# On Conic Sections and the General Equation of the Second Degree.

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## Some linear algebra.

Consider the $2\times2$ symmetric matrix

$$\begin{bmatrix} a & h \\ h & b \end{bmatrix}$$

Characteristic equation:

$$\lambda^2 - (a+b)\lambda + (ab - h^2) = 0.$$

$$\lambda = \frac{(a+b) \pm \sqrt{(a+b)^2 - 4ab + 4h^2}}{2}.$$

$$(a+b)^2 - 4ab + 4h^2 = (a-b)^2 + 4h^2 \geq 0.$$

Thus both eigenvalues are real.

Coincident $\Leftrightarrow$ $a = b,\ h = 0$.

Let $\underline{u} = (u_1, u_2)$ be an eigenvector for $\lambda$

$\underline{v} = (v_1, v_2)$ 〃 〃 $\mu$.

$$\lambda\langle \underline{u}, \underline{v}\rangle = \langle \lambda\underline{u}, \underline{v}\rangle = \langle A\underline{u}, \underline{v}\rangle$$
$$= au_1v_1 + h(u_1v_2 + u_2v_1) + bu_2v_2$$
$$= \langle \underline{u}, A\underline{v}\rangle = \mu\langle \underline{u}, \underline{v}\rangle$$

$\lambda \neq \mu = \langle \underline{u}, \underline{v}\rangle = 0$. and we can

If $\lambda = \mu$ then matrix is $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ and again

choose $\underline{e_1} = (1,0)$ and $\underline{e_2} = (0,1)$

$\langle \underline{u}, \underline{v}\rangle = 0$.

Choose $\underline{u}, \underline{v}$ such that $|u_1|^2 + |u_2|^2 = 1 = |v_1|^2 + |v_2|^2$

Then $P = \begin{bmatrix} u_1 & v_1 \end{bmatrix}$ is s.t. $P^TP = PP^T = I$.

(orthog. matrix).

$$A\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = \begin{pmatrix} \lambda u_1 & \mu v_1 \\ \lambda u_2 & \mu v_2 \end{pmatrix} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

i.e. $AP = P\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.

$$P^T A P = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

or $A = P^T \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} P$.

If $\lambda = \mu = 0$ then $A = 0$ i.e. $a = b = h = 0$.

$\lambda + \mu = a + b = tr(A)$

$\lambda \mu = ab - h^2 = det(A)$.

Example: $\begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$.

$\lambda^2 - 10\lambda + 16 = 0$.

$\lambda^2 - 8\lambda - 2\lambda + 16 = 0$

$\lambda(\lambda - 8) - 2(\lambda - 8) = 0 \Rightarrow \lambda = 2, \lambda = 8$.

$5x - 3y = 2x \Rightarrow 3x - 3y = 0 \Rightarrow x = y$.

$-3x + 5y = 2y$.

$(1, 1) \Longleftrightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

$5x - 3y = 8x \Rightarrow 3x + 3y = 0 \quad x = -y$.

$-3x + 5y = 8y$

$(1, -1) \quad \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}.$$

I. Homogeneous equation of second degree.

$$ax^2 + 2hxy + by^2 = 0.$$

i.e. $(x\ y)\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

i.e. $[x, y]\, P \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} P^T \begin{bmatrix} x \\ y \end{bmatrix} = 0.$

Set $P^T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \iff \begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x' \\ y' \end{bmatrix}.$

i.e. $x' = u_1 x + u_2 y$
$y' = v_1 x + v_2 y.$

Then eqn. becomes $\lambda x'^2 + \mu y'^2 = 0.$

$(\lambda, \mu) \neq (0,0)$,

① If $\lambda > 0, \mu > 0$ then only solution is $x' = y' = 0$
i.e. $x = y = 0$.

② So also for $\lambda < 0, \mu < 0$.

③ $\lambda > 0, \mu < 0$ or $\lambda < 0, \mu > 0$.

$$\lambda x'^2 = -\mu y'^2$$

$$x' = \pm\sqrt{\frac{-\mu}{\lambda}}\, y'$$

We have two lines.

