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κ-DEFORMED OSCILLATORS: DEFORMED MULTIPLICATION, DEFORMED FLIP OPERATORS AND MULTIPARTICLE CLUSTERS

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GENERAL FRAMEWORK:

$$[\hat{x}_\mu, \hat{x}_\nu] = \underset{①}{\frac{i}{\kappa^2}\theta^{(0)}_{\mu\nu}} + \underset{②}{\frac{i}{\kappa}\theta^{(1)}_{\mu\nu}{}^{\rho}\hat{x}_\rho} + i\,\theta^{(2)}_{\mu\nu}{}^{\rho\kappa}\hat{x}_\rho\hat{x}_\kappa + \dots$$

canonical (DFR)

Lie-algebraic

quadratic algebra

Example of ②: κ-Minkowski space-time

$$[\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa}\hat{x}_i \qquad [\hat{x}_i, \hat{x}_j] = 0$$

① $\longrightarrow$ ②

Moyal-Weyl star product

BCH star product (for κ-Minkowski - special case)

$\Rightarrow$ κ-Poincare $\rightarrow$ algebra
$\rightarrow$ group (translations $\equiv$ κ-Minkowski algebra)

$\Rightarrow$ κ-Poincare algebra can not be obtained by a standard Drinfeld twist (only nonstandard)

1. INTRODUCTION

We shall consider the fourmomentum basis of κ-deformed algebra with the non-Abelian coproducts

$$\Delta(P_i) = P_i \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes P_i$$

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0$$

and κ-deformed mass Casimir

$$C^2_\kappa(P_\mu) = \left(2\kappa \sinh\left(\frac{P_0}{2\kappa}\right)\right)^2 - \vec{P}^2 = M^2$$

(modified Majid-Ruegg basis)

We introduce the κ-deformed bosonic oscillators $a_\kappa(p) \equiv a_\kappa(\vec{p}, p_0)$ satisfying the relation

$$P_\mu \triangleright a_\kappa(p) \equiv \mathrm{ad}_{P_\mu} a_\kappa(p) = p_\mu a_\kappa(p)$$

(quantum adjoint: $\mathrm{ad}_A B = A_{(1)} B S(A_{(2)})$
where $\Delta(A) = A_{(1)} \otimes A_{(2)}$)

$$p_0 = \omega_\kappa = 2\kappa \,\mathrm{arcsinh} \frac{(\vec{p}^2 + M^2)^{\frac{1}{2}}}{2\kappa}$$

($p_0 > 0$ – creation operators)

Hopf-algebraic action formula:

$$P_\mu \triangleright (a_\kappa(p)\, a_\kappa(q)) = (P_{\mu(1)} \triangleright a_\kappa(p))(P_{\mu(2)} \triangleright a_\kappa(q)) =$$
$$= p_\mu^{(1+2)}\, a_\kappa(p)\, a_\kappa(q)$$

One gets

$$p_i^{(1+2)} = p_i\, e^{\frac{q_0}{2\kappa}} + q_i\, e^{-\frac{p_0}{2\kappa}}$$
$$p_0^{(1+2)} = p_0 + q_0$$

Exchanging $p_\mu \leftrightarrow q_\mu$ (flipping operation) one gets

$$p_i^{(2+1)} = q_i\, e^{\frac{p_0}{2\kappa}} + p_i\, e^{-\frac{q_0}{2\kappa}} \neq p_i^{(1+2)}$$
$$p_0^{(2+1)} = q_0 + p_0 = p_0^{(1+2)}$$

Conclusion: in κ-deformed theory the standard exchange relation

$$a_\kappa(p)\, a_\kappa(q) = a_\kappa(q)\, a_\kappa(p) \Leftrightarrow [a_\kappa(p), a_\kappa(q)] = 0$$

is nonconsistent with κ-deformed addition law of three-momenta.

Solution: κ-deformed multiplication
κ-deformed flip operator

2. BINARY ALGEBRA OF $\varkappa$-OSCILLATORS

Let us assume that the effect of $\varkappa$-multiplication on the algebra $\mathcal{A}(a, a^+)$ of standard creation and annihilation operators consists in the replacement

$$m(A \otimes B) = A \cdot B \xrightarrow[\varkappa < \infty]{} m_\varkappa(A \otimes B) = A \circ B$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

standard multiplication $\qquad$ $\varkappa$-multiplication

We introduce

$$a_\varkappa(p) \circ a_\varkappa(q) = a_\varkappa(e^{-\frac{q_0}{2\varkappa}} \vec{p}, p_0)\, a_\varkappa(e^{\frac{p_0}{2\varkappa}} \vec{q}, q_0)$$

