Intrinsic Parity of Neutral Pion

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Contents

1 Symmetry

What does it mean by 'a certain law of physics is symmetric under certain transfor- mations' ? To be specific, consider the statement 'classical mechanics is symmetric under mirror inversion' which can be defined as follows: take any motion that satisfies the laws of classical mechanics. Then, reflect the motion into a mirror and imagine that the motion in the mirror is actually happening in front of your eyes, and check if the motion satisfies the same laws of classical mechanics. If it does, then classical mechanics is said to be symmetric under mirror inversion. Or more precisely, if all motions that satisfy the laws of classical mechanics also satisfy them after being re- flected into a mirror, then classical mechanics is said to be symmetric under mirror inversion. In general, suppose one applies certain transformation to a motion that follows certain law of physics, if the resulting motion satisfies the same law, and if such is the case for all motion that satisfies the law, then the law of physics is said to be symmetric under the given transformation.

It is important to use exactly the same law of physics after the transformation is applied. For example, I could use my right hand to specify the law physics, say, to state the direction of the force felt by a moving charge inside a magnetic field. Then, I have to use the same hand to see the law is still satisfied by the mirror-inverted motion. If I also mirror-invert my right hand to apply the law, then the law would be trivially satisfied by the transformed motion.

In the context of quantum mechanics, the above criterion for the symmetry of physical law can be stated as follows: for the state vectors $|i\rangle$ and $|f\rangle$ representing certain initial and final states, there exist $|i'\rangle = U|i\rangle$ and $|f'\rangle = U|f\rangle$ that represent the corresponding states reflected into a mirror, where U is an operator in the Hilbert space that corresponds to mirror inversion. Then, if the laws of physics are symmetric under mirror inversion, the transition probability is the same before and after the transformation:

$$|\langle f'|S|i'\rangle|^2 = |\langle f|S|i\rangle|^2 \tag{1}$$

Note that the same S operator, not $S' \equiv USU^\dagger$[Recall, under the change of basis from $\{|\alpha\rangle\} \to \{|i\rangle\}$ through $\{|i\rangle\} = U\{|\alpha\rangle\}$, then an operator $O \to UOU^\dagger$], is used for the transformed states.

In fact, if S' is used instead of S, the transition probability is trivially invariant:

$$|\langle f| \quad S \quad |i\rangle|^2 = |\langle f'|S'|i'\rangle|^2 \quad [always] \tag{2}$$

Just insert $U^\dagger U$ in the two above specified gaps.

In general, a transformation $|\psi'\rangle = U|\psi\rangle$ is called a symmetry transformation if it preserves inner products between any physical states

$$|\langle \psi_1'|\psi_2'\rangle|^2 = |\langle \psi_1|\psi_2\rangle|^2 \tag{3}$$

2 Parity Transformation

The parity transformation P is the operation to change the signs of the three space coordinates, which is equivalent to:
(a)the mirror inversion followed by
(b) a rotation by π radian.
To be specific, the mirror inversion of the z-axis(say) : $z \to -z$ followed by a rotation around the z axis by π ($x \to -x$ & $y \to -y$) flips the sign of all three coordinates. Since rotating a whole motion by a certain angle does not affect the motion(isotropy of space), the parity transformation and the mirror inversion are equivalent.

Let's take an example of a Coulomb scattering of an electron by a nucleus N where an incoming e is polarized right-handed (helicity +) and the outgoing e is also right-handed.

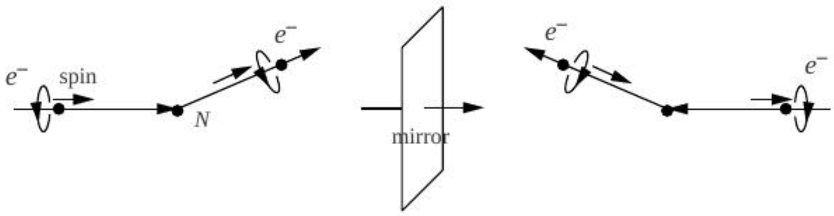

Figure 1: Mirror reflection

The nucleus could recoil, but we will focus on the motion of the electron. After the mirror inversion, the electrons in the reflected process are left-handed for both the initial and final state. In fact, by rotating the reflected process by 180 degrees around the vertical axis, one can make it completely overlap with the original process except that the spins are in the opposite direction. If the physics involved is symmetric under parity, the original and reflected processes should occur with the same cross section. This is experimentally confirmed. As far as we know, every process caused by QED and its mirror inversion occur with same probability, and thus we believe that QED is symmetric under parity.

2.1 Parity in CM

The effect of a parity transformation is defined as the inversion of the spatial coordinates with respect to the origin

$$\vec{x} \xrightarrow{\mathcal{P}} \vec{x}_p = -\vec{x} \quad [\text{in } 3D] \tag{4}$$

and its passive interpretation corresponds to the reversal of the three spatial axes, under which a right handed coordinate system becomes a left-handed one(and vice-versa). Note that space inversion cannot be obtained through any rotation and, therefore, it is not a continuous transformation.

