 2 by 2 or 3 by 3 elementary steps in a Motzkin path, in relation with terms of degree 4 or 6 in the continued fraction.

It seems that the problem remains open to give bijections between the two interpretations of Dixmionian functions, which would give a combinatorial proof of Conrad continued fraction. The same problem seems to remain to give a correspondence between the interpretation of Jacobi elliptic functions given in [12] in term of alternating permutations, and the original combinatorial interpretations given by Dumont [40] or Viennot [57] in terms of some permutations related to the system of differential equations.

In the spirit of this beautiful paper, PF continues in the paper [17] with R.Bacher, his study of continued fraction related to elliptic function. It is not frequent to find new continued fractions. Such fraction related to special functions are rare (perhaps less than 100 from Perron and Wall). PF gave a new and deep continued fraction related to some elliptic function coming from "pseudo-factorials", which can be express from the Weierstass function. We are close to lattice, periodic complex function and number theory.

The methodology is not using combinatorics but using the classical equivalence, due to Rogers and Stieltjes, between expansion into continued functions and some addition formula of the following type. Let $\Phi(z)$ be a function expressed as an exponential generating series $\Phi(z) = 1 + \sum_{n\geq 1} \Phi_n z^z/n!$ and suppose there exist coefficients $\{\omega_i\}_{i\geq 1}$ and functions $\{\varphi_r\}_{r\geq 0}$ of the form $\varphi_r(z) = z^r/r! + O(z^{r+1})$. The Rogers-Stieltjes addition formula is of the type

$$\Phi(x+y) = \varphi_0(x)\varphi_0(y) + \sum_{r\geq 1} \omega_r \varphi_r(x)\varphi_r(y). \tag{12}$$

Remark the PF gave a combinatorial of this equivalence in his thesis [24] using the geometry of Motzkin paths.

If J(t) is the expansion into Jacobi continued fraction (type (3)) of the ordinary generating function $1 + \sum_{n\geq 1} \Phi_n z^n$ (i.e., the Laplace transform of $\Phi(z)$), then there are simple expressions giving the coefficients $c_k$, $\lambda_k$ of $J(t)$ in terms of the $\omega_i$ and of the coefficients of $z^{r+1}$ in the power series $\varphi(z)$. By a heavy use of computer algebra and heuristic guesses, PF and R.Bacher manage to produce such addition formula which leads to a beautiful new continued fraction. Moreover, again with deep techniques, they give the exponential generating function for the associated orthogonal polynomials, which belong the family called "elliptic polynomials" (see the book [49]).

In papers [16], [17], PF is opening a new deep field of researches about elliptic functions and polynomials. The relation with combinatorics has to be explored. The very deep new continued fractions appearing in his two Happy New Year cards for 2009 and 2010 underlie this promising and unachieved period.

# 10 Operators, physics and orthogonal polynomials

Finally the paper [18] with P.Blasiak is one of the very last papers of PF. All along this introduction we insist on the underlying concept of "histories" appearing in most of the 16 papers of this chapter. Histories can be viewed as sequences of operators with some extra informations. In the toy example of Hermite histories (section 3), there are two operators satisfying the commutation relation $UD = DU + I$ familiar to physicists in quantum mechanics (creation and annihilation of particles). The algebra generated by this relation has a basis forms by the monomial $D^iU^j$ and any word $w(U, D)$ in non-commutative variables can be expressed in a unique way $w = c_{i,j}(w) \sum_{i,j\geq 0} D^iU^j$. This is the well known *normal ordering* in physics. The paper of P.Blasiak and PF follows a long list of papers by a group combinatorics and physicists with P.Blasiak (G.Duchamp, K.Penson, A. Horzela, I.Solomon) about combinatorial properties of the normal ordering, in relation with some combinatorial Hopf algebra. If $Q(U, D)$ is a polynomial in $U$ and $D$, one is interested in the generating function of the coefficients in the normal ordering of the word $Q(U, D)^n$. The most known example is the normal ordering of $(UD)^n$ with the appearance of the Stirling numbers, Bell numbers and set partitions.

