On the general equation of the second degree

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Abstract

We give a unified treatment of the general equation of the second degree in two real variables in terms of the eigenvalues of the matrix associated to the quadratic terms and describe the solution sets in all cases.

1 Introduction

The study of the general equation of the second degree in two variables used to be a major chapter in a course on analytic geometry in the undergraduate mathematics curriculum for a long time. The equation usually represents a pair of straight lines or a conic. In the latter case the method of tracing a conic was to compute the trigonometric ratios of the angle that the axes of the conic make with the coordinate axes and then rotate the coordinate axes to reduce the equation to the normal form. These computations could be tedious. Further in most classical text books the treatment is rather incomplete and the cases when the solution set is degenerate (especially when it contains a single point or is empty) are not carefully explained.

The aim of this note is to study all cases of the equation in a unified manner. By just computing the eigenvalues and eigenvectors of the 2×2 real symmetric matrix associated to the quadratic terms, we can just read off the properties of the solution set and also write down the equations of various features of the set very easily.

This approach neatly brings out some of the connections between linear algebra and geometry.

2 Some linear algebra

Consider the following 2×2 real symmetric matrix:

$$A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}.$$

Its characteristic equation is

$$\lambda^2 - (a+b)\lambda + (ab - h^2) = 0.$$

The discriminant is

$$(a+b)^2 - 4ab + 4h^2 = (a-b)^2 + 4h^2 \geq 0$$

and so both its eigenvalues are real and are given by

$$\frac{(a+b) \pm \sqrt{(a-b)^2 + 4h^2}}{2}.$$

These eigenvalues are coincident if, and only if, $a = b$ and $h = 0$.

Let us denote these eigenvalues by λ_1 and λ_2. Then we can find an eigenvector $\mathbf{u} = (u_1, u_2)$ associated to λ_1 and an eigenvector $\mathbf{v} = (v_1, v_2)$ associated to λ_2. If $(.,.)$ denotes the usual euclidean scalar product in $\mathbb{R}^2$, we have

$$\begin{aligned}\lambda_1(\mathbf{u},\mathbf{v}) &= (\lambda_1\mathbf{u},\mathbf{v}) \;=\; (A\mathbf{u},\mathbf{v}) \\ &= au_1v_1 + h(u_2v_1 + u_1v_2) + bu_2v_2 \\ &= (\mathbf{u}, A\mathbf{v}) \;=\; \lambda_2(\mathbf{u},\mathbf{v}).\end{aligned}$$

If $\lambda_1 \neq \lambda_2$, it then follows that

$$(\mathbf{u},\mathbf{v}) = 0. \tag{2.1}$$

If $\lambda_1 = \lambda_2$, then, as we already observed, $A = aI$, where I is the 2×2 identity matrix, and so every vector in $\mathbb{R}^2$ is an eigenvector and we can always choose the two eigenvectors $\mathbf{u} = (1,0)$ and $\mathbf{v} = (0,1)$ so that (2.1) is still valid.

Thus, we always have a pair of orthogonal eigenvectors. Let us normalize them so that their euclidean norms are unity. Thus let $\mathbf{u}$ and $\mathbf{v}$ satisfy

$$u_1^2 + u_2^2 \;=\; 1 \;=\; v_1^2 + v_2^2.$$

Now define

$$P \;=\; \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}.$$

Then we have $PP^T = P^TP = I$ (where by B^T we denote the transpose of a given matrix B). Thus P is an orthogonal matrix. Now

$$AP \;=\; \begin{bmatrix} \lambda_1 u_1 & \lambda_2 v_1 \\ \lambda_1 u_2 & \lambda_2 v_2 \end{bmatrix} \;=\; \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Thus, if

$$D \;=\; \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

we have

$$AP \;=\; PD, \text{ or equivalently } A \;=\; PDP^T.$$

Remark This is a particular case (when $n = 2$) of the following general result: If A is an $n\times n$ real symmetric matrix, then there exists an orthonormal basis of eigenvectors; if P is the orthogonal matrix whose columns are these eigenvectors and if D is the diagonal matrix whose diagonal entries are the eigenvalues of A (in the same order corresponding to the column vectors of P), then $A = PDP^T$. The same result is true if A is a complex hermitian (*i.e.* self-adjoint) matrix, in which case we replace P^T in the preceding relation by P^*, the conjugate transpose of P (and P will be a unitary matrix). ■

