1) Quantum group of isometries

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(most of the work done jointly
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Ref: (i) D. Goswami; Quantum gp of isometries in classical and noncomm. geometry; to appear in Comm Math Phys (2009)

(ii) J. Bhowmick & D. Goswami: Quantum isometry groups: examples and computations; ~~to appear~~ in Comm Math Phys 285 (2009)

(iii) ~~Qtm Gp of Orien~~ J.B + DG: Qtm gp of orientation preserving Rie. isometries; arXiv 0806.3687

(iv) JB + DG: Qtm iso Gp of Podles spheres: arXiv ...

2)

We generalize various isometry gps, e.g. $\mathrm{ISO}(X,d)$ for a metric space, $\mathrm{ISO}(M)$ for a Rie manifold M, $\mathrm{ISO}^+(M) = SO(M)$ for a Rie. spin manifold etc. and define (and prove existence) their qtm gp analogous.

Basic principle: First characterize any such iso. gp as a univ. obj in a suitable category, and then consider a similar but bigger category replacing gps by qtm gps... and try to see if $\exists$ univ obj?

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~~Part I~~ : Formulation

Motivation / background:

Wang defined 'qtm automorphism gp', following a suggestion by Connes. The idea is as follows

For a finite set X, one may think of the gp of all bijections of X, i.e. permutations of X as the universal obj in the following category $\mathcal{C}_X$:

$$\mathrm{Obj}(\mathcal{C}_X) = \Big\{ (G,\pi) \;\Big|\; G \text{ is a gp}, \cancel{\pi: G \to \mathrm{Aut}(X)} \; \pi: G \to \text{set of all bijections of } X, \text{ st } \pi \text{ is a gp-homo. \& faithful} \Big\}$$

Morphisms from (G,π) to (G',π') will be gp-homomorphisms $\varphi: G \to G'$ s.t

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\varphi"'] & \text{Set of bijections of } X. \\ G' \arrow[ur, "\pi'"'] & \end{tikzcd}$$

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One can do even better; by considering a bigger category $\widetilde{\mathcal{C}}_X = \{ (F, \pi) \mid F$ is a set & $\pi : F \to$ Set of bijections of X is 1-1 map$\}$

Morphisms $(F, \pi) \to (F', \pi')$ are now set-maps φ from F to F' s.t.

$$\begin{tikzcd} F \arrow[d, "\varphi"'] \arrow[r, "\pi"] & \text{Set of bijections of } X. \\ F' \arrow[ur, "\pi'"'] & \end{tikzcd}$$

One can easily show that $\exists!$ universal obj (F_0, π_0) in this category s.t. F_0 has a canonical gp structure for which π_0 is a gp isomorphism between F_0 & $Aut(X)$.

Keeping this 'abstract' characterization of the gp $Aut(X)$, one can 'quantize' it by considering the following category:

$$Q\mathcal{C}_X = \{ (\mathcal{Q}, \alpha) \mid \mathcal{Q} \text{ is a cpt qtm gp; } \alpha : C(X) \to C(X) \otimes \mathcal{Q} \text{ action} \}$$

5)

Here, let me quickly recall a few def^{n}s:

<u>Cpt Qtm Gp (CQG)</u> is a pair (Q,Δ), where Q is a unital C^* alg, $\Delta : Q \to Q \otimes Q$ (~~minimal~~ injective tensor product), co-associative, i.e. $(\Delta \otimes id)\Delta = (id \otimes \Delta)\cdot\Delta$; and one has

$\overline{\text{Span}\, \Delta(Q)(1 \otimes Q)} = \overline{\text{Span}\, \Delta(Q)(Q \otimes 1)} = Q \otimes Q$.

From this, one can prove the existence of a dense, unital $*$-subalg $Q_0 \subseteq Q$ s.t. $\Delta(Q_0) \subseteq Q_0 \otimes_{alg} Q_0$, & Q_0 is a Hopf $*$ alg, with some antipode & counit.