Further in consistency with $a^+_\varkappa(p) = a(-p)$

$$a^+_\varkappa(p) \circ a^+_\varkappa(q) = a^+_\varkappa(e^{\frac{q_0}{2\varkappa}} \vec{p}, p_0)\, a_\varkappa(e^{-\frac{p_0}{2\varkappa}} \vec{q}, q_0)$$

$$a^+_\varkappa(p) \circ a_\varkappa(q) = a^+_\varkappa(e^{-\frac{q_0}{2\varkappa}} \vec{p}, p_0)\, a_\varkappa(e^{-\frac{p_0}{2\varkappa}} \vec{q}, q_0)$$

$$a_\varkappa(p) \circ a^+_\varkappa(q) = a_\varkappa(e^{\frac{q_0}{2\varkappa}} \vec{p}, p_0)\, a^+_\varkappa(e^{\frac{p_0}{2\varkappa}} \vec{q}, q_0)$$

$\varkappa$-deformed oscillators algebra

$$[a_\varkappa(p), a_\varkappa(q)]_\circ = [a^+_\varkappa(p), a^+_\varkappa(q)] = 0$$

$\varkappa$-statistics!

$\varkappa$-statistics

$$[a^+_\varkappa(p), a_\varkappa(q)]_\circ = \delta^3(\vec{p} - \vec{q})$$

bosonic with respect to $\circ$-multiplication

Let us introduce κ- deformed Fock space ($p_0 = \omega_\kappa(\vec{p})$)

$$|0\rangle \qquad a^+_\kappa(\vec{p}, p_0)|0\rangle = 0$$

$$|\vec{p}\rangle = a_\kappa(\vec{p}, p_0)|0\rangle$$

$$\|\vec{p}, \vec{q}\rangle\!\rangle = a_\kappa(\vec{p}, p_0) \circ a_\kappa(\vec{q}, q_0)|0\rangle$$

$$\vdots$$

We get

$$P_\mu \|\vec{p}, \vec{q}\rangle\!\rangle = P_\mu \triangleright \left(a_\kappa(\vec{p}e^{-\frac{q_0}{2\kappa}}, p_0)\, a_\kappa(\vec{q}e^{\frac{p_0}{2\kappa}}, q_0)\right)|0\rangle$$

$$= (p_\mu + q_\mu)\, \|\vec{p}, \vec{q}\rangle$$

Abelian addition law !

Due to the relation

$$a_\kappa(p) \circ a_\kappa(q) = a_\kappa(q) \circ a_\kappa(p) \quad \kappa\text{-statistics}$$

one gets

$$\|\vec{p}, \vec{q}\rangle\!\rangle = \|\vec{q}, \vec{p}\rangle\!\rangle$$

standard bosonic symmetry!

The construction has been extended to n- particle states $\|\vec{p}_1 \cdots \vec{p}_n\rangle\!\rangle$ hep-th

3. MULTIPLICATION VERSUS FLIP OPERATION

The standard commutator can be described as follows:

$$[A,B] = (1-\hat{\tau}_0)\, A\cdot B$$

where

$$\hat{\tau}_0\,(A\cdot B) = B\cdot A \qquad \hat{\tau}_0^2 = 1$$

One can introduce *x-deformed commutator* as follows:

$$[A,B] \xrightarrow[x<\infty]{} [A,B]_x = (1-\hat{\tau}_x)\, A\cdot B$$

where $\hat{\tau}_x$ is *x-deformed flip operator*

$$\hat{\tau}_x(A\cdot B) = (\hat{\tau}_x^{(1)} B)\cdot \hat{\tau}_x^{(2)}(A)$$

in such a way that

$$\hat{\tau}_x^2 = 1$$

The *consistency* with fourmomentum *addition law* requires

$$[\hat{\tau}_x\,,\,\Delta(P_\mu)] = 0$$

The standard bosonic exchange relation is modified as follows

x-statistics $\Rightarrow$

(standard multiplication) $$a_x(p)\, a_x(q) = \hat{\tau}_x [a_x(p)\, a_x(q)] \Leftrightarrow [a_x(p), a_x(q)]_x =$$

One can introduce the following two 2-particle states:

$$|\vec{p}', \vec{q}'\rangle = a_x(\vec{p}', p_0')\, a_x(\vec{q}', q_0')\, |0\rangle$$

$$|\vec{q}', \vec{p}'\rangle_{\tau_x} = \tau_x \left(a_x(\vec{p}', p_0')\, a_x(\vec{q}', q_0')\, |0\rangle \right. \Leftarrow x\text{-flipped}$$

The x-statistics leads to the identification of states

$$|\vec{p}', \vec{q}'\rangle = |\vec{q}', \vec{p}'\rangle_{\tau_x}$$

One can introduce the symmetric states:

$$|\vec{p}', \vec{q}'\rangle_s = \frac{1}{2}\left(|\vec{p}', \vec{q}'\rangle + |\vec{p}', \vec{q}'\rangle_{\tau_x} \right)$$

