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# Enjoying mathematics: only for the select few[1]?

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I would like begin by thanking Prof Ramanujam for inviting me to this workshop, and Dr Viswanathan for all the arrangements that have made my visit very pleasant. I am planning to speak for about 50 mins and then would like to devote the rest of the session to a discussion on the ideas presented .

**Beauty and mathematics**

I would like you to stare at the Euler equation for a few seconds.

$$e^{i\pi} + 1 = 0$$

Now stare at the following picture (the audience were asked to stare at Escher's Circle Limit III [http://mcescher.com/wp-content/uploads/2013/10/LW434-MC-Escher-Circle-Limit-III-1959.jpg ], but any beautiful picture would do).

If you found the Euler equation beautiful, then a recent study [http://journal.frontiersin.org/article/10.3389/fnhum.2014.00068/full] found that the following areas [http://www.frontiersin.org/files/Articles/74738/fnhum-08-00068-r2/image_m/fnhum-08-00068-g003.jpg] in the 'aesthetic' part of your brain, the medial orbito-frontal cortex, lit up when you were watching this slide! Amazingly, if you found the picture beautiful, then the same areas light up!

Of course mathematicians have been saying this for ages. The following quote by Russell is quite famous:

*Mathematics, rightly viewed, possesses not only truth, but supreme beauty.*

**Illumination**

[1] The talk had several cartoons, which have not been included because of copyright issues.

The experience of mathematical beauty goes along with the 'high' one gets when one has an 'aha!' moment, or in other words, when one experiences deep insight or understanding.

My favourite quote about a heightened aesthetic experience bordering on the mystic is by the French Fields medallist Alain Connes.

*But the moment illumination occurs, it engages the emotion in such a way that it's impossible to remain passive or indifferent. On those rare occasions when I've actually experienced it, I couldn't keep tears from coming to my eyes.*

I suspect that this experience of 'illumination' is what drove and continues to drive mathematicians. But this kind of experience of mathematics may be accessible to very few. For most ordinary souls like us, while we may not experience 'illumination', our experience of mathematics can probably simply be described as intensely enjoyable. I don't want to get into a debate about what motivates each of us to do mathematics. I guess for me the bottom line is that, whatever our motivation, we do enjoy doing mathematics intensely.

**Fear and mathematics**

For a very large majority of human beings who have learned mathematics, their experience of it has been far from enjoyable. On the contrary it may even have been traumatic.

Most illustrative of this trauma are the results of a study by Susan Picker and John Berry entitled 'Investigating Pupils' Images of Mathematicians'. The researchers asked children between the ages of 12 and 13 from the US, UK, Finland, Sweden and Romania to draw a picture of "a mathematician at work". Here are example of children's drawings [reproduced with kind permission from Dr Susan Picker and Dr John Berry].


*Figure 2.* Finland – Female pupil.

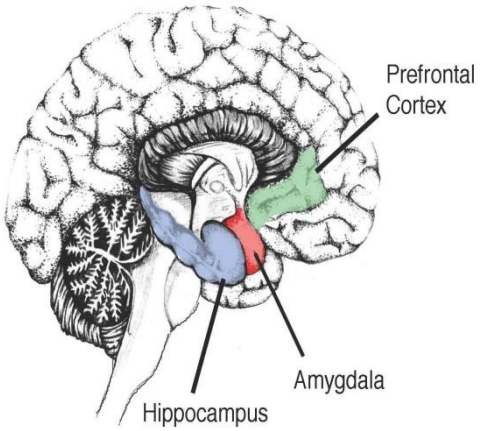

*Figure 3.* Finland – male pupil.

If we were to do a study where we scanned the brains of these students as they were learning mathematics, you will all agree that for most of them the last thing we would expect is for their medial orbito-frontal cortices to light up! What centres will light up?

*Figure 6.* U.K. – male pupil.

**The psychology of fear**

From my limited understanding, this is what neurobiologists are saying about how fear is affecting the human brain. It is a bit of a detour into the psychology of fear, but since the phrase 'math phobia' is actually used quite often, I felt it is relevant to study the effects of fear on the brain.




