# Universal $\mathcal{T}$ matrix for quantum groups and its applications

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## 1 Introduction

Universal $\mathcal{T}$ matrix : the concept [Fronsdal, Galindo]

(a) Lie algebra $\stackrel{\text{exponentiation}}{\longrightarrow}$ Lie groups

Coproduct for the group elements $(G)$

$$\Delta(G) = G \otimes G.$$

Can the proceess of exponentiation be generalized for a Hopf algebra?

(b) Hopf duality
Universal enveloping algebra $(\mathcal{U}) \longleftrightarrow$ Function Hopf Algebra $(\mathcal{A})$
$\sim$ Counterpart of Lie algebra $\qquad\qquad$ $\sim$ Counterpart of Lie group
$(u, v) \in \mathcal{U}$ $\qquad\qquad$ $(a, b) \in \mathcal{A}$

Nondegenerate bilinear form $\langle , \rangle : \mathcal{A} \otimes \mathcal{U} \to \mathbb{C}$

$$\begin{aligned} \langle a, uv \rangle &= \langle \Delta_{\mathcal{A}}(a), u \otimes v \rangle, & \langle ab, u \rangle &= \langle a \otimes b, \Delta_{\mathcal{U}}(u) \rangle, \\ \langle a, I_{\mathcal{U}} \rangle &= \epsilon_{\mathcal{A}}(a), & \langle I_{\mathcal{A}}, u \rangle &= \epsilon_u(u). \end{aligned}$$

$\{E_A\}, \{e^A\} \Longrightarrow$ fintely generated basis elements of $\mathcal{U}$ and $\mathcal{A}$ algebras, respectively.
Multiplication map of the $\mathcal{U}$ algebra is associated with the comultiplication map of the $\mathcal{A}$ algebra, and vice versa.

$$\begin{aligned} E_A E_B &= \sum_{A,B} f_{AB}^C E_C, & \Delta_{\mathcal{U}}(E_A) &= \sum_{B,C} g_A^{BC} E_B \otimes E_C, \\ \Delta_{\mathcal{A}}(e^A) &= \sum_{B,C} f_{BC}^A e^B \otimes e^C, & e^A e^B &= \sum_{A,B} g_C^{AB} e^C. \end{aligned}$$

(c) Universal $\mathcal{T}$ matrix (canonical element) is the capstone of the concept of duality.
$\mathcal{T} = \sum_A e^A E_A \sim$ closed form expression for the finitely generated basis elements.

Classical $q \to 1$ limit $\Rightarrow$ Lie group element

Group behaviour

$$\begin{aligned}
(\Delta_{\mathcal{A}} \otimes \mathrm{id})\mathcal{T} &= \sum_A \Delta_{\mathcal{A}}(e^A)\ E_A = \mathcal{T} \otimes \mathcal{T}, \\
(\mathrm{id} \otimes \Delta_{\mathcal{U}})\mathcal{T} &= \sum_A e^A\ \Delta_{\mathcal{U}}(E_A) = \mathcal{T} \otimes \mathcal{T}.
\end{aligned}$$

(d) Maps to the universal $\mathcal{R}$ matrix
Two noninvertible maps exist.

$$\begin{aligned}
\phi : \mathcal{A} \to \mathcal{U} \Rightarrow (\phi \otimes \mathrm{id})\mathcal{T} = \mathcal{R}, \\
\psi : \mathcal{A} \to \mathcal{U} \Rightarrow (\psi \otimes \mathrm{id})\mathcal{T} = \mathcal{R}_{21}^{-1}.
\end{aligned}$$

Each Borel subalgebra of $\mathcal{A}$ is mapped to the conjugate Borel subalgebra of $\mathcal{U}$.