One gets classical bosonic symmetry

$$|\vec{p}', \vec{q}'\rangle_s = |\vec{q}', \vec{p}'\rangle_s \Leftarrow \text{due to } \hat{\tau}_x^2 = 1$$

The states $|\vec{p}\,', \vec{q}\,'\rangle$ and $|\vec{q}\,', \vec{p}\,'\rangle_{\tau_x}$ carry the same fourmomentum, but the addition law is non-Abelian

$$P_\mu |\vec{p}\,', \vec{q}\,'\rangle = P_\mu \triangleright (a_x(\vec{p}\,', p_0')\, a_x(\vec{q}\,', q_0'))|0\rangle$$
$$= \tilde{p}_\mu^{(1+2)} |\vec{p}\,', \vec{q}\,'\rangle$$

where

$$\tilde{p}_i^{(1+2)} = p_i' e^{\frac{q_0'}{2x}} + q_i' e^{-\frac{p_0'}{2x}}$$

Non Abelian addition law!

Because we assumed $[\hat{\tau}_x, \Delta(P_\mu)] = 0$ one obtains as well after flipping

$$P_\mu |\vec{p}\,', \vec{q}\,'\rangle_{\tau_x} = \tilde{p}_\mu^{(1+2)} |\vec{p}\,', \vec{q}\,'\rangle_{\tau_x}$$

Problem: can one link two statistics with Abelian and nonAbelian addition law?

Answer: Yes, if we use the transformations in 2-particle fourmomentum space (p_μ, q_μ)

4. BINARY ALGEBRA VIA κ-DEFORMED FLIP

If in the relation $a_\kappa(p) \circ a_\kappa(q) = a_\kappa(q) \circ a_\kappa(p)$ we introduce

$$\vec{p}\,' = \vec{p}\, e^{-\frac{q_0}{2\kappa}} \qquad \vec{q}\,' = \vec{q}\, e^{\frac{p_0}{2\kappa}}$$

one gets the relation ($p_0' = p_0$, $q_0' = q_0$)

$$\underline{a_\kappa(\vec{p}\,', p_0')\, a_\kappa(\vec{q}\,', q_0') = a_\kappa(\vec{q}\,' e^{-\frac{p_0'}{\kappa}}, q_0')\, a_\kappa(\vec{p}\,' e^{\frac{q_0'}{\kappa}}, p_0')}$$

or (explicite formula for τ_κ)

$$\tau_\kappa(a_\kappa(\vec{p}\,', p_0')\, a_\kappa(\vec{q}\,', q_0')) = a_\kappa(\vec{q}\,' e^{-\frac{p_0'}{\kappa}}, q_0')\, a_\kappa(\vec{p}\, e^{\frac{q_0'}{\kappa}}, p_0')$$

<u>The energy-shell:</u>

$$p_0' = p_0 = \omega_\kappa(\vec{p}) = \omega_\kappa(\vec{p}\,' e^{\frac{q_0'}{2\kappa}}) = \omega_\kappa^{(+)}(\vec{p}\,')$$

$$q_0' = q_0 = \omega_\kappa(\vec{q}) = \omega_\kappa(\vec{q}\,' e^{-\frac{p_0'}{2\kappa}}) = \omega_\kappa^{(-)}(\vec{q}\,')$$

We get the system of coupled nonlinear algebraic equations for p_0', q_0', or

$$p_0' = \omega_\kappa\left(\vec{p}\,' \exp \frac{1}{2\kappa}\left\{\omega_\kappa(\vec{q}\,' e^{-\frac{p_0'}{2\kappa}})\right\}\right)$$

<u>Complicated nonlinear eq. for p_0'</u>; analogous one gets for q_0' – it can be solved perturbatively

2 - particle non-Abelian summ of three-momenta can be calculated from explicite formula of $\hat{\tau}_x$, and one gets

$$\vec{p}^{\,\prime(1+2)} = \vec{p}^{\,\prime} e^{\frac{q_0'}{2x}} + \vec{q}^{\,\prime} e^{-\frac{p_0'}{2x}} = \vec{p}^{\,\prime(2+1)}$$

and

$$p_0^{\prime(1+2)} = \omega_x^{(-)}(\vec{p}^{\,\prime}) + \omega_x^{(+)}(\vec{q}^{\,\prime}) = p_0^{\prime(2+1)}$$

where ($\epsilon = \pm 1$, $\eta = \pm 1$)

$$\omega_x^{(\epsilon)}(\vec{p}^{\,\prime}) = \omega_x(\vec{p}^{\,\prime} e^{\epsilon \frac{q_0'}{2x}}) \qquad \omega_x^{(\eta)}(\vec{q}^{\,\prime}) = \omega_x(\vec{q}^{\,\prime} e^{\eta \frac{p_0'}{2x}})$$