Scalar $\Rightarrow$ +1, pseudo-scalar $\Rightarrow$ -1
vector $\Rightarrow$ -1, pseudo-vector $\Rightarrow$ +1

Let's note the transformation properties of some of the well known classical variables under parity:

$$\begin{aligned}
\mathbf{x} &\xrightarrow{\mathcal{P}} -\mathbf{x}, \\
\mathbf{p} &\xrightarrow{\mathcal{P}} -\mathbf{p}, \\
\mathbf{L} = \mathbf{x} \times \mathbf{p} &\xrightarrow{\mathcal{P}} (-\mathbf{x}) \times (-\mathbf{p}) = \mathbf{x} \times \mathbf{p} = \mathbf{L}, \\
\frac{\mathbf{L} \cdot \mathbf{p}}{|\mathbf{p}|} &\xrightarrow{\mathcal{P}} \frac{\mathbf{L} \cdot (-\mathbf{p})}{|\mathbf{p}|} = -\frac{\mathbf{L} \cdot \mathbf{p}}{|\mathbf{p}|}, \\
\mathbf{J}(\mathbf{x}, t) &\xrightarrow{\mathcal{P}} -\mathbf{J}(-\mathbf{x}, t), \\
J^0(\mathbf{x}, t) = \rho(\mathbf{x}, t) &\xrightarrow{\mathcal{P}} \rho(-\mathbf{x}, t) = J^0(-\mathbf{x}, t).
\end{aligned}$$

Applying parity transformation twice, returns the coordinates to their original value, namely $\mathcal{P}^2(\vec{x}) = \vec{x}$, so that even classically parity operation defines a group with two elements, namely, I and $\mathcal{P}$ with $\mathcal{P}^2 = I$.

Classically, the dynamical laws of physics(for example, Newton's law) remain invariant under parity transformation, hence it's a symmetry. However, in CM this does not result any constant of motion, but it puts constraint on the form the solutions.

2.2 Parity in QM

The concept of parity is introduced in quantum theory through the correspondence principle(Ehrenfest theorem $\Rightarrow$ expectation values of quantum operators behave like classical objects). For simplicity, let us consider an one dimensional quantum mechanical system. In this case, the parity transformation would result in

$$\langle X \rangle \xrightarrow{\mathcal{P}} -\langle X \rangle \tag{5}$$

$$\langle P \rangle \xrightarrow{\mathcal{P}} -\langle P \rangle \tag{6}$$

As in standard QM, Parity can be analysed in two equivalent ways: active and passive.

2.2.1 Active viewpoint

- States $\Rightarrow$ transform
  operators $\Rightarrow$ do not transform
  such that (5) and (6) hold. Namely,

$$|\psi\rangle \xrightarrow{\mathcal{P}} |\psi^{\mathcal{P}}\rangle \equiv \mathcal{P}|\psi\rangle \quad \text{such that(in 1D)} \tag{7}$$

$$\langle\psi|X|\psi\rangle \xrightarrow{\mathcal{P}} \langle\psi^{\mathcal{P}}|X|\psi^{\mathcal{P}}\rangle = \langle\psi|\mathcal{P}^\dagger X\mathcal{P}|\psi\rangle = -\langle\psi|X|\psi\rangle \tag{8}$$

  Similarly for momentum operator.
  Since parity inverts space coordinates :

$$|x\rangle \xrightarrow{\mathcal{P}} |x^{\mathcal{P}}\rangle \equiv \mathcal{P}|x\rangle = |-x\rangle \tag{9}$$

  See, this satisfies (8). Hence,

$$\langle x^{\mathcal{P}}|y^{\mathcal{P}}\rangle = \langle x|\mathcal{P}^\dagger\mathcal{P}|y\rangle \tag{10}$$
$$\Rightarrow \langle -x|-y\rangle = \langle x|\mathcal{P}^\dagger\mathcal{P}|y\rangle \tag{11}$$
$$\Rightarrow \delta(x-y) = \langle x|\mathcal{P}^\dagger\mathcal{P}|y\rangle \tag{12}$$
$$\Rightarrow \mathcal{P}^\dagger\mathcal{P} = I \tag{13}$$
$$\Rightarrow \mathcal{P} \ \text{ is unitary.} \tag{14}$$

  Furthermore,

$$\mathcal{P}^2|x\rangle = I|x\rangle \tag{15}$$
$$\Rightarrow \mathcal{P}^2 = I \Rightarrow \text{eigenvalues can be} \pm 1 \tag{16}$$

  Also, unitarity and idempotent $\Rightarrow$ Hermitian. Hence,

$$\boxed{\mathcal{P}^\dagger = \mathcal{P} = \mathcal{P}^{-1} \quad \text{with} \quad \mathcal{P}^2 = I} \tag{17}$$

- Any arbitrary state :

$$|\psi\rangle \xrightarrow{\mathcal{P}} |\psi^{\mathcal{P}}\rangle = \mathcal{P}|\psi\rangle = \mathcal{P}\int dx|x\rangle\langle x|\psi\rangle \tag{18}$$
$$= \mathcal{P}\int dx\psi(x)|x\rangle \tag{19}$$
$$= \int dx\psi(x)|-x\rangle \tag{20}$$
$$\Rightarrow \langle x|\psi^{\mathcal{P}}\rangle = \psi^{\mathcal{P}}(x) = \psi(-x) \tag{21}$$

  Hence,

$$\psi(x) \xrightarrow{\mathcal{P}} \psi^{\mathcal{P}}(x) = \psi(-x) \tag{22}$$

- If $|\psi\rangle$ is an eigenstate of $\mathcal{P}$ then,

$$\mathcal{P}|\psi\rangle = \pm|\psi\rangle \tag{23}$$

$$\Rightarrow \psi(x) \xrightarrow{\mathcal{P}} \psi^{\mathcal{P}}(x) = \psi(-x) = \pm\psi(x) \tag{24}$$