All along this introduction, some representations of the algebra defined by $UD = DU + I$ appear in the form of the operations of a priority queue in computer science, some Polya urns or with the combinatorial construction with chord diagrams and Hermite histories. Here, PF and P.Blasiak start with the classical representation using the algebra of polynomial in one variable and the two operators on polynomials: $X$ (multiplying by $x$) or $D$ (derivative). They introduce the concept of *gates*, and for different polynomials $Q(U, D)$ give generating function, related to classical objects in enumerative combinatorics and related continued fractions. The spirit of this paper is in fact combinatorics of differential calculus in relation with continued fraction. This theory developed from the concept of *gates* appears to be strongly connected to the combinatorial theory of differential equations developed by P.Leroux and X.G.Viennot based on *enriched increasing trees*. Gates appear as an extension of some *weighted* increasing tree. Further researches and extensions should be developed.

# 11 Further researches

Following PF seminal paper [6], X.G.Viennot [58] (summary in [59]) made a complete "combinatorial" theory of (formal) orthogonal polynomials, starting from scratch, "rewriting" the classical theory (section 3) using only combinatorial arguments based on weighted paths and histories, in particular starting with a pure combinatorial proof of the orthogonality of polynomials defined by the recurrence (6) according to the moments defined by (7). The FV bijection leads to complete combinatorial interpretation of the moments of the class of Sheffer orthogonal polynomials.

Convergents of order $k$ of $J-$ and $S-$ continued fractions are particular cases of the socalled Padé approximants introduced by Padé at the end of 19th century. The idea is to approximate a power series $\Phi(x)$ by a rational fraction $N_r/D_s$ where $N_r$ (resp. $D_s$) is a polynomial of degree $r$ (resp. $s$) and where the expansion of the rational fraction coincides with $\Phi(x)$ as far as possible (in general for the first $r+s+1$ terms). Extending the monograph [58], E.Roblet gave in [52] a complete combinatorial theory of the classical (in analysis) theory of Padé approximants, with combinatorial extensions to other kinds of continued fraction ($L-$ and tree-like continued fraction). Summary of this work is given in the abstract [20] of Roblet's talk.

Other extensions have been made by Roblet in [53] with the use in combinatorics of Tfractions (equivalent to "Padé approximants in two points"). In the same spirit of Roblet's monograph [52], D.Drake continues such combinatorial theory for multiple orthogonal polynomials [39].

In the above papers, intensive use is made of Hankel determinants interpreted by configurations of non-crossing Motzkin and Dyck paths, using the so-called classical LGV lemma (see for example [31]) relating determinants and weighted non-crossing paths. These determinants play a key role in the so-called qd-algorithm (or "quotient-difference" algorithm), a classical algorithm in numerical analysis for computing the expansion into S-fraction of a power series. A simple combinatorial proof, using again the geometry of Motzkin paths, has been made by Viennot [60], with application to the enumeration of some Young tableaux, as explained in the abstract [19] of Gouyou-Beauchamps's talk.

Many works about combinatorial theory of orthogonal polynomials has been made in the last 30 years. There are two dual points of view: interpretation of the polynomials (coefficients, generating function, combinatorial proof of various formulae, ...) or combinatorial interpretation of the moments of the polynomials, in the spirit of the work of PF and Viennot [58]. Following this moments interpretation, many works has been done, in particular by J.Zeng and his students in Lyon and co-authors. For example, mention the beautiful paper [46] about addition theorems via continued fractions, ultra spherical and $q$-Jacobi (and more) polynomials.

Recently, some works have been made in other parts of theoretical physics using the combinatorics of continued fraction with the geometry of paths. Mention the work of J.Bouttier and E. Guitter [32] about continued fractions appearing in the context of random planar maps, and the paper [38] (the first of a series) by Di Francesco and R.Kedem about Q-systems in relation with the new domain of *cluster algebra* introduced by S.Fomin and S.Zelevinsky. Deep combinatorial manipulations of paths and continued fractions lead to a proof a positivity conjecture.