$u_1x + u_2y = \pm\sqrt{\frac{-\mu}{\lambda}}(v_1x + v_2y)$

II. The equation

$$ax^2 + 2hxy + by^2 = 1.$$

As before

$$\lambda x'^2 + \mu y'^2 = 1.$$

(1) ~~λ~~ $a = b, h = 0 \Rightarrow$ circle.
$\quad > 0$

$\Rightarrow \lambda = \mu > 0$

(2) $a = b, h = 0 \Rightarrow \phi$.
$\quad < 0$

$\lambda = \mu < 0 \Rightarrow \phi$.

(3) $\lambda \overset{<0}{\neq} \mu < 0 \Rightarrow \phi$.

(4) $\lambda > 0, \mu > 0 \Rightarrow$ ellipse.

Centre $= (0,0)$.

~~semi~~ axes are $x' = 0, y' = 0$

i.e. $u_1 x + u_2 y = 0, \; v_1 x + v_2 y = 0$.

Lengths of semi axes $\frac{1}{\sqrt{\lambda}}, \frac{1}{\sqrt{\mu}}$.

$$\text{Area} = \frac{\pi}{\sqrt{\lambda\mu}}.$$

(5) $\lambda > 0, \mu < 0$ or $\lambda < 0, \mu > 0$.

$\Rightarrow$ hyperbola

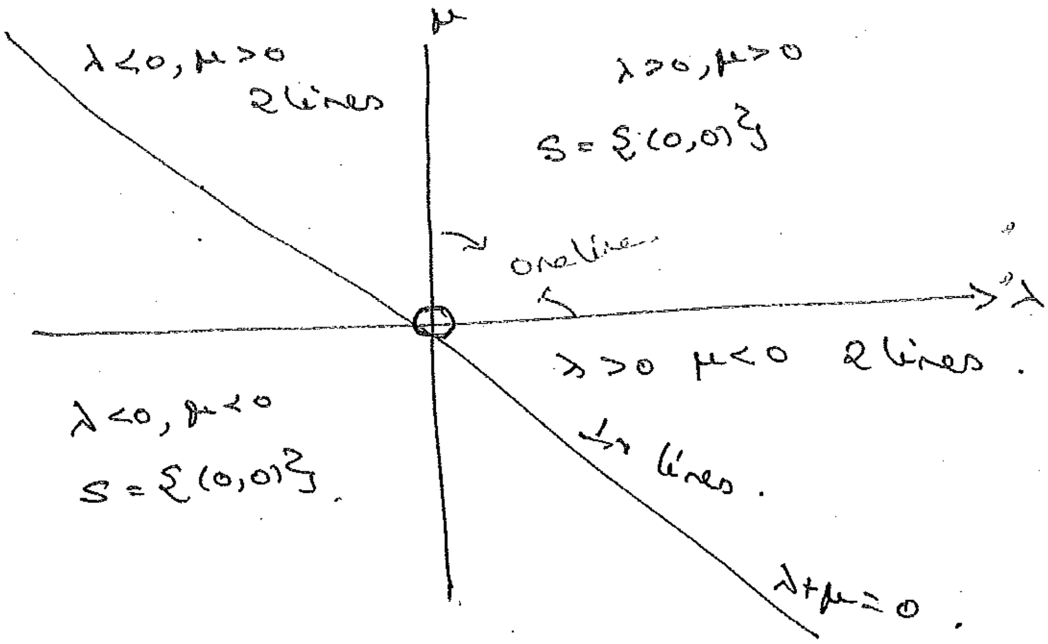
cases as before.

(4) If $\lambda = 0$, $\mu \neq 0$ we have

$$\mu y'^2 = 0 \quad \text{i.e.} \quad y' = 0$$

i.e. One line, $v_1 x + v_2 y = 0$.