Wave function associated with the eigenstate of a parity operator is either even or odd.
We have seen, eigenvalues of a parity operator are just phase factors. Hence, we denote:

$$\mathcal{P}|\psi\rangle = \eta_{\mathcal{P}}|\psi\rangle \tag{25}$$

$$\text{with } \eta_p = \pm 1 \text{ as a phase factor.} \tag{26}$$

2.2.2 Passive viewpoint

- states $\Rightarrow$ do not transform
operators $\Rightarrow$ transform
such that (5) and (6) hold. Namely,

$$\mathcal{O} \xrightarrow{\mathcal{P}} \mathcal{O}^{\mathcal{P}} \equiv \mathcal{P}^{\dagger}\mathcal{O}\mathcal{P} \quad \text{such that in 1D} \tag{27}$$

$$\langle\psi|X|\psi\rangle \xrightarrow{\mathcal{P}} \langle\psi|X^{\mathcal{P}}|\psi\rangle = \langle\psi|\mathcal{P}^{\dagger}X\mathcal{P}|\psi\rangle = -\langle\psi|X|\psi\rangle \tag{28}$$

Hence,

$$X \xrightarrow{\mathcal{P}} X^{\mathcal{P}} = \mathcal{P}^{\dagger}X\mathcal{P} = -X \tag{29}$$

$$\Rightarrow \{\mathcal{P}, X\} = 0 \tag{30}$$

Similarly,

$$\{\mathcal{P}, P\} = 0 \tag{31}$$

Note that, only states with zero momentum can possibly be eigenstates of parity operator.
In general,

$$\mathcal{O}(X,P) \xrightarrow{\mathcal{P}} \mathcal{O}^{\mathcal{P}}(X,P) = \mathcal{O}(-X,-P) \tag{32}$$

- Note that, if the Hamiltonian of a theory remains invariant under parity then the quantum theory would be parity invariant.

$$H(X,P) \xrightarrow{\mathcal{P}} \mathcal{P}^{\dagger}H(X,P)\mathcal{P} = H(-X,-P) = H(X,P) \tag{33}$$

$$\Rightarrow [\mathcal{P}, H] = 0 \tag{34}$$

$$\Rightarrow \text{eigenstates of the Hamiltonian would be either even or odd.} \tag{35}$$

- Also if the Hamiltonian is time independent and commutes with parity operator then

$$|\psi(t)\rangle = exp(-iHt)|\psi(0)\rangle = U(t)|\psi(0)\rangle \tag{36}$$

$$\Rightarrow [\mathcal{P}, U(t)] = 0 \tag{37}$$

$$\Rightarrow \text{parity of the state remains unchanged upon time evolution.} \tag{38}$$

- Though only states with zero momentum can only possibly be eigenstates of a parity operator, nevertheless state with non-zero angular momentum can be an eigenstate of a parity operator since angular momentum and parity operator do commute with each other:

  $[L,\mathcal{P}] = 0 \Leftrightarrow \mathcal{P}^\dagger L \mathcal{P} = L$

  $\Rightarrow$ angular momentum eigenstates are also parity eigenstates

  Consider AM eigenstates: $\psi_l(r,\theta,\phi) = f_l(r) Y_{lm}(\theta,\phi)$

  Now, $\vec{x} \xrightarrow{\mathcal{P}} -\vec{x}$

  $\Rightarrow r \xrightarrow{\mathcal{P}} r, \theta \xrightarrow{\mathcal{P}} \pi - \theta, \phi \xrightarrow{\mathcal{P}} \pi + \phi$

  $\Rightarrow Y_{lm}(\theta,\phi) \xrightarrow{\mathcal{P}} Y_{lm}(\pi-\theta,\pi+\phi) = (-1)^l Y_{lm}(\theta,\phi)$

  $\Rightarrow \mathcal{P} Y_{lm}(\theta,\phi) = (-1)^l Y_{lm}(\theta,\phi)$

  So, under parity transformation the wave function of a state with AM quantum number l acquires a sign $(-1)^l$ with respect to the $l = 0$ state.

  And the effect of parity on the spin operator S is defined to be

  $$S \to S_{\mathcal{P}} \equiv \mathcal{P}^\dagger S \mathcal{P} = S \tag{39}$$

  in analogy with the behaviour of orbital AM L. More formally, since rotations and space inversion are commuting operations, $\mathcal{P}$ and the generator of an infinitesimal rotation(the total AM J)must commute too, and since $J = L + S$, the above transformation is obtained.

2.3 Parity in Relativistic QM

There is no such consistent relativistic quantum mechanics, but QFT.

2.4 Parity in QFT

- In quantum field theory the parity transformation is represented by a unitary operator $\mathcal{P}$ in Hilbert space, whose effect on the creation or annihilation operator of definite momentum $\vec{p}$ and spin s should be:

  $$\mathcal{P}^\dagger a(\vec{p},s)\mathcal{P} = \eta_{\mathcal{P}} a(-\vec{p}) \tag{40}$$
  $$\mathcal{P}^\dagger a^\dagger(\vec{p},s)\mathcal{P} = \eta_{\mathcal{P}}^* a^\dagger(-\vec{p},s) \tag{41}$$
  $$\text{with} \quad |\eta_{\mathcal{P}}| = 1; \tag{42}$$

  $\eta_{\mathcal{P}}$ is called the intrinsic parity of the particle created by this creation operator.