Example 2.1 Let

$$A = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}.$$

Then, its characteristic equation is $\lambda^2 - 10\lambda + 16 = 0$ and its eigenvalues are, therefore, $\lambda_1 = 2$ and $\lambda_2 = 8$. Corresponding to $\lambda_1 = 2$, we get the equation $5x - 3y = 2x$ or, equivalently, $x = y$. Thus the eigenvectors corresponding to λ_1 are scalar multiples of (1,1). Normalizing, we get

$$(u_1, u_2) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Corresponding to the eigenvalue $\lambda_2 = 8$, we get the equation $5x-3y = 8x$, or, equivalently, $x = -y$. Thus all eigenvectors corresponding to this eigenvalue are scalar multiples of $(1, -1)$ and, normalizing, we get

$$(v_1, v_2) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Thus

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and it is easy to verify that $A = PDP^T$ where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}. \blacksquare$$

3 The homogeneous equation

Consider the following homogeneous equation of the second degree in two real variables:

$$ax^2 + 2hxy + by^2 = 0. \tag{3.1}$$

Let us denote by S the set of all points (x, y) in the plane which satisfy this equation. Our aim is to determine this set.

We will set

$$A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$$

and use the notations developed in the preceding section. The above equation can be written in matrix form as

$$[\ x \quad y\ ] \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Writing $A = PDP^T$ as in the previous section, let us define

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = P^T \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus

$$\begin{aligned} x' &= u_1 x + u_2 y, \\ y' &= v_1 x + v_2 y. \end{aligned}$$

Then the equation (3.1) reduces to

$$[\ x' \quad y'\ ] D \begin{bmatrix} x' \\ y' \end{bmatrix} = \lambda_1 x'^2 + \lambda_2 y'^2 = 0. \tag{3.2}$$

We can now easily determine S from (3.2).

- If $\lambda_1 > 0$ and $\lambda_2 > 0$ or if $\lambda_1 < 0$ and $\lambda_2 < 0$, then the only solution will be $x' = y' = 0$. Then it follows that $x = y = 0$ and so $S = \{(0,0)\}$.
- Let $\lambda_1 > 0$ and $\lambda_2 < 0$, or $\lambda_1 < 0$ and $\lambda_2 > 0$. Then $-\frac{\lambda_2}{\lambda_1} > 0$ and we get from (3.2) that

  $$x' = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}} y'.$$

  Thus S consists of a pair of lines passing through the origin given by

  $$\begin{aligned} u_1 x + u_2 y &= \sqrt{-\tfrac{\lambda_2}{\lambda_1}}(v_1 x + v_2 y), \\ u_1 x + u_2 y &= -\sqrt{-\tfrac{\lambda_2}{\lambda_1}}(v_1 x + v_2 y). \end{aligned}$$

- If $\lambda_1 = 0$ and $\lambda_2 \neq 0$ or $\lambda_1 \neq 0$ and $\lambda_2 = 0$, we get the solution as $y' = 0$ or $x' = 0$, respectively. Thus we have S to consist of a single line given by

  $$v_1 x + v_2 y = 0, \text{ if } \lambda_1 = 0, \lambda_2 \neq 0,$$

and

$$u_1x + u_2y \;=\; 0, \text{ if } \lambda_1 \;\neq\; 0, \lambda_2 \;=\; 0.$$

- The case $\lambda_1 = \lambda_2 = 0$ is excluded since in this case we have $a = b = h = 0$ and so the equation is vacuous.

- Finally, if $\lambda_1 = -\lambda_2 \neq 0$, the equation (3.2) reduces to

  $$x'^2 - y'^2 \;=\; 0.$$

  In this case we get two perpendicular lines $x' + y' = 0$ and $x' - y' = 0$ which are given in the original coordinates by

  $$\begin{aligned}(u_1x + u_2y) + (v_1x + v_2y) &= 0,\\ (u_1x + u_2y) - (v_1x + v_2y) &= 0.\end{aligned}$$

Thus by computing the eigenvalues and normalized eigenvectors of the matrix A, we can immediately explicitly describe the solution set S.