Given a unital C^* alg A, we say that <u>Q acts on A</u> if $\exists$ C^* homo. (unital) $\alpha : A \to A \otimes Q$ s.t. $\overline{\text{sp}\, \alpha(A)(1 \otimes Q)} = A \otimes Q$

6)

Back to qt. automorphism gp :

We have the category $\mathcal{QC}_X = \{ (Q, \alpha) : Q$ is a cpt qtm gp, $\alpha : C(X) \to C(X) \otimes Q$ faithful action $\}$. Morphisms from (Q, α) to (Q', α') are given by qtm gp homomorphism $\pi : Q \to Q'$ st

$$\begin{tikzcd} C(X) \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & C(X) \otimes Q \arrow[d, "id \otimes \pi"] \\ & C(X) \otimes Q' \end{tikzcd}$$

[Note: qtm gp homo means a C^*-homo $\pi : Q \to Q'$ st $\Delta_{Q'} \circ \pi = (\pi \otimes \pi) \circ \Delta_Q$, where $\Delta_Q, \Delta_{Q'}$ are coproduct of Q & Q' resp.],

α faithful means $\nexists$ any proper C^* subalg. ~~$B \subset A$, st $\alpha(B$~~ $Q_1 \subset Q$ st. $\Delta(Q_1) \subseteq Q_1 \otimes Q_1$, $\alpha(A) \subseteq A \otimes Q_1$.

7)

Wang proved that for $|X|=n$ the category QG_X has a universal obj, which is CQG generated by symbols x_{ij}, $i,j=1,\dots,n$ satisfying

$$\sum_i x_{ij} = 1, \ \forall j$$

$$x_{ij}^2 = x_{ij}^* = x_{ij} \quad \forall i j$$

$$\sum_j x_{ij} = 1 \quad \forall i$$

Such univ. Qtm gp is called $\boxed{\text{Qtm Permutation Grp}}$ on n symbols.

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~~For~~ Similarly, one can try to get ~~the~~ QTm Auto. gp of $M_n(\mathbb{C})$. One should define a category consting of pairs (Q, α), where Q is a CQG & $\alpha: M_n \to M_n \otimes Q$ is a faithful action of Q. Morphisms are natural to define.

However, it is remarkable that:

There does NOT exist a universal obj in this category,

even though the subcat. consisting of gp action has Aut (M_n) ~~obj~~ as its univ. obj.

Thus qtm aut- gp of M_n does NOT exist!!

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Nevertheless, univ. obj do exist if one fixes a 'ref functional', e.g. take the usual trace tr on M_n, and look at the sub-cat. consisting of (Q, α) where α is tr-preserving in the sense

$$(\mathrm{tr} \otimes \mathrm{id}) \cdot \alpha(a) = \mathrm{tr}(a) 1_Q \quad \forall a \in M_n.$$

Then $\exists$ univ obj, which is in fact the CQG generated by $\{u_{ij}, i,j = 1, \dots, n\}$ st $u \equiv ((u_{ij}))$ satisfies

$$u^* u = I_n = u u^*, \quad u' \bar{u} = I_n = \bar{u} u',$$

where $u^* = ((u_{ji}^*))$, $\bar{u} = ((u_{ij}^*))$, $u' = ((u_{ji}))$.

with $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ being the coprod.

More generally, if we ~~change to~~ replace tr by some other ~~faithful state on M_n~~ funcl on M_n, say a faithful state,

the following 'Wang algs' will appear as the univ obj of the correspon-ding category:

$$A_u(T) \equiv C^*\{ u_{ij},\ i,j=1,\dots,n \mid uu^* = I = u^*u,\ u^t T \bar{u} T^{-1} = T \bar{u} T^{-1} u^t \}$$

where $T \in GL_n$,

Such CQG are studied extensively by Wang, Bichon, Banica and others ...

11)

All the above constructions of universal autm gps are for 'finite' structures ~~ie~~ eg finite set / matrix algs / finite graph etc. From the viewpoint of geometry or topology, it is more interesting to extend such constructions to 'continuous' or 'smooth structures' like top spaces or manifolds.

Now, we'll consider quantum generalization of group of isometries ... for metric spaces / Riemannian manifolds.

~~For a finite metric space~~, ~~Banica~~

~~above~~

Classically: Isometry gps can be ~~thought of~~ considered at different levels:

(I) <u>Metric space</u>:

- $ISO(X,d)$, for a compact ~~set~~ metric space:

This is the universal objects in the following category:

$$\text{Obj class} = \left\{ (G,\pi) \;\middle|\; \begin{array}{l} G \text{ cpt 2nd ctble gp,} \\ \pi: G \to Aut(X) \\ \text{cont. action of } G \\ \text{s.t. } d(\pi(g)x, \pi(g)y) \\ \quad = d(x,y) \\ \quad \forall x,y,g \end{array} \right\}$$

Equivalently, the objs can be described as follows. View d as a fn on $X \times X$, i.e. $d \in C(X \times X)$, G-action induces 'diagonal action'