  Clearly vacuum should transform to itself under parity up to a phase(because $\mathcal{P}$ does not interchange creation and annihilation operators)

  $$\mathcal{P}|O\rangle = \eta_{vac}|O\rangle \tag{43}$$

  We assume the intrinsic parity of the vacuum is +1 i.e. $\mathcal{P}|O\rangle = |O\rangle$

- For a non-Hermitian (charged) scalar field the effect of a parity transformation is(followed from the earlier definitions (44) & (45))

$$\mathcal{P}^\dagger \psi(t,\vec{x})\mathcal{P} = \eta_{\mathcal{P}}\psi(t,-\vec{x}) \tag{44}$$
$$\mathcal{P}^\dagger \psi^\dagger(t,\vec{x})\mathcal{P} = \eta^*_{\mathcal{P}}\psi^\dagger(t,-\vec{x}) \tag{45}$$

- Take an n-particle state(free field theory)

$$|\vec{p}_1,...,\vec{p}_n\rangle = a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)...a^\dagger(\vec{p}_n)|O\rangle \tag{46}$$

Under parity this state transforms as

$$\mathcal{P}|\vec{p}_1,...,\vec{p}_n\rangle = \mathcal{P}a^\dagger(\vec{p}_1)\ \ a^\dagger(\vec{p}_2)\ \ ...\ \ a^\dagger(\vec{p}_n)\ \ |O\rangle \tag{47}$$
$$\Rightarrow \mathcal{P}^\dagger|\vec{p}_1,...,\vec{p}_n\rangle = \mathcal{P}a^\dagger(\vec{p}_1)\ \ a^\dagger(\vec{p}_2)\ \ ...\ \ a^\dagger(\vec{p}_n)\ \ |O\rangle \tag{48}$$

Now, insert $\mathcal{P}\mathcal{P}^\dagger$ at every gap:

$$\mathcal{P}^\dagger|\vec{p}_1,...,\vec{p}_n\rangle = \mathcal{P}^\dagger a^\dagger(\vec{p}_1)\ \ a^\dagger(\vec{p}_2)\ \ ...\ \ a^\dagger(\vec{p}_n)\ \ |O\rangle \tag{49}$$
$$\Rightarrow \mathcal{P}^\dagger|\vec{p}_1,...,\vec{p}_n\rangle = \eta^*_1\eta^*_2...\eta^*_n a^\dagger(-\vec{p}_1)a^\dagger(-\vec{p}_2)...a^\dagger(-\vec{p}_n)|O\rangle \tag{50}$$
$$\Rightarrow \mathcal{P}^\dagger|\vec{p}_1,...,\vec{p}_n\rangle = \eta^*_{total} a^\dagger(-\vec{p}_1)a^\dagger(-\vec{p}_2)...a^\dagger(-\vec{p}_n)|O\rangle \tag{51}$$

See, intrinsic parity is a multiplicative quantum number.

So, if there is a state corresponding to

$$\psi^{\mathcal{P}}_l(\vec{x},t) = \eta_\psi\psi_l(-\vec{x},t) = \eta_\psi(-1)^l\psi_l(\vec{x},t) \tag{52}$$

then, we can define the total parity for the state(with orbital AM quantum number l) as

$$\eta_{tot} = \eta_\psi(-1)^l$$

- If parity is a symmetry of the theory, the total parity quantum number must be conserved in a physical process and this leads to the fact that for a decay (in the rest frame of A) of a spin zero particle into two spin zero particles

$$A \to B + C$$

We must have

$$\eta_A = \eta_B\eta_C(-1)^l$$

where η_A, η_B, η_C are the intrinsic parities of the particles A, B and C respectively and l is the orbital angular momentum quantum number of the B-C system.

- First note that (48) $\Rightarrow \mathcal{P}^\dagger\psi(t,\vec{0})\mathcal{P} = \eta_{\mathcal{P}}\psi(t,\vec{0}) \Rightarrow$ intrinsic parity indeed corresponds to the phase factor(or sign) which the field acquires at the origin of the coordinates under parity transformation.

Also, this $\eta_{\mathcal{P}}$ denotes the parity eigenvalue of the single particle state at rest(Remember that parity and the momentum operators do not commute and, therefore, cannot have simultaneous eigenstates unless the eigenvalue of momentum vanishes.).

Thus we see that $\eta_{\mathcal{P}}$ really measures the intrinsic behaviour (and not the space part) of the single particle state under space inversion.

- The concept of intrinsic parity only plays a role in QFT, where particles can be created and destroyed: in single-particle quantum mechanics the intrinsic parities are identical in the initial and final states for any physical process and therefore need not be taken into account. In QFT, however, the $\eta_{\mathcal{P}}$ factor in the field transformation can have observable consequences, as it defines the transformation law for an operator which may appear in interaction terms of the Lagrangian together with other different fields.

2.4.1 Parity in Photon field

- But, how do we know the parity of a particle?
  $\Rightarrow$ By convention we assign positive intrinsic parity (+) to spin 1/2 fermions: +parity: proton, neutron, electron, muon.**The sign here is simply due to convention**, because baryons are conserved and the nucleon parities cancel in any reaction. While the intrinsic parity of a fermion is a matter of convention, the relative parity of a fermion and anti-fermion is not. The Dirac theory of fermions requires particle and antiparticle to have opposite intrinsic parity.

  Bosons and their anti-particles have the same intrinsic parity.