# 2 $\mathcal{T}$-matrix for $SL_q(2)$ algebra

[Fronsdal, Galindo]

(a) $\mathcal{U} \sim \mathcal{U}_q(sl(2))$ algebra

$$\begin{aligned}
&[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = [2J_0]_q = \frac{q^{2J_0} - q^{-2J_0}}{q - q^{-1}}. \\
&\Delta(J_0) = J_0 \otimes 1 + 1 \otimes J_0, \qquad \Delta(J_\pm) = J_\pm \otimes q^{J_0} + q^{-J_0} \otimes J_\pm.
\end{aligned}$$

$$\text{Basis}\ \ E_{k\ell m} = J_+^k\, J_0^\ell\, J_-^m,\ \ (k, \ell, m = 0, 1, 2, ...).$$

(b) Dual $\mathcal{A} \sim SL_2(2)$ algebra
Dual basis of quantum group $\mathcal{A} \sim SL_q(2) \sim e^{k\ell m}$

$$\langle\, e^{k\ell m},\ E_{k'\ell'm'}\, \rangle = \delta^k_{k'}\, \delta^\ell_{\ell'}\, \delta^m_{m'}.$$

Generating element $e^{100} = x, \qquad e^{010} = z, \qquad e^{001} = y$
Dual algebra

$$[x, y] = 0, \qquad [z, x] = -2 \ln q\ x, \qquad [z, y] = -2 \ln q\ y.$$

Full basis set

$$e^{k\ell m} = \frac{x^k}{[k]_q!}\frac{(z-(k-m)\ln q)^\ell}{\ell!}\frac{y^m}{[m]_q!}$$

(c) Universal $\mathcal{T}$-matrix

$$\begin{aligned}
\mathcal{T} &= \sum_{k\ell m} e^{k\ell m} E_{k\ell m}, \\
&= \left(\sum_{k=0}^{\infty} \frac{(x\otimes J_+ q^{-J_0})^k}{(k)_{q^{-2}}!}\right) e^{z\otimes J_0} \left(\sum_{m=0}^{\infty} \frac{(y\otimes q^{J_0} J_-)^m}{(m)_{q^2}!}\right) \\
(n)_q &= \frac{1-q^n}{1-q}.
\end{aligned}$$

$$\mathcal{T}|_{q\to 1} = e^{x\otimes J_+}\, e^{z\otimes J_0}\, e^{y\otimes J_-} \sim \quad SL(2) \text{ group element}$$

(d) Fundamental rep

$$\mathcal{T}|_{j=\frac{1}{2}} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e^{z/2} + xe^{-z/2}y & q^{1/2}xe^{-z/2} \\ q^{-1/2}e^{-z/2}y & e^{-z/2} \end{pmatrix},$$

$\sim$ Gauss decomposition of $SL_q(2,\mathbb{C})$.
$a, b, c, d$ satisfy quantum group relation.

$$\begin{array}{lll}
ab = q^{-1}ba, & ac = q^{-1}ca, & bd = q^{-1}db \\
cd = q^{-1}dc, & bc = cb, & [a,d] = (q^{-1}-q)\,bc, \\
ad - q^{-1}bc = 1. & &
\end{array}$$

$$\Rightarrow \qquad x = q^{-1/2}bd^{-1}, \qquad y = q^{1/2}d^{-1}c, \qquad e^{z/2} = d^{-1}.$$

(e) $*$ -involution ($SU_q(2)$) $\quad q \sim$ real

$$J_\pm^* = J_\mp, \quad J_0^* = J_0$$

$$\Rightarrow \qquad SU_q(2) = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}$$

$$\begin{aligned}
\Rightarrow x^* &= -\frac{1}{1-\zeta}\, ye^{-z}, \qquad & e^{z^*/2} &= \frac{1}{1-\zeta}\, e^{-z/2}, \\
y^* &= -\frac{1}{1-\zeta}\, e^{-z}x, \qquad & \zeta &= -qxe^{-z}y = -qbc.
\end{aligned}$$