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Then, the G-action is isometric iff the induced diagonal action, $\pi^{(2)}$ say, leaves $d \in C(X \times X)$ invariant, i.e.

$$\pi_g^{(2)}(d) = d$$

This has obvious qtm analogue:
Consider the category with objects $(\mathcal{Q}, \alpha)$, $\mathcal{Q}$ CQG (cpt qtm gp),

$\alpha : C(X) \to C(X) \otimes \mathcal{Q}$ $\mathcal{Q}$-action

s.t. $\alpha^{(2)} \equiv (id \otimes m_{\mathcal{Q}}) \circ \sigma_{23} \cdot (\alpha \otimes \alpha)$

$: C(X) \otimes C(X) \to C(X) \otimes C(X) \otimes \mathcal{Q}$

satisfies $\alpha^{(2)}(d) = d \otimes 1$,

where σ_{23} flips 2nd & 3rd tensor components, $m_{\mathcal{Q}} : \mathcal{Q} \otimes \mathcal{Q} \to \mathcal{Q}$ mult.

Remark: $\alpha^{(2)}$ is indeed CQG action on $C(X) \otimes C(X) \cong C(X \times X)$

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<u>Question</u>: Does this category have universal obj?

Banica answered it in the affirmative for <u>finite set X</u> ...

we've extended Banica's result to a general cpt metric space, thus define QISO(X,d) ≡ Qtm gp of isometries of (X,d).

~~Interesting~~ Computations for some cases are done: for X = [0,1], or S^1 (usual metric), QISO ≡ ISO. i.e. commutative! But in general, we have proved the existence of (X,d) st QISO(X,d) is <u>genuine</u> CQG!

15) (II): ~~Rie. manifold~~: approach via Laplacian

Now, we consider geometric set-up.

M Rie (cpt) manifold.

dvol Rie. vol form.

$\mathcal{H} = L^2(M, dvol)$, $C(M) \subset \mathcal{B}(\mathcal{H})$.

$\mathcal{H}_0, \mathcal{H}_1$ resp L^2-space of 0 & 1-form.

$d : \mathcal{H}_0 \to \mathcal{H}_1$ unbdd closable map,

$\mathcal{L} \equiv -d^* d$ "Laplacian"

Then, ISO(M) is the univ. obj in the category of groups G acting smoothly on M satisfying that the action commutes with $\mathcal{L}$.

More generally, one may consider (~~ISO(M) first~~ the category :

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with objects being Cpt metrizable space X, equipped with map $\psi : X \times M \to M$ s.t. $\forall x \in X$, $\psi_x : M \to M$ is smooth 1-1 map, and its induced map on $C^\infty(M)$ commutes with $\mathcal{L}$.

~~There exists~~ ISO(M) is indeed the univ. obj in this category as well — and then one can obtain the group structure of ISO(M) from its universality.

We'll generalize this to define QISO for a possibly noncomm. manifold, i.e. for a spectral triple, satisfying certain regularity conditions.

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We'll give some details of our construction for QISO of a Laplacian later, but now we just recall.

Def^n: Spectral triple: (Connes)

It is given by (A^∞, H, D),

where $\mathcal{H}$ is sep. Hil. space,

$A^\infty \subset B(H)$ $*$-subalg,

D s.a. operator

(typically unbdd) satisfying:

- $[D, a] \in B(H)$ $\forall a \in A^\infty$
- $\forall \lambda \in$ Resolvent of D, $a(D-\lambda)^{-1}$ is compact.

Sp. triple is called 'compact type' if $1 \in A^\infty$ (and hence $(D-\lambda)^{-1}$ cpt)
ie D has cpt resolvents.

~~Noncomm. analogue~~

classical sp. triple : M Rie. spin manifold, $\mathcal{H} = L^2(\text{Spinors})$, $\mathcal{A}^\infty = C^\infty(M)$, $D =$ usual Dirac operator.

Remark: Unlike classical case, ~~there is no good theory~~ the 'Laplacian' for a general noncomm manifold (i.e. sp. triple) may not have good properties. So, a def^n of QISO for sp. triples should ideally be given directly in terms of D. We've also achieved this ... however, it's not the analogue of $ISO(M)$ but that of $ISO^+(M)$.

19) This leads to ...

III Qtm gp of orientation preserving isometries : $QISO^{+}(A^{\infty}, H, D)$;

Classically, for the usual Dirac op on the spinor bundles, the following is true:

given a subgp $G \subseteq ISO^{+}(M)$ (orientation pres. isometries $\equiv SO(M)$;
here M is assumed to be spin manifold, so orientable !)