In the same spirit of section 10 about the relation between combinatorics, physics, operators and orthogonal polynomials, a series of researches is done around the PASEP (partially asymmetric exclusion model). It is a toy model in the physics of dynamical systems far from equilibrium. The computation of the stationary probabilities is based on the PASEP algebra, i.e., the algebra with by two generators $D$ and $E$ satisfying the commutation relation $DE = qED + E + D$. Many recent works have done by combinatorists (for example: [35], [36], [61]), giving an interpretation of these probabilities in terms of some "tableaux" (*permutations tableaux, alternative, staircase and tree-like* tableaux). For the PASEP model with 3 parameters $\alpha$, $\beta$, $q$, some $q$-Laguerre polynomials play a crucial role. The moments of these orthogonal polynomials are exactly the weight of these tableaux (with $\alpha$, $\beta$, $q$). Another approach uses the q-Hermite introduced above. Behind this approach are related continued fractions. These $q$-Laguerre polynomials (and also $q$-Hermite) are exactly the same as the polynomials introduced by PF for the integrated cost of linear lists [11] and mentioned in sections 4 and 7 as the “natural” $q$-weight for Laguerre and Hermite histories.

Thus, the spirit of PF work on continued fractions is underlying all these works over a period of 30 years, with these unexpected apparitions of continued fractions in algebra, analysis, physics and probability theory, with strong connections with classical orthogonal polynomials.

# References


**papers PF of this chapter**

[1] Analyse d’algorithmes de manipulation de fichiers. Technical Report 321, Institut de Recherche en Informatique et en Automatique, 1978.

[2] (with Jean Françon and Jean Vuillemin) Towards analyzing sequences of operations for dynamic data structures, *In Proc. of the 20th Annual Symposium on Foundation of Computer Science*, pp 183–195, IEEE Computer Society Press, 1979.

[3] (with Jean Françon and Jean Vuillemin) Computing integrated costs of sequences of operations with application to dictionaries. In *Proceedings of the 11th Annual ACM Symposium on Theory of Computing, pp 49–61*. Association for Computing Machinery

[4] (with Laurent Cheno, Jean Françon, Claude Puech and Jean Vuillemin) Dynamic data structures: finite files, limiting profiles and variance analysis. In *Eighteenth Annual Conference on Communication, Control and Computing*, pp 223–232. The University of Illinois at Urbana- Champaign, 1980

[5] Combinatorial aspects of continued fractions. *Annals of Discrete Mathematics*, 8:217–222, 1980. Extended abstract. Proceeding of “Colloque Franco-Canadien de Combinatoire”, Montreal, 1979.

[6] Combinatorial aspects of continued fractions. *Discrete Mathematics*, 32:125–161, 1980. Reprinted in the 35th Special Anniversary Issue of *Discrete Mathematics*, Volume 306, Issue 10-11, pp 992–1021 (2006).

[7] (avec Jean Françon) Structures de données dynamiques en réservoir borné, in J.Morgenstern, editor, *IIIémes Rencontres R.C.P. Algorithmiques*. Université de Nice, 1980. 14 p. (Proceeding of a meeting, June 1980).

[8] (with Jean Françon and Jean Vuillemin) Sequence of operations analysis for dynamic data structures, Journal of Algorithms, 1:111–141, 1980.

[9] (with C.Puech) Analyse de structures de données dynamiques et histoires de fichiers. *Questiió*, 5:31–48, 1981.

[10] On congruences and continued fractions for some classical combinatorial quantities. Discrete *Mathematics*, 41:145–153, 1982.

[11] (with Claude Puech and Jean Vuillemin) The analysis of simple list structures. *Information Sciences*, 38:121–146, 1986.

[12] (with Jean Françon) Elliptic functions, continued fractions and doubled permutations. *European Journal of Combinatorics*, 10:235–241, 1989.

[13] (with René Schott) Non-overlapping partitions, continued fractions, Bessel functions and a divergent series. *European Journal of Combinatorics*, 11:421–432, 1990.

[14] (with Fabrice Guillemin) The formal theory of birth-and-death processes, lattice path combinatorics and continued fractions. *Advances in Applied Probability*, 32:750–778, 2000

[15] (with Philippe Chassaing) Hachage, arbres, chemins, et graphes. *Gazette des Mathématiciens*, pages 29–49, 2003.

