Noncommutative Gravity and the Cosmological Constant Puzzle

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I. Introduction

- Observed notion of time is external to quantum mechanics. It is part of a classical spacetime geometry, which comprises of a spacetime manifold and the metric. The metric is determined by classical matter fields.
- In principle, the Universe could be in a state in which there are no classical matter fields, but only quantum fields. In such a situation, the metric of the Universe will in general no longer be classical, but will undergo quantum fluctuations.
- It is known from the Einstein hole argument that in order for the spacetime manifold to have a physically meanigful point structure, a well-determined classical metric (which is a solution of the Einstein equations) must reside on the manifold. When the metric is undergoing quantum fluctuations, the point structure of the spacetime manifold is destroyed.

I. Introduction

- Nonetheless, one should be able to describe the dynamics of a quantum system, even if an external classical time is not available. It can be shown that standard linear quantum theory is a limiting case of a more general quantum theory which is nonlinear on the Planck mass/energy scale.
- A natural mathematical structure which foregoes the point structure of spacetime is a noncommutative spacetime. We construct the reformulation by pursuing the following proposal : in the reformulation, relativistic quantum mechanics is the same theory as noncommutative special relativity.
- The physical principle is that the basic laws are invariant under 'inertial' coordinate transformations of noncommuting coordinates. One is naturally led to attach an antisymmetric part to the Minkowski metric. The theory is supposed to describe dynmaics when gravity can be neglected.

I. Introduction

- The nonlinear generalization of this reformulation describes the dynamics of the system when its energy becomes comparable to Planck energy. The Schrödinger equation becomes nonlinear and the gravitational dynamics is now a noncommutative general relativity. The physical principle now is that basic laws are invariant under general coordinate transformations of noncommuting coordinates.
- The presence of the nonlinearity has two important consequences. Firstly, the antisymmetric part of the gravitational field associated with this nonlinearity suggests the existence of a quantum-classical duality, as a consequence of which one can match a dominantly quantum sector of the theory to a dominantly classical sector. In turn this helps us understand why the cosmological constant should be non-zero and yet have the very small value it does.

Introduction

- The second important consequence of the nonlinearity has to do with the nonlinearity in the Schrödinger equation, which becomes relevant in the vicinity of the Planck mass scale. This can lead to a breakdown of quantum superposition, and could lead to the collapse of the wavefunction during a quantum measurement.
- The parameters influencing the collapse of the wavefunction are in principle measureable in the laboratory. These are the same parameters which are responsible for the existence of the quantum classical duality, and for the non-zero value of the comological constant. Thus our explanation for the origin of the dark energy is in principle testable experimentally, via the quantum measurement process.

II. Quantum mechanics as a noncommutative special relativity

- In this reformulation, the quantum dynamics of a relativistic particle of mass $m \ll m_{Pl}$ is described as a noncommutative special relativity. In the illustrative 2-d case, the noncommutative spacetime has the line element

$$d\hat{s}^2 = \hat{\eta}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu \equiv d\hat{t}^2 - d\hat{x}^2 + d\hat{t}d\hat{x} - d\hat{x}d\hat{t}, \qquad (1)$$

- and the noncommuting coordinates $\hat{t}, \hat{x}$ obey the commutation relations

$$[\hat{t}, \hat{x}] = f^{-1}(\hat{p}^t, \hat{p}^x) \qquad [\hat{p}^t, \hat{p}^x] = f(\hat{p}^t, \hat{p}^x). \qquad (2)$$

Quantum mechanics as a noncommutative special relativity

- Dynamics is described by defining a velocity $\hat{u}^i = d\hat{x}^i/d\hat{s}$, a momentum $\hat{p}^i = m\hat{u}^i$, and by defining momenta as the gradients of a complex action $\hat{S}$, in the generalized Casimir relation
$$(\hat{p}^t)^2 - (\hat{p}^x)^2 + \hat{p}^t\hat{p}^x - \hat{p}^x\hat{p}^t = m^2, \tag{3}$$
in the spirit of the Hamilton-Jacobi equation.
- If an external classical spacetime (x, t) becomes available, the Klein-Gordon equation of standard linear quantum mechanics is recovered from this reformulation via the correspondence
$$(\hat{p}^t)^2 - (\hat{p}^x)^2 + \hat{p}^t\hat{p}^x - \hat{p}^x\hat{p}^t = (p^t)^2 - (p^x)^2 + i\hbar\frac{\partial p^\mu}{\partial x^\mu} \tag{4}$$
and by defining the wave-function as $\psi \equiv e^{iS/\hbar}$.

