QUANTUM SPACETIME

AND

NONCOMMUTATIVE GEOMETRY

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joint work with

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(in progress)

BASED ON

– , K. FREDENHAGEN , J.E. Roberts
'94
'95

FURTHER REFS:

D. BAHNS , – , K. FREDENHAGEN ,
G. PIACITELLI '02
'03
'05
+ in progress

SURVEYS & OUTLOOK:

S. D. , Arxiv hep-th '01
'06

Heisenberg Uncertainty Principle:

sharp localization requires concentration of energy;

CLASSICAL GENERAL RELATIVITY:

that energy might give rise to CLOSED TRAPPED SURFACES, HIDING THE EVENT ITSELF;

$\Downarrow$

UNCERTAINTY RELATIONS AMONG COORDINATES OF AN EVENT

S T U R

$\Uparrow$

COMMUTATION RELATIONS

$$[q^\mu, q^\nu] = i\, Q^{\mu\nu}$$

e.g. Δx_0 UNLIMITED, (STATIONARY)

$\Delta x_1 \sim \Delta x_2 \sim \Delta x_3 \sim a$,
spherical symmetry, THEN

$a \gtrsim$ Schwarzschild radius $= E \sim \frac{1}{a}$

i.e. $a \gtrsim 1$ ($\hbar = G = c = 1$)

= PLANCK LENGTH, $\sim 1.6 \times 10^{-33}$ cm.

BUT IF $\Delta x_0 = \infty$ (stationary field),

$\Delta x_j, \Delta x_k$ ARBITRARILY SMALL BUT FIXED, $\Delta x_\ell \sim L \to \infty$,

THE CLASSICAL POTENTIAL GENERATED $\to 0$ as $L \to \infty$.

- A SINGLE COORDINATE CAN BE MEASURED WITH ARBITRARY PRECISION
- SPACETIME UNCERTAINTY RELATIONS MUST HOLD.

i.e.

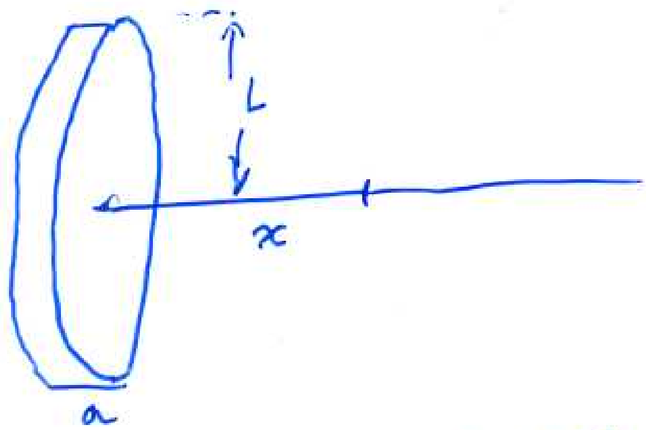
$$a \gtrsim \lambda_p = \left(\frac{G\hbar}{c^3}\right)^{1/2} \sim 1.6 \times 10^{-33}\ \mathrm{cm}$$

($\rightarrow$ PICTURE)

BUT, if $\Delta x \sim a$ possibly very small

$\Delta y \sim \Delta z \sim L$ possibly very large

$E \sim \frac{1}{a}$ MAY SPREAD OVER DISC OF RADIUS L; uniform distribution generates potential $-\varphi_a$, $\varphi_a \leq \varphi_0$



$$\varphi_0 = \frac{2GE}{c^2} \frac{(x^2+L^2)^{1/2} - x}{L^2} \xrightarrow[L\to\infty]{} 0$$

for any distance x from the disc.

$\boxed{\text{SAME}}$ CONCLUSION IF

$\Delta x \sim \Delta y \sim a$ (SMALL) $\Delta z \sim L \to \infty$.

$$\Delta q_0 (\Delta q_1 + \Delta q_2 + \Delta q_3) \gtrsim 1$$

$$\Delta q_1 \Delta q_2 + \Delta q_2 \Delta q_3 + \Delta q_3 \Delta q_1 \gtrsim 1$$

$$\hbar = c = G = 1$$

UNIT OF LENGTH:

$$\lambda_P \sim 1.6 \times 10^{-33} \text{ cm}$$

THESE ST UR

ARE MINIMAL RESTRICTIONS

IMPLIED BY QM + GR;

MUST EXPECT DEPENDENCE ON

$g_{\mu\nu}$ DETERMINED BY THE

BACKGROUND QUANTUM STATE

$\Longrightarrow$

$$[q^\mu, q^\nu] = i\, Q_{\mu\nu}(g)$$

COUPLED TO E.E. +

FIELD EQ. $\Longrightarrow$

GR: $\boxed{\text{GEOMETRY} \sim \text{DYNAMICS}}$

QG: $\boxed{\text{ALGEBRA} \sim \text{DYNAMICS}}$

UNCERTAINTY RELATIONS

$\simeq$ NON COMMUTATIVITY

$$\Delta_\omega A \cdot \Delta_\omega B \geq \frac{1}{2} |\omega([A,B])|$$

BY SCHWARZ INEQUALITY.