$\exists$ gp $\tilde{G}$ acting by unitaries on L^2(spinors)
st $\tilde{G}$ action commutes with D,
& $\exists$ 2-covering $\tilde{G} \to G$ ...

It is important to note that G may not ~~lift to~~ have unitary rep^n on L^2(spinors)... Ex : $SO(3) \equiv ISO^{+}(S^2)$

20)

This motivates the following:

given a sp triple (cpt type) (A^∞, H, D), consider the category whose objects are $(\tilde{\mathcal{Q}}, U)$, where $\tilde{\mathcal{Q}}$ is CQG having unitary (co)-representation U on $\mathcal{H}$ st :

(i) U commutes with D

(ii) $\alpha_U(x) \equiv U(x \otimes 1)U^*$ satisfies
$$(\mathrm{id} \otimes \varphi)(\alpha_U(A^\infty)) \subseteq (A^\infty)''$$
$\forall$ state φ on $\tilde{\mathcal{Q}}$

(Note: In the classical case, cond^n (ii) means that the $\tilde{G}$-action is measurable ... but Sobolev's Thm ensures that the action is actually smooth.)

21)

Morphisms are CQG morphisms which intertwin the unitary rep^ns

If this category has a univ. obj, we should call ~~it QIm gp of~~ ~~orient~~ the Woronowicz subalg of the univ obj making α_u 'faithful' to be the QIm gp of orientation preserving isometries. denoted by $QISO^+(D)$.

But it may NOT exist in general

To ensure existence of univ obj, we need to fix a choice of 'volume' i.e. some +ve (unbdd) R in $\mathcal{H}$, commuting with D, & $\tau_R(X) = Tr(XR)$, for $X \in$ $*$-subalg spanned by ~~eigen-vectors~~ $|\xi\rangle\langle\eta|$, ξ, η eigenvector of D.

22)

In fact, when $Tr(Re^{-tD^2}) < \infty \ \forall t > 0$, τ_R is essentially same as the final $Tr(\cdot\, Re^{-tD^2})$, so τ_R-preservation is ~~same as~~ equivalent to the fact α_U preserves $Tr(\cdot\, Re^{-tD^2})$.

Thm $\exists!$ univ obj in the above subcategory, for any fixed choice de R. This univ obj is denoted by $\widetilde{QISO}^+_R(D)$, & its maximal Woronowicz subalg for which α_U is faithful is denoted by $QISO^+_R(D)$.

We should mention that given any equivariant sp. triple (A^∞, H, D), wrt a CQG action, we can always find such an (not unique) R, which is I if the CQG is uni-modular, but o.w. $R \neq I$, ...

Some important qns:

(i) When does there exist a univ obj in the 'global' category of orientation preserving qlm isometries, i.e. without fixing any R ??

(ii) When does α_U maps A^∞ into itself and is a $\mathbb{C}^*$-action ??

24)

We have so far obtained only partial answers to these qns:

(i) has an affirmative ans. ~~for~~ when D has an eigenvalue which has 1-dim espace spanned by a cyclic sep. vector. However, there are examples when the ans. to (i) is ~~n~~ affirmative, without this assumption about cyclic sep e. vector.

(ii) We could prove: ~~class~~ for all classical sp triples, $R=I$, ans. is yes. Also, in all the examples for ~~classical~~ which we've done explicit computations we've got an affirmative ans. to this qn.

Concrete examples:

(1) Sp. triple on $A_\theta \equiv C^*(U, V \mid UV = e^{2\pi i\theta} VU,\ U, V \text{ unitary})$

θ irr. no.

$\mathcal{H} = L^2(A_\theta, \tau) \otimes \mathbb{C}^2$, $\tau(U^m V^n) = \delta_{m0}\delta_{n0}$

$$D = \begin{pmatrix} 0 & \delta_1 + i\delta_2 \\ \delta_1 - i\delta_2 & 0 \end{pmatrix}, \quad \begin{array}{l} \delta_1(U^m V^n) = m U^m V^n \\ \delta_2(U^m V^n) = n U^m V^n \end{array}$$

$\mathcal{A}_\theta^{fin} \equiv$ ~~fin~~ 'polynomials' in U, V

then $(\mathcal{A}_\theta^{fin}, \mathcal{H}, D)$ is a sp triple.