- What about the photon?
  $\Rightarrow$ Strictly speaking, we can not assign a parity to the photon since it is never at rest. But, it is necessary to assign an intrinsic parity to photon.For example, in case of electric dipole transitions between atomic states(s,p,d,f..which are characterized by various values of l) are characterised by the selection rule $\Delta l = \pm 1$, so that as a result of the transition, the parity of the atomic state must change. The electromagnetic (El) radiation (photons) emitted in this case must have an intrinsic parity such that the parity of the whole system (atom + photon) is conserved(being EM interaction). And for this reason, by convention, the parity of the photon is given by the radiation field involved.

  We determine the parity of other particles using the above conventions and assuming parity is conserved in the strong and electromagnetic interaction. Usually we need to resort to experiment to determine the parity of a particle.

- First let's look at the electromagnetic current $U(1)$ associated with a charged KG system.

$$J^\mu(t, \vec{x}) = i\psi^\dagger(t, \vec{x})\partial^\mu\psi(t, \vec{x}) - (\partial^\mu\psi^\dagger(t, \vec{x}))\psi(t, \vec{x})$$

Under Parity

$$J^\mu(t,\vec{x}) \xrightarrow{\mathcal{P}} J^{\mathcal{P}\mu}(t,\vec{x}) \equiv \mathcal{P}^\dagger J^\mu \mathcal{P} \tag{53}$$
$$= i\mathcal{P}^\dagger \psi^\dagger(t,\vec{x})\mathcal{P}\mathcal{P}^\dagger \partial^\mu \psi(t,\vec{x})\mathcal{P} - \mathcal{P}^\dagger(\partial^\mu \psi^\dagger(t,\vec{x}))\mathcal{P}\mathcal{P}^\dagger \psi(t,\vec{x})\mathcal{P} \tag{54}$$
$$= i\psi^{\mathcal{P}\dagger}(t,\vec{x})\partial^\mu \psi^{\mathcal{P}}(t,\vec{x}) - (\partial^\mu \psi^{\mathcal{P}\dagger}(t,\vec{x}))\psi^{\mathcal{P}}(t,\vec{x}) \tag{55}$$
$$= i|\eta_\psi|^2[\psi^\dagger(t,-\vec{x})\partial^\mu \psi(t,-\vec{x}) - (\partial^\mu \psi^\dagger(t,-\vec{x}))\psi(t,-\vec{x})] \tag{56}$$

which leads

$$\vec{J}^{\mathcal{P}}(t,\vec{x}) = -\vec{J}(t,-\vec{x}), \quad J^{\mathcal{P}0}(t,\vec{x}) = J^0(t,-\vec{x})$$

Symbolically we denote

$$\boxed{J^\mu(t,\vec{x}) \xrightarrow{\mathcal{P}} J_\mu(t,-\vec{x})}$$

Now, experiments $\Rightarrow$ electrodynamics respects parity symmetry $\Rightarrow$ Maxwell equations should also respect.So,

$$\nabla.\vec{E} = \rho = J^0, \quad \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{J}$$

To make these invariant

$$\vec{E}(t,\vec{x}) \xrightarrow{\mathcal{P}} -\vec{E}(t,-\vec{x}), \quad \vec{B}(t,\vec{x}) \xrightarrow{\mathcal{P}} \vec{B}(t,-\vec{x})$$

Furthermore,

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla A^0, \quad \vec{B} = \nabla \times \vec{A}$$

Hence,

$$\vec{A}(t,\vec{x}) \xrightarrow{\mathcal{P}} -\vec{A}(t,-\vec{x}), \quad A^0(t,\vec{x}) \xrightarrow{\mathcal{P}} A^0(t,-\vec{x})$$

Symbolically we denote

$$\boxed{A^\mu(t,\vec{x}) \xrightarrow{\mathcal{P}} A_\mu(t,-\vec{x})}$$

Photon has odd intrinsic parity.

- For the electromagnetic field if we take the temporal gauge :
  $A_0 = 0$ & $\nabla.\vec{A} = 0 \Rightarrow \Box A^\mu = 0 \Rightarrow \Box A^i = 0 \Rightarrow \vec{A}(x) \propto \vec{\epsilon}(\vec{k})e^{\pm ik.x}$.

$$\vec{A}(x) = \sum_{\lambda=1}^{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} [\epsilon(\vec{k},\lambda)e^{-ik.x}a(\vec{k},\lambda) + \epsilon^*(\vec{k},\lambda)e^{ik.x}a^\dagger(\vec{k},\lambda)] \tag{57}$$

inverting (58)

$$a(\vec{k},\lambda) = i\int d^3x \frac{1}{\sqrt{2k^0}}[e^{ik.x}\overleftrightarrow{\partial_0}\vec{A}(x).\epsilon^*(\vec{k},\lambda) \tag{59}$$
$$a^\dagger(\vec{k},\lambda) = -i\int d^3x \frac{1}{\sqrt{2k^0}}[e^{-ik.x}\overleftrightarrow{\partial_0}\vec{A}(x).\epsilon(\vec{k},\lambda) \tag{60}$$

Now, $\nabla.\vec{A}(x) = 0 \Rightarrow \vec{k}.\vec{\epsilon}(\vec{k}) = 0 \Rightarrow$polarization vector characterizing the vector potential must be transverse to the propagation of the plane wave $\Rightarrow$ two independent polarization vector : $\vec{\epsilon}(\vec{k},\lambda)$ with $\lambda = 1, 2$. We normalize these such they obey

$$\frac{\vec{k}}{|\vec{k}|} = \vec{\epsilon}(\vec{k},1) \times \vec{\epsilon}(\vec{k},2) \tag{61}$$

$$\text{[We can think of} \quad \epsilon(1), \epsilon(2) \quad \text{along x \& y axis respectively.]} \tag{62}$$

$$\Rightarrow -\frac{\vec{k}}{|\vec{k}|} = \vec{\epsilon}(-\vec{k},1) \times \vec{\epsilon}(-\vec{k},2) \tag{63}$$