Deep in the brain is a small structure called the **amygdala**. Its function is to detect threats in the environment, and send out commands to other parts of the brain and body to release the stress hormone cortisol. This causes us to experience fear. The amygdala also activates (via cortisol) another small area of the brain called the **hippocampus**, to record the context in which this threat occurred so as to remember it for next time. The amygdala is also connected to the **prefrontal cortex**, the large area of brain just behind your forehead, responsible for thoughts about fearful stimuli. These are exactly the thoughts that make it so difficult to extinguish or suppress fears of any kind, however irrational.

If the amygdala repeatedly signals fear, in response to chronically threatening situations (such as math classes?) the brain becomes flooded with too much cortisol. Both the hippocampus and the prefrontal cortex are particularly sensitive to this hormone, and can become damaged as a result. The result? Functions of the brain that rely heavily on the hippocampus, such as learning and memory, will be affected negatively by stress. A particular ability of the prefrontal cortex is called **executive function**. It is the ability to plan, schedule, multitask, prioritise, focus on relevant tasks and tune out distractions. It turns out to be very useful in school...and it is this very capacity that is damaged when a child experiences chronic stress.

**Mathematics for all**

If we believe that mathematics is a core subject and that all students up to a certain age must study mathematics, don't we need to understand why, world over, there is so little joy in learning mathematics?

I would like to spend some time sharing what I believe are some of the factors.

The first and foremost is fear. I would to like to spend a little time talking about this, dividing it into two categories: the fear that is induced by the system and the environment, and the fear that is particular to the learning of mathematics.

## Landscape of Fear

For a variety of reasons, social and cultural, fear has become endemic to our school system. There is a famous national park in the US called the Yellowstone National Park. It had a large population of herbivores like moose and elk. This large population had a negative effect on the ecology of the park, because of large-scale grazing. Recently, park authorities decided to try and control the population of herbivores naturally, so they introduced grey wolves into the park. An ecologist called John Laundre studied the effects of the presence of these wolves on the elks, and discovered that the wolves impacted the elks in a profound manner. The very existence of wolves put the elks in a state of constant anxiety.

Laundre describes this rather dramatically:
*In the eyes of an elk, the physical terrain is overlaid with a mental map of risk - a landscape of fear.*

Can you imagine, in the eyes of a child, a school as a physical terrain overlaid with a mental map of risk and fear?

## Mathematics and fear

Fear is detrimental to all learning. We saw what happens in the brain each time we respond to a situation through fear. However, there is something about mathematics that has a special relationship to fear and we need to understand what this is. One sad reflection of this fact—and almost all math educators are pointing to it—is a common experience we share.

I am alluding to the fact that when we meet perfect strangers in a social situation and reveal that we teach mathematics, almost as an automatic reflex they have to tell us how good or bad they were at mathematics when they were young. My favourite example is the story of a visitor from Germany. He was rather intense looking, short with thick glasses. When he heard that I teach mathematics, he stared at me and waggled his finger and said *"I am not scared of mathematics teachers, I was good at maths!"*

So what is about the nature of mathematics that makes it so scary to learn?

I could do a thought experiment right now that will give us an insight into this. The experiment would be to propose a challenging problem and ask you all to solve it in a timed situation, right now. I can however be lazy and not do that but just ask you to recall a situation in which you experienced this. Let me share what happens to me when I am put in such a situation. Remember, the problem must be something we have not seen before, for which we do not have prior practice and knowledge, but yet be doable by high school math teachers.

### On encountering a mathematical problem: success

Here is what I experience. Let me paint two scenarios - one of failure and the other of success!
Let us talk about success first.

- Initially a sense of excitement at doing the problem
- While I have a sense of tension associated with not knowing if I will be able to solve the problem or not, there is a slight adrenalin rush, perhaps cortisol is also released
- I play with it in a variety of ways bringing in all the problem solving strategies I have picked up
- There is a flash of insight and I see the solution
- I feel very good about myself and this success reinforces a picture that I have built up, that I am very clever and good at mathematics
- If I am aware that my fellow problem solver has failed, then this makes me feel even better, and reinforces the image that I am more intelligent than others because I am good at mathematics

### On encountering a mathematical problem: failure

- Initially a sense of excitement at doing the problem
- If I do not see the solution immediately, slowly a sense of fear and dread of failure starts creeping in
- Then, there is a feeling that I am dumb and not smart enough to do the problem
- Then thoughts like, I am a teacher, I should be able to do this, what will my colleagues and students think of me
- If I know someone else has managed to solve the problem, I feel even more bad about myself and even resent this person
- Since I know there is a specific solution, I feel that whatever thoughts I have had in reasoning and wrestling with the problem are not worth sharing
- Eventually I may give up!