We have:

~~Edm~~ $QISO^+(D)$ exists & $\cong C(\mathbb{T}^2)$

(so classical!)

However, if we take the Laplacian based approach, where $\mathcal{L}(U^m V^n) = -(m^2+n^2) U^m V^n$

~~we have $QISO(\mathcal{L}) \cong \ldots(\mathbb{T}^3)$~~

PTO

24-(ii)

$QISO(\mathcal{L}) \cong$ direct sum of 4 copies of $C(\mathbb{T}^2)$ & four copies of $\mathcal{A}_{2\theta}$. (as a C^* alg.)

(2) Sp. triple on $S^2_{\mu c} \equiv$ Podles sphere

$$\mathcal{C} \equiv C^*\big(A, B \mid AB = \mu^{-2} BA,\ AB^* = \mu^2 B^* A,$$
$$B^*B = A - A^2 + cI,$$
$$BB^* = \mu^2 A - \mu^4 A^2 + cI,$$
$$A = A^*\big)$$

There's some canonical sp triple on this, & we get the corresponding $QISO^+_R$, for a suitable R, to be $SO_\mu(3)$.

$(0 \leq \mu \leq 1)$.

25) Some technical details:
Quantum Isometry Gp of noncomm. manifolds:: (approach via Laplacian)

~~Let (A~~

Preparatory results:

Free product of CQG (cpt qtm gp):

Let (A_1, Δ_1), (A_2, Δ_2) be CQG, which are matrix CQG, with $U_i \in M_{n_i}(A_i)$ being the fundamental unitary.

Then $A = A_1 * A_2$ (free prod. C^* alg)

View $A_1, A_2 \subset A$ as subalgs in A; and ~~take~~ then view U_1, U_2 as unitaries in $M_{n_1}(A)$ & $M_{n_2}(A)$ resp.

Let $U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \in M_{n_1+n_2}(A)$. Then U is the fundamental unitary for a CQG structure on A.

28)

In fact, this can be generalized for arbitrary family of CQG (not necessarily matrix CQG): and we have

<u>Thm</u> (Wang):
For a family $\{(A_i, \Delta_i)\}_{i\in I}$ of CQG's

$\exists$ canonical CQG structure on

$A = \underset{i}{*} A_i$, s.t. the canonical

imbedding $A_i \hookrightarrow A$ is a CQG

-morphism $\forall i$.

In case $A_i = C^*_{r}(\Gamma_i)$, Γ_i discrete gp,
we get $A = C^*_{r}(\Gamma)$, $\Gamma = \underset{i}{*} \Gamma_i$ (free product of gps)

We shall also need:

<u>Thm</u> Given unitary rep's V_i on H_i (Hil. sp.) of (A_i, Δ_i), $\exists!$ unitary rep^n $V \equiv \underset{i}{*} V_i$ of $A = \underset{i}{*} A_i$ on $\mathcal{H} = \underset{i}{\bigoplus} H_i$ s.t. $V|_{H_i} = V_i$.

Note that here $V_i : H_i \to H_i \otimes A_i$, satisfying $(id \otimes \Delta_i)V_i = (V_i \otimes id) \cdot V_i$,
& $\tilde{V}_i : H_i \otimes A_i \to H_i \otimes A_i$ given by
$\tilde{V}_i(\xi \otimes a) = V_i(\xi)(1 \otimes a)$ is A_i-linear
unitary, $\tilde{V}_i \in M(K(H_i) \otimes A_i)$.

We shall call the above unitary repⁿ of $A = *_i A_i$ the free product of the representation V_i of A_i, $i \in I$.

Now, let us assume that we are given a spectral triple (A^∞, H, D), which is of cpt type, i.e. A^∞ is unital & D has cpt resolvent. Moreover, assume that D is Θ-summable, i.e. $Tr(e^{-tD^2}) < \infty$ $\forall t > 0$, and $\tau(x) = \underset{t \downarrow 0+}{Lim} \frac{Tr(e^{-tD^2} x)}{Tr(e^{-tD^2})}$ be the canonical ~~trace~~ the final. ('Lim' suitable Banach limit)

29) We'll call τ the 'volume form' on the noncomm manifold A^∞.

Suppose also that A^∞, $[D, A^\infty] \subseteq$ domain of all powers of the derivation $[|D|, \cdot]$. Then τ is a trace on the $*$-subalg of $B(H)$ gen by A^∞, $[D, A^\infty]$

Let $H^0_D = L^2(A^\infty, \tau)$,

~~Ω^1_D = A^∞-bimodule gen by~~

$\Omega^1_D = \mathrm{sp}\{a_0[D, a_1],\ a_0, a_1 \in A^\infty\}$, $\subseteq B(H)$

viewed as A^∞-A^∞ bimodule in the obvious way.