The results derived above can be summarized in the following figure.

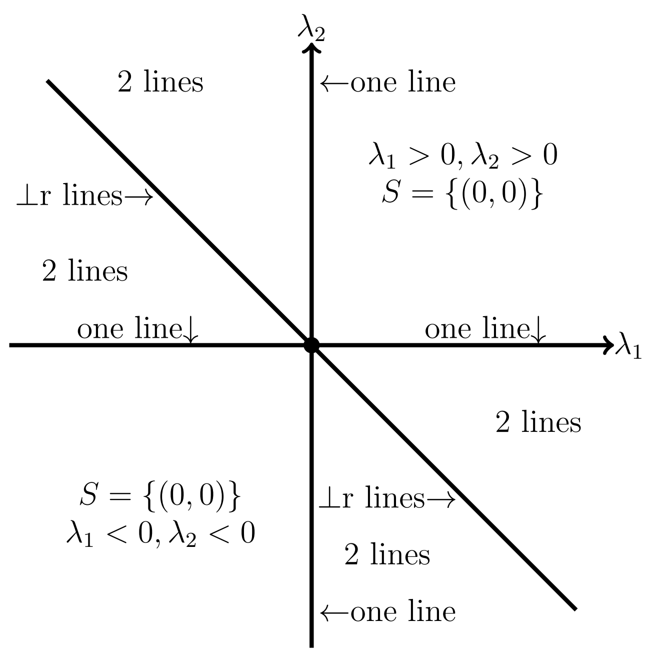

Figure 1

4 The inhomogeneous equation

Let us now consider the inhomogeneous equation

$$ax^2 + 2hxy + by^2 = 1. \tag{4.1}$$

With the notations established in the preceding sections, this equation reduces to

$$\lambda_1 x'^2 + \lambda_2 y'^2 = 1. \tag{4.2}$$

If S is the solution set, then we have the following cases.

- Once again the case $\lambda_1 = \lambda_2 = 0$ is excluded since then we have that $a = b = h = 0$ and the equation (4.1) is meaningless.
- If $\lambda_1 = \lambda_2 > 0$, *i.e* if $a = b > 0$ and $h = 0$, then S is a circle centered at the origin with radius $\frac{1}{\sqrt{a}}$.
- If $\lambda_1 = \lambda_2 < 0$, *i.e.* $a = b < 0, h = 0$, then $S = \emptyset$.
- More generally, if $\lambda_1 \leq 0$ and $\lambda_2 \leq 0$, we have $S = \emptyset$.
- If $\lambda_1 > 0$ and $\lambda_2 > 0$, then the equation represents an ellipse with centre at the origin. The lengths of the semi-axes of the ellipse are $\frac{1}{\sqrt{\lambda_1}}$ and $\frac{1}{\sqrt{\lambda_2}}$. The equations of the respective axes are $y' = 0$ and $x' = 0$, which can be written in the original coordinates as

$$\begin{aligned} v_1 x + v_2 y &= 0, \\ u_1 x + u_2 y &= 0. \end{aligned}$$

- If $\lambda_1 > 0$ and $\lambda_2 < 0$ or if $\lambda_1 < 0$ and $\lambda_2 > 0$, then we have a hyperbola. The lengths and equations of the axes are given as in the case of the ellipse above.
- If $\lambda_1 = -\lambda_2 \neq 0$, then the equation (4.2) reduces to

$$x'^2 - y'^2 = \frac{1}{\lambda_1}$$

which is a rectangular hyperbola with axes given by $x' = \pm y'$. In the original coordinates, this reduces to