## The nature of mathematics and the self

Some of you may feel that I am exaggerating! But I don't think so. I think the point I am trying to make is that as teachers of mathematics, we must be aware that there is something about the nature of mathematics that has the potential to create fear in our students. I would like to suggest that this is somewhat unique to mathematics and perhaps some of the allied subjects like physics. We need to be alert to this fact and tailor our education appropriately.

Notice that in both scenarios there is something happening with the 'self-image' of the problem solver. In one case there is a potential for arrogance and in the other, a sense of low self-worth. When a student repeatedly experiences a sense of failure and associates it with mathematics, surely there is a long term damage to the psyche of such an individual, and their relationship with the subject will remain distorted, well into adulthood. If a student has repeated success with mathematics, unless the adults caring for such a student help to put it into perspective, the very ability and how society responds to this ability can distort his or her life.

It may be appropriate to comment at this point, that only when one somehow transcends these self-referential loops, can one be joyfully engaged with mathematics or any pursuit for that matter.

## Testing and grading as the main tool for assessment

Fear is one major reason why mathematics ceases to be enjoyable. I would like to go into some of the other reasons, some of which will lead us back to fear.

We all know that assessment in most schools, starting at a very young age, has been reduced to tests accompanied by marks. I am told horror stories of how schools have entrance tests for kindergarten. Guess what they are testing? Mathematics!

What does this overemphasis on testing do? Apart from wasting precious time from classroom transaction, it is distorting the learning process. For those who have learned to 'crack' these tests, it is giving a very misleading message. It reduces learning to the ability to memorize and answer rather routine questions, and it discourages a deep engagement with the subject. Moreover, the motivation to learn comes from the pride of having done well, and better than others. For those who routinely perform poorly on these tests, the message about being 'bad' at math is further reinforced.

Notice, in both cases a strong message is going out that one should avoid making mistakes. Mistakes have a high cost! However, mistakes are a serious source of learning. Mathematics educators Jo Boaler and Carol Dweck have worked extensively on how to use 'mistakes' in a creative way. In fact,

Carol Dweck claims that every time we make a mistake and learn from it, new neural networks are being created in our brain. We should be rewarding students for their mistakes, rather than punishing them for it! I will go into this later.

### Speed thrills but kills!

Mathematics has come to be associated with quick thinking. The way we have translated this is to test students under extremely tight time constraints. The well known mathematician Laurent Schwartz, often felt that he was stupid in school, because he was not very quick at doing mathematics as compared to his peers. He says,

*Rapidity doesn't have a precise relation to intelligence. What is important is to deeply understand things and their relations to each other. This is where intelligence lies. The fact of being quick or slow isn't really relevant.*

In India, we have taken this need to solve problems at high speed to another extreme. I am talking here about our entrance exams. I have an untested hypothesis that making mathematics a 'gate keeper', one that decides whether or not to allow young people access to certain opportunities, has had serious detrimental effects on mathematics education and particularly in creating a broader base of individuals who enjoy doing mathematics, perhaps lifelong.

### Does competition promote enjoyment?

This is a tricky one, because competition seems such a great motivator of learning. How can it kill enjoyment? But when you are constantly trying to out-perform others, where is the space for enjoyment?

We have all heard how during its heydays, the Soviet Union had a great culture of mathematics and the whole so-called 'Soviet bloc' has certainly produced some great mathematicians. Mathematicians like Isreal Gelfand and others were involved in interacting with school students and wrote textbooks at the primary school level. I am given to understand that competition within the so-called 'mathematical circles' played a key part in keeping the interest in mathematics alive in young people. I am sure that competition is an excellent motivator. But what happens with competition in the long run? Even if it can keep us motivated to do mathematics and better mathematics than our colleagues lifelong, what does it do to us as human beings? If all my life I feel I need to ‘beat’ other people at what I am doing, what kind of relationships do I have? How can I be secure? Someone younger and smarter is going to come along and be better at the game than me...

Somehow, beyond the initial thrill of beating my competitor, I don't see how it can be enjoyable in the long run. Especially, if engaged in a deep and beautiful subject like mathematics, should not my motivation come from the engagement with this beauty rather than wanting one-upmanship? Can we communicate all this to our students?