Flip operator $\hat{\tau}_x$ for other binary products:

i) $a_x^+(p) \circ a_x^+(q) = a_x^+(q) \circ a_x^+(p)$

We introduce

$$\vec{p}^{\,\prime} = \vec{p} e^{\frac{q_0}{2x}} \qquad \vec{q}^{\,\prime} = \vec{q} e^{-\frac{p_0}{2x}}$$

One gets

$$a_x^+(\vec{p}^{\,\prime}, p_0') a^+(\vec{q}^{\,\prime}, q_0') = a_x^+(\vec{q}^{\,\prime} e^{\frac{p_0'}{x}}, q_0') \, a_x^+(\vec{p}^{\,\prime} e^{-\frac{q_0'}{x}}, p_0')$$

where

$$p_0' = \omega_x^{(-)}(\vec{p}^{\,\prime}) \qquad q_0' = \omega_x^{(+)}(\vec{q}^{\,\prime})$$

ii) $[a_x^+(p), a_x(q)]_0 = \delta^3(\vec{p}-\vec{q})$

If
$$\vec{p}\,' = \vec{p}\, e^{-\frac{q_0}{2x}} \qquad \vec{q}\,' = \vec{q}\, e^{-\frac{p_0}{2x}}$$

we get

$$a_x^+(\vec{p}\,', p_0') a_x(\vec{q}\,', q_0') - \overbrace{a_x(\vec{q}\,' e^{\frac{p_0'}{x}}, q_0') a^+(\vec{p}\,' e^{\frac{q_0'}{x}}, p_0')}^{\hat{\tau}_x} =$$
$$= \delta^3(\vec{p}\,' e^{\frac{q_0'}{2x}} - \vec{q}\,' e^{\frac{p_0'}{2x}})$$

with

$$p_0' = \omega_x^{(+)}(\vec{p}\,') \qquad q_0' = \omega_x^{(+)}(\vec{q}\,')$$

The choice of the energy shell conditions depends on the character (creation or annihilation) of other oscillator in binary relation. The rule:

$$a(\vec{p}, p_0) = a^{(1)}(\vec{p}, p_0) \qquad a^+(\vec{p}, p_0) = a^{(-1)}(\vec{p}, p_0)$$

Then the choice of mass-shell conditions in binary product is

$$a^{(\varepsilon)}(\vec{p}, \omega_x^{(\eta)}(\vec{p}))\, a^{(\eta)}(\vec{q}, \omega_x^{(-\varepsilon)}(\vec{q}))$$

5. NONFACTORIZABLE $\varkappa$-DEFORMED 2-PARTICLE CLUSTERS

a) Standard Fock space

$$\mathcal{F} = \sum_{n=0}^{\infty} \mathcal{H}_n^{(0)}$$

$$\mathcal{H}_n^{(0)} = \underset{\substack{\Uparrow \\ \text{symmetrizator}}}{S_n} \big(\overbrace{\mathcal{H}_1^{(0)} \otimes \cdots \otimes \mathcal{H}_1^{(0)}}^{n}\big)$$

If $a^{(0)}(p)$ are standard creation operators then

$$\mathcal{H}_n^{(0)}: \quad |p_1 \cdots p_n\rangle^{(0)} = a^{(0)}(p_1) \cdots a^{(0)}(p_n)|0\rangle$$

b) $\varkappa$-deformed Fock space

$$\mathcal{F}^{\varkappa} = \sum_{n=0}^{\infty} \mathcal{H}_n^{\varkappa}$$

For $n=2$ the general formula

$$\mathcal{H}_2^{\varkappa} = S_2^{\varkappa}(\mathcal{H}_1 \otimes_{\varkappa} \mathcal{H}_1)$$

Two special choices:

i) <u>$\varkappa$ - deformed multiplication</u>

$$S_2^\varkappa = S_2 \qquad m(\mathcal{H}_1 \otimes_\varkappa \mathcal{H}_1) = \mathcal{H}_1 \circ \mathcal{H}_2$$

ii) <u>$\varkappa$ - deformed flip operator $\hat{\tau}_\varkappa$</u>

$$S_2^\varkappa = \frac{1}{2}(1 \otimes 1 + \hat{\tau}_\varkappa) \qquad \otimes_\varkappa = \otimes$$

The introduction of $\varkappa$-deformed multiplication does not permit the <u>factorization</u> of 2-particle state into 1-particle wave packets, because

$$\left(\int d^3\vec{p}\, f^{(1)}(\vec{p})\, a(\vec{p}, p_0) \right) \circ \left(\int d^3\vec{q}\, g^{(1)}(\vec{q})\, a(\vec{q}, q_0) \right)$$