But the LHS

$$-\frac{\vec{k}}{|\vec{k}|} = -\vec{\epsilon}(\vec{k},1) \times \vec{\epsilon}(\vec{k},2) \tag{64}$$

So, (58) & (59) should match $\Rightarrow$ one choice could be

$$\vec{\epsilon}(-\vec{k},1) = -\vec{\epsilon}(\vec{k},1), \quad \vec{\epsilon}(-\vec{k},2) = \vec{\epsilon}(\vec{k},2) \tag{65}$$

In order to have $\vec{A}(t,\vec{x}) \xrightarrow{\mathcal{P}} \mathcal{P}^\dagger \vec{A}(t,\vec{x})\mathcal{P} = -\vec{A}(t,-\vec{x})$ under parity one needs the following transformation law of creation and annihilation operators under parity:

$$\boxed{\mathcal{P}^\dagger a(\vec{k},1)\mathcal{P} = +a(-\vec{k},1) \quad \& \quad \mathcal{P}^\dagger a(\vec{k},2)\mathcal{P} = -a(-\vec{k},2)} \tag{66}$$

These follow from (59),(60) & (64).

- Since we have chosen $\epsilon(1)$ & $\epsilon(2)$ along x and y axis respectively, we can identify $a^\dagger(1)$ as creation operator of a photon polarized along the x-axis and $a^\dagger(2)$ as creation operator of a photon polarized along the y-axis.Then the operators

$$a^\dagger(\vec{k},R) = \frac{1}{\sqrt{2}}(a^\dagger(\vec{k},1) + ia^\dagger(\vec{k},2)) \tag{67}$$

$$a^\dagger(\vec{k},L) = \frac{1}{\sqrt{2}}(a^\dagger(\vec{k},1) - ia^\dagger(\vec{k},2)) \tag{68}$$

would create photons which are left and right circularly polarized respectively[These one photon states are eigenstates of helicity operator with eigenvalue ± 1].
Now, what would be the corresponding polarization vectors? To answer this let's do the following:

$$\sum_\lambda \vec{\epsilon}^*(\vec{k},\lambda)a^\dagger(\vec{k},\lambda) = \vec{\epsilon}^*(\vec{k},1)a^\dagger(\vec{k},1) + \vec{\epsilon}^*(\vec{k},2)a^\dagger(\vec{k},2) \tag{69}$$

$$= \frac{1}{2}(\vec{\epsilon}(1)^* + i\vec{\epsilon}(2)^*)(a^\dagger(1) - ia^\dagger(2)) \tag{70}$$

$$+ \frac{1}{2}(\vec{\epsilon}(1)^* - i\vec{\epsilon}(2)^*)(a^\dagger(1) + ia^\dagger(2)) \tag{71}$$

$$\equiv \vec{\epsilon}^*(\vec{k},L)a^\dagger(\vec{k},L) + \vec{\epsilon}^*(\vec{k},R)a^\dagger(\vec{k},R) \tag{72}$$

So,

$$\vec{\epsilon}(\vec{k},R) = \frac{1}{\sqrt{2}}[\vec{\epsilon}(\vec{k},1) + i\vec{\epsilon}(\vec{k},2)] \tag{73}$$

$$\vec{\epsilon}(\vec{k},L) = \frac{1}{\sqrt{2}}[\vec{\epsilon}(\vec{k},1) - i\vec{\epsilon}(\vec{k},2)] \tag{74}$$

Now, under parity transformation:

$$\mathcal{P}^\dagger a^\dagger(\vec{k},R)\mathcal{P} = \frac{1}{\sqrt{2}}(\mathcal{P}^\dagger a^\dagger(\vec{k},1)\mathcal{P} + i\mathcal{P}^\dagger a^\dagger(\vec{k},2)\mathcal{P}) \tag{75}$$

$$= \frac{1}{\sqrt{2}}[a^\dagger(-\vec{k},1) - ia^\dagger(-\vec{k},2)] \tag{76}$$

$$= a^\dagger(-\vec{k},L) \tag{77}$$

and

$$\mathcal{P}^\dagger a^\dagger(\vec{k},L)\mathcal{P} = \frac{1}{\sqrt{2}}(\mathcal{P}^\dagger a^\dagger(\vec{k},1)\mathcal{P} - i\mathcal{P}^\dagger a^\dagger(\vec{k},2)\mathcal{P}) \tag{78}$$

$$= \frac{1}{\sqrt{2}}[a^\dagger(-\vec{k},1) + ia^\dagger(-\vec{k},2)] \tag{79}$$

$$= a^\dagger(-\vec{k},R) \tag{80}$$

Hence,

$$\boxed{\mathcal{P}^\dagger a^\dagger(\vec{k},R)\mathcal{P} = a^\dagger(-\vec{k},L) \quad \& \quad \mathcal{P}^\dagger a^\dagger(\vec{k},L)\mathcal{P} = a^\dagger(-\vec{k},R)} \tag{81}$$

Same for annihilation operators.
If we operate (81) on the vacuum then we see under the parity transformation

$$|R\rangle \leftrightarrow |L\rangle \tag{82}$$

Specifically,

$$\boxed{\mathcal{P}|\vec{k},R\rangle = |-\vec{k},L\rangle \quad \mathcal{P}|\vec{k},L\rangle = |-\vec{k},R\rangle} \tag{83}$$

Physically this is expected if we see the below figure: under parity $\vec{p} \to -\vec{p} \Rightarrow L \leftrightarrow R$.