As a race we have become so used to competing, that when there is a crying need for unprecedented cooperation to solve problems like global warming, we are ending up unable to really cooperate.

## Proof by intimidation

Have you noticed how in the world of mathematics, almost everyone is intimidated by someone else who is more knowledgeable or cleverer?

Junior school math teachers are scared of senior school math teachers, senior school teachers are scared of college lecturers, and college lecturers are in turn scared of university professors, who may be scared of their brightest graduate students! I don't know if things have changed, but when I went to college in India, students hardly mingled with their professors and neither did we students discuss or do mathematics together. Not only is there a hierarchy among practitioners of mathematics, but there seems to be an atmosphere of intimidation, rather than a feeling of collegiality, where we are all learning together.

I am not all sure that an intimidating environment promotes enjoyment. This does not mean that I do not acknowledge our different capacities, and that some human beings seem to have exceptionally deep insight into mathematical reality, and can reveal deep truths which seem to have 'infinite order' and beauty. Surely, such an acknowledgement should bring about humility and respect, but why fear and intimidation?

## Myths and beliefs: mathematics is all about calculating

I would like to now spend some time going into some of the myths and beliefs about mathematics which not only give math a ‘bad rap’, but also kill students' enjoyment.

Certainly the equating of mathematics to numeracy and accountancy has done mathematics a disservice. I am reminded of a funny joke here. My friend who was doing her PhD in mathematics, was asked by her mother, "What should I say when people ask me, what you are doing for your PhD?" My friend answered, "Just tell them I am calculating a big number." The mother then went to a party, where she met a stranger, who promptly asked her this question. So pat came the answer, "She

is calculating big numbers." The stranger looked very perplexed, because as it turned out she was a mathematician herself!

I love this quote by Sofia Kovalevskaia:
*Many who have never had the occasion to discover more about mathematics confuse it with arithmetic and consider it a dry and arid science. In reality, however, it is a science which demands the greatest imagination.*

I do hope a large number of students get to experience mathematics with all its depth and imagination.

### Myths and beliefs: the teacher knows everything

In most teaching situations all information flows from the teacher, and students immediately set him or her up as an authority. There is no room for discussion, where the student can contribute at a level comparable to the teacher. Students are rarely exposed to problems to which the teacher does not already know the solution, or where the teacher is ready to admit that he does not know and is willing to learn along with the student. All this further promotes the sense of intimidation that I talked about earlier.

### Myths and beliefs: mathematics does not afford experimentation and exploration

Another way to put this is that there is only one right answer to a question in mathematics and there is no scope for interpretation. In other subjects, not just the humanities, even in the sciences, students can offer an opinion or a hypothesis as a solution, which can be respected for what it is and then further explored perhaps leading to a more nuanced understanding of the question in hand.
In mathematics students may feel that it is all about remembering the right technique or formula, or that they just have to get the right algorithm to solve a problem.

However we know that actually in mathematical discovery, there are many guesses, conjectures, false steps and a lot of scope for exploration. What is key is however that at the core is a coherent reasoning process, which needs to be constantly tested, lest one is making wrong assumptions or logical fallacies. To enable students to build this capacity may require a great deal of patience on our part, because for many it may not be 'natural'.

### Engagement with mathematics

I have not counted the number of times I have used the word enjoyment, and you may be feeling, "Enough with this enjoyment business, why is he making such a big deal about it?"

Suppose we start with the premise that in an increasingly technological world, we would like a significant majority of the citizens to be comfortable with and competent in mathematics. In an ideal world, this would translate to people being engaged with mathematics at various levels:

- Stimulating this capacity of the human brain (seeing patterns, playing with numbers)
- Meeting the demands of daily life (commerce, construction, dealing with data)
- Harnessing its power to improve the living conditions of all humans (access to basic services, engineering, medicine )
- Using it as a language to understand the universe (scientific understanding of the universe)
- Teaching and sharing this faculty with the next generation
- Creating new mathematics

To me, the only sustainable way this is going to happen is if the people who engage with mathematics see it as meaningful and enjoyable, and society at large supports it from this standpoint.