τ is a faithful trace on Ω^1_D too, so we can equip Ω^1_D with an inner product $\langle \cdot, \cdot \rangle_D$ given by $\langle \omega, \eta \rangle_D = \tau(\omega^* \eta)$, $\omega, \eta \in \Omega^1_D \subseteq B(H)$

Let H^1_D be the ~~completed~~ Hil sp. obtained from Ω_D wrt $\langle \cdot, \cdot \rangle_D$.

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$d_D : H_D^0 \to H_D^1$, densely defined lin. map, $d_D(a) = [D, a]$, $a \in A^\infty$.

We assume:

(i) d_D closable, (closure denoted by d_D again)

$A^\infty \subseteq \mathcal{D}(\mathcal{L} = -d_D^* d_D)$

(ii) $\mathcal{L} = -d_D^* d_D$ has cpt resolvents

(iii) $\mathcal{L}(A^\infty) \subseteq A^\infty$

(iv) eigenvectors of $\mathcal{L}$ belongs to A^∞

(v) ' ~~[illegible]~~ '

$Sp\{e_{ij}\} \equiv A_0^\infty$ is norm-dense in $\bar{A}^\infty$,

where $\{e_{ij}, j = 1..., d_i; i \geq 0\}$ are e. vectors of $\mathcal{L}$.

Remark : All these are valid for a classical cpt Rie manifold

We also argue that these are not unnatural in the noncomm situation. Indeed, we have

Thm If the sp. triple satisfies the additional cond^n that ~~$\forall a \in A^\infty$,~~ $t \mapsto e^{itD} a e^{-itD}$ is norm-differentiable at $t=0$ for $a \in A^\infty \cup [D, A^\infty]$, then

- d_D is closable
- $d_D^* d_D (A^\infty) \subseteq (A^\infty)''$

We omit the pf here, but this tells us that some of the assumption in the list (i)–(v) are true ~~in~~ under very mild cond^n. ~~The main~~ The other assumption regarding cpt resolv. of $\mathcal{L}$ & density of

~~28~~) 31)

eigenvectors have to be checked case-by-case, and this can be done for ~~the~~ most of the well-known examples, like the standard sp-triple on A_θ on those on qtm Heisenberg manifolds or more generally, obtained from Rieffel-type deformation.

We ~~hope~~ conjecture that these assumption can be proven under the cond^n $x \mapsto e^{itD} x e^{-itD}$ norm-diffble for $x \in A^\infty$; $[D, A^\infty]$.

<u>Def^n</u> If the sp triple satisfies (i)–(v) and also the additional assumption that $\operatorname{Ker} \mathcal{L}$ is one-dimensional ($= \mathbb{C}1$) then we say that the sp. triple is <u>admissible</u>.

32)

Def^n & existence of q. iso. gp:

Fix a sp triple as above satisfying (i)–(ii)

Def^n A <u>quantum family of smooth</u> <u>isometries</u> of (A^∞, H, D) is a pair (S, α), where S is a unital separable C^* alg, $\alpha : A \to A \otimes S$ unital C^* homo; s.t.

A = closure of A^∞ in $B(H^0_D)$

- $\overline{Sp(\alpha(A)(1 \otimes S))} = A \otimes S$
- $\alpha_\phi \equiv (id \otimes \phi) \circ \alpha$ maps A^∞_0 into itself & commutes with $\mathcal{L}$ on A^∞_0 $\forall$ state ϕ on S.

~~5~~ 39) Motivation for the above def^n:

Let Y be a cpt 2nd ctble space with cont map $\theta : M \times Y \to M$.

We write $\theta(m,y)$ simply as my; $\xi_y : m \mapsto my$. Let $\alpha : C(M) \to C(M) \otimes C(Y)$ be given by $\alpha(f)(m,y) = f(my)$.

For a state ϕ on $C(M)$, we denote by α_ϕ the map $(id \otimes \phi) \circ \alpha$ on $C(M)$.