$|||^{ly}$ $\lambda \neq 0$, $\mu = 0$, $u_1 x + u_2 y = 0$.




If $\lambda + \mu = 0$ then $\mu = -\lambda$ so that we get

$$\lambda(x'^2 - y'^2) = 0$$

i.e. $x' + y' = 0$ or $x' - y' = 0$.

$$(u_1 x + u_2 y) + (v_1 x + v_2 y) = 0$$

$$(u_1 x + u_2 y) - (v_1 x + v_2 y) = 0$$

Pair of $\perp$r lines.

(6) $\lambda = 0, \ \mu \neq 0$ or $\lambda \neq 0, \ \mu = 0$.

$$\mu y'^2 = 1 \quad \text{or} \quad \lambda x'^2 = 1.$$

$\mu < 0 \Rightarrow \phi$. $\quad \lambda < 0 \Rightarrow \phi$.

$\mu > 0 \Rightarrow y' = \pm \frac{1}{\sqrt{\mu}}$ $\quad \lambda > 0 \Rightarrow x' = \pm \frac{1}{\sqrt{\lambda}}$.

$\therefore$ Pair of lines $\quad u_1 x + u_2 y = \pm \frac{1}{\sqrt{\mu}}$.

or $\quad u_1 x + u_2 y = \pm \frac{1}{\sqrt{\lambda}}$

(7) If $\lambda + \mu = 0$ then

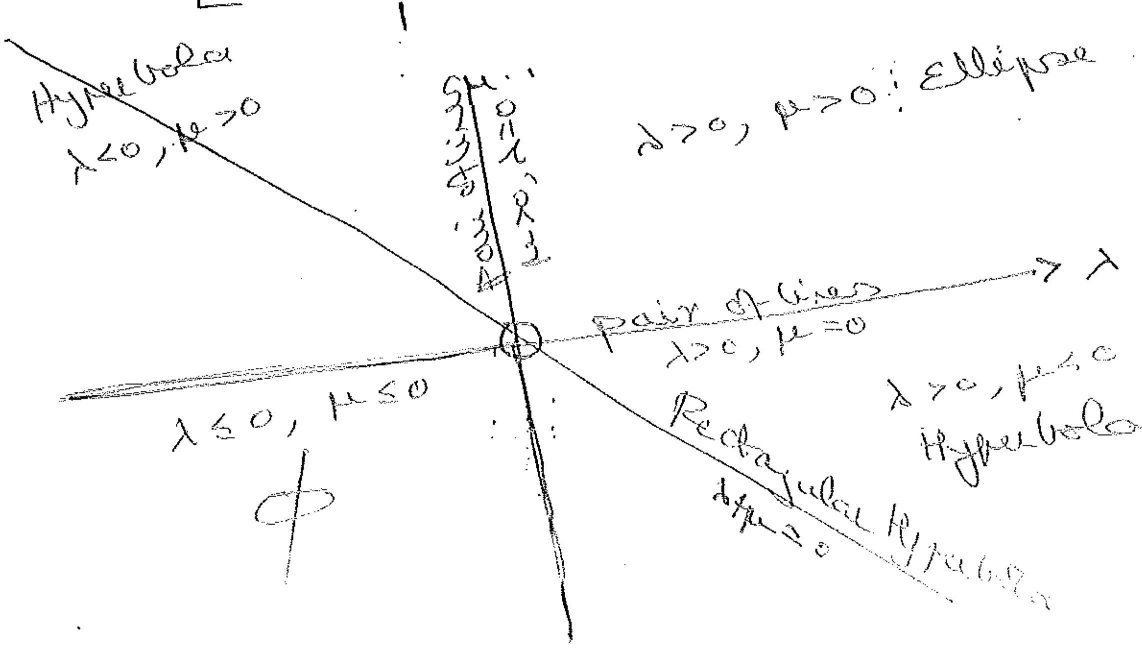
$\lambda > 0$

$$x'^2 - y'^2 = \frac{1}{\lambda}.$$

$$(x' + y')(x' - y') = \frac{1}{\lambda}$$

Rectangular hyperbola.

$$\left[(u_1 x + u_2 y) + (v_1 x + v_2 y)\right]\left[(u_1 x + u_2 y) - (v_1 x + v_2 y)\right] = \frac{1}{\lambda}.$$



## III. General Equation of $2^{nd}$ degree.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

(1) Assume it represents a pair of $x^{ing}$ lines.

