SPONTANEOUS SYMMETRY BREAKING IN TWISTED NC QUANTUM THEORIES

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Based on work with

- A P Balachandran & T R Govindarajan
  arXiv 0901 xxx
- Balachandran, Pinzul, Qureshi & S.V.
  arXiv 0708.0069

- Motivation(s)
- Twisted QFTs
- Gauge theories
- Spontaneous symmetry breaking
- (Towards) Standard Model
- $(LSZ)_\theta$
- Outlook

- Arguments for noncommutativity

DFR (1995)

Uncertainty relations from standard QM, when taken together with considerations from classical GR, inexorably lead to black hole formation.

Postulate emergence of new physics near Planck scale : spacetime coordinates give a noncommutative algebra

$$[x_\mu, x_\nu] = i Q_{\mu\nu}$$

For $Q_{\mu\nu} = \theta_{\mu\nu} + \cdots$

To leading order, we get the GM plane.

- From string theory : "Low energy" dynamics of certain brane configurations is given fields theories on nc spaces.

- In QHE – guiding centre coordinates of the electron become noncommutative:

$$[x, y] = -\frac{i}{B}$$

We restrict to Moyal algebra $\mathcal{A}_\theta(\mathbb{R}^N)$ henceforth :

Elements of the Moyal algebra can be represented by functions on $\mathbb{R}^N$, but with a new multiplication rule :

$$f * g \equiv f \, e^{i/2\, \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} g$$

$$= m_0 \left[\underbrace{e^{i/2\, \theta^{\mu\nu} \partial_\mu \otimes \partial_\nu}}_{\mathcal{F}} f \otimes g \right] \equiv m_\theta (f \otimes g)$$

Lorentz (or Poincaré) transformations are implemented by a deformed coproduct

$$\Delta_\theta(\Lambda) = \mathcal{F}^{-1} \underbrace{\Delta_0(\Lambda)}_{\hookrightarrow\ \Lambda \otimes \Lambda} \mathcal{F}$$

Easy to check that

$$m_\theta(\Delta_\theta(\Lambda) \, f \otimes g) = \rho(\Lambda) \, f * g$$

Under infinitesimal transformations,

$$\Delta_\theta(M_{\mu\nu}) = M_{\mu\nu} \otimes \mathbb{1} + \mathbb{1} \otimes M_{\mu\nu}$$
$$- \frac{1}{2} \Big((p \cdot \theta)_\mu \otimes p_\nu - p_\nu \otimes (p \cdot \theta)_\mu \quad - (\mu \leftrightarrow \nu) \Big)$$

Here $(p \cdot \theta)_\mu \equiv p_\lambda \theta^\lambda{}_\mu$

This modified Leibnitz rule has an alternative explanation :

Since $\mathcal{A}_\theta(\mathbb{R}^N)$ is noncommutative, we must distinguish between left and right multiplication :

$$\hat{x}_\mu^L f(x) = x_\mu * f,$$

$$\hat{x}_\mu^R f(x) = f * x_\mu$$

These satisfy

$$[\hat{x}_\mu^L, \hat{x}_\nu^L] = i\theta_{\mu\nu} = -[\hat{x}_\mu^R, \hat{x}_\nu^R], \text{ and}$$

$$[\hat{x}_\mu^L, \hat{x}_\nu^R] = 0$$

Define a (commuting) $\hat{x}_\mu^c$ as

$$\hat{x}_\mu^c = \frac{1}{2}(\hat{x}_\mu^L + \hat{x}_\mu^R)$$

These satisfy $[\hat{x}_\mu^c, \hat{x}_\nu^c] = 0$!!

$$\hat{x}_\mu^c f(x) = \frac{1}{2}(x_\mu * f + f * x_\mu) = x_\mu \cdot f(x)$$

So :

1. $\hat{x}_\mu^c$ generate the commutative algebra $\mathcal{A}_0(\mathbb{R}^N)$
2. $\mathcal{A}_0(\mathbb{R}^N)$ has a natural action on $\mathcal{A}_\theta(\mathbb{R}^N)$.