We have:

<u>Thm</u>: Let $\mathcal{C}_0$ be the subspace spanned by $\alpha(f)(1 \otimes \psi)$, $f \in C(M)$, $\psi \in C(Y)$. Then:

(i) ξ_y is C^∞ $\forall y \iff \alpha_\phi(C^\infty(M)) \subseteq C^\infty(M)$ $\forall$ state ϕ

(ii) $\mathcal{C}_0$ is norm-dense in $C(M) \otimes C(Y)$ iff $\forall y \in Y$, ξ_y is 1-1

(iii) Under the hyp (ii), ξ_y is isometry $\forall y \iff \alpha_\phi, \mathcal{L}$ commutes

3)

We say that (S,α) is <u>volume-preserving</u> if $(\tau\otimes id)(\alpha(a)) = \tau(a)1_S$, $\forall a \in A^\infty$.

In case S has a CQG structure given by a coproduct Δ, and α is an action of (S,Δ) on A, we say (S,Δ) acts smoothly & isometrically on the noncomm manifold (A^∞, H, D).

Let $\mathcal{Q}$ be the category whose obj. are the qtm families of smooth isometries (S,α), with the obvious morphism.

Let $\mathcal{Q}'$ be the cat. of CQG (S,Δ), having action α on A st $(S,\alpha)\in Obj(\mathcal{Q})$.

Clearly, we have the forgetful functor $F: \mathcal{Q}' \to \mathcal{Q}$, and $F(\mathcal{Q}')$ is a subcategory of $\mathcal{Q}$.

<u>Lemma</u> In case (A^∞, H, D) is admissible, i.e. $\ker R = \mathbb{C}1$, ~~we have~~ any qtm family of isometry is volume-preserving.

~~Pf. is easy by a lemma mentioned in the last talk ...~~

Let $\mathcal{Q}_0$ = subcat. of $\mathcal{Q}$ consisting of volume-preserving qtm families of smooth isometries;

$\mathcal{Q}'_0$ = similar subcat. of $\mathcal{Q}'$.

We have $\mathcal{Q} = \mathcal{Q}_0$, $\mathcal{Q}' = \mathcal{Q}'_0$ for <u>admissible sp. trip.</u>

~~However, in gener~~

- Also, we've seen that if the sp triple is classical, $Q = Q_0$, $Q' = Q_0'$.

- However, in general, they may differ! For example, take $A = M_n(\mathbb{C})$, $D = I$, $H = \mathbb{C}^n$, and note that there are many CQG actions which don't preserve tr of M_n.

- On the other hand, admissibility $\equiv$ connectedness for classical sp. triples. But even for disconnected classical sp. triples, we have automatic volume preserving; ~~so the Lemma stated before is does not m~~

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For the sp triple (A^∞, H, D) satisfying (i)-(v),

let $H_D^0 = \bigoplus_{i=0}^{\infty} H_i$, $H_i = \mathrm{sp}\{e_{ij}, j=1,\dots,d_i\}$

let $V_i : H_i \to H_i \otimes A_u(I_{d_i}) \equiv H_i \otimes \mathcal{U}_i$,

(where $\mathcal{U}_i = A_u(I_{d_i})$ = CQG gen. by $u = ((u_{kl}^{(i)}))_{k,l=1}^{d_i}$ st u, u' are unitaries)

$$V_i(e_{ij}) = \sum_{k=1}^{d_i} e_{ik} \otimes u_{kj}^{(i)}$$

This is unitary rep^n of $\mathcal{U}_i$ on H_i,

so $\exists!$ unitary rep^n of $\mathcal{U} \equiv *_i \mathcal{U}_i$ on $\mathcal{H}_D^0$ as described before; ~~call that~~ denote it by $V : H_D^0 \to H_D^0 \otimes \mathcal{U}$.

We also need:

Def^n A Woronowicz C^* ideal $\mathcal{J}$ of a CQG (S, Δ) is C^* subalg st $\Delta \mathcal{J} \subseteq \mathcal{J} \otimes S + S \otimes \mathcal{J}$.

~~On the other hand, ∃ non-admiss~~

Thm For the sp triple (A^∞, H, D) satisfying (i)–(v) (but not necessarily admissible) let $(S,\alpha) \in \mathrm{Ob}_g(\mathcal{Q}_0)$. Assume also that α is faithful, i.e. $\nexists$ proper C^*-subalg $S_1 \subset S$ st $\alpha(A^\infty) \subseteq A^\infty \otimes S_1$. Let $\tilde{\alpha} : A^\infty \otimes_{alg} S \to A^\infty \otimes S$ given by

$$\tilde{\alpha}(a \otimes x) = \alpha(a)(1 \otimes x), \quad a \in A^\infty, \; x \in S.$$