# 3 Generalized coherent states

[Aizawa, RC]
(a) spin-$j$ representation of $SU_q(2)$

$$J_\pm \, |jm\rangle = \sqrt{[j \mp m]_q [j \pm m + 1]_q} \;\; |j \; m \pm 1\rangle \, ,$$
$$J_0 \, |jm\rangle = m \, |jm\rangle \, ,$$

(b) Coherent state

$$\begin{aligned} |x, z\rangle &= \mathcal{T} \, |j \; -j\rangle \, , \\ &= e^{-jz} \sum_{n=0}^{2j} q^{-nj} \left[ \begin{array}{c} 2j \\ n \end{array} \right]_q^{1/2} x^n \, |j \; -j+n\rangle \, . \end{aligned}$$

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

classical case ($q = 1$) $\Rightarrow$ coherent state $\longleftrightarrow$ coset space $SU(2)/U(1)$
Usual choice $z \sim$ real $\quad e^z = 1 + |x|^2, \quad x^* = -y, \quad y^* = -x.$
Not possible in the $q \neq 1$ case
Contradiction $\Rightarrow \quad (1 - q^{-2})\zeta = 0 \quad x^{**} = q^{-2} x \quad y^{**} = q^{-2} y.$

(c) Unit norm
$*$-involution requires $\mathcal{T}$ to be unitary operator $\quad \mathcal{T}^\dagger \mathcal{T} = I.$
$\Rightarrow$ coherent state is of unit norm $\quad \langle x, z | x, z\rangle = 1$
explicit check

$$\begin{aligned} \langle x, z | x, z\rangle &= e^{-jz^*} e^{-jz} \;\; {}_1\phi_1 \left[ \begin{array}{c} q^{-4j} \\ q^{-4j}\zeta \end{array} ; q^2 ; \zeta \right] \sim \text{Basic hypergeometric function,} \\ &= e^{-jz^*} e^{-jz} \; \frac{(\zeta; q^2)_\infty}{(q^{-4j}\zeta; q^2)_\infty} = 1. \end{aligned}$$

(d) Resolution of unity
Normalized bi-invariant integral [Chari, Pressley]
$H : \mathcal{A} \longrightarrow \mathbb{C}$

(1) $H[\, 1_{\mathcal{A}} \,] = 1,$

(2) for any $f \in \mathcal{A}$

$$(H \otimes id)[\, \Delta(f) \,] = (id \otimes H)[\, \Delta(f) \,] = H[\, f \,].$$

$$\begin{aligned} \Rightarrow \qquad & H[\, a^\alpha b^\beta c^\gamma d^\delta \,] \neq 0, \quad \text{only if} \quad \alpha = \delta \text{ and } \beta = \gamma, \\ & H[\, b^\beta c^\gamma d^{-\delta} a^{-\alpha} \,] \neq 0, \quad \text{only if} \quad \alpha = \delta \text{ and } \beta = \gamma. \end{aligned}$$

$$\Rightarrow H[\,\zeta^n\,] = \frac{q^{2n}}{(n+1)_{q^2}}.$$

$$\begin{aligned}
&H[\ |x,z\rangle\langle x,z|\ ] \\
&\quad = \sum_{n=0}^{2j} q^{2jn-n(n+2)} \begin{bmatrix} 2j \\ n \end{bmatrix}_q H[\,(\zeta;q^2)_{2j-n}\zeta^n\,]\,|j\ \ -j+n\rangle\langle j\ \ -j+n|\,, \\
&\quad = \frac{1}{(2j+1)_{q^2}} \sum_{n=0}^{2j} |j\ \ -j+n\rangle\langle j\ \ -j+n|\,.
\end{aligned}$$

$$\Rightarrow (2j+1)_{q^2}\ H[\ |x,z\rangle\langle x,z|\ ] = \mathbb{I}$$

$\Rightarrow$ Any state in the spin-$j$ rep can be expanded in the coherent state basis