$$u_1 x + u_2 y = \pm(v_1 x + v_2 y).$$

- Finally if $\lambda_1 = 0, \lambda_2 > 0$ or if $\lambda_1 > 0, \lambda_2 = 0$, then the equation (4.2) reduces to
$$\lambda_2 {y'}^2 \; = \; 1 \text{ or } \lambda_1 {x'}^2 \; = \; 1$$
respectively. In these cases, each time we get a pair of parallel lines given respectively by
$$y' \; = \; \pm\frac{1}{\sqrt{\lambda_2}}, \text{ or } x' \; = \; \frac{1}{\sqrt{\lambda_1}}.$$
Thus the equations to the lines in the original coordinates are
$$v_1 x + v_2 y \; = \; \pm\frac{1}{\sqrt{\lambda_2}}, \text{ or } u_1 x + u_2 y \; = \; \pm\frac{1}{\sqrt{\lambda_1}}.$$

Thus, once again, by computing the eigenvalues and eigenvectors of the matrix A, we can explicitly describe the solution set S of (4.1). In this case also we can summarize the results obtained above in the following figure.

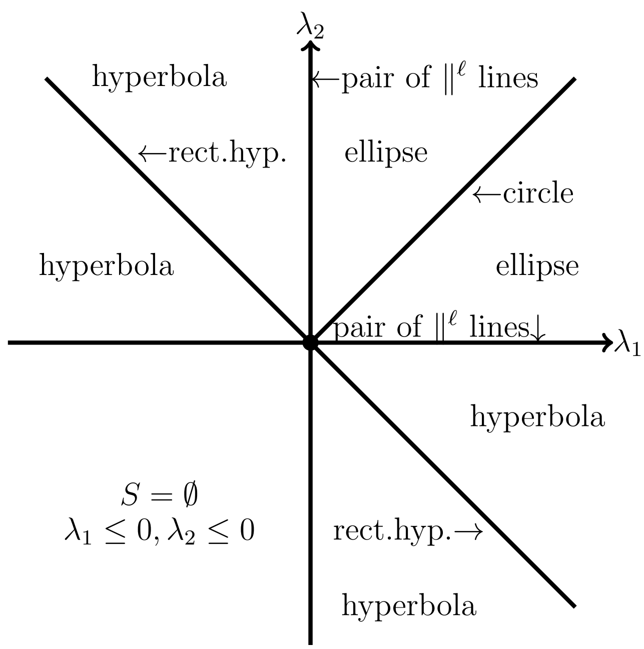

Figure 2

5 The general equation of the second degree: straight lines

Let us now consider the general equation of the second degree in two variables given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c \;=\; 0. \tag{5.1}$$

We will try to completely describe the solution set S of this equation.

Certain computations will repeatedly occur and so it will be useful for us to do them once and for all. Set $x = X + \alpha$ and $y = Y + \beta$. Then (5.1) becomes

$$\left.\begin{array}{r} aX^2 + 2hXY + bY^2 \\ +2(a\alpha + h\beta + g)X + 2(h\alpha + b\beta + f)Y \\ +(a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c) \end{array}\right\} \;=\; 0. \tag{5.2}$$

Notice that the constant term in the last line on the left-hand side of the above equation can also be rewritten as

$$\alpha(a\alpha + h\beta + g) + \beta(h\alpha + b\beta + f) + (g\alpha + f\beta + c). \tag{5.3}$$

We will also set

$$\Delta \;=\; \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

We will denote by A the symmetric matrix associated to the quadratic terms, *i.e.*

$$A \;=\; \begin{bmatrix} a & h \\ h & b \end{bmatrix}.$$

We will use the notations of the previous sections. In particular λ_1 and λ_2 will stand for its two eigenvalues. An associated pair of normalized eigenvectors will be denoted respectively by (u_1, u_2) and (v_1, v_2) as before. Notice that

$$ab - h^2 \;=\; \det(A) \;=\; \lambda_1\lambda_2.$$

Theorem 5.1 *The general equation of the second degree in two variables given by (5.1) defines a pair of intersecting lines if, and only if,* $ab - h^2 < 0$ *and* $\Delta = 0$.