III. A noncommutative general relativity

- When the mass of the particle is comparable to Planck mass, the noncommutative line-element (1) is modified to the curved noncommutative line-element

$$ds^2 = \hat{h}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu \equiv \hat{g}_{tt} d\hat{t}^2 - \hat{g}_{xx} d\hat{x}^2 + \hat{\theta}[d\hat{t}d\hat{x} - d\hat{x}d\hat{t}]. \quad (5)$$

- Correspondingly, the Casimir relation (3) is generalized to

$$\hat{g}_{tt}(\hat{p}^t)^2 - \hat{g}_{xx}(\hat{p}^x)^2 + \hat{\theta}\left(\hat{p}^t\hat{p}^x - \hat{p}^x\hat{p}^t\right) = m^2 \quad (6)$$

and the correspondence rule (4) to

$$\hat{g}_{tt}(\hat{p}^t)^2 - \hat{g}_{xx}(\hat{p}^x)^2 + \hat{\theta}\left(\hat{p}^t\hat{p}^x - \hat{p}^x\hat{p}^t\right) = g_{tt}(p^t)^2 - g_{xx}(p^x)^2 + i\hbar\theta\frac{\partial p^\mu}{\partial x^\mu}. \quad (7)$$

- This leads, in the simplest case, where θ is a function of m/m_{Pl}, to the equation of motion

$$\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial x}\right)^2 - i\hbar\theta(m/m_{Pl})\left(\frac{\partial^2 S}{\partial t^2} - \frac{\partial^2 S}{\partial x^2}\right) = m^2 \quad (8)$$

  which is equivalent to a nonlinear Klein-Gordon equation.
- The noncommutative metric is assumed to obey a noncommutative generalization of Einstein equations, with the property that $\theta(m/m_{Pl})$ goes to one for $m \ll m_{Pl}$, and to zero for $m \gg m_{Pl}$. Also, as $\theta(m/m_{Pl}) \to 0$ one recovers classical mechanics, and in the limit $\theta \to 1$ standard linear quantum mechanics is recovered.
- In the mesoscopic domain, where θ is away from these limits and the mass m is comparable to Planck mass, both quantum and gravitational features can be defined simultaneously, and new physics arises. The antisymmetric component θ of the gravitational field plays a crucial role in what follows.

IV. A proposed quantum-classical duality

Motivation for the duality

- In general relativity, the Schwarzschild radius $R_S = 2Gm/c^2$ of a particle of mass m can be written in Planck units as $R_{SP} \equiv R_S/L_{Pl} = 2m/m_{Pl}$, where L_{Pl} is Planck length and $m_{Pl} \sim 10^{-5}$ gm is the Planck mass.
- If the same particle were to be treated, not according to general relativity, but according to relativistic quantum mechanics, then one-half of the Compton wavelength $R_C = h/mc$ of the particle can be written in Planck units as $R_{CP} \equiv R_C/2L_{Pl} = m_{Pl}/2m$.
- The fact that the product $R_{SP}R_{CP} = 1$ is a universal constant cannot be a coincidence; however it cannot be explained in the existing theoretical framework of general relativity (because herein $h = 0$) and quantum mechanics (because herein $G = 0$).

Proposal

- We propose and justify the following quantum-classical duality:
  The weakly quantum, strongly gravitational dynamics of a particle of mass $m_c \gg m_{Pl}$ *is dual to the strongly quantum, weakly gravitational dynamics of a particle of mass* $m_q = m_{Pl}^2/m_c \ll m_{pl}$.
- It follows that the dimensionless Schwarzschild radius R_{SP} of m_c is four times the dimensionless Compton-wavelength R_{CP} of m_q.
- The origin of this duality lies in the requirement that there be a reformulation of quantum mechanics which does not refer to an external classical spacetime manifold.