"MEASURES" OF NONCOMMUTATIVITY

$$[q^\mu, q^\nu] \equiv \det\begin{pmatrix} q^\mu & q^\nu \\ q^\mu & q^\nu \end{pmatrix} = q^\mu q^\nu - q^\nu q^\mu;$$

$$\equiv i\, Q^{\mu\nu};$$

$$[q^0, q^1, q^2, q^3] \equiv \frac{1}{2} \det\begin{pmatrix} q^0 & q^1 & q^2 & q^3 \\ q^0 & q^1 & q^2 & q^3 \\ q^0 & q^1 & q^2 & q^3 \\ q^0 & q^1 & q^2 & q^3 \end{pmatrix}$$

$$= \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho}\, q^\mu q^\nu q^\lambda q^\rho;$$

$$= \frac{1}{4} Q_{\mu\nu} (*Q)^{\mu\nu};$$ with

$[[q^\mu, q^\nu], q^\rho]$. Q. C. :

1) $\hookrightarrow \equiv 0$. ($\Leftrightarrow Q^{\mu\nu}$ are CENTRAL).

1) $[q_\mu, Q_{\nu\lambda}] = 0$

2) $Q_{\mu\nu} Q^{\mu\nu} = 0$

3) $[q^0, q^1, q^2, q^3]^2 = I$

NOTE $\wedge 2$ BECAUSE

$[\ ,\ ,\ ,\ ]$ is a PSEUDOSCALAR,

while $Q^{\mu\nu} Q_{\mu\nu}$ is a scalar.

THIS IS ONLY THE ALGEBRAIC PART OF OUR "BASIC MODEL" OF QUANTUM SPACETIME.

COMMENTS:

- WE ARE SEARCHING FOR A QUANTUM MINKOWSKI SPACE: NO COVARIANCE UNDER GEN. COORD. TR.: SAKE OF EL. PART. PHYS. LARGE SC. DISTR. OF OUTSIDE MASSES IRRELEVANT (EACH COLLISION CAN BE IDEALIZED AS OCCURRING IN VACUUM).

- THE CLASSICAL POINCARÉ GROUP SHOULD ACT AS GLOBAL ST SYMM, AS THE GALILEI GR. ACTS ON Q MECHANICAL OPS $\vec{q}$, $\vec{p}$.

- QST SHOULD PROVIDE A BETTER GEOM. BACKGROUND FOR QG (LOCAL MODEL).

REGULAR REPS, s.t. WEYL REL. HOLD

$$e^{i\alpha q} e^{i\beta q} = e^{\frac{i}{2}\alpha Q \beta} e^{i(\alpha+\beta)q}$$

via their linearization

$$\pi(f) = \int f(Q, \alpha) e^{i\alpha q} d^4\alpha$$

are 1-1 with the reps of their ENVELOPING C^* ALGEBRA $\mathcal{E}$.

IT TURNS OUT THAT

$$\mathcal{E} = \mathcal{C}_0(\Sigma, \mathcal{K});$$

$$\Sigma = \{ \sigma \text{ antisym real 2-tensors s.t. } \sigma_{\mu\nu}\sigma^{\mu\nu} = 0, \ \left(\tfrac{1}{4}\sigma_{\mu\nu}(*\sigma)^{\mu\nu}\right)^2 = I \}$$

= joint sp $\{ Q_{\mu\nu}, \ \mu, \nu = 0, .., 3 \}$ in fen. reps.

$$Q_{\mu\nu} = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & m_3 & -m_2 \\ -e_2 & -m_3 & 0 & m_1 \\ -e_3 & m_2 & -m_1 & 0 \end{pmatrix}$$

(CENTRAL!), QUANTUM CONDITIONS AMOUNT TO

$$e^2 = m^2,$$

$$(\vec{e}\cdot\vec{m})^2 = 1.$$

DEFINE MANIFOLD Σ (joint spectrum $Q_{\mu\nu}$):

$$\Sigma = \Sigma_+ \cup \Sigma_-, \quad \Sigma_+ \sim \Sigma_- \sim$$

$$\sim L_+^\uparrow \Big/ \begin{matrix}\text{boost along } z \\ \text{rot around } z\end{matrix} \sim SL(2,\mathbb{C})/\text{diag}$$

$$\sim TS^2; \ \Sigma \sim \{ z \in \mathbb{C}^3 / z^4 + 1 = 0 \}.$$

FULL DISCUSSION IN DFR 1995. RESULT:

$$\mathcal{E} \sim \mathcal{C}_0(\Sigma) \otimes \mathcal{K},$$

$\mathcal{K}$ = compact ops on separable ∞-dim H. S.