Proof: Let us assume that the equation (5.1) represents a pair of intersecting lines and that the point of intersection is (α, β). Then, if we set $x = X +$

$\alpha,\ y = Y + \beta$, the equation represents a pair of lines intersecting at the origin in the XY-plane. Since (α, β) obviously satisfies the equation, we have

$$a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c \;=\; 0. \tag{5.4}$$

Further since the equation in the XY variables must be homogeneous, we must have

$$\begin{aligned} a\alpha + h\beta + g &= 0, \\ h\alpha + b\beta + f &= 0, \end{aligned} \tag{5.5}$$

and (5.2) reduces to

$$aX^2 + 2hXY + bY^2 \;=\; 0.$$

Since this represents a pair of lines intersecting at the origin, we deduce that $ab - h^2 < 0$. This then implies that the equations (5.5) admit a unique solution (α, β) which is the point of intersection of these lines in the xy-plane. Now it follows from (5.4), the formulation of the left-hand side of this equation given in (5.3) and the system of equations in (5.5) that we also have

$$g\alpha + f\beta + c \;=\; 0. \tag{5.6}$$

Thus $(\alpha, \beta, 1)$ is a non-trivial solution to the system of equations

$$\begin{aligned} a\alpha + h\beta + g &= 0, \\ h\alpha + b\beta + f &= 0, \\ g\alpha + f\beta + c &= 0. \end{aligned} \tag{5.7}$$

This is possible only if $\Delta = 0$.

Conversely, if $ab - h^2 < 0$ and $\Delta = 0$, then choose (α, β) as the unique solution of (5.5). Then, since $\Delta = 0$, it automatically follows that (5.6) is also satisfied. Then, taking into accont (5.3), we get that (5.2) reduces to

$$aX^2 + 2hXY + bY^2 \;=\; 0$$

under the transformation $x = X + \alpha,\ y = Y + \beta$. Now we know that this represents a pair of straight lines intersecting at the origin in the XY-plane and hence (5.1) represents a pair of straight lines intersecting at (α, β). This completes the proof. ■

Remark 5.1 The unique solution of (5.5) gives the point of intersection of the two lines. From the discussion in Section 2, we can explicitly write down the equations of the two lines. They are given by

$$\begin{aligned}(u_1(x-\alpha)+u_2(y-\beta)) &= \sqrt{-\tfrac{\lambda_2}{\lambda_1}}(v_1(x-\alpha)+v_2(y-\beta)),\\ (u_1(x-\alpha)+u_2(y-\beta)) &= -\sqrt{-\tfrac{\lambda_2}{\lambda_1}}(v_1(x-\alpha)+v_2(y-\beta)).\end{aligned}$$

■

Remark 5.2 If, in addition, we also have that $a+b=0$, then the lines will be perpendicular to each other. ■

Example 5.1 Consider the equation

$$x^2-y^2+x-3y-2 = 0.$$

Then $ab-h^2=-1<0$. Further

$$\Delta = \begin{vmatrix} 1 & 0 & \frac{1}{2} \\ 0 & -1 & -\frac{3}{2} \\ \frac{1}{2} & -\frac{3}{2} & -2 \end{vmatrix} = 0.$$

Thus this equation represents a pair of intersecting straight lines. The system (5.5) reduces to

$$\begin{aligned}\alpha+\tfrac{1}{2} &= 0,\\ -\beta-\tfrac{3}{2} &= 0.\end{aligned}$$

Thus the point of intersection is $(-\frac{1}{2},-\frac{3}{2})$. Now

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus its eigenvalues and eigenvectors are $\lambda_1=1$, with $(u_1,u_2)=(1,0)$ and $\lambda_2=-1$ with $(v_1,v_2)=(0,1)$. Thus the pair of lines are

$$\begin{aligned}(x+\tfrac{1}{2}) &= (y+\tfrac{3}{2})\\ (x+\tfrac{1}{2}) &= -(y+\tfrac{3}{2}),\end{aligned}$$

or, equivalently, $x - y - 1 = 0$ and $x + y + 2 = 0$. Notice that since $a + b = 0$, we also have that these lines are perpendicular to each other. ■

Theorem 5.2 *With the notations established above, assume that $\Delta = 0$ and $ab - h^2 > 0$. Then the solution set S of the equation (5.1) consists of a single point.*