<u>is not well defined</u>. Only if we introduce

$$f^{(2)}(\vec{p}, \vec{q}) = f^{(1)}(\vec{p})\, f^{(2)}(\vec{q})$$

then

$$(a_\varkappa \circ a_\varkappa)[f^{(2)}] = \int d^3p \int d^3\vec{q}\, f^{(2)}(\vec{p}, \vec{q})\, a_\varkappa(p) \circ a_\varkappa(q)$$

<u>Firstly multiply oscillators, then integrate!!</u>

In particular one can consider the nonlinear transformations of 2-particle wave packet function by change of momenta

$$p_i = p_i(\vec{q}, \vec{Q}) \qquad q_i = q_i(\vec{q}, \vec{Q})$$

Two-particle wave packet transforms as follows:

$$\tilde{f}^{(2)}(\vec{q}, \vec{Q}) = J\left[\begin{matrix} p_i, q_i \\ q_i, Q_i \end{matrix}\right](\vec{q}, \vec{Q})\, f^{(2)}(\vec{p}(\vec{q}, \vec{Q}), \vec{q}(\vec{q}, \vec{Q}))$$

where J is the Jacobian

The transformations linking

{ x-multiplication framework $\leftrightarrow$ { x-flipping framework

are related by such transformations.

In $\mathcal{H}_n^x$ we have clusters depending on n three momenta $(\vec{p}_1, \dots \vec{p}_n)$.

The momentum description of clusters can be subjected to the transformations

$$\vec{p}_i = \vec{p}_i(\vec{q}_1, \dots, \vec{q}_n)$$

6. NONFACTORIZABLE BINARY $*_\varkappa$ - PRODUC[T] OF FIELDS

a) For simplest canonical noncommutativ[ity]

$$[\hat{x}_\mu, \hat{x}_\nu] = i\,\frac{\theta_{\mu\nu}}{\varkappa^2} \qquad \theta_{\mu\nu} = \text{const}$$

the Moyal-Weyl star product is not factorizable because

$$\varphi_0(x) *_\theta \varphi_0(y) = \int d^4p \int d^4q\, \phi(p,q)\, e^{ipx+q[y]}$$

where

$$\phi(p,q) = \phi(p)\,\phi(q)\, e^{\frac{i}{\varkappa}\theta_{\mu\nu}p^\mu q^\nu} \neq \phi_1(p)\cdot\phi_2(q)$$

however field equations remain not modi[fied]

$$(\Box_x - m^2)(\Box_y - m^2)(\varphi_0(x) *_\theta \varphi(y)) = 0$$

b) We introduce $\varkappa$-deformed fields

$$\varphi_\varkappa^{(+)}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{p}}{\Omega_\varkappa(\vec{p})}\, a_\varkappa(\vec{p}, \omega_\varkappa(\vec{p}))\, e^{i(\vec{p}\vec{x} - \omega_\varkappa(\vec{p})t)}$$

for simplicit[y] (only positive frequen[cy] p[art])

where

$$\Omega_\varkappa = 2\varkappa \sinh \frac{\omega_\varkappa(\vec{p})}{\varkappa} \xrightarrow[\varkappa\to\infty]{} 2\omega(\vec{p})$$

and introduce $\varkappa$-deformed star product $*_\varkappa$

$$e^{ipx} *_\varkappa e^{iqy} = e^{i\left(e^{\frac{q_0}{2\varkappa}}\vec{p}\cdot\vec{x} + e^{-\frac{p_0}{2\varkappa}}\vec{q}\vec{y}\right)} e^{i(p_0x_0+q_0y_0)}$$

The $*_\varkappa$-product of fields is understood that we firstly perform the $*_\varkappa$-product, and then integrate over momenta

$$\varphi_\varkappa^{(+)}(x) *_\varkappa \varphi_\varkappa^{(+)}(y) = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{\Omega_\varkappa(\vec{p})} \frac{d^3\vec{q}}{\Omega_\varkappa(\vec{p})} \cdot$$

$$\cdot\, a_\varkappa(\vec{p}, p_0)\, a_\varkappa(\vec{q}, q_0) \underbrace{\left(e^{ipx} *_\varkappa e^{iqy}\right)}_{\text{firstly this operation!}}\Bigg|_{\substack{p_0 = \omega_\varkappa(\vec{p}) \\ q_0 = \omega_\varkappa(\vec{q})}}$$

In fact we obtain bilocal field

$$\varphi_\varkappa^{(+)}(x) *_\varkappa \varphi_\varkappa^{(+)}(y) \equiv \varphi_\varkappa^{(+,+)}(x,y)$$