$$\begin{aligned}
|c\rangle &= \sum_{m=-j}^{j} c_m\,|jm\rangle = \sum_{m=-j}^{j} c_m\,(2j+1)_{q^2} H[\ |x,z\rangle\langle x,z|\ ]\,|jm\rangle\,, \\
&= (2j+1)_{q^2}\ H[\ |x,z\rangle \sum_{m=-j}^{j} c_m\,\langle x,z\,|\,jm\rangle\ ].
\end{aligned}$$

$$\langle x,z\,|\,jm\rangle = q^{j(j+m)} \begin{bmatrix} 2j \\ j+m \end{bmatrix}_q^{1/2} e^{-jz^*}(x^*)^{j+m}.$$

(e) Coherent state $\longrightarrow SU_q/U(1) \sim S_q^2$
$\sim$ Podleś $q$ sphere
$\sim$ quantum homogenous space.

$$\begin{aligned}
x_1 &= -q\frac{\sqrt{[2]_q}}{[2j]_q}\langle x,z|\,J_+ q^{-J_0}\,|x,z\rangle = -q\sqrt{[2]_q}(1-\zeta)x^*, \\
x_0 &= 1 - q\frac{[2]_q}{[2j]_q}(\langle x,z|\,q^{-J_0}\,[J_0]_q\,|x,z\rangle + q^j\,[j]_q) = 1 - q^{-1}\,[2]_q\,\zeta, \\
x_{-1} &= \frac{\sqrt{[2]_q}}{[2j]_q}\langle x,z|\,q^{-J_0} J_-\,|x,z\rangle = \sqrt{[2]_q}x(1-\zeta).
\end{aligned}$$

Noncommutative algebra

$$\begin{aligned}
&x_0^2 - q^{-1}x_1x_{-1} - qx_{-1}x_1 = 1, \\
&(1-q^{-2})\,x_0^2 + q^{-1}x_{-1}x_1 - q^{-1}x_1x_{-1} = (1-q^{-2})\,x_0, \\
&x_{-1}x_0 - q^{-2}x_0x_{-1} = (1-q^{-2})\,x_{-1}, \\
&x_0x_1 - q^{-2}x_1x_0 = (1-q^{-2})\,x_1.
\end{aligned}$$

$\sim$ special case of Podleś $q$ sphere.
$*$-involution $x_1^* = -q x_{-1}, x_0^* = x_0, x_{-1}^* = -q^{-1} x_1$.
$\sim$ left-covariant diffferential structure may also be derived.

# 4 Universal $\mathcal{T}$-matrix for $OSp_q(1/2)$ quantum supergroup

(a) Universal enveloping algebra $\mathcal{U} \sim \mathcal{U}_q(osp(1/2))$
generating elements $H$ (parity even) $V_\pm$ (parity odd)

$$[H, V_\pm] = \pm \frac{1}{2} V_\pm, \qquad \{V_+, V_-\} = -\frac{q^{2H} - q^{-2H}}{q - q^{-1}} \equiv - [2H]_q \, .$$

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \qquad \Delta(V_\pm) = V_\pm \otimes q^{-H} + q^H \otimes V_\pm,$$

Universal $\mathcal{R}$ -matrix [Kulish, Reshetikhin]

$$\mathcal{R} = q^{4H \otimes H} \sum_{k \geq 0} \frac{(q - q^{-1})^k q^{-k/2}}{\left(k\right)_{q^{-1}}!} (q^H V_+ \otimes q^{-H} V_-)^k,$$

$$\left(k\right)_q = \frac{1 - (-1)^k \, q^k}{1 + q}$$

$\Rightarrow$ satisfies Yang-Baxter equation

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}$$

Basis set for $\mathcal{U}$ $\Rightarrow$ $E_{klm} = V_+^k H^l V_-^m, (k, l, m) = 0, 1, 2, \ldots$