[16] (with Eric van Fossen Conrad) The Fermat cubic, elliptic functions, continued fractions, and a combinatorial excursion. *Séminaire Lotharingien de Combinatoire*, 54(B54g):1–44, 2006.

[17] (with Roland Bacher) Pseudo-factorials, elliptic functions, and continued fractions. *Ramanujan Journal*, 21:71–97, 2010

[18] (with Pawel Blasiak) Combinatorial models of creation-annihilation, *Séminaire Lotharingien de Combinatoire*, Issue 65, B65c, 2011, 78 pp.

[19] ALGO Seminar, year ??, summary of "Enumeration of semi-standard Young tableaux", Dominique Gouyou-Beauchamps.

[20] ALGO Seminar, June 1994, summary of "Combinatorial interpretation of continued fractions", Emmanuel Roblet.

[21] (with F.Guillemin) ALGO seminar, February 1998, summary of "Birth-Death processes, lattice path, combinatorics, continued fraction and orthogonal polynomials", Fabrice Guillemin.

[22] PF Happy New year Card 2009

[23] PF Happy New year Card 2010

**[PF secondary papers]**

Vol 7, Ch 2

[24] Analyse d'algorithmes de manipulation d'arbres et de fichiers. Thése Doctorat ès sciences. Université de Paris XI, Orsay, 1979.

Vol 5, Ch 4 On random walks, Lattice paths

[25] The evolution of two stacks in bounded space ands random walk in a triangle, 1986.

Vol 2, Ch 1, On Gaussian limit law

[26] (with Brigitte Vallée) Continued fraction algorithms, functional operators and structure constants, 1998.

Vol 2, Ch 3, On dynamical systems

[27] (with Brigitte Vallée) Continued fraction algorithms, comparison algorithms and fine structure constants. 2000.

Vol 5, Ch 5, on Urns

[28] (with Joaquim Gabarró and Helmut Pekari) Analytic urns. 2005

[29] (with Philippe Dumas andVincent Puyhaubert) Some exactly solvable model of urn process theory, 2006.

[30] (with Thierry Huillet) Analytic combinatorics of Mabinogion urn. 2008.

Vol 5, Ch 6, on Number theory, see the introduction

**other references**

[31] M. Aigner and G.M.Ziegler, *Proofs from the* BOOK, Springer Verlag, Heidelberg,1998, 4th ed. 2009.

[32] J. Bouttier and E. Guitter, Planar maps and continued fractions, *Comm. Math. Phys*. 309:3 (2012) 623–662.

[33] T.S. Chihara, *An introduction to Orthogonal Polynomials*, Gordon and Breach, New-York, 1978.

[34] E. van Fossen Conrad, *Some continued fraction expansions of Laplace transforms of elliptic, functions*, PhD Thesis, The Ohio State University, 2002.

[35] S. Corteel, R. Stanley, D. Stanton and L. Williams, Formulae for Askey-Wilson moments and enumeration of staircase tableaux, *Trans. of the AMS*, to appear (2011).

[36] S. Corteel, M. Josuat-Vergès L.K. Williams, The Matrix Ansatz, Orthogonal Polynomials, and Permutations, *Adv. in Appl. Maths*, 46 (2011), 209–225.

[37] A. de Medicis and X.G.Viennot *Moments des q-polynômes de Laguerre et la bijection de Foata- Zeilberger*, Advances in Applied Maths., 15 (1994) 262–304.

[38] P. Di Francesco and R. Kedem, Q-systems, heaps, paths and cluster positivity, *Comm. Math. Phys*. 293, n°3 (2010) 727–802.

[39] D.A. Drake, *Towards a combinatorial theory of multiple orthogonal polynomials*, Ph.D. thesis, Univ. of Minnesota, USA, August 2006.

[40] D.Dumont, Une approche combinatoire des fonctions elliptiques de Jacobi, *Adv. in Maths* 1 (1981), 1–39

[41] L.Euler, De seriebus divergentibus, [E.247], Novi Commentarii academiae scientiarum Petropolitanae 5, 1760, pp. 205–237.

[42] J. Françon, Histoires de fichiers, *RAIRO Informatique Théorique* 12 (1978) 49–62.