All of our attempts so far at trying to improve the state of mathematics education because of its commercial utilitarian value, because of fear that our rivals are better than us, or that somehow greater knowledge will save mankind, have failed because these ideas inherently contain the very seeds that destroy the enjoyment of doing mathematics.

## Not a feel-good know-nothing approach

I know many of you may have this lurking fear that what I am suggesting is a fuzzy feel-good know-nothing approach. Not at all. I am all for rigour and excellence and domain competence. I am aware that one of the major causes for poor education all over the world, not just in India, is that teachers themselves are not very proficient and lack competence in the areas they are teaching.

My favourite story in this regard is that when a student of ours was struggling with mathematics, the parent was surprised. She said, in his previous school, he had no trouble with maths. She said they had such a nice approach to teaching mathematics, and very innocently said, "You know till Class V, they learnt mathematics without using numbers at all!"

I am talking about a sound understanding of mathematics. But, I am also talking about making sure that this is happening without damaging the psyche. For the rest of the talk, I would like to first go into what this sound understanding of mathematics means and explore ways we can do so keeping in mind all that has been said so far.

## Mathematical disposition

There seems to be broad consensus among educators that being competent in mathematics involves moving towards a mathematical disposition. This disposition has been broken down into five components.

- A well-organized and flexibly accessible **domain-specific knowledge base** involving the facts, symbols, algorithms, concepts, and rules that constitute the contents of mathematics as a subject-matter field.
- **Heuristic methods**, i.e., search strategies for problem solving, which do not guarantee, but significantly increase the probability of finding the correct solution: for instance, decomposing a problem into subgoals.

The first two in other words say - students should know their stuff! This is what all of us teachers are aiming at. One obvious thing to keep in mind is that this is never a finished project, and there is always more to be learned. A key point that is being made here by several educators is that students not only learn expertise, but are able to use this expertise flexibly and creatively. So it is not enough that they are able to solve the so-called routine exercises, they should be able apply their expertise to solve new and unfamiliar problems. I think the point about heuristic methods is self evident, and I am sure you have been exposed to several such techniques in your work.

The last three points have to with the culture of learning we are creating in our schools. This is what the bulk of this talk has been about.

So let us go through these three slowly and see what they mean and how they can come about in our classes.

- **Meta knowledge:** this involves knowledge about one's cognitive functioning on the one hand, and knowledge about one's motivation and emotions on the other hand (e.g., becoming aware of one's fear of failure when confronted with a complex mathematical task or problem).

The point being made here is that it is not enough for a learner to just have a knowledge base; they also need to be aware of what gaps they have in their knowledge, and the emotional forces and motivations underlying their approach to the subject.

- **Positive mathematics-related beliefs**, which include implicitly and explicitly held subjective conceptions about mathematics education, about the self as a learner of mathematics, and about the social context of the mathematics classroom. It also importantly includes *not* having the false belief that girls are weaker at math!

We have talked at length about the fact that the beliefs one holds about mathematics and its nature, and beliefs about one's own capacity to learn, seriously affect the learning of mathematics. A sad fact is that girls and boys, men and women, seem to believe that males are better than females at math. This has been studied scientifically and the consensus is that the difference is only on a few select tasks, and is slight.

- **Self-regulatory skills**. These include the self-regulation of one's cognitive processes (planning and monitoring one's problem-solving processes) on the one hand, and skills for regulating one's volitional processes/activities on the other hand (keeping up one's attention and motivation to solve a given problem).

This one is actually about helping and guiding a student to stay focussed while doing mathematics.

## What can we do?

I would like to share what we have attempted in our school to bring about mathematical disposition. First, a disclaimer. We are not churning out mathematicians by the dozen. We can say humbly that most of our students enjoy mathematics, few are scared of it and many positively love it. I am also aware that we have several advantages in our working environment, not the least of which is that we are very small school and our students receive support from their parents. In contrast, I am acutely conscious that many of you are working under difficult circumstances, with large classrooms, and perhaps tremendous pressure from parents and the management. What the rest of the talk offers, therefore, is not a blueprint or a solution, but shows what we are trying in a small way to address the daunting issue of maths education.

## Creating a learning culture:

I will begin by first of all sharing what the nature of learning environment in our school is.

- The relationship between the teacher and student is based on mutual trust and affection, not on authority and fear
- The classroom environment is kept light, yet rigour is not sacrificed for informality.
- The students' collective attention has to be constantly held on what is being learned, the teacher keeping track of each child as the lesson progresses.