(b) Universal $\mathcal{T}$-matrix [Aizawa, RC, Naina Mohammed, Segar]
Generating elements of the dual $\mathcal{A} = OSp_q(1/2)$ algebra:
$x = e^{100}, y = e^{001}$ and $z = e^{010}$,
Basis of $\mathcal{A}$ algebra $e^{k\ell m}$.

$$\langle e^{k\ell m}, E_{k'\ell'm'} \rangle = \delta^k_{k'} \delta^\ell_{\ell'} \delta^m_{m'}.$$

Algebra of $\mathcal{A}$

$$\{x, y\} = 0, \qquad [z, x] = 2 \ln q \; x, \qquad [z, y] = 2 \ln q \; y.$$

$$\text{Set of dual basis} \quad e^{k\ell m} = \frac{x^k}{[k]_q!} \frac{(z - (k - \ell) \ln q)^\ell}{\ell!} \frac{y^m}{[m]_{q^{-1}}!}.$$

$$\{m\}_q = \frac{q^{-m/2} - (-1)^m q^{m/2}}{q^{-1/2} + q^{1/2}}.$$

Universal $\mathcal{T}$ matrix for $OSp_q(1/2) \Rightarrow$

$$\begin{aligned}
\mathcal{T}_{e,E} &= \sum_{k\ell m} e^{k\ell m} \otimes E_{k\ell m} \\
&= \left( \sum_{k=0}^{\infty} \frac{1}{\Big(k\Big)_q!} (x \otimes V_+ q^H)^k \right) e^{z\otimes H} \left( \sum_{m=0}^{\infty} \frac{1}{\Big(m\Big)_{q^{-1}}!} (y \otimes q^{-H} V_-)^m \right).
\end{aligned}$$

(c) Classical $q \to 1$ limit of the universal $\mathcal{T}$ matrix

Odd generators are nilpotent

$$\begin{array}{ll}
x^2 \stackrel{q\to 1}{\longrightarrow} 0, & y^2 \stackrel{q\to 1}{\longrightarrow} 0, \\
\lim_{q\to 1} (2n)_q \to n(1-q), & \lim_{q\to 1} (2n+1)_q \to 1, n = 0, 1, 2, \ldots.
\end{array}$$

$$\lim_{q\to 1} \frac{x^2}{q-1} = \mathfrak{x}, \qquad \lim_{q\to 1} \frac{y^2}{q^{-1}-1} = \mathfrak{y},$$

$\mathfrak{x}, \mathfrak{y} \sim$ parity-even elements
Classical limit $\Rightarrow$

$$\mathcal{T}_{e,E}\Big|_{q\to 1} = (1\otimes 1 + x \otimes V_+) \, \exp(\mathfrak{x} \otimes V_+^2) \, \exp(z \otimes H) \exp(\mathfrak{y} \otimes V_-^2) \, (1 \otimes 1 + y \otimes V_-).$$

The elements $(V_\pm^2, H)$ of the classical $osp(1/2)$ algebra form a $sl(2)$ subalgebra.

(d) Maps to the universal $R$-matrix

Two maps $\Rightarrow$

(1) $\Phi : \mathcal{A} \to \mathcal{U}$

$$\Phi(x) = 0, \qquad \Phi(z) = (4 \ln q) H, \qquad \Phi(y) = q^{-1/2}(q - q^{-1}) q^H V_+.$$

$(\Phi \otimes \mathrm{id})(\mathcal{T}_{E,e}) = \mathcal{R}$,

(2) $\Psi : \mathcal{A} \to \mathcal{U}$

$$\Psi(x) = (q^{-1} - q) \, q^{-H} V_-, \qquad \Psi(z) = (-4 \ln q) H, \qquad \Psi(y) = 0.$$

$(\Psi \otimes \mathrm{id})(\mathcal{T}_{E,e}) = \mathcal{R}_{21}^{-1}$,
Maps are *not* invertible.
Each Borel subalgebra of $\mathcal{A}$ is mapped to the conjugate Borel subalgebra of $\mathcal{U}$.