[43] J. Françon and G. Viennot, Permutations selon les pics, creux, doubles montées et doubles descentes, nombres d'Euler et nombres de Genocchi, *Discrete Maths*., 28 (1979) 21–35.

[44] M.E.H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Encyclopedia of Mathematics and its Applications, no. 98, Cambridge University Press, 2005.

[45] M.E.H. Ismail, D. Stanton and G. Viennot, The combinatorics of q-Hermite polynomials and the Askey-Wilson integral, *Europ. J. of Combinatorics*, 8 (1987) 379–392.

[46] M.E.H. Ismail and J. Zeng, Addition theorems via continued fractions, *Trans. Amer. Math. Soc*. 362 (2010), n°2, 957–983.

[47] W. B. Jones and W.J. Thron, (1990). *Continued fractions: analytic theory and applications*. In Encyclopedia of Mathematics and its Applications. Vol. 11. Addison-Wesley, New York.

[48] S. Karlin and J.L. McGregor, The differential equation of birth and death processes, and the Stieltjes moment problem. *Trans. Amer. Math. Soc.* **85** (1957),489–546.

[49] J.S. Lomont and J. Brillhart,: *Elliptic polynomials*, Chapman and Hall/CRC, Boca Raton, 2001. 16

[50] P. Leroux and G.Viennot, Combinatorial resolution of system of differential equations, I. Ordinary differential equations, in ≪ *Combinatoire énumérative* ≫, eds. G. Labelle et P. Leroux, , *Lecture Notes in Maths*. n° 1234, Springer-Verlag, Berlin, 1986, p. 210–245.

[51] O. Perron, *Die Lehre den Kettenbrüchen*, vol2, Teubner, Stuttgart, 1954.

[52] E. Roblet, *Une interprétation combinatoire des approximants de Padé*, Publication du LACIM, n°17, 1994, Montréal, http://lacim.uqam.ca/publications_pdf/17.pdf

[53] E. Roblet and X.G.Viennot, Théorie combinatoire des T-fractions et approximants de Padé en deux points, *Discrete Maths*., 153 (1996) 271–288.

[54] L.J. Rogers, On the representation of certain asymptotic series as contineud fractions, *Proc. London math. Soc*. (2) 4 (1907) 72–89.

[55] T.J. Stieltjes, Recherches sur les fractions continues, *Ann. Fac. Sc. Toulouse*, 8, n°4 (1894) J1-J122; [suite et fin] 9, n°1 (1895) A5–A47.

[56] J. Touchard, Sur un probléme de configurations et sur les fractions continues, *Canad. J. Math*. 4 (1952) 2–25.

[57] G. Viennot, Une interprétation combinatoire des coefficients des développements en série entière des fonctions elliptiques de Jacobi, *J. Combinatorial Th. A*, 29 (1980) 121–133.

[58] G. Viennot, *Une théorie combinatoire des polynômes orthogonaux*, Lecture Notes UQAM, 219p., Publication du LACIM, Université du Québec à Montréal, 1984, available on the website of the author.

[59] G. Viennot, A combinatorial theory for general orthogonal polynomials with extensions and applications, in *Orthogonal polynomials and applications* (Bar-le-Duc, 1984), Lecture Notes in Math., vol. 1171, Springer, Berlin, 1985, pp. 139–157.

[60] X.G.Viennot, A combinatorial interpretation of the quotient-difference algorithm, Proc. 12th FPSAC (*Formal Power Series and Algebraic Combinatorics*), Moscou, eds D.Krob, A.A et A.V.Mikhalev, Springer-Verlag, 2000, pp 379–390.

[61] X.G.Viennot, Alternative tableaux, permutations and partially asymmetric exclusion process, in Workshop *"Statistical Mechanics & Quantum-Field Theory Methods in Combinatorial Enumeration"*, Isaac Newton Institute for Mathematical Science, Cambridge, 23 April 2008, video and slides on http://www.newton.ac.uk/webseminars/pg+ws/2008/csm/csmw04/0423/viennot/

[62] H.S. Wall, *Analytic Theory of Continued Fractions*, Chelsea, New-York, 1967