Then $\tilde{\alpha}$ extends to the Hilbert S-module $H_D^0 \otimes S$ as S-lin. unitary.
and $\exists$ C^*-ideal (not necessarily Woronowicz ideal!) $\mathcal{J}$ of $\mathcal{U} \equiv *_i \mathcal{U}_i$ and C^* isomorphism $\phi : \mathcal{U}/\mathcal{J} \to S$ st $\alpha = (id \otimes \phi) \circ (id \otimes \pi_{\mathcal{J}}) \circ V$ on $A^\infty \subseteq H_D^0$. ($\pi_{\mathcal{J}} : \mathcal{U} \to \mathcal{U}/\mathcal{J}$.)

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Moreover, if S has CQG structure s.t. it acts smoothly isometrically & vol-preserving way on A^∞, the above ideal can be chosen to be a Woronowicz ideal & ϕ is then isomorphism of CQG between S and $\mathcal{U}/\mathcal{J}$. (quotient CQG)

Pf: Clearly, $\langle \alpha(x), \alpha(y) \rangle_S = (\tau \otimes \mathrm{id})\alpha(x^* y)$
$= \tau(x^* y) 1_S$
$= \langle x, y \rangle 1_S$

So $\tilde{\alpha}$ is clearly isometry.
It's ~~…~~ has dense range since $\alpha(A^\infty)(1 \otimes S)$ is norm dense so dense in Hil module sense.

Now, ~~…~~ $\tilde{\alpha}(H_i) \subseteq H_i \otimes S$ unitary
& $\tilde{\alpha}(e_{ij}) = \sum e_{ik} \otimes s_{kj}$, say $s_{kj} \in S$
we have $\tilde{\alpha}(e_{ij}^*) = \alpha(e_{ij})^* = \sum e_{ik}^* \otimes s_{kj}^*$.

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Note now that (easy to prove!)

$\rho(x^*) = \rho(x)^*$, so $e_{ij}^* \in$ eigensp at same e-value | i.e. $sp\{e_{ij}^*\}_{j=1}^{d_i} = sp\{e_{ij}\}_{j=1}^{d_i} = H_i$,

and τ being tracial, $\{e_{ij}^*\}_{j=1}^{d_i}$ orthonormal basis too.

Thus, (s_{kj}^*) is unitary too!

This gives the C^* homo. from $\mathcal{U}_i$ to S, given by $u_{kj}^{(i)} \mapsto s_{kj}$

Faithfulness of α tells us that

$C^*\{s_{kj}, k, j\}$ ~~is~~ $= S$, so

the map $\mathcal{U} \to S$ is onto.

Rest of the arguments are routine.

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Thm :

For a st triple (A^∞, H, D) satisfying (i)–(v), the category $\mathcal{Q}_0$ of volume-preserving qtm families of smooth isometries has a (unique) universal obj, say $(\mathcal{G}, \alpha_0)$. Moreover, $\mathcal{G}$ has a coproduct Δ_0 st $(\mathcal{G}_0, \Delta_0)$ is CQG, α_0 is CQG action and $(\mathcal{G}_0, \Delta_0, \alpha_0)$ is universal in $\mathcal{Q}_0'$. The action α_0 is faithful.

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Pf (Sketch):

Lemma: For a family of closed two-sided ideals $\{J_i\}_i$ of a C^* alg $\mathcal{C}$, we have

$$\|x + J\| = \sup_i \|x + J_i\|$$

where $J = \bigcap_i J_i$,

$x + J_i \in A/J_i$

Now, we consider the family $\mathcal{F}$ of all C^* ideals of $\mathcal{U}$ s.t. $\forall g \in \mathcal{F}$, the map $(id \otimes \pi_g) V : A_0^\infty \to A_0^\infty \otimes \mathcal{U}/g$ extends to C^*-homo. from A to $A \otimes \mathcal{U}/g$.

$\mathcal{F} \neq \phi$ as trivial $\mathcal{U}$ give a member of $\mathcal{F}$.

Then, take $\mathcal{J}_0 = \bigcap_{\mathcal{J} \in \mathcal{F}} \mathcal{J}$, and check by the Lemma ~~that~~ & norm-density of A_0^∞ that $\mathcal{J}_0 \in \mathcal{F}$ too.

Thus $\mathcal{G}_0 = \mathcal{U}/\mathcal{J}_0$ clearly is the univ obj in $\mathcal{Q}_0$.

Rest are standard. $\blacksquare$