Look at Lorentz transformations again:

$$f(x) \longrightarrow f^{\wedge}(x) = f(\Lambda^{-1} x)$$

For $\Lambda \simeq \mathbb{1} - i\,\epsilon^{\mu\nu} M_{\mu\nu}$,

$$f^{\wedge}(x) \simeq f(x) - i\,\epsilon^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu) f$$

no star here!

So $M_{\mu\nu}$ must be identified as

$$M_{\mu\nu} = \hat{x}_\mu^{\,*} p_\nu - \hat{x}_\nu^{\,*} p_\mu$$

Immediate consequences:

- $M_{\mu\nu}$ satisfy the standard Poincaré comm. relations (structure constants are same). Recover usual representation theory. Wigner's classification of particles according to UIRs is available again.

- But $M_{\mu\nu}$ as a vector field, obeys modified Leibnitz rule:

$$M_{\mu\nu}(\alpha * \beta) = (M_{\mu\nu}\alpha) * \beta + \alpha * (M_{\mu\nu}\beta) - \frac{1}{2}\Big((\hat{p}\cdot\theta)_\mu \alpha * \hat{p}_\nu \beta - \hat{p}_\nu \alpha * (p\cdot\theta)_\mu \beta - (\mu \leftrightarrow \nu)\Big)$$

Exactly same as Drinfel'd twist!!

The use of $\hat{x}^c$ is a possible starting point for discussing gravity:

Inf. diffeos are generated by vector fields

$$\hat{v}\, f(x) = v^\mu(x)\, \partial_\mu f \quad \text{(in usual case)}$$

$$= v^\mu(\hat{x}^c)\, \partial_\mu f \quad \text{for us.}$$

Algebra of diffeomorphisms = same as before.

In standard treatment, when gauge fields are present, the group of 'symmetries' is

$$\underbrace{\mathcal{G}}_{\text{gauge group}} \ltimes \underbrace{\mathcal{D}_0}_{\text{diffeos.}}$$

We take the position that this semi-direct product structure should be maintained when $\theta \neq 0$ as well.

Require that elts of $\mathcal{G}$ depend on $\hat{x}^c$ only.

i.e. The abstract group $\mathcal{G}$ is independent of $\theta^{\mu\nu}$.

Charged matter fields (scalars, fermions, ..)
are modules over $\mathcal{A}_\theta(\mathbb{R}^n)$. (or $\mathcal{A}_\theta \otimes \mathbb{C}^q$)

The coproduct

$$\Delta_\theta(g(\hat{x}^i)) \equiv \mathcal{F}^{-1}\,\Delta_0(g(\hat{x}^i))\,\mathcal{F}$$

implements gauge tr^{n}s in a manner compatible with the module str.

We can also form gauge scalars, composite operators (like Yukawa terms) in a manner suitable with twisted gauge tr^{n}s.

What about covariant derivatives ?

$$\phi(x) = \int d\mu_p \,(a_p\, e^{-ip\cdot x} + b_p^+\, e^{ip\cdot x})$$

charged scalar field

a's, b's are twisted

$$a_p a_q = e^{ip\wedge q}\, a_q a_p \qquad \text{etc.}$$

Using the Fock space momentum operator, we can write this in terms of the com. commutative charged field :

$$P_\mu = \int d\mu_q \, q_\mu \left(a_q^+ a_q + b_q^+ b_q \right)$$

Then

$$\Phi(x) = [\Phi_c(x)] e^{\frac{1}{2}\overleftarrow{\partial} \wedge P}$$

If Φ_1 & Φ_2 are two such fields, then

$$(\Phi_1 * \Phi_2)(x) = [\Phi_{1c}(x) \cdot \Phi_{2c}(x)] \, e^{\frac{1}{2}\overleftarrow{\partial} \wedge P}$$

On a single field Φ, the cov. derivative acts as

$$D_\mu \Phi = \partial_\mu \Phi + A_\mu \Phi$$

no star!