$\exists$! $\tau : \mathcal{P} \to \mathrm{Aut}\,\mathcal{E}$ s.t.

$$\tau_L(q) = L^{-1} q,$$

q^μ "AFFILIATED" TO $\mathcal{E}$,

FULFILLING STUR.

STATES ON $\mathcal{E}$:

$$\omega : \mathcal{E} \longrightarrow \mathbb{C}$$

LIN, POS, NORMALIZED,

REPLACE POINTS.

"POINTS", DISTANCE, AREA AND VOLUMES

IRREPS:

$$[q^\mu, q^\nu] = i\,\sigma^{\mu\nu} \cdot I$$

$$\sigma \in \Sigma = \text{jsp}\{Q_{\mu\nu};\ \mu,\nu = 0,\dots,3\}$$

FULL LORENTZ GROUP L ACTS TRANSITIVELY ON Σ ($L_+^\uparrow$ ON $\Sigma_\pm$); SPECIAL σ:

$\sigma_0 = (\vec{e}, \vec{m})$ with $\vec{e} = \vec{m} = (1,0,0)$, i.e.

$$\sigma_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix};$$

THEN:

$$q^\mu = \begin{pmatrix} Q \otimes I \\ P \otimes I \\ I \otimes Q \\ I \otimes P \end{pmatrix}$$

Q, P : SCHRÖDINGER OPS
1 - d. of f.

~ multipl by s , $-i\frac{d}{ds}$ on $L^2(\mathbb{R})$.

$$\sum_{\mu=0}^{3} (q^\mu)^2 = 2\,(H \otimes I + I \otimes H),$$

H = Hamiltonian of harmonic osc.

$\geq \frac{1}{2}$,

$$\Rightarrow \quad \sum_{\mu=0}^{3} (\Delta q^\mu)^2 \geq 2$$

IN THIS REP; BUT HOLDS ALWAYS TRUE, ≡ IFF : ~~AS~~ STATED, i.e.

$$\sum_{\mu=0}^{3} (\Delta_\omega q^\mu)^2 = 2 \quad \text{(MIN)}$$

$$\Updownarrow$$

$$\omega = \mu \circ \bar{\eta} \circ \rho$$

$\rho =$ quotient map

$$\mathcal{E} \equiv C_0(\Sigma, \mathcal{K}) \to C(\Sigma_1, \mathcal{K})$$
$$\equiv \mathcal{E}_1$$

$\bar{\eta} \equiv$ UNIVERSAL LOCALIZATION CONDITIONAL EXPECTATION

$$\mathcal{E}_1 \longrightarrow C(\Sigma_1) \equiv Z(M(\mathcal{E}_1))$$

$\mu =$ ANY REGULAR PROBABILITY MEASURE ON Σ_1.

15'

$\eta \equiv \tilde{\eta} \circ e$:

$$\left\{ \sigma \in \Sigma \longrightarrow \int f(\sigma,\alpha) e^{i\alpha q_\sigma} d^4\alpha \right\}$$

$$\longrightarrow \left\{ \sigma \in \Sigma_1 \to \int f(\sigma,\alpha) e^{-\frac{1}{4}|\alpha|^2} d^4\alpha \right\}$$

LARGE SCALE LIMIT

(compared to Planck):

$X \longrightarrow$ ORDINARY CONVOLUTION

$= \mathcal{Y}$ (pointwise prod)

$\mathcal{E} \longrightarrow \mathcal{C}_0(\mathbb{R}^4 \times \Sigma)$

$QST \longrightarrow \mathbb{R}^4 \times \Sigma$

IF PROBED WITH OPTIMALLY LOCALIZED STATES:

$\longrightarrow \mathbb{R}^4 \times \Sigma_1 \sim \mathbb{R}^4 \times S^2 \times \{\pm\}$

CONNES-LOTT
2 dim

REPS. OPT. LOC. STATES:

$(q^\mu) \rightarrow$ SCHROEDINGER OPS

PHASE SPACE: CELLS OF VOLUME

$\sim \lambda_P^4$.

CALCULUS ON QST

(DFR '94, '95)

$$f(q) \equiv \int \check{f}(\alpha) e^{i\alpha q} d^4\alpha$$

$f(q)\, g(Q)$ span $\mathcal{E}$ linearly, $f \in L^1$, $g \in \mathcal{C}_0(\Sigma)$.

$$f(q)\, g(q) = (f_1 \times f_2)(q),$$

$$(f_1 \times f_2)(Q, \alpha) = \int f_1(\alpha') f_2(\alpha - \alpha') e^{\frac{i}{2}\alpha Q \alpha'} d^4\alpha'$$

IN IRREPS: MOYAL. WARNING!