Then $ax^2 + 2hxy + by^2 = 0$ represents a pair of $\parallel^{el}$ lines $x^{ing}$ at $0$.

( $lx + my + n = 0$, $l'x + m'y + n' = 0$ ).

compare coeffs.

$\Rightarrow \quad ab - h^2 < 0.$

Let $(\alpha, \beta)$ be the pt. of $x^n$.

$x = X + \alpha, \quad y = Y + \beta.$

$$\Rightarrow aX^2 + 2hXY + bY^2 + 2(a\alpha + h\beta + g)X + 2(h\alpha + b\beta + f)Y + \underbrace{a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c}_{=0} = 0$$

$= 0$ ∵ it lies on $x^n$.

$\alpha, \beta \quad \begin{cases} a\alpha + h\beta + g = 0 \\ h\alpha + b\beta + f = 0 \end{cases}$ | ∵ pair of lines $x^{ing}$ at origin.

$\exists!$ soln. ∵ $ab - h^2 < 0$.

This gives the pt. of $x^n$.

lines are $aX^2 + 2hXY + bY^2 = 0$

i.e. $u_1 X + u_2 Y = 0$

$v_1 X + v_2 Y = 0$.

i.e. $u_1(x - \alpha) + u_2(y - \beta) = 0$

$v_1(x - \alpha) + v_2(y - \beta) = 0$ $\Big\}$.

Also $a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0$

$\Rightarrow \alpha(a\alpha + h\beta + g) + \beta(h\alpha + b\beta + f)$

$+ g\alpha + f\beta + c = 0$.

i.e $g\alpha + f\beta + c = 0$

$$a\alpha + h\beta + g = 0$$
$$h\alpha + b\beta + f = 0$$
$$g\alpha + f\beta + c = 0$$

i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

(2) $\|^{el}$ lines $lx + my + n = 0$
$lx + my + n' = 0$.

$l^2 = a,\ m^2 = b \quad ml = h$

$\Rightarrow ab = h^2$.

$g = \frac{l}{2}(n+n')$

$f = \frac{m}{2}(n+n')$

$c = nn'$.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} l^2 & lm & l/2\,(n+n') \\ lm & m^2 & m/2\,(n+n') \\ l/2\,(n+n') & \frac{m}{2}(n+n') & nn' \end{vmatrix}$$

$$= lm \begin{vmatrix} l & m & \frac{n+n'}{2} \\ l & m & \frac{n+n'}{2} \\ \frac{l}{2}(n+n') & \frac{m}{2}(n+n') & nn' \end{vmatrix} = 0$$

(3) Converse

Case 1. Let $ab - h^2 < 0$ and

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

Choose $\alpha, \beta$ s.t. $a\alpha + h\beta + g = 0$
$h\alpha + b\beta + f = 0$

$\exists !$ soln $\because ab - h^2 \neq 0$.

$\Delta = 0 \implies g\alpha + f\beta + c = 0$.

$\therefore$ $x = X + \alpha$, $y = Y + \beta$ $\implies aX^2 + 2hXY + bY^2 = 0$

which is a pair of st. lines $X^{ing}$ at $X = 0, Y = 0$,

i.e. $(x, y) = (\alpha, \beta)$.

Example:- ~~Again if~~ $x^2 - y^2 + x - 3y - 2 = 0$.

$$\begin{vmatrix} 1 & 0 & 1/2 \\ 0 & -1 & -3/2 \\ 1/2 & -3/2 & -2 \end{vmatrix}.$$

$(-2)(-1) + \frac{3}{2}(-3/2) + \frac{1}{2}(+\frac{1}{2}) = 2 - \frac{9}{4} + \frac{1}{4} = 2 - 2 = 0$.