D_μ transports Φ, and we want to preserve the above property - we want the quantum D_μ to preserve statistics

A natural choice is

$$D_\mu \Phi = (D_\mu^c \, \Phi_c) \, e^{\frac{1}{2}\overleftarrow{\partial} \wedge P}$$

$[D_\mu, D_\nu]$ gives the curvature :

$$[D_\mu, D_\nu]\,\phi = \left([D_{\mu c}, D_{\nu c}]\,\phi_c\right) e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$$
$$= \left(F_{\mu\nu,c}\,\phi_c\right) e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$$

We can use this $F_{\mu\nu,c}$ to construct the Hamiltonian of gauge theory.

Pure YM theory in Moyal plane is same as its comm. counterpart.

Not so when matter is included.

Interaction Hamiltonian

$$H_\theta^I = \int d^3x \left[\mathcal{H}_\theta^{M-G} + \mathcal{H}_\theta^{G} \right]$$

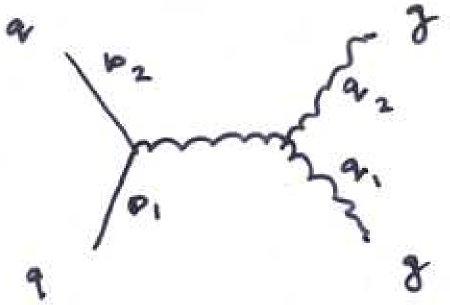
terms like $\bar{\psi} * A\!\!\!/ \psi$

terms like A^3, A^4

$$\mathcal{H}_\theta^G = \mathcal{H}_0^G \quad \text{but}$$
$$\mathcal{H}_\theta^{M-G} \neq \mathcal{H}_0^{M-G}$$

So scattering processes involving both terms will show effects of noncommutativity.

Eg $qg \to qg$ in QCD



We get a frame dependent propagator - breaking of Lorentz invariance!

Seems to be a generic feature of non-Abelian gauge theories.

To study this better, as well as look for signatures from phenomenology, we need to look at the Standard Model.

Understand spontaneous symmetry breaking.

Develop LSZ to do scattering theory.

Let us first see how global symmetries break:

$\phi_i(x)$ multiplet transforming under $D(g)$ as

$$\phi_i(x) \longrightarrow \phi_i^g(x) = D_{ij}(g)\,\phi_j(x)$$

If this is a symmetry, then the quantum charges Q^a commute with the Hamiltonian H:

$$[\,Q^a, H\,] = 0, \qquad a = 1, \dots \dim G.$$

These charges arise from conserved currents J_μ^a

$$\partial_\mu J^{\mu a} = 0\,, \qquad J^{\mu a} = J^{\mu a}(\phi_i, \partial\phi_i)$$

Given a commutative theory with conserved currents $J_{(0)}^{\mu a}$, the corres. nc. theory (obtained by replacing $\phi_{ic}(x) \rightarrow \phi_i(x) = \phi_{ic}(x)\, e^{\frac{1}{2}\overleftarrow{\partial}\wedge\overrightarrow{\partial}}$)

has currents $J_{(\theta)}^{\mu a} = J_{(0)}^{\mu a}\, e^{\frac{1}{2}\overleftarrow{\partial}\wedge\overrightarrow{\partial}}$

that are also conserved:

$$\partial_\mu J_{(\theta)}^{\mu a} = \partial_\mu \left(J_{(0)}^{\mu a}\, e^{\frac{1}{2}\overleftarrow{\partial}\wedge\overrightarrow{\partial}}\right) = \left(\partial_\mu J_0^{\mu a}\right) e^{\frac{1}{2}\overleftarrow{\partial}\wedge\overrightarrow{\partial}}$$
$$= 0$$

As a result, there are conserved charges

$$Q_{\theta}^{a} \equiv \int d^3x \; J_{\theta}^{0a}$$

$$= \int d^3x \; J_{(0)}^{0,a} \, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P} = \int d^3x \; J_{(0)}^{0,a} = Q_{(0)}^{a}$$

$Q_{(0)}^{a}$ generate infinitesimal symmetry transform^n's:

$$[\, Q_{(0)}^{a},\; \phi_i(x)\,] = \sum_j T_{ij}^{a}\, \phi_j(x)$$

just as in the comm. case.