- $$dA = \sum_\mu \frac{\partial}{\partial \alpha_\mu} \tau_\alpha(A)\, d\alpha_\mu \Big|_{\alpha=0};$$

- $$\int f(q)\, d^4q = \operatorname{Tr} f(q) = \check{f}(0),$$

- $$\int_{q_0 = t} f(q)\, d^3q = \lim_n \operatorname{Tr} g_n(q)^* f(q)\, g_n(q) = \int d\alpha_0\, e^{i\alpha_0 t}\, \check{f}(\alpha_0, \vec{0})$$

QUANTUM FIELDS ON QST

With ϕ a Wightman field on ordinary ST ($\mathbb{R}^4$),

$$f \longrightarrow \phi(f) = \text{“}\int \phi(x) f(x)\, d^4x\text{”},$$

can define

$$\phi(q) \equiv (2\pi)^{-2} \int e^{ikq} \otimes \check{\phi}(k)\, d^4k$$

BASIS FOR PERTURBATION THEORY (DFR 95, BDFP 02, 03); $\boxed{\text{ULTRAVIOLET FINITE}}$ IF BASED ON "QUANTUM WICK PRODUCT" (BDFP 03).

LOCAL ALGEBRAS ?

MAIN RESULT

(D. BAHNS, K. FREDENHAGEN, G. PIACITELLI, S.D.) :

WITH

$$:\phi(q)^n:_Q \equiv$$

$$E^{(n)}(:\phi(q_1)\dots\phi(q_n):),$$

$$H_I(t) = \int_{q_0=t} d^3q \; g(t) :\phi(q)^n:_Q ,$$

$g \in \mathcal{S}(\mathbb{R}^1)$ ADIAB. SWITCHED OFF COUPLING,

the Gell-mann Low Dyson exp of the Scattering Matrix

$$T \exp\{i \int_{-\infty}^{\infty} H_I(t)\,dt\} / \langle \quad \rangle_{vac-vac}$$

IS <u>ULTRAVIOLET FINITE</u> IN ITS PERTURBATION EXPANSION.

HOWEVER:

- HARD TO TAKE THE ADIABATIC LIMIT $g \longrightarrow 1$;
- NEEDS FINITE RENORMALIZATION TO HAVE PHYSICAL SIGNIFICANCE (should give back usual ren. theory, if any, in the limit $\lambda_P \sim 0$);
- LORENTZ COVARIANCE IS BROKEN BY $:\ :_{CQ}:$

"$\sum (dq^\mu)^2 = \text{MIN}$" DEPENDS UPON CHOICE OF REFERENCE SYSTEM!

ALL KNOWN APPROACHES TO INTERACTING QFT ON QST BREAKS LORENTZ COV.!!

BUT: FLAMINIO BRUNELLI, w.i.p.

$$\int_{q_0 > t} d^3q \ \lambda :\phi^n(q):_{Q} =$$

$$= \int_{x_0 = t} d^3x \ H^I_{eff}(x) \ ;$$

$$H^I_{eff}(x) =$$

$$= \int_{\mathbb{R}^{4n}} d^4x_1 \dots d^4x_n \ K_n(x - x_1, \dots, x - x_n) \ \lambda :\phi(x_1) \dots \phi(x_n): \ ,$$

$$K_n(y_1, \dots, y_n) =$$

$$= C_n \, e^{-\frac{1}{2} \sum_{\mu, j} (y^\mu_j)^2} \, \delta^{(4)}\left(\sum_{j=1}^{n} y_j\right).$$

(BDFP, CMP 237, 221-241 (2003))
hep-th/0301100

FAST (or Gaussian $\overset{\text{polin } x}{}$) VANISHING OF CROSS SECTIONS AT TRANSPLANCKIAN ENERGIES (~STRING TH)

$20'$

n INDEPENDENT EVENTS :

$$\mathcal{E} \otimes \mathcal{E} \otimes \ldots \otimes \mathcal{E} \quad n \text{ times}$$

$$q_j = I \otimes I \otimes \ldots \otimes q \otimes \ldots \otimes I$$

$\uparrow$ j th place.

REQUIRE :

$$[q_j^\mu, q_j^\nu] = [q_k^\mu, q_k^\nu]$$

i.e. $\otimes \longrightarrow \otimes_{\mathcal{Z}}$ of

$\mathcal{Z}$ – BIMODULES ,

$$\mathcal{Z} = Z(M(\mathcal{E})) = \mathcal{C}_B(\Sigma);$$

e.g. $A, B \in \mathcal{E}$, $f \in \mathcal{Z}$:

$$Af \otimes B = A \otimes f B$$

$\Longleftrightarrow$ (cf LATER)

$$dQ = 0.$$

WILL USE A

"REDUCED" UNIVERSAL

DIFFERENTIAL CALCULUS:

$\mathcal{O}$ alg with I,

$$\Lambda(\mathcal{O}) = \bigoplus_{n=0}^{\infty} \underbrace{\mathcal{O} \otimes \mathcal{O} \otimes \ldots \otimes \mathcal{O}}_{n\text{-times}}$$

with product

$$(a_1 \otimes \ldots \otimes a_n)(b_1 \otimes \ldots \otimes b_m) \equiv$$

$$a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n b_1 \otimes b_2 \otimes \ldots \otimes b_m$$

$$dA \equiv I \otimes A - A \otimes I$$

$\Omega(\mathcal{O})$ = subalg of $\Lambda(\mathcal{O})$

gen by $\mathcal{O}$, $d\mathcal{O}$.