$ab - h^2 = -1 < 0$.

$\therefore$ Pair of lines.

$\alpha + 1/2 = 0 \implies \alpha = -1/2$

$-\beta - 3/2 = 0 \implies \beta = -3/2$

$x = X + \alpha$, $y = Y + \beta$.

Pt. of $X^n = (-1/2, -3/2)$

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\lambda = 1$, $\mu = -1$ $(1, 0)$ $(0, 1)$.

$X^2 - Y^2 = 0$

$x + 1/2 = y + 3/2$

Case (ii)

~~[illegible]~~ $\Delta = 0$ $ab = h^2$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \Rightarrow \text{indep of } c$$

$$\Rightarrow \begin{pmatrix} a & h & g \\ h & b & f \end{pmatrix} \text{ lin dep.}$$

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix} \longrightarrow \text{eigenvalues } 0, \mu (\neq 0)$$
eigen vect - $(u_1, u_2)$ $(v_1, v_2)$

$X = u_1 x + u_2 y$ $\quad x = u_1 X + v_1 Y$

$Y = v_1 x + v_2 y.$ $\quad y = u_2 X + v_2 Y.$

$$\mu Y^2 + 2GX + 2FY + C = 0.$$

$G = g u_1 + f u_2$ $\quad a u_1 + h u_2 = 0$

$F = g v_1 + f v_2$ $\quad h u_1 + b u_2 = 0.$

$\Rightarrow g u_1 + f u_2 = 0.$

$\Rightarrow G = 0.$

$$\therefore \mu Y^2 + 2FY + C = 0$$

$$\mu\left(Y + \frac{F}{\mu}\right)^2 + C - \frac{F^2}{\mu} = 0.$$

$$\left(Y + F/\mu\right)^2 = \frac{F^2}{\mu^2} - \frac{C}{\mu}.$$

$\Rightarrow$ $\|^{el}$ lines $\cdot$ ~~B~~ if $\frac{1}{\mu^2}\left(F^2 - \mu C\right) > 0.$ $\quad F^2 - \mu C > 0$

line if $F^2 = \mu C$.

~~Case~~

Case (iii) $ab - h^2 > 0$ $\Delta = 0$,

Again $\exists ! \, (\alpha, \beta)$ s.t. $a\alpha + h\beta + g = 0$

$h\alpha + b\beta + f = 0$

$\Longrightarrow g\alpha + f\beta + c = 0$.

$\Longrightarrow x = X + \alpha, \; y = Y + \alpha$

$aX^2 + 2hXY + bY^2 = 0$

$\Rightarrow$ Set is $\{(\alpha, \beta)\}$.

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(5) Conics $\Delta \neq 0$.

Case (i) $ab - h^2 < 0$. Now $g\alpha + f\beta + c \neq 0$.

Choose $\alpha, \beta$ as before.

Thus $x = X + \alpha$, $y = Y + \beta$ $\Rightarrow$

$aX^2 + 2hXY + bY^2 + (g\alpha + f\beta + c) = 0$ . i.e. $\lambda X'^2 + \mu Y'^2 = C$

Then we get a hyperbola with centre at $(\alpha, \beta)$; lengths of semi-axes $\frac{1}{\sqrt{|\lambda|}}, \frac{1}{\sqrt{|\mu|}}$.

Semi-axes: $u_1 X + u_2 Y = 0$

$v_1 X + v_2 Y = 0$

i.e. $u_1(x - \alpha) + u_2(y - \beta) = 0$

$v_1(x - \alpha) + v_2(y - \beta) = 0$.

<u>Case (ii)</u> $ab - h^2 > 0$.

Again $(\alpha, \beta)$ exists uniquely and we get

$$aX^2 + 2hXY + bY^2 + C = 0 \qquad C = g\alpha + f\beta + c.$$

i.e. $\lambda X'^2 + \mu Y'^2 + C = 0$.