We need this to prove Goldstone's th^m in nc spacetime.

Consider

$$\langle 0 | [\, J_{\theta}^{a,\mu}(y),\; \phi_i(x)\,] | 0 \rangle$$

Using the map $J_{\theta}^{a,\mu} = e^{\frac{1}{2}\overrightarrow{\partial}\wedge P} J_{0}^{a,\mu}$

& $\phi_i = \phi_{ic}\, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$, we can

write this as

$$\langle 0 | [\, e^{\frac{1}{2}\overrightarrow{\partial}_y \wedge P} J_{(0)}^{a,\mu}(y),\; \phi_{ic}(x)\, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}\,] | 0 \rangle$$

Since $P_\mu|0\rangle = 0$, it is easy to show

that

$$\langle 0|[J^{a,\mu}_{(0)}(y),\ \phi_i(x)]|0\rangle = \langle 0|[J^{a,\mu}_{(0)}(y),\phi_{ic}(x)]|0\rangle$$

$$\equiv \text{ comm. case.}$$

We can now use standard arguments using spectral density & Poincaré invariance, to show that there are massless bosons in the broken phase.

Write the $\langle 0|$commutator$|0\rangle$ as

$$\langle 0|[J^{a,\mu}_{(0)}(y),\ \phi_{ic}(x)]|0\rangle = \int d\mu_p \Big[\rho_i^{a,\mu}(p) e^{-ip\cdot(y-x)} - \tilde{\rho}_i^{a,\mu}(p)\, e^{ip\cdot(y-x)} \Big]$$

where spectral densities ρ, $\tilde{\rho}$ are

$$\rho_i^{a,\mu}(p) = \sum_N \langle 0|J^{a,\mu}_{(0)}(0)|N\rangle\langle N|\phi_{ic}(0)|0\rangle\, \delta^4(p-p_N)$$

$$\tilde{\rho}_i^{a,\mu}(p) = \sum_N \langle 0|\phi_{ic}(0)|N\rangle\langle N|J^{a,\mu}_{(0)}(0)|0\rangle\, \delta^4(p-p_N)$$

By Lorentz invariance,

$$\rho_i^{a,\mu}(p) = p^\mu \rho_i^a(p^2)\,\theta(p^0)$$

$$\tilde{\rho}_i^{a,\mu}(p) = p^\mu \tilde{\rho}_i^a(p^2)\,\theta(p^0)$$

So

$$\langle 0 | [\quad J | 0 \rangle = \frac{\partial}{\partial y_\mu} \int dM^2 \Big(\rho_i^a(M^2)\, D(y-x; M^2) + \tilde{\rho}_i^a(M^2)\, D(x-y, M^2) \Big)$$

standard causal propagator $\int d\mu_p\, e^{-ip\cdot x}$

But $D(x; M^2) = D(-x; M^2)$ for x spacelike

so

$$\langle 0 | [\quad J | 0 \rangle = \frac{\partial}{\partial y_\mu} \int dM^2 \Big(\rho_i^a(M^2) + \tilde{\rho}_i^a(M^2) \Big) D(y-x; M^2)$$

$= 0$ for spacelike separations

so $\rho_i^a(M^2) = -\tilde{\rho}_i^a(M^2)$

$$\langle 0 | [\quad J | 0 \rangle = \frac{\partial}{\partial y_\mu} \int dM^2 \, \rho_i^a(M^2) \Big(-D(x-y; M^2) + D(y-x; M^2) \Big)$$