If Z is a subalg of $Z(\mathcal{O})$

later will further specify to

$\otimes_Z$.

HERE

$$dA = I \otimes A - A \otimes I$$

in $\mathcal{E} \otimes_{\mathcal{Z}} \mathcal{E}$, where

$$\mathcal{Z} = Z(M(\mathcal{E})) \sim C_B(\Sigma),$$

so that

$$I \otimes Q_{\mu\nu} = Q_{\mu\nu} \otimes I,$$

with $q_1^\mu \equiv q^\mu \otimes I$, $q_2^\mu \equiv I \otimes q^\mu$.

$$[q_1^\mu, q_1^\nu] = [q_2^\mu, q_2^\nu] = i\, Q^{\mu\nu};$$

$\frac{1}{\sqrt{2}}(q_2^\mu - q_1^\mu) \equiv \frac{1}{\sqrt{2}}\, dq^\mu$ fulfill

$$\left[\frac{1}{\sqrt{2}}\, dq^\mu, \frac{1}{\sqrt{2}}\, dq^\nu\right] = i\, Q^{\mu\nu}.$$

MAIN RESULTS

20''''

MINIMAL DISTANCE:

$$\sum_{\mu=0}^{3} |dq^\mu|^2 \geq 4$$

THE AREA AND VOLUME OPERATORS ARE NORMAL, AND

- $$\sum_{1\leq j<k\leq 3} |dq_j \wedge dq_k|^2 \geq 1,$$

  $$\sum_{j=1}^{3} |dq_0 \wedge dq_j|^2 \geq 1;$$

- $$\sigma(dq^1 \wedge dq^2 \wedge dq^3) = \mathbb{C}$$

- $dq\wedge dq\wedge dq\wedge dq$ IS A LORENTZ PSEUDOSCALAR OPERATOR WITH PURE POINT SPECTRUM

$$\sigma_p(dq\wedge dq\wedge dq\wedge dq) = \pm 2 + \mathbb{Z}ab + i(\mathbb{Z}a + \mathbb{Z}b)$$
$$= \pm 2 + \mathbb{Z}\sqrt{5} + i(\mathbb{Z}\sqrt{5-2\sqrt{5}} + \mathbb{Z}\sqrt{5+2\sqrt{5}}),$$

so that $|dq\wedge dq\wedge dq\wedge dq| \geq \sqrt{5}-2$.

BUT $\sigma(|3\text{-VOLUME OP}|)$ HAS NO GAP.

NOTE $\bigwedge_{\mathcal{Z}}(\mathcal{E})$ has

two alg. structures:

- that of Ω-dimens. tensor alg
- that of $\bigoplus_n \mathcal{E}^{\otimes_{\mathcal{Z}} n}$;

$dq \wedge .. \wedge dq$ has to be computed with the first prod; it produces elements whose norms, spectra, .. are evaluated in the (C*-completion of the) second.

Area: $j = 1, 2, 3$;

$$dq^j \wedge dq^k = (I \otimes q^j - q^j \otimes I)(I \otimes q^k - q^k \otimes I) - (j \rightleftarrows k) =$$

$$= I \otimes q^j \otimes q^k - I \otimes q^j q^k \otimes I - q^j \otimes I \otimes q^k + q^j \otimes q^k \otimes I - (j \rightleftarrows k) =$$

$$\text{s.a.} \quad -i\, Q^{jk}$$

Hence, as an operator in

$\mathcal{E} \otimes_{\mathcal{Z}} \mathcal{E} \otimes_{\mathcal{Z}} \mathcal{E}$,

$$|dq^j \wedge dq^k|^2 = (\text{s.-a.})^2 + (Q^{jk})^2$$
$$\geq (Q^{jk})^2,$$

and

$$\sum_{\substack{j,k,l \text{ cyclic} \\ \text{perm of } 1,2,3}} |dq^j \wedge dq^k|^2 \geq \vec{m}^2 \geq I;$$

Similarly for the "timelike area"

$$\sum_{j=1}^{3} |dq^0 \wedge dq^j|^2 \geq \vec{e}^2 \geq I.$$

Space Volume:

$$d\vec{q} \wedge d\vec{q} \wedge d\vec{q} = \varepsilon_{jkl}\, dq^j dq^k dq^l =$$

$$= \varepsilon_{jkl} (I \otimes q^j_{\mathcal{E}} - q^j \otimes I)(I \otimes q^k - q^k \otimes I)(I \otimes q^l - q^l \otimes I)$$

$$dq^1 \wedge dq^2 \wedge dq^3 =$$

$$= \det \begin{pmatrix} I & q_1^1 & q_1^2 & q_1^3 \\ I & q_2^1 & q_2^2 & q_2^3 \\ I & q_3^1 & q_3^2 & q_3^3 \\ I & q_4^1 & q_4^2 & q_4^3 \end{pmatrix} + i m_e \hat{Q}^e$$

IS NORMAL: SUFFICES TO CHECK IN IRREPS. ROTATION INVARIANCE: SUFFICES TO CHECK IN THOSE WHERE

$$\vec{m} = (m, 0, 0)$$

i.e. $[q^1, q^2] = [q^1, q^3] = 0$

BUT THEN

$$m_e Q^e = m Q^1 =$$

$$= \frac{m}{2} \sum_{j=1}^{4} (-1)^{j-1} q_j^1 \text{ COMMUTES}$$

WITH $\det(\quad)$.