Because

$$p_i\left(e^{ipx} *_\varkappa e^{iqy}\right) = e^{i\frac{\partial_0^y}{2\varkappa}}\, \partial_i^x \left(e^{ipx} *_\varkappa e^{iqy}\right)$$

$$q_i\left(e^{ipx} *_\varkappa e^{iqy}\right) = e^{-i\frac{\partial_0^x}{2\varkappa}}\, \partial_i^y \left(e^{ipx} *_\varkappa e^{iqy}\right)$$

one gets bilocal field equation

$$\left[\Delta^x e^{i\frac{\partial_0^y}{\varkappa}} - \left(2\varkappa \sinh\frac{\partial_0^x}{2\varkappa}\right)^2\right]\left[\Delta^y e^{-i\frac{\partial_0^x}{\varkappa}} - \left(2\varkappa \sinh\frac{\partial_0^y}{2\varkappa}\right)^2\right] \cdot$$
$$\cdot\, \varphi_\varkappa^{(+,+)}(x,y) = 0$$

The exponential differential operator introduces nonlocality in time

$$e^{-i\frac{\partial_0^x}{\varkappa}} \varphi_\varkappa^{(+,+)}(x;y) = \varphi_\varkappa^{(+,+)}(\vec{x}, x_0 + \tfrac{1}{\varkappa}; \vec{y}, y_0)$$

Noncommutative space-time coordinates $\Leftrightarrow$ geometric non-local interaction

In [illegible] ([illegible])

we did show that

$$\underbrace{\varphi^{(+)}(x) \,\underset{\Uparrow}{\hat{*}_\varkappa}\, \varphi^{(+)}(y)}_{\varkappa\text{-multiplication of of exponentials}} = \underbrace{\varphi^{(+)}(x) \,\underset{\Uparrow}{\circ_{rel}}\, \varphi^{(+)}(y)}_{\varkappa\text{-multiplication of oscillators}} \qquad \text{(A)}$$

provided

1) $$a(p) \circ_{rel} a(q) = \underbrace{e^{\frac{3}{2\varkappa}(\omega_\varkappa(\vec{p}) - \omega_\varkappa(\vec{q}))}}_{\text{numerical factor coming from Jacobian}} a(p) \circ a(q)$$

2) On lhs of Ⓐ one modifies the κ-deformed mass-shell condition

$$\begin{matrix} p_0 = \omega_\kappa(\vec{p}) \\ q_0 = \omega_\kappa(\vec{q}) \\ \Uparrow \\ \text{decoupled} \end{matrix} \quad \approx\!> \quad \begin{matrix} p_0 = \omega_\kappa(\vec{p}\, e^{-\frac{q_0}{2\kappa}}) \\ q_0 = \omega_\kappa(\vec{q}\, e^{\frac{p_0}{2\kappa}}) \\ \Uparrow \\ \text{coupled} \end{matrix}$$

If 1) and 2) are introduced one can obtain the c-number commutator

$$[\varphi_\kappa(x), \varphi_\kappa(y)]_{*_\kappa} = [\varphi_\kappa(x), \varphi_\kappa(y)]_{0_{rd}} = i\,\Delta_\kappa(x-y; M^2)$$

where

$$\Delta_\kappa(x; M^2) = -\frac{i}{(2\pi)^3} \int \frac{d^3\vec{p}}{\Omega_\kappa(\vec{p})} \sin(\omega_\kappa(\vec{p})\, x_0)\, e^{i\vec{p}\vec{x}}$$

obtained by replacement

$$\delta(p^2 - M^2) \longrightarrow \delta(C^2_\kappa(p) - M^2)$$

$\Uparrow$ κ-deformed mass Casimir

7. TRILINEAR ALGEBRAIC SECTOR: ASSOCIATIVITY AND FACTORIZATION OF BINARY RELATIONS

We define general binary o-product:

$$\left(a_\varkappa(p_1)\cdot\ldots\cdot a_\varkappa(p_n)\right)\circ\left(a_\varkappa(q_1)\cdots a_\varkappa(q_m)\right)=$$

$$=\prod_{i=1}^{n} a_\varkappa\Big(\vec{p}_i e^{\frac{1}{2\varkappa}\sum_{k=1}^{n} q_k^0}, p_i^0\Big)\prod_{j=1}^{m} a_\varkappa\Big(\vec{q}_j e^{-\frac{1}{2\varkappa}\sum_{k=1}^{m} p_k^0}, q_j^0\Big)$$

plus sign minus sign

We propose the triple product of $\varkappa$-oscillators

$$a_\varkappa(p)\circ\left(a_\varkappa(q)\circ a_\varkappa(r)\right)=a_\varkappa(p)\circ\Big(a_\varkappa(\vec{q}e^{\frac{r_0}{2\varkappa}},q_0)\cdot$$

$$\cdot\, a(\vec{r}e^{-\frac{q_0}{2\varkappa}},r_0)\Big)=a_\varkappa(\vec{p}e^{\frac{1}{2\varkappa}(q_0+r_0)},p_0)\cdot$$

$$\cdot\, a_\varkappa(\vec{q}e^{\frac{1}{2\varkappa}(-p_0+r_0)},q_0)\cdot a_\varkappa(\vec{r}e^{-\frac{1}{2\varkappa}(p_0+q_0)},r_0)$$