Proof: If $ab - h^2 > 0$, then the system (5.5) has a unique solution (α, β). Since $\Delta = 0$, this implies that (5.6) is also satisfied. Then the transformation $x = X + \alpha,\ y = Y + \beta$ yields

$$aX^2 + 2hXY + bY^2 \ = \ 0$$

and we have seen that in this case the solution set consists of only the origin in the XY-plane. Thus $S = \{(\alpha, \beta)\}$. ■

Let us now turn to the case where $ab - h^2 = 0$ and $\Delta = 0$. In this case the matrix A has $\lambda_1 = 0$ and $\lambda_2 \neq 0$. (If $\lambda_2 = 0$ as well, then we saw that $a = b = h = 0$ and the second degree terms disappear altogether.) Let us define

$$x \ = \ u_1X + v_1Y,\ y \ = \ u_2X + v_2Y,$$

or, equivalently,

$$X \ = \ u_1x + u_2y,\ Y \ = \ v_1x + v_2y.$$

Then, as in Section 2, the quadratic terms reduce to λ_2Y^2. The equation (5.1) now becomes

$$\lambda_2Y^2 + 2GX + 2FY + c \ = \ 0$$

where

$$G \ = \ gu_1 + fu_2, \text{ and } F \ = \ gv_1 + fv_2.$$

Now, consider the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Developing this by the third row (or third column) and taking into account the fact that $ab - h^2 = 0$, we immediately observe that the determinant is independent of the value of c. Thus if $\Delta = 0$, it follows that the vectors (a, h, g) and (h, b, f) must be linearly dependent. Since (u_1, u_2) is an eigenvector corresponding to $\lambda_1 = 0$, we have

$$au_1 + hu_2 \;=\; 0 \text{ and } hu_1 + bu_2 \;=\; 0.$$

This then implies that $G = gu_1 + fu_2 = 0$. Thus the equation (5.1) now reduces to

$$\lambda_2 Y^2 + 2FY + c \;=\; 0.$$

Completing the square, we get

$$\lambda_2 \left(Y + \frac{F}{\lambda_2} \right)^2 + c - \frac{F^2}{\lambda_2} \;=\; 0$$

or, equivalently

$$\left(Y + \frac{F}{\lambda_2} \right)^2 \;=\; \frac{1}{\lambda_2^2}(F^2 - \lambda_2 c).$$

This leads us to the following conclusions.

- If $F^2 > \lambda_2 c$, then the solution set S of the equation (5.1) consists of two parallel lines. They are given by

$$v_1 x + v_2 y + \frac{gv_1 + fv_2}{\lambda_2} \;=\; \pm \frac{1}{|\lambda_2|} \sqrt{(gv_1 + fv_2)^2 - \lambda_2 c}.$$

- If $F^2 = \lambda_2 c$, then S consists of a single line given by

$$v_1 x + v_2 y \;=\; -\frac{gv_1 + fv_2}{\lambda_2}.$$

- If $F^2 < \lambda_2 c$, then $S = \emptyset$.

Example 5.2 Consider the equation

$$2x^2 + 8xy + 8y^2 + 2gx + 2fy + c \;=\; 0.$$

Then

$$\Delta \;=\; \begin{vmatrix} 2 & 4 & g \\ 4 & 8 & f \\ g & f & c \end{vmatrix}.$$

Then Δ vanishes if, and only if, $f = 2g$. Further,

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}.$$

Its characteristic equation is $\lambda^2 - 10\lambda = 0$. The eigenvalues are, therefore, $\lambda_1 = 0$ and $\lambda_2 = 10$. Correspondiong to $\lambda_1 = 0$, we get $2x + 4y = 0$, and so every eigenvector is a multiple of $(2, -1)$. Thus we can take

$$(u_1, u_2) = \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right).$$

Corresponding to $\lambda_2 = 10$, we get $2x + 4y = 10x$, and so every eigenvector is a constant multiple of $(1, 2)$ so that we can take