$=$ $\sum$ terms with q's at 3 different tensor places (s.a.) $+$

$+$ $\sum$ terms with $q_k q_l$ at one place and q_j at another (skew adjoint by antisymmetrization).

$$\equiv \quad U + i\, m_l \hat{Q}^l ,$$

$$m_l = \frac{1}{2} \varepsilon_{ljk} Q^{jk} ,$$

$$\hat{Q}_l = \frac{1}{2} \sum_{j=1}^{4} (-1)^{j+1} q_j^l ,$$

with $q_j = \underset{j\text{th place}}{I \otimes .. q \otimes .. I}$.

CLAIM: $O \in \mathrm{Sp}\,[d\vec{q} \wedge d\vec{q} \wedge d\vec{q}]$.

Suffices to work with irreps $\sigma \in \Sigma$.

Choose $\sigma = (\vec{e}, \vec{m})$, with

$\vec{m} = \vec{e} = (1, 0, 0)$;

The associated regular irrep acts on

$H \otimes H$, $H \equiv L^2(\mathbb{R}, ds)$ as

$$\begin{pmatrix} q^0 \\ \vdots \\ q^3 \end{pmatrix} = \begin{pmatrix} q \otimes I \\ p \otimes I \\ I \otimes q \\ I \otimes p \end{pmatrix},$$

q = multipl by s,

$p = -i \frac{d}{ds}$.

We identify, by the obvious $\cong$, $(H \otimes H)^{\otimes 4}$ with $H^{\otimes 4} \otimes H^{\otimes 4}$;

Then, in the chosen irrep,

$$m_\ell \hat{Q}_\ell = m_1 \hat{Q}_1 = \frac{1}{2} \sum (-1)^{j+1} p_j \otimes I,$$

$$V = \sum_{m=1}^{4} (-1)^{m+1} \varepsilon_{jk\ell}\, q_1^{j} \cdots q_4^{\ell}$$

I at ~~m~~ m-th place

$$= \det \begin{pmatrix} I & p_1 \otimes I & I \otimes q_1 & I \otimes p_1 \\ I & p_2 \otimes I & I \otimes q_2 & I \otimes p_2 \\ I & p_3 \otimes I & I \otimes q_3 & I \otimes p_3 \\ I & p_4 \otimes I & I \otimes q_4 & I \otimes p_4 \end{pmatrix}$$

$$= \sum \varepsilon^{jklm}\, p_k \otimes q_l p_m =$$

$$= \frac{1}{4} \sum \varepsilon^{jklm} (p_k - p_j) \otimes M_{lm} ,$$

so that

$$d\vec{q} \wedge d\vec{q} \wedge d\vec{q} = \frac{1}{2} \sum (-1)^{j+1} p_j \otimes I +$$

$$+ \frac{1}{4} \sum \varepsilon^{jklm} (p_k - p_j) \otimes M_{lm} .$$

Choose $x, y \in H^{\otimes 4}$, $\|x\| = \|y\| = 1$, s.t. $y \in H^{\otimes 4} = L^2(\mathbb{R}^4, d^4 s)$ depends upon $\sum_{j=1}^{4} s_j^2$ only, and $\hat{x}$ has supp in a sphere of radius $\varepsilon/2$ around 0; then

$$\| d\vec{q} \wedge d\vec{q} \wedge d\vec{q}\; x \otimes y \| \le \varepsilon .$$

The Spacetime Volume Operator

$$dq \wedge dq \wedge dq \wedge dq =$$

$$= \varepsilon_{\mu\nu\lambda\rho}\, dq^\mu dq^\nu dq^\lambda dq^\rho =$$

$$= \varepsilon_{\mu\nu\lambda\rho} (I \otimes q^\mu - q^\mu \otimes I) \ldots (I \otimes q^\rho - q^\rho \otimes I)$$

(products in $\Lambda_{\mathcal{Z}}(\mathcal{E})$!).

= (no pair of q's in same place) + S. Q.

A $\equiv$ (two pairs of q's in same places) + S. Q.