- The Planck mass demarcates the dominantly quantum domain $m < m_{Pl}$ from the dominantly classical domain $m > m_{Pl}$ and is responsible for the quantum-classical duality.
- As is evident from (8), the effective Planck's constant is $\hbar\theta(m/m_{Pl})$, going to zero for large masses, and to $\hbar$ for small masses, as expected. Similarly, the effective Newton's gravitational constant is likely to be $G(1-\theta(m/m_{Pl}))$, going to zero for small masses, and to G for large masses.
- Thus the parameter space $\theta \approx 1$ is strongly quantum and weakly gravitational, whereas $\theta \approx 0$ is weakly quantum and strongly gravitational. The Compton wavelength R_{CP} for a particle of mass m_q gets modified to $R_{CE} \equiv R_{CP}\theta(m_q/m_{Pl})$ and the Schwarzschild radius R_{SP} for a mass m_c gets modified to $R_{SE} \equiv R_{SP}(1-\theta(m_c/m_{Pl}))$.

- We propose that the dynamics of a mass $m_q \ll m_{Pl}$ is dual to the dynamics of a mass $m_c \gg m_{Pl}$ if $R_{SE}(m_c) = 4R_{CE}(m_q)$. This holds if $m_c = m_{Pl}^2/m_q$ and

$$\theta(m/m_{Pl}) + \theta(m_{Pl}/m) = 1. \tag{9}$$

- If (9) holds, the solution for the dynamics for a particle of mass m_c can be obtained by first finding the solutions of (8) for mass m_q, and then replacing $\theta(m_q/m_{Pl})$ by $1 - \theta(m_{Pl}/m_q)$, and finally writing m_c instead of m_q, wherever m_q appears.

- We can deduce the functional form of $\theta(m/m_{Pl})$ by noting that the contribution of the symmetric part of the metric, g_{ik}, to the curvature, grows as m, whereas the contribution of the antisymmetric part θ must fall with growing m. This suggests that $1/\theta$ grows linearly with m; thus

$$\frac{1}{\theta(m/m_{Pl})} = a(m/m_{Pl}) + b, \tag{10}$$

  and $\theta(0) = 1$ implies $b = 1$; and we set $a = 1$ since this simply defines m_{Pl} as the scaling mass. Hence we get $\theta(m/m_{Pl}) = 1/(1 + m/m_{Pl})$, which satisfies (9) and thus establishes the duality.
- The mapping $m \to 1/m$ interchanges the two fundamental length scales in the solution : Compton wavelength and Schwarzschild radius.

- Quantum-classical duality has previously been observed in string theory. Our results suggest one of two possibilities : (i) such a duality is a property of quantum gravity, independent of string theory; or (ii) we have identified a key physical principle underlying string theory, namely, invariance of physical laws under general coordinate transformations of noncommuting coordinates.
- The duality we observe is holographic, by virtue of the above-mentioned relation $R_{SE}(m_c) = 4R_{CE}(m_q)$. Thus, the number of degrees of freedom N that a quantum field associated with the particle m_q possesses (bulk property) should be of the order of the area of the horizon of the dual black hole in Planck units (boundary property), i.e. $N \sim m_{Pl}^2/m_q^2$.

The Cosmological Constant Puzzle

- The quantum-classical duality helps understand why there should be a cosmological constant of the order of the observed matter density; the most likely explanation for the observed cosmic acceleration.
- If there is a non-zero cosmological constant term Λ in the Einstein equations, of the standard form Λg_{ik}, it follows from symmetry arguments that in the noncommutative generalization of gravity, a corresponding term of the form $\Lambda\theta$ should also be present. This latter term vanishes in the macroscopic limit $m \gg m_{Pl}$ but is present in the microscopic limit $m \ll m_{Pl}$.
- However, when $m \ll m_{Pl}$, the effective gravitational constant goes to zero, so Λ cannot be sourced by ordinary matter. Its only possible source is the zero-point energy associated with the quantum particle $m \ll m_{Pl}$. Since this zero-point energy is necessarily non-zero, it follows that Λ is necessarily non-zero.