One can show that

$$\underline{a_\varkappa(p)\circ\left(a_\varkappa(q)\circ a_\varkappa(r)\right)=\left(a_\varkappa(p)\circ a_\varkappa(q)\right)\circ a_\varkappa(r)}$$

We obtain <u>associativity</u> – no need of marking the binary brackets!

Let us write

$$R: \quad a_x(r) \circ \left(a_x(p) \circ a_x(q) - a_x(q) \circ a_x(p)\right) = 0$$

Explicitely

$$a_x(\vec{r}\, e^{\frac{p_0+q_0}{2x}}, r_0)\left(a_x(\vec{p}\, e^{-\frac{r_0+q_0}{2x}}, p_0)\, a_x(\vec{q}\, e^{-\frac{(p_0+r_0)}{2x}}, q_0) - \right.$$

$$\left. - a_x(\vec{q}\, e^{-\frac{r_0+q_0}{2x}}, q_0)\, a_x(\vec{p}\, e^{-\frac{(r_0+q_0)}{2x}}, p_0)\right) = 0$$

We see that we factorize the binary relation

$$R: \quad a_x(\vec{p}(r_0), p_0) \circ a_x(\vec{q}(r_0), q_0) = a_x(\vec{q}(r_0), q_0) \circ a_x(\vec{p}(r_0), p_0)$$

where

$$\vec{p}(r_0) = \vec{p}\, e^{-\frac{r_0}{2x}} \qquad \vec{q}(r_0) = \vec{q}\, e^{-\frac{r_0}{2x}}$$

and we recall that (the mass-shell is changed)

$$p_0 = \omega_x(\vec{p}) = \omega_x(\vec{p}(r_0)\, e^{\frac{r_0}{2x}}) \qquad q_0 = \omega_x(\vec{q}) = \cdots$$

Factorizing binary relation from

$$L: \quad \left(a_x(p) \circ a_x(q) - a_x(q) \circ a_x(p_0)\right) \circ a_x(r) = 0$$

leads to

$$L: \quad a_x(\vec{p}(-r_0), p_0) \circ a_x(\vec{q}(-r_0), q_0) = a_x(\vec{q}(-r_0), q_0) \circ a_x(\vec{p}(-r_0), p_0)$$

Extra factors is due to the derivation of x-multiplication in $\mathcal{H}_2^x$ from x-multipl. in $\mathcal{H}_3^x$ – the energy of third particle is remembered.

8. N-LINEAR PRODUCTS OF $\varkappa$-OSCILLATORS AND PRODUCTS OF FIELDS

We should define the $\varkappa$-multiplication of n elements of algebra $\mathfrak{A}(a, a^+)$

$$\prod_{i=1}^{n} a_\varkappa(p_i) \circ \dots \circ \prod_{j=1}^{m} a_\varkappa(q_j) \circ \dots \circ \prod_{k=1}^{s} a_\varkappa(r_k) =$$

$$= a_\varkappa\Big(\vec{p}_1\, e^{\frac{1}{2\varkappa}\left(\sum_{i=2}^{n} p_i^0 + \dots + \sum_{j=1}^{m} q_j^0 + \dots \sum_{k=1}^{s} r_k^0\right)}, p_0\Big) \cdot \dots$$

$$\dots \cdot a_\varkappa\Big(\vec{q}_{\tilde{j}}\, e^{-\frac{1}{2\varkappa}\left(\sum_{i=1}^{n} p_i^0 + \dots \sum_{j=1}^{\tilde{j}-1} q_j^0 - \sum_{j=\tilde{j}+1}^{m} q_j^0 + \dots + \sum_{k=1}^{s} r_k^0\right)}, q_{\tilde{j}}^0\Big) \cdot \dots$$

$$\dots \cdot a_\varkappa\Big(\vec{r}_s\, e^{-\frac{1}{2\varkappa}\left(\sum_{i=1}^{n} p_i^0 + \dots + \sum_{j=1}^{m} q_j^0 + \dots + \sum_{k=1}^{s-1} r_k^0\right)}, r_s^0\Big)$$

<u>The rule</u>:

- any oscillator located in the product <u>to the left</u> produces the term $\underline{e^{-\frac{1}{2\varkappa} p_0}}$
- any oscillator located <u>to the right</u> produces the term $\underline{e^{\frac{1}{2\varkappa} p_0}}$

Having the formula above one can show the associativity: multiplication can be done in arbitrary steps $\Rightarrow$ <u>brackets not needed</u>!