(one pair of q's in same place) skew adjoint

$\equiv$ B

$$\varepsilon_{\mu\nu\lambda\rho}\, I \otimes q^\mu q^\nu \otimes I \otimes q^\lambda q^\rho \otimes I =$$

$$= \frac{1}{4} \varepsilon_{\mu\nu\lambda\rho}\, I \otimes [q^\mu, q^\nu] \otimes I \otimes [q^\lambda, q^\rho] \otimes I =$$

$$= -\frac{1}{4}\, Q \wedge \cdot\, Q = -2\eta, \underline{\in \mathcal{Z}},$$

$\eta = \pm I$ on $\Sigma_\pm$.

$$dq \wedge dq \wedge dq \wedge dq = A - 2\eta + i\mathcal{B},$$

$$A = \sum_{j=1}^{5} (-1)^j \bigwedge_{i \neq j} q_i \equiv \sum_{j=1}^{5} (-1)^j A_j,$$

$$\mathcal{B} = \frac{1}{2} \sum_{i<j} (-1)^{i-j} Q \wedge q_i \wedge q_j \equiv \frac{1}{2} \sum (-1)^{i-j} \mathcal{B}_{ij}.$$

Compute:

$$[q_k^\mu, \mathcal{B}_{ij}] = \delta_{ik} \underbrace{Q^{\mu\cdot} \wedge Q \wedge q_j}_{} - \delta_{jk} Q^{\mu\cdot} \wedge Q \wedge q_i$$

$$" \qquad \varepsilon_{\nu\lambda\rho\sigma} Q^{\mu\nu} Q^{\lambda\rho} q_j^\sigma =$$

$$= \quad 2 \left(Q^{\mu\nu} (*Q)_{\nu\sigma} \right) q_j^\sigma$$

But antisymmetry + centrality of $Q \Rightarrow$

$$Q^{\mu\nu} (*Q)_{\nu\sigma} = \frac{1}{4} Q^{\nu\rho} (*Q)_{\nu\rho} \cdot \delta_{\mu\sigma}$$

$$= \eta\, \delta_{\mu\sigma}; \quad \text{hence}$$

$$[q_k^\mu, \mathcal{B}_{ij}] = \eta \left(\delta_{ik} q_j^\mu - \delta_{jk} q_k^\mu \right). \quad *$$

Thus $\mathrm{Ad}\, B_{ij}$ act on q_k's as ($\mathbb{Z}$.) Lie Algebra gen of $SO(5)$; by $*$,

$$(\mathrm{Ad}\, B)\left(\sum_{j=1}^{5} q_j\right) = 0$$

hence $\mathrm{Ad}\, B$ acts as a generator of 1 par sbgp in $\mathcal{N} \equiv$ stabilizer in $SO(5)$ of $(1,1,1,1,1)$.

Now $A = \sum_{j=1}^{5} (-1)^j \bigwedge_{i \neq j} q_i =$

$$= \det \begin{pmatrix} I & q_1^0 & q_1^1 & \cdots & q_1^3 \\ I & q_2^0 & q_2^1 & \cdots & q_2^3 \\ - & - & - & - & - \\ I & q_5^0 & q_5^1 & \cdots & q_5^3 \end{pmatrix} =$$

$$= \det R \cdot \left(\quad \right)$$

and, if $R \in \mathcal{N}$:

$$= \det \begin{pmatrix} I & q'^{0}_{1} & q'^{1}_{1} & \cdots & q'^{3}_{1} \\ I & q'^{0}_{2} & q'^{1}_{2} & \cdots & q'^{3}_{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ I & q'^{0}_{5} & q'^{1}_{5} & \cdots & q'^{3}_{5} \end{pmatrix},$$

if
$$q'^{\mu}_{j} = R^{jk} q^{\mu}_{k},$$
$$R \in N,$$

and for any generator D of a 1-par. subgp in N,

$$(\mathrm{Ad}\, D)(A) = 0.$$

Hence $[A, B] = 0$ and

$dq \wedge dq \wedge dq \wedge dq = A - 2z + iB$

is NORMAL (z is central!), and

$$|dq \wedge dq \wedge dq \wedge dq|^2 = (A - 2z)^2 + B^2$$
$$\geq (A - 2z)^2 .$$

Now as a field of operators on Σ, by LORENTZ invariance, $dq \wedge dq \wedge dq \wedge dq$ is CONSTANT and of opposite signs on $\Sigma_\pm$; it suffices to compute at $\sigma \in \Sigma$, $\sigma = (\vec{e}, \vec{m})$, $\vec{e} = \vec{m} = (1,0,0)$ as before: q^j_μ act on $H^{\otimes 5} \otimes H^{\otimes 5}$; if we let q_j, p_j denote Schroedinger's q, p acting on the j-th place in $H^{\otimes 5}$, we have

$$q_\mu^j = \begin{pmatrix} q_j \otimes I \\ p_j \otimes I \\ I \otimes q_j \\ I \otimes p_j \end{pmatrix},$$

and

$$A = \det \begin{pmatrix} I & q_1 \otimes I & p_1 \otimes I & I \otimes q_1 & I \otimes p_1 \\ I & q_2 \otimes I & p_2 \otimes I & I \otimes q_2 & I \otimes p_2 \\ - & - & - & - & - \\ I & q_5 \otimes I & p_5 \otimes I & I \otimes q_5 & I \otimes p_5 \end{pmatrix}$$

$$= \frac{1}{4} \sum_1^5 \varepsilon_{ijklm} M_{jk} \otimes M_{lm},$$

with $M_{jk} = q_j p_k - q_k p_j$ gen. of rotations in (j,k) plane in $SO(5)$,

$$[M_{jk}, M_{lm}] = i\big(\delta_{jl} M_{km} - \delta_{jm} M_{kl} + \delta_{km} M_{jl} - \delta_{kl} M_{jm}\big),$$