The set is either $\phi$ or an ellipse depending on C.

We have semiaxes ~~and~~ etc. as before.

<u>Example:</u>

$$5x^2 - 6xy + 5y^2 + 22x - 26y + 29 = 0.$$

$$\begin{vmatrix} 5 & -3 & 11 \\ -3 & 5 & -13 \\ 11 & -13 & 29 \end{vmatrix} \neq 0.$$

$ab - h^2 > 0$.

$$\left.\begin{array}{l} 5\alpha - 3\beta + 11 = 0 \\ -3\alpha + 5\beta - 13 = 0 \end{array}\right| \Rightarrow \alpha = -1, \beta = 2.$$

$$g\alpha + f\beta + c = -11 - 26 + 29 = -8$$

$$x = X + \alpha, \; y = Y + \beta$$

$$\Rightarrow \quad 5X^2 - 6XY + 5Y^2 = 8.$$

$$\frac{5}{8}X^2 - \frac{6}{8}XY + \frac{5}{8}Y^2 = 1.$$

$$\frac{1}{8}\begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda - 8)(\lambda - 2) = 0.$$

$$\lambda = 8, \mu = 2.$$

$\lambda = 8$. $\quad 5x - 3y = 8x$ $\quad 3x = 3y$

$\Rightarrow -3y = 4x$. $\quad (1, -1)$. $\quad \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.

$\lambda = 2$ $\quad 5x - 3y = 2x$ $\quad 3x = 3y$.

$(1, 1)$ $\quad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Centre: $(-1, 2)$.

Semi-major axis 2

Semi-minor axis 1.

Major axis. $X = Y$ $\quad x+1 = y-2$ or $x - y + 3 = 0$

Minor axis $X = -Y$ $\quad x+1 = 2-y$ or $x + y - 1 = 0$

If we change $2g$ to $3g$

Then $g\alpha + f\beta + c = 2$.

and we get $5X^2 - 6XY + 5Y^2 = -2$

$8X'^2 + 2Y'^2 = -2$

which is the empty set.

Case (iii) $\Delta \neq 0$ $ab - h^2 = 0$.

Now, since $\Delta \neq 0$ $\quad a\alpha + h\beta + f = 0$
$\quad h\alpha + b\beta + g = 0$

will not have a solution in general.

So we ~~directly~~ first rotate the axes:

$$X = u_1 x + u_2 y \qquad \Longrightarrow \qquad x = u_1 X + v_1 Y.$$
$$Y = v_1 x + v_2 y. \qquad\qquad y = u_2 X + v_2 Y.$$

to get

$$\lambda X^2 + \mu Y^2 + 2GX + 2FY + C = 0.$$

with $\begin{Bmatrix} \lambda = 0 \\ \mu \neq 0 \end{Bmatrix}$ or $\begin{Bmatrix} \mu = 0 \\ \lambda \neq 0 \end{Bmatrix}$.

Let $\lambda = 0$. $\mu \neq 0$.

$G = gu_1 + fu_2$

$F = gv_1 + fv_2$.

Since $au_1 + hu_2 = 0$
$hu_1 + bu_2 = 0$

$G = 0 \Longrightarrow \Delta = 0$ X.

$\therefore G \neq 0$.

So $\mu Y^2 + 2GX + 2FY + C = 0$.

$$\mu\left(Y + \frac{F}{\mu}\right)^2 = -2GX - C + \frac{F^2}{\mu}$$

$$\left(Y + \frac{F}{\mu}\right)^2 = -\frac{2G}{\mu}\left(X + \frac{C}{2G} - \frac{F^2}{2G\mu}\right)$$

$$x' = X - \frac{F^2}{2G\mu} + \frac{c}{2G}, \quad y' = Y + \frac{F}{\mu}.$$

we get $\quad y'^2 = -\frac{2G}{\mu} x'.$

Which is a parabola.