- This same Λ manifests itself on cosmological scales, where Λg_{ik} is non-vanishing, because g_{ik} is non-vanishing, even though $\Lambda\theta$ goes to zero on cosmological scales, because θ goes to zero.
- The value of Λ can be estimated by appealing to the deduced quantum-classical duality. The total mass in the observable Universe is $m_c \sim c^3(GH_0)^{-1}$, where H_0 is the present value of the Hubble constant. The mass dual to this m_c is $m_q = m_{Pl}^2/m_c \sim hH_0/c^2$, and $m_q c^2$ is roughly the magnitude of the zero-point energy.
- We associate this much zero point energy with each of the N degrees of freedom, where, according to the holographic arguments alluded to above, N is of the order $(cH_0^{-1})^2/L_P^2$.

- The vacuum energy density, and hence the value of the cosmological constant, is $(m_q c^2)N/(cH_0^{-1})^3 \sim (cH_0)^2/G$ which is of the order of the observed value of Λ.
- Clearly, nothing in this argument singles out today's epoch; hence it follows that there is an ever-present Λ, of the order $(cH)^2/G$, at any epoch, with H being the Hubble constant at that epoch. This solves the cosmic coincidence and fine-tuning problems; and difficulties related to an ever-present Λ can possibly be addressed.

Understanding Λ

- The holographic value for the allowed number of degrees of freedom plays a crucial role in the argument. The minimum value of the zero point energy, $m_q c^2 \sim h H_0$, corresponds to a frequency H_0, which being the inverse of the age of the Universe, is a natural minimum frequency (infra-red cut-off).
- This corresponds to a contribution Λ_f to the cosmological constant, per degree of freedom, given by

$$\Lambda_f = \left(\frac{L_P}{c H_0^{-1}} \right)^4 L_P^{-2} \tag{11}$$

and a corresponding energy density

$$\rho_f = \left(\frac{L_P}{c H_0^{-1}} \right)^4 \rho_{Pl}. \tag{12}$$

- We recall that the observed Λ and its associated energy density can be written as

$$\Lambda_{obs} = H_0^2/c^2 = \left(\frac{L_P}{cH_0^{-1}}\right)^2 L_P^{-2} \tag{13}$$

$$\rho_{\Lambda obs} = c^2 H_0^2/G = \left(\frac{L_P}{cH_0^{-1}}\right)^2 \rho_{Pl} \tag{14}$$

  where ρ_{Pl} is Planck energy density.
- One could artificially introduce a cut-off to the total zero point energy of the dual quantum field, for example simply by saying that the maximum allowed frequency is Planck frequency. In this case, the number of degrees of freedom N is E_{Pl}/E_q, which is equal to cH_0^{-1}/L_{Pl}. This gives $\Lambda = N\Lambda_f = \left(L_P/cH_0^{-1}\right)^3 L_P^{-2}$ which does not match with observations.

- Now consider what values of Λ result from other choices of N, by writing $N = (cH_0^{-1}/L_P)^n$. Our deduction has been the holographic value $n = 2$, which reproduces the correct Λ. The choices $n = 4$ and $n = 3$, which correspond to the four volume and the three volume, give wrong values of Λ (too high), whereas $n = 1$ also gives a wrong value of Λ (too low).
- Put another way, the natural minimum frequency is $\omega_{min} = H_0$. Our choice of N is such that $\omega_{max} = \omega_{Pl}(\omega_{Pl}/H_0)$, ($N = \omega_{max}/\omega_{min}$). Thus the maximum frequency is scaled up from Planck frequency by the same factor by which the minimum frequency is scaled down with respect to Planck frequency. It is also the frequency corresponding to the rest mass of the observed Universe, which is of the order H_0^{-1}. Thus the UV cut-off is not at Planck energy, but at the observed rest mass of the Universe.

- With hindsight, it seems rather natural that the observed value of the cosmological constant is reproduced when the infra-red and ultra-violet cut-offs for the zero point energy are taken at the cosmological values H_0 and H_0^{-1}, respectively. The quantum-classical duality proposed here provides the reason as to why quantum zero point energy contributes to gravity in the first place.

VI. Can there be an ever-present Λ?

- Open problem. If one stays within the framework of Friedmann equations, one has to make modifications so as to allow $\dot{\Lambda} \neq 0$.
- Eventually, one must allow for spatial inhomogeneity and anisotropy, in a manner consistent with observations.