$$(v_1, v_2) = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right).$$

Thus $g = gu_1 + fu_2 = g(u_1 + 2u_2) = 0$. Similarly,

$$F = gv_1 + fv_2 = g(v_1 + 2v_2) = 5gv_1.$$

Then

$$F^2 - \lambda_2 c = 25g^2v_1^2 - 10c = 5g^2 - 10c.$$

(i) Let us take $g = 2,\ c = 1$. Then $F^2 > \lambda_2 c$. In this case, the equation is

$$2x^2 + 8xy + 8y^2 + 4x + 8y + 1 = 0.$$

Dividing throughout by 2, we get

$$(x + 2y + 1)^2 = \frac{1}{2} \ i.e.\ x + 2y + 1 = \pm\frac{1}{\sqrt{2}}.$$

It can be easily seen that this expression agrees with the abstract equations of the lines given earlier.
(ii) Let us take $g = 2,\ c = 2$. Then $F^2 = \lambda_2 c$. In this case, the equation is

$$2x^2 + 8xy + 8y^2 + 4x + 8y + 2 = 0$$

which can be rewritten as

$$(x + 2y + 1)^2 = 0.$$

Thus we get a single line $x + 2y + 1 = 0$ which can be seen to agree with the abstract expression given earlier.
(iii) Let us take $g = 1,\ c = 1$. Then $F^2 < \lambda_2 c$. In this case, the equation is

$$2x^2 + 8xy + 8y^2 + 2x + 4y + 1 \ = \ 0.$$

This can be rewritten as

$$(x + 2y)^2 + (x + 2y + 1)^2 \ = \ 0.$$

It is easy to see that this equation has no solution.

Thus, all three cases can occur.

We now summarize the results of this section as follows.

- If $\Delta = 0$ and if $ab - h^2 < 0$, then the equation (5.1) represents a pair of intersecting straight lines.
- If $\Delta = 0$ and if $ab - h^2 > 0$, then the solution set of the equation (5.1) consists of a single point.
- If $\Delta = 0$ and if $ab - h^2 = 0$, then the solution set is one of the following:
  - a pair of parallel lines;
  - a single line;
  - the empty set.

  Starting from the coefficients of the equation and the eigenvalues and eigenvectors of the matrix A associated to the quadratic terms, we can determine which of these cases occurs.

6 The general equation of the second degree: conics

We will consider the equation (5.1) when $\Delta \neq 0$.

Case 1 $\Delta \neq 0$ and $ab - h^2 < 0$.
Since $ab - h^2 < 0$, there exists a unique solution (α, β) to the system (5.5).

However, since $\Delta \neq 0$, we also have that $g\alpha + f\beta + c \neq 0$. Setting $x = X + \alpha,\ y = Y + \beta$, the equation now reads as

$$aX^2 + 2hXY + bY^2 + (g\alpha + f\beta + c) \ = \ 0.$$

Let us set $C = g\alpha + f\beta + c$. Then as in Section 3, we can easily see that the solution set is a hyperbola with centre at (α, β) with the lengths of the semi-axes being

$$\sqrt{\left|\frac{C}{\lambda_1}\right|} \text{ and } \sqrt{\left|\frac{C}{\lambda_2}\right|}.$$

The equations to the axes are given by

$$\begin{aligned} v_1(x-\alpha) + v_2(y-\beta) &= 0, \\ u_1(x-\alpha) + u_2(y-\beta) &= 0, \end{aligned}$$

respectively. If, in addition, $a + b = 0$, this will be a rectangular hyperbola.

Case 2 $\Delta \neq 0$ and $ab - h^2 > 0$.
Once again, there exists a unique solution (α, β) to the system (5.5) and setting $x = X + \alpha,\ y = Y + \beta$, the equation reduces to

$$aX^2 + 2hXY + bY^2 + C \ = \ 0$$

where $C = g\alpha + f\beta + c \neq 0$. The solution set is either an ellipse or is empty, depending on the signs of C, λ_1 and λ_2. The centre, lengths of the semi-axes and the equations of the axes are exactly as in the previous case.