(e) Irrep $V^{(\ell)}$ and the matrix elements

$V^{(\ell)} \Rightarrow \{\ e^\ell_m(\lambda) \mid m = \ell, \ell-1, \cdots, -\ell\ \}$,
$\lambda = 0, 1$ $\sim$ Parity of the highest weight state

$$H e^\ell_m(\lambda) = \frac{m}{2} e^\ell_m(\lambda),$$
$$V_+ e^\ell_m(\lambda) = \left( \frac{1}{\{2\}_q} \{\ell - m\}_q \{\ell + m + 1\}_q \right)^{1/2} e^\ell_{m+1}(\lambda),$$
$$V_- e^\ell_m(\lambda) = (-1)^{\ell-m-1} \left( \frac{1}{\{2\}_q} \{\ell + m\}_q \{\ell - m + 1\}_q \right)^{1/2} e^\ell_{m-1}(\lambda),$$

Matrix elements of the $\mathcal{T}$-matrix $\Rightarrow$

$$\begin{aligned} T^\ell_{m'm}(\lambda) &= (e^\ell_{m'}(\lambda), \mathcal{T}_{e,E}\, e^\ell_m(\lambda)) \\ &= N^l_{m'm}(q)\ x^{m'-m}\ e^{mz/2}\ P^\ell_{m'm}(\zeta) \end{aligned}$$

The polynomial reads

$$P^\ell_{m'm}(\zeta) = \sum_c (-1)^{c(\ell-m)+c(c-1)/2} q^{-c(m'+m-1)/2} \times \frac{\{m'-m\}_q!\,\{l+m\}_q!\,\{\ell-m+c\}_q!}{\{m'-m+c\}_q!\,\{\ell+m-c\}_q!\,\{\ell-m\}_q!\,\{c\}_q!} \zeta^c$$

Polynomial is expressed in terms of basic hypergeometric functions $\Rightarrow$

$$\begin{aligned} P^\ell_{m'm}(\zeta) &= \sum_a \frac{((-q)^{-\ell-m}; -q)_a\, ((-q)^{\ell-m+1}; -q)_a}{((-q)^{m'-m+1}; -q)_a\, (-q; -q)_a} (-q\zeta)^a \\ &= {}_2\phi_1 \left( (-q)^{(-l-m)}, (-q)^{(l-m+1)}; (-q)^{(m'-m+1)}; -q; -q\zeta \right) \end{aligned}$$

$\sim$ little $(-q)$ - Jacobi polynomial.
$q$-Jacobi polynomial appears for the matrix elements of the $sl_q(2)$ algebra.
$(-q)$ - Jacobi Polynomial appears for the matrix elements of the $OSp_q(1/2)$ algebra.

(f) Gaussian decomposition of the fundamental representation of $OSp_q(1/2)$

$$\begin{aligned} \mathcal{T}_{e,E}\big|_{l=1} &= \begin{pmatrix} 1 & 0 & 0 \\ -q^{-1/2}x & 1 & 0 \\ -\frac{1}{q\{2\}_q}x^2 & x & 1 \end{pmatrix} \begin{pmatrix} e^{-z/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{z/2} \end{pmatrix} \begin{pmatrix} 1 & q^{1/2}y & -\frac{q}{\{2\}_q}y^2 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \\ &\sim \begin{pmatrix} a & \alpha & b \\ \gamma & e & \beta \\ c & \delta & d \end{pmatrix} \quad \sim \text{quantum supermatrix} \end{aligned}$$