3 Decay of The Neutral Pion

- The spinless neutral pion decays to two photons: $\pi^0 \to \gamma\gamma$.
- This decay is driven by electromagnetic interaction which obeys the parity symmetry.Hence,

$$\mathcal{P}(\pi^0) = \mathcal{P}(\gamma_1)\mathcal{P}(\gamma_2)$$

Now, we observe the decay in CM frame of the pion $\Rightarrow \mathcal{P}(\pi^0) = \mathcal{P}(intrinsic)$.
And, total $\mathcal{P}$ of the RHS : $\boxed{\mathcal{P}^{\gamma\gamma} = (-1)^{L_{\gamma\gamma}}\mathcal{P}^2_{intr}\mathcal{P}^{\gamma\gamma}_{spin}}$

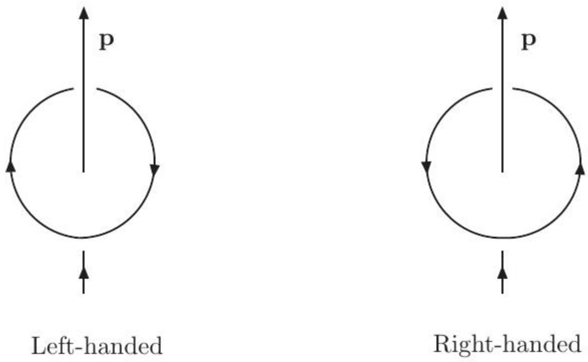

Figure 2: circular polarization

- Consider two-photon decay(one along +z axis and another along -z axis) in the CM frame of the neutral pion in full generality. For two photons with momenta $\vec{k}$ & $-\vec{k}$, one can define the states:

$$|RR\rangle \equiv a^\dagger(\vec{k},R)a^\dagger(-\vec{k},R)|0\rangle, \quad |LL\rangle \equiv a^\dagger(\vec{k},L)a^\dagger(-\vec{k},L)|0\rangle \tag{84}$$
$$|RL\rangle \equiv a^\dagger(\vec{k},R)a^\dagger(-\vec{k},L)|0\rangle, \quad |LR\rangle \equiv a^\dagger(\vec{k},L)a^\dagger(-\vec{k},R)|0\rangle \tag{85}$$

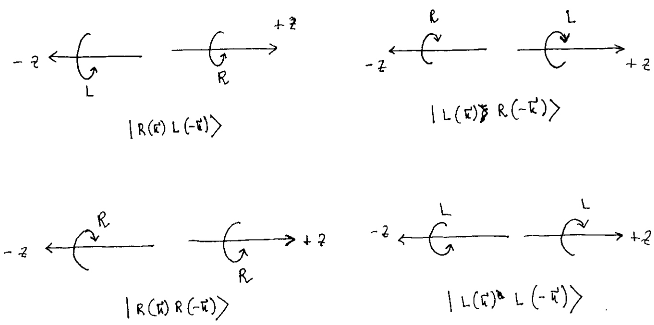

Figure 3: polarization of two photons

- This system has certain symmetries:
  1. Rotation about the z-axis$[R^z_\phi]$,
  2. Rotation about the x-axis through an angle $\pi[R^x_\pi] \Rightarrow$ interchange of +z & -z directions $\Rightarrow$ interchange of two γ-rays.
  3. Parity symmetry since the decay is governed by the EM interaction.

  Hence the final two photon state would be a simultaneous eigenstate of these three operators.

- Consider the **symmetry for the rotation around z-axis**. Photon has helicity $\pm 1 \Rightarrow J_z = \vec{J}.\frac{\vec{k}}{|\vec{k}|} = (\vec{S}+\vec{L}).\frac{\vec{k}}{|\vec{k}|} = \vec{S}.\frac{\vec{k}}{|\vec{k}|} = S_z = \pm 1$. Thus for the two photons the total z-component of the AM is

$$J_z = S_z = \pm 2 \quad \text{for LR(-2) \& RL(+2) : parallel spins(see fig. 3)} \tag{86}$$
$$J_z = S_z = 0 \quad \text{for LL \& RR : anti-parallel spins(see fig. 3)} \tag{87}$$

$\rightsquigarrow$ So, all four states are the eigenstates of R^z_ϕ.
Now, initial AM, $J_{\pi_0} = 0 \Rightarrow J_{final} = 0 = L^{\gamma\gamma} + S^{\gamma\gamma}$.
Also from (86) & (87) $S^{\gamma\gamma} = 0, 2$.
If $S^{\gamma\gamma} = 0$ then $L^{\gamma\gamma}$ also has to be zero to satisfy AM conservation.
If $S^{\gamma\gamma} = 2$ then

  1. let $L^{\gamma\gamma} = 0 \Rightarrow J^{\gamma\gamma} = 2 \Rightarrow$ not possible.
  2. let $L^{\gamma\gamma} = 1 \Rightarrow J^{\gamma\gamma} = 1, 2, 3 \Rightarrow$ not possible.
  3. let $L^{\gamma\gamma} = 2 \Rightarrow J^{\gamma\gamma} = 0, 1, 2, 3, 4 \Rightarrow$ possible since $J^{\gamma\gamma} = 0$ is a possible solution.
  4. Higher $L^{\gamma\gamma}$ is not possible since they cannot render $J^{\gamma\gamma} = 0$.