There is tremendous emphasis in allowing students participation in class and hence:

- Students speak as much as, if not more than, the teacher. They spontaneously explain to each other what they have learned, and answer each other's questions. Some comments may seem

tangential or even irrelevant to a discussion, but, if followed up, often yield unexpected connections and ways of understanding.

- They are encouraged to talk about what is happening to them as they are learning mathematics, what their fears are, what is blocking their learning, what their implicit and explicit beliefs about learning are.
- They are also encouraged to think aloud and reveal what their thinking process is, how they are going about attacking a problem and so on.

We believe learning is a social activity. Thus:

- The student who finds math easiest is not the star of the math class! Everyone feels equally important in class, in terms of attention, appreciation and affection.
- Students often work in groups and learn cooperatively, making mathematics a social activity. They engage in thinking together in solving problems and help each other to build the solution without a sense of competition.
- Every attempt is made to demonstrate that mathematics is a human endeavour. This is done by talking about the history of mathematics and stories of mathematicians, trying to discover why humans might have needed/developed the mathematics that is being taught.

I would like to share some of our teaching practices:

- From the very beginning, there is a great emphasis on understanding, rather than rote learning. Teachers spend a significant amount of class time explaining concepts, and children are often called upon to articulate what they have learnt, as far as possible in precise language.

Teachers will find that in the beginning this takes forever, and it may seem easier to just give the student the algorithm or help them memorize the formula or procedure. However, understanding builds upon understanding and you will find the gains accruing over time, students gain a sense of security because of their capacity to understand and the meta-knowledge of what they have understood and what they have not. Memorizing only makes students lose the capacity to think and creates fear of not having really understood.

- There is an attempt to create ‘number sense.’
- Reasoning skills are taught through math games and ‘thinking stories.’
- As I said before mistakes are valuable aids to learning, so students are encouraged to examine their mistakes and see what led them to pursue a certain track.

- As and when possible, the teacher will try and connect what appear to be different parts of mathematics, so that the child's learning is not compartmentalised.

We emphasize experimental mathematics and project work. Yes! Mathematics is a theoretical subject, but there is great scope for experimentation within mathematics. Archimedes, considered the greatest mathematician from antiquity has this to say:
*…(it is) possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded no less useful even for proof of the theorems themselves; for certain things first become clear to me by a mechanical method…*

Experiments can be done with hands-on activities or nowadays with computer software like Geogebra. We also spend time within our curriculum to do projects with our students. For example, doing tessellations, nets for 3D solids, creating Platonic solids, playing with the Mobius strip, computing the digits of $\pi$, topics from the history of mathematics and so on. We find projects have the following in-built valuable features:

- Excellent opportunities for cooperative learning
- Enjoyment of mathematics as a social activity
- Dispelling several of the myths about mathematics
- Multi-disciplinary, bringing in art, craft, history and allied sciences
- Experience of topics not traditionally taught at school
- Pattern recognition, which is indeed the core of mathematics
- Concrete ways of showing how mathematics is beautiful

Mathematics is part of the overall consciousness of the school, for example in whole school assemblies students present what they have learnt in their classes. Every two years the whole school is engaged in common theme that culminates as a mela. We had a mela completed devoted to mathematics, and in many other melas, like astronomy and magic, mathematics played a key part.

At CFL our approach to assessment is as follows. We save a lot of classroom time because we have done away with tests and exams. Our small class size allows for a keen observation and understanding of where each child is. We break down concepts into various components to figure out difficulties that students have with these. We believe that understanding is a cumulative process and we can continually refine and improve it through feedback and correction. The students give us written work in assignments which are corrected but not graded. There is no comparative evaluation, so that students can focus on their own learning and not be troubled by their peers' performance. However,

they do give feedback to each other in classroom discussions, and help each other spontaneously. Finally, parents are involved in supporting their children's learning process.

## Enjoying mathematics: not only for the select few!

I feel with Jo Boaler that:

*I want to live in a world where everyone can learn and enjoy math, and where everyone receives encouragement regardless of the color of their skin, their gender, their income, their sexuality, or any other characteristic.*

I would like to end this talk by going back to the beauty of mathematics and enjoyment by quoting the French mathematician Henri Poincare:

*The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful.*