Since $\frac{\partial}{\partial y_\mu} J^\mu = 0$, we get

$$0 = \int dM^2 \, M^2 \, \rho_i^a(M^2) \Big(D(y-x; M^2) - D(x-y; M^2) \Big)$$

$$\Rightarrow \quad M^2 \rho_i^a(M^2) = 0$$

When the symm. is broken:

take $\mu = 0$, $x^0 = y^0 = t$

$$\langle 0 | [J_{(0)}^{a,0} (\vec{y}, t), \phi_{ic}(\vec{x}, t)] | 0 \rangle = i \delta(\vec{y} - \vec{x}) \cdot \int dM^2 \, \rho_i^a(M^2)$$

Integrate wrt y, to get the charge Q.

Then commutator gives the change in ϕ_{ic}

$$- \langle 0 | \sum_j T^a_{ij} \phi_{jc}(x) | 0 \rangle = i \int \rho_i^a (M^2)$$

This is OK only if

$$\rho_i^a(M^2) = i \, \delta(M^2) \sum_j T^a_{ij} \langle 0 | \phi_{jc}(0) | 0 \rangle$$

As long as symmetry is broken,

$\rho_i^a \propto \delta(M^2)$

$\Rightarrow$ Theory has massless particles.

Simple counting shows that we need massless particle(s) with same quantum no.s as $J_{(0)}^{a,0}$, and as many as the no. of broken symm.

What about spontaneously broken local symmetries?

Basic idea – use the "twist map"

$$\varphi = \varphi_c \, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$$

& just push it thru' the general SSB arg^t.

$$D_\mu \varphi = (D_{\mu c}\, \varphi_c)\, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$$

&

$$(D_\mu \varphi) * (D_\mu \varphi) = (D_{\mu c}\, \varphi_c)^2\, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$$

For SSB,

$$\varphi_c = g_c \langle \varphi_c \rangle \quad (\text{for } U(1) \text{ case})$$

$$D_{\mu c}\varphi_c = (D_{\mu c}\, g_c) \langle \varphi_c \rangle$$

$$(D_\mu \varphi) = (D_{\mu c}\, g_c) \langle \varphi_c \rangle\, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$$

W boson:

$$W_{\mu c} = (g_c^{-1} D_{\mu c}\, g)$$

$$W_\mu = (\quad)\, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$$

Kinetic term for W_μ comes from

$$(D_\mu W_\nu - D_\nu W_\mu)^2 = (D_{\mu c} W_{\nu c} - D_{\nu c} W_{\mu c})^2 e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$$

So massive vector boson that correctly deformed statistics.

What about observable effects?

In usual scattering theory, we split H as

$$H = H_0 + H_I$$

$H_0 \equiv$ used to define states in distant past or far future.

$$e^{-iHT_-} |\psi, in\rangle \xrightarrow{T_- \to -\infty} e^{-iH_0T_-} |\psi; free\rangle$$

$$e^{-iHT_+} |\psi, out\rangle \xrightarrow{T_+ \to \infty} e^{-iH_0T_+} |\psi; free\rangle$$

So

$$|\psi; out\rangle = \Omega_- \Omega_+^+ |\psi; in\rangle$$

$$\Omega_\pm \equiv e^{iHT_\mp} e^{-iH_0T_\pm}$$ Moller operators

Scattering amplitude $\langle out | in \rangle$

$$= \langle in | \underbrace{\Omega_+ \Omega_-^+}_{S \text{ matrix.}} | in \rangle$$

LSZ gives S directly i.t.o. in- & out- states

(Imp because Haag's th^m says that in QFT, there are no $\Omega_\pm$).