(g) Vector representation ($\ell = 2$) and $OSp_q(1/2)$ sphere

$$T^2(0) = \begin{pmatrix} a^2 & \kappa_1 a\alpha & \kappa_3 ab & \kappa_1 \alpha b & b^2 \\ \kappa_1 a\gamma & ae + q^{-1}\gamma\alpha & \kappa_2(a\beta + q^{-1}\gamma b) & -\alpha\beta + q^{-1}eb & \kappa_1 b\beta \\ \kappa_3 ac & \kappa_2(a\delta + c\alpha) & ad + q^{-1}[2]\alpha\delta + q^{-2}bc & \kappa_2(\alpha d + \delta b) & \kappa_3 bd \\ \kappa_1 \gamma c & \gamma\delta + q^{-1}ce & \kappa_2(\gamma d + q^{-1}c\beta) & ed + q^{-1}\beta\delta & \kappa_1 \beta d \\ c^2 & \kappa_1 c\delta & \kappa_3 cd & \kappa_1 \delta d & d^2 \end{pmatrix},$$

where

$$\kappa_1 = \sqrt{\frac{[4]}{q[2]}} \qquad \kappa_2 = \sqrt{q^{-1}[3]} \qquad \kappa_3 = \kappa_1\kappa_2.$$

$\mathcal{A}$-covariant algebra $\Rightarrow$
Algebraic relations are covariant under the right coaction of $\mathcal{A}$ on the vector space $V$.
$\varphi_R : V \to V \otimes \mathcal{A}$

$$(\varphi_R \otimes \mathrm{id}) \circ \varphi_R = (\mathrm{id} \otimes \Delta) \circ \varphi_R, \qquad (\mathrm{id} \otimes \epsilon) \circ \varphi_R = \mathrm{id}.$$

One way of constructing $\mathcal{A}$-covariant spaces $\Rightarrow$

- Assume a multiplication map on $V^{(\ell)}$

$$\mu(f \otimes g) = fg; \qquad f, g \in V^{(\ell)}.$$

- Consider Clebsch-Gordan decomposition

$$e^L_M(\ell, \ell, \Lambda) = \sum_{m_1 m_2} C^{\ \ell\ \ \ \ell\ \ \ L}_{m_1\, m_2\, M} e^\ell_{m_1}(\lambda) \otimes e^\ell_{m_2}(\lambda)$$
$$\Rightarrow E^L_M \equiv \mu(e^L_M(\ell, \ell, \Lambda)) = \sum_{m_1, m_2} C^{\ \ell\ \ \ \ell\ \ \ L}_{m_1\, m_2\, M} e^\ell_{m_1}(\lambda) e^\ell_{m_2}(\lambda),$$

- right coaction on $E^L_M(\Lambda)$ is

$$\varphi_R(E^L_M) = \sum_{M'} E^L_{M'} \otimes T^L_{M'M}(\Lambda).$$

- $L = 0 \ \Rightarrow$ Scalar relation

$$\begin{aligned} \varphi_R(E^0_0) &= E^0_0, \\ \Rightarrow E^0_0 &= \sum_{m_1, m_2} C^{\ \ell\ \ \ \ell\ \ \ 0}_{m_1\, m_2\, 0} e^\ell_{m_1}(\lambda) e^\ell_{m_2}(\lambda) = r \end{aligned}$$

- $L = l \ \Rightarrow$ Identical transformation for $E^\ell_m(\lambda)$ and $e^\ell_m(\lambda)$

$$\Rightarrow E^\ell_m(\lambda) = \sum_{m_1, m_2} C^{\ \ell\ \ \ \ell\ \ \ L}_{m_1\, m_2\, M} e^\ell_{m_1}(\lambda) e^\ell_{m_2}(\lambda) = \xi e^\ell_m(\lambda),$$

$\xi \to 0$ when $q \to 1$.

- $L \neq 0, l \Rightarrow$ $\quad E^\ell_m(\Lambda)$ and $e^\ell_m(\lambda)$ transform differently.

$$\Rightarrow E^\ell_m(\Lambda) = \sum_{m_1, m_2} C^{\ \ell\ \ \ \ell\ \ \ L}_{m_1\, m_2\, M} e^\ell_{m_1}(\lambda) e^\ell_{m_2}(\lambda) = 0,$$

$\Rightarrow$ At most two parametric $(r, \xi)$ deformed space.