30'

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \to \sqrt{5} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$A = \sqrt{5} \det \begin{pmatrix} 0 & q_1' \otimes I & p_1' \otimes I & I \otimes q_1' & I \times p_1' \\ 0 & q_2' \otimes I & p_2' \otimes I & I \otimes q_2' & I \otimes p_2' \\ 0 & \cdots & & \text{---} & \\ 0 & \text{--} & \text{--} & \text{------} & \\ I & q_5' \otimes I & p_5' \otimes I & I \otimes q_5' & I \otimes p_5' \end{pmatrix}$$

$$= \sqrt{5} \det \begin{pmatrix} q_1' \otimes I & p_1' \otimes I & I \otimes q_1' & I \otimes p_1' \\ \cdots & \text{--} & \text{--} & \text{--} \\ \text{--} & \text{--} & \text{--} & \text{---} \\ q_4' \otimes I & p_4' \otimes I & I \otimes q_4' & I \otimes p_4' \end{pmatrix}$$

$= \sqrt{5}\ .$ det (minors in the first 2 ~~rows~~ columns)

$\cdot$ det (compl. minors) $=$

UNCHANGED IF WE ROTATE TO ANOTHER ORTHONORMAL BASIS $\xi_1, .. \xi_5$ IN $\mathbb{R}^5$: choosing

$$\xi_5 = \frac{1}{\sqrt{5}} (1,1,1,1,1)$$

we get $\longrightarrow 30'$

$$A = \frac{1}{4} \sqrt{5} \sum_1^4 \varepsilon_{ijkl} M'_{ij} \otimes M'_{kl} .$$

Now with

$$\vec{B} \equiv (M'_{23}, M'_{31}, M'_{12}),$$

$$\vec{D} \equiv (M'_{14}, M'_{24}, M'_{34}),$$

we have that

$$\vec{L}^{(\pm)} \equiv \frac{1}{2} (\vec{B} \pm \vec{D})$$

are mutually commuting generators of $SU(2)$ and

$$A = 2\sqrt{5} (\vec{L}^{(+)} \otimes \vec{L}^{(+)} - \vec{L}^{(-)} \otimes \vec{L}^{(-)})$$

Now

$$\vec{J} \otimes \vec{J} = \frac{1}{2}\left\{ \left(\vec{J} \otimes I + I \otimes \vec{J}\right)^2 - \vec{J}^2 \otimes I - I \otimes \vec{J}^2 \right\}$$

by Clebsch-Gordan has eigenvalues

$$\frac{1}{2}\left(s(s+1) - u(u+1) - v(v+1) \right),$$

$$u, v \in \frac{1}{2}\cancel{\mathbb{N}}_0 = \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots\}$$

$$s = u+v,\ u+v-1, \dots |u-v|;$$

keeping track of the fact that eigenvalues $s^+(s^++1)$ of $\vec{L}^{(+)2}$ and $s^-(s^-+1)$ of $\vec{L}^{(-)2}$ arising from reps of $SO(4)$ must be with s^+, s^- simultaneously integers or half integers, we see that

$$|dq \wedge \delta q \wedge \delta q \wedge \delta q| \geq |A - 2q| \geq \sqrt{5} - 2.$$

असतो मा सद्गमय
तमसो मा ज्योतिर्गमय
मृत्योर् मा अमृतं गमय
ॐ शांति शांति शांति

Asato Ma Sat Gamaya
Tamaso Ma Jyotir Gamaya
Mrityor Ma Amritam Gamaya
Om Shanti Shanti Shanti.

Lead Us From the Unreal To Real,
Lead Us From Darkness To Light,
Lead Us From Death To Immortality

Om (the universal sound of God)
Let There Be Peace Peace Peace.

Brihadaranyaka Upanishad 1.3.28.

1.2.1.:

NAIVEHA KIMCANAGRA ASIT MRTYUNA
VEDAM AVRTAM ASIT_ASANAYAYA,
ASANAYA HI MRTYUH; TAN MANO'
KURUTA, ATMANI SYAM ITI.

FIRST DOWN HERE THERE
WAS NOTHING. ALL THIS
UNIVERSE WAS WRAPPED IN
MRTYU, HUNGER, FOR
HUNGER IS DEATH.
THEN HE THOUGHT:
LET ME EXIST.