Hence calculation the parity of the final state becomes $\boxed{\mathcal{P}^{\gamma\gamma} = \mathcal{P}^{\gamma\gamma}_{spin}}$

- Now take the **rotation of π about the x-axis** $\Rightarrow$ interchange of +z & -z directions $\Rightarrow$ interchange of two γ-rays $\Rightarrow$ system should be remained invariant. Then all angles θ measured from +z become $(\pi - \theta), L(\vec{k}) \leftrightarrow L(-\vec{k})$ & $R(\vec{k}) \leftrightarrow R(-\vec{k})$.
Also, from the last physical argument:

$$R^x_\pi|L(\vec{k})R(-\vec{k})\rangle = |L(-\vec{k})R(\vec{k})\rangle \Rightarrow \quad \text{not an eigenstate of} \quad R^x_\pi \tag{88}$$
$$R^x_\pi|R(\vec{k})L(-\vec{k})\rangle = |R(-\vec{k})L(\vec{k})\rangle \Rightarrow \quad \text{not an eigenstate of} \quad R^x_\pi \tag{89}$$
$$R^x_\pi|L(\vec{k})L(-\vec{k})\rangle = |L(-\vec{k})L(\vec{k})\rangle \Rightarrow \quad \text{an eigenstate of} \quad R^x_\pi \tag{90}$$
$$R^x_\pi|R(\vec{k})R(-\vec{k})\rangle = |R(-\vec{k})R(\vec{k})\rangle \Rightarrow \quad \text{an eigenstate of} \quad R^x_\pi \tag{91}$$

- Now, let us consider the **effect of the parity** transformation which corresponds to a rotation of π about z-axis plus inversion of the z-axis. From (83):

$$\mathcal{P}|L(\vec{k})R(-\vec{k})\rangle = |R(-\vec{k})L(\vec{k})\rangle \Rightarrow \quad \text{an eigenstate of} \quad R^x_\pi \tag{92}$$
$$\mathcal{P}|R(\vec{k})L(-\vec{k})\rangle = |L(-\vec{k})R(\vec{k})\rangle \Rightarrow \quad \text{an eigenstate of} \quad R^x_\pi \tag{93}$$
$$\mathcal{P}|L(\vec{k})L(-\vec{k})\rangle = |R(-\vec{k})R(\vec{k})\rangle \Rightarrow \quad \text{not an eigenstate of} \quad R^x_\pi \tag{94}$$
$$\mathcal{P}|R(\vec{k})R(-\vec{k})\rangle = |L(-\vec{k})L(\vec{k})\rangle \Rightarrow \quad \text{not an eigenstate of} \quad R^x_\pi \tag{95}$$

However, the following combinations are eigenstates of parity operator:

$$\mathcal{P}[|R(\vec{k})R(-\vec{k})\rangle + |L(\vec{k})L(-\vec{k})\rangle] = [|R(\vec{k})R(-\vec{k})\rangle + |L(\vec{k})L(-\vec{k})\rangle] \tag{96}$$
$$\mathcal{P}[|R(\vec{k})R(-\vec{k})\rangle - |L(\vec{k})L(-\vec{k})\rangle] = -[|R(\vec{k})R(-\vec{k})\rangle - |L(\vec{k})L(-\vec{k})\rangle] \tag{97}$$

$\rightsquigarrow$ Hence, the simultaneous eigenstates of 3-symmetry operators can only be $|R(\vec{k})R(-\vec{k})\rangle \pm |L(\vec{k})L(-\vec{k})\rangle$.

- Now, it is straightforward to show

$$\frac{1}{\sqrt{2}}(|R(\vec{k})R(-\vec{k})\rangle + |L(\vec{k})L(-\vec{k})\rangle) \propto (\vec{A}_1.\vec{A}_2)|0\rangle \quad \text{parallel polariz} \tag{98}$$

$$\frac{1}{\sqrt{2}}(|R(\vec{k})R(-\vec{k})\rangle - |L(\vec{k})L(-\vec{k})\rangle) \propto (\vec{A}_1 \times \vec{A}_2).\vec{k}|0\rangle \quad \text{perpend polariz} \tag{99}$$

So, we conclude that **information on the neutral pion parity can be obtained from the polarization of the decay photons by measuring the state of linear polarization of both photons.**

$$\text{parallel} \quad \Rightarrow \quad \mathcal{P}^{intr}_{\pi^0} = +1 \Rightarrow scalar \tag{100}$$

$$\text{perpendicular} \quad \Rightarrow \quad \mathcal{P}^{intr}_{\pi^0} = -1 \Rightarrow pseudoscalar. \tag{101}$$

π^0 can't be a vector or pseudo-vector being a spin zero particle.

- **Experimental Result:** The direct measurement of the relative polarization of low energy photons(67 MeV for a π^0 decaying at rest) is experimentally challenging; however, the photon polarization plane is highly correlated to the plane of an e^+e^- pair which it can produce. The plane of each pair is predominantly is that of $\vec{E}$, so that the measurement of the angle between the plane of pairs allows one to infer about pions parity.

  In the experiment performed by Steinberger mostly relative perpendicular polarization was obtained $\Rightarrow \mathcal{P}^{intr}_{\pi^0} = -1 \Rightarrow pseudoscalar$.

4 Bibliography

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