Example 1: $\ell = 1$ case $z_m = e^1_m(0)$, $z_{\pm 1} \to$ parity even, $z_0 \to$ parity odd
$1 \otimes 1 = 2 \oplus 1 \oplus 0$

$$q^{1/2} z_{-1} z_1 + z_0^2 - q^{-1/2} z_1 z_{-1} = r.$$

$$\begin{aligned}
&-q^{1/2} z_0 z_1 + q^{-1/2} z_1 z_0 = \xi z_1, \\
&z_{-1} z_1 + (q^{-1/2} + q^{1/2}) z_0^2 - z_1 z_{-1} = \xi z_0, \\
&q^{1/2} z_{-1} z_0 - q^{-1/2} z_0 z_{-1} = \xi z_{-1}.
\end{aligned}$$

$\Rightarrow$ Covariant under the coaction of $OSp_q(1/2)$
Consistency requirements $\Rightarrow$

(a) The constant $r$ commutes with all generators

(b) Product of three generators, say $z_1 z_0 z_{-1}$, has two ways of reversing its ordering:

$$\begin{array}{ccccccc}
 & & z_1 z_0 z_{-1} & \longrightarrow & z_1 z_{-1} z_0 & & \\
 & \nearrow & & & & \searrow & \\
z_0 z_1 z_{-1} & & & & & & z_{-1} z_1 z_0. \\
 & \searrow & & & & \nearrow & \\
 & & z_0 z_{-1} z_1 & \longrightarrow & z_{-1} z_0 z_1 & &
\end{array}$$

Both possibilities must give identical answers.

Satisfying the second requirement needs the choice $\xi = 0$
$\Rightarrow$ Corresponding noncommutative space

$$\begin{aligned}
&z_1 z_0 = q z_0 z_1, \qquad z_0 z_{-1} = q z_{-1} z_0, \\
&z_1 z_{-1} = q^2 z_{-1} z_1 - q(q^{-1/2} + q^{1/2}) r, \\
&z_0^2 = -q^{-1}[2] z_1 z_{-1} - q^{-1} r.
\end{aligned}$$

Example 2: $L = 2$ case $\Rightarrow$ Decomposition $2 \otimes 2 = 4 \oplus 3 \oplus 2 \oplus 1 \oplus 0$
Basis $Y_m = e^2_m(0)$, where $m = 0, \pm 1, \pm 2$; $m = 0, \pm 2$ even parity, $m = \pm 1$ odd parity
$\Rightarrow$ radial element $L = 0$

$$\Rightarrow q^{-1} Y_2 Y_{-2} - q^{-1/2} Y_1 Y_{-1} - Y_0^2 + q^{1/2} Y_{-1} Y_1 + q Y_{-2} Y_2 = r,$$

$\Rightarrow$ (1) To define the algebra 10 commutation relations, 2 equations specifying $Y^2_{\pm 1}$, and 3 relations coming from consistency conditions are needed. They appear from $L = 3, 2, 1$ relations. The $L = 4$ relations are rejected as they do not have the correct classical limits.
$\Rightarrow$ (2) $OSp_q(1/2)$ supersphere algebra can be embedded in the function algebra $\mathcal{A}$

# 5 Conclusion

• Universal $\mathcal{T}$-matrix for the quantum groups allows for $q$-generalizations of the Perelomov type of coherent states. For the $SU_q(2)$ case the decomposition of unity and a new complex structure on the quantum homogeneous space have been obtained explicitly. It should be posible to obtain star product structure on complexified Podleś sphere using these coherent states.

• In the general case the braiding relation puts extra conditions on the covariant noncommutative spaces. The role of the modified braid relations needs to be investigated in this context.