One can say that the change of three-momentum dependence in the multiplication formula describes a <u>long-range coupling to all particles present in the system</u>

- n-linear multiplication in $\mathcal{H}^{\tau}_{n}$ depends on energies of remaining $n-1$ particles for a given oscillator
- if we <u>factorize m-linear relation from n-linear</u>, the m-linear multiplication <u>will "remember"</u> that it was obtained from n-linear – one gets

<u>m-linear relation in the presence of remaining $n-m$ particles</u> ⇐ spectators

These "<u>spectator particles</u>" will modify the explicite form of m-linear relation by <u>the geometric interactions with $K = n-m$ "spectators".</u>

Nonfactorizability of $*_\varkappa$-star product of n fields:

$$\varphi^{(+)}_\varkappa(\hat{X}_1)\cdots\varphi^{(+)}_\varkappa(\hat{X}_n) \xrightarrow[\text{MAP}]{\text{WEYL}} \varphi^{(+)}_\varkappa(x_1) *_\varkappa \cdots *_\varkappa \varphi^{(+)}_\varkappa(x_n)$$

noncommutative Minkowski spaces

commutative Minkowski spaces

where

$$[\hat{X}^\mu_i, \hat{X}^\nu_j] \neq 0 \quad \text{for } \underline{\text{any pair}}\ i \neq j$$

In consistency with the coinciding coordinates limit

$$\hat{X}^\mu_i \longrightarrow \hat{x}^\mu \Leftarrow \varkappa\text{-Minkowski space}$$
$$i = 1 \cdots n$$

one can assume

$$[\vec{\hat{X}}_i, \vec{\hat{X}}_j] = 0 \qquad [\hat{X}^0_i, \hat{X}^0_j] = 0$$

$$[\vec{\hat{X}}_i, \hat{X}^0_j] = \frac{i}{\varkappa}\vec{\hat{X}}_i \quad \text{for all } i,j$$

The Weyl map leads to the $*_\varkappa$-product of n fields depending on the derivatives of all fields – all n fields are coupled!

One can relate the <u>$\circ$-multiplication of oscillators</u> in $\mathcal{H}_n^x$ and <u>$*_x$-products of fields</u>

$$\varphi_x^{(+)}(x_1) *_x \cdots \varphi_x^{(+)}(x_n) = \varphi_x^{(+)}(x_1) \circ_{rel} \cdots \circ_{rel} \varphi_x^{(+)}(x_n)$$

<u>geometric interaction between all fields</u> — <u>clustering of n-particle states in $\mathcal{H}_n^x$</u>

The $*_x$-product $\varphi_x^{(+)}(x_1) *_x \cdots *_x \varphi^{(+)}(x_n)$ describes <u>n-local nonfactorizable multi-field $\varphi^{(+,\cdots+)}(x_1,\cdots x_n)$</u> on n-fold product of classical Minkowski spaces:

	<u>oscillators</u>:	<u>fields</u>:
$\mathcal{H}_0^x$	$c \cdot \mathbb{1}$	$c \cdot \mathbb{1}$
$\mathcal{H}_1^x$	$a_x(p)$	$\varphi_x^{(+)}(x)$
$\mathcal{H}_2^x$	$a_x(p) \circ a_x(q)$	$\varphi_x^{(+,+)}(x,y)$
$\vdots$	$\vdots$	$\vdots$
$\mathcal{H}_n^x$	$a_x(p_1) \circ \cdots \circ a_x(p_n)$	$\varphi_x^{(+\cdots+)}(x_1,\cdots x_n)$
	$\Uparrow$	$\Uparrow$
	<u>cluster creation operator</u>	<u>n-cluster field</u>

9. FINAL REMARKS

i) To describe κ-deformation one can use $\circ$-multiplication or κ-deformed flip operators. The $\circ$-multiplication in $\mathcal{H}_n^\kappa$ can be represented by products of "2-particle" flips $\hat{\tau}_\kappa$ (← recent result)

ii) Important problem: κ-Poincaré covariance of κ-statistics. Following Young + Zegers (arXiv: 0711.2206 [hep-th], arXiv: 0807.2745 [hep-th]) such statistics is described by generalized flip changing energies of flipped particles. ← in c-numbers commutator.
Problem: is c-number commutator and κ-covariance incompatible?

iii) MAIN ISSUE:

noncommutative space-time ↔ geometric interactions

This link should be better understood!