- Create states $|k_1, k_2 \ldots \begin{smallmatrix} in \\ out \end{smallmatrix}\rangle$

- Assume that $|0\rangle$ & $|k\rangle$ are unique upto a phase

- Assume that there is an interpolating field $\varphi \longrightarrow \varphi_{\begin{smallmatrix} in \\ out \end{smallmatrix}}$

$$(\varphi - \varphi_{\begin{smallmatrix} in \\ out \end{smallmatrix}}) \longrightarrow 0 \quad \text{at} \quad t \to \mp \infty.$$

Then :

$\langle k_1' \ldots k_{n'}'; out | k_1 \ldots k_n \, in \rangle =$

$$\int \prod d^4 x_i' \int \prod d^4 x_i \; e^{i k_i' \cdot x_i' - i k_i \cdot x_i} \; {}_i(\partial_i'^2 + m^2)\,{}_i(\partial_j^2 + m^2) .$$

$$G(x_1, x_1'; \ldots x_N \, x_{N'})$$

$$\hookrightarrow \langle 0 | T(\varphi(x_1)\, \varphi(x_1') \ldots \varphi(x_N) \, \varphi(x_{N'})) | 0 \rangle$$

For twisted quantum fields,

$$\phi = \phi_c \, e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$$

We wish to compute Green's functions

$$G_N^\theta(x_1, \dots x_N) = \langle 0 | T (\varphi(x_1) \dots \varphi(x_N) | 0 \rangle$$

$$= T \, e^{\frac{i}{2}\sum_{I<J} \partial_{x_I} \wedge \partial_{x_J}} \, W_N^{(0)}(x_1, \dots x_N)$$

Wightman functions of untwisted fields.

Translational invariance $\Rightarrow$ W_N & G_N^θ depends only on $x_I - x_J$.

In the $\langle k_1' \dots | k_1, \dots \rangle$ amplitude, we take a Fourier transform, as a result of which the θ_{ij} part can be partially integrated. This gives $e^{i/2 \, k_{Ii} \theta_{ij} k_{Jj}}$

What about θ_{0i} part ?

Consider a typical term in the T product

$$\theta(x_1^0 - x_2^0) \dots \theta(x_{N-1}^0 - x_N^0)\, e^{1/2 \sum_{I<J} \partial_{x_I} \wedge \partial_{x_J}} W_N$$

Twist $e^{1/2 \sum \partial_{x_I} \wedge \partial_{x_J}}$ is of the form

$$e^{1/2 \left[\partial_{x_I^0} \theta \cdot \nabla_J - (\theta \cdot \nabla_I) \partial_{x_J^0} \right]} \quad I < J$$

Coeff of ∂x_I^0 is $\sum_{J>I} \theta \cdot \nabla_J - \sum_{J<I} \theta \cdot \nabla_J$

On partial integratn.

$$\longrightarrow e^{-\frac{1}{2}\left(\sum_{J>I} \theta \cdot k_J - \sum_{J<I} \theta \cdot k_J \right) \partial_{x_I^0}}$$

x^0_I's get translated:

$$x^0_{I-1} \longrightarrow x^0_{I-1} - \frac{1}{2}\left(\sum_{J>I-1} \theta \cdot k_J - \sum_{J<I-1} \theta \cdot k_J \right)$$

$$x^0_I \longrightarrow x^0_I - \frac{1}{2}\left(\sum_{J>I} \theta \cdot k_J - \sum_{J<I} \theta \cdot k_J \right)$$

$$x^0_{I+1} \longrightarrow \quad \dots\dots$$

Each time x_I^0 is shifted to

$x_I^0 + \delta x_I^0$ $\longrightarrow$ depends on $k_1 \ldots k_N$.

Once we know the W_N's we can thus compute the scattering amplitudes.

Rules for computing W_N's in pert^n th^y were given by Ostendorf. We use his rules to get $(LSZ)_\theta$.

Outlook :

- Charges Q are the same as in the corresponding $\theta=0$ th^y.
- The non perturbative proof of Goldstone's th^m continues to hold.
- For spontaneously broken local symmetries, we get new effects.
- The LSZ theory gives answers that depends explicitly on θ, even for cross-sections.
- Need to do some explicit computations for predictions to excite our phenomenology colleagues.