Example 6.1 Consider the equation

$$5x^2 - 6xy + 5y^2 + 22x - 26y + 29 \ = \ 0.$$

In this case

$$\Delta \ = \ \begin{vmatrix} 5 & -3 & 11 \\ -3 & 5 & -13 \\ 11 & -13 & 29 \end{vmatrix} \neq \ 0.$$

Further $ab - h^2 > 0$. The system (5.5) reads as

$$\begin{aligned} 5\alpha - 3\beta + 11 &= 0 \\ -3\alpha + 5\beta - 13 &= 0 \end{aligned}$$

which admits the unique solution $(\alpha, \beta) = (-1, 2)$. Further

$$g\alpha + f\beta + c \;=\; -8.$$

Thus, setting $x = X + \alpha,\; y = Y + \beta$, we get

$$5X^2 - 6XY + 5Y^2 \;=\; 8.$$

The eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$$

were computed in Example 2.1. They are

$$\lambda_1 = 2,\; (u_1, u_2) \;=\; \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

and

$$\lambda_2 = 8,\; (v_1, v_2) \;=\; \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Then setting

$$\begin{aligned} X' &= u_1 X + u_2 Y \\ Y' &= v_1 X + v_2 Y, \end{aligned}$$

the equation reduces to

$$2X'^2 + 8Y'^2 \;=\; 8$$

or, equivalently,

$$\frac{X'^2}{4} + \frac{Y'^2}{1} \;=\; 1.$$

Thus the solution set is an ellipse with centre at $(-1, 2)$ with the lengths of the semi-major axis being 2 and that of the semi-minor axis being 1. The equation of the major axis is $Y' = 0$ which becomes $X = Y$, or,

$$x + 1 \;=\; y - 2,\; i.e.\; x - y + 3 \;=\; 0$$

and that of the minor axis is $X' = 0$, which becomes $X = -Y$, or,

$$x + 1 \;=\; 2 - y \; i.e.\; x + y - 1 \;=\; 0.$$

If we change the value of the constant c from 29 to 39, then $g\alpha+f\beta+c=2$ and in this case the equation reduces to

$$2{X'}^2+8{Y'}^2 \;=\; -2$$

and clearly the solution set is empty. ■

Case 3 $\Delta \neq 0$ and $ab-h^2=0$.
Since the matrix A is singular, one of its eigenvalues will be zero. Without loss of generality, let $\lambda_1=0$ and let $\lambda_2 \neq 0$. Let us make the usual change of coordinates using the normalized eignevectors of A:

$$\begin{aligned} x &= u_1x'+v_1y' \\ y &= u_2x'+v_2y'. \end{aligned}$$

The equation then transforms to

$$\lambda_2{y'}^2+2Gx'+2Fy'+c \;=\; 0$$

where

$$F \;=\; gu_1+fu_2 \text{ and } F \;=\; gv_1+fv_2.$$

Since we already have

$$\begin{aligned} au_1+hu_2 &= 0, \\ hu_1+bu_2 &= 0 \end{aligned}$$

if we also have $G=0$, then it will imply that $\Delta=0$, which is not the case. Thus $G \neq 0$. We complete squares and rewrite the equation as follows:

$$\lambda_2\left(y'+\frac{F}{\lambda_2}\right)^2 \;=\; -2Gx'-c+\frac{F^2}{\lambda_2}.$$

Now set

$$X=x'+\frac{c}{2G}-\frac{F^2}{2G\lambda_2},\; Y \;=\; y'+\frac{F}{\lambda_2}$$

to get

$$Y^2 \;=\; -\frac{2G}{\lambda_2}X$$

which is a parabola.

We summarize the results of this section as follows.

- If $\Delta \neq 0$ and if $ab - h^2 < 0$, the equation (5.1) represents a hyperbola. If, in addition, $a + b = 0$, it represents a rectangular hyperbola.
- If $\Delta \neq 0$ and if $ab - h^2 > 0$, then the equation (5.1) either represents an ellipse or the solution set is empty. If $a = b$ and $h = 0$, then the equation represents a circle provided the solution set is non-empty.
- If $\Delta \neq 0$ and if $ab - h^2 = 0$, then the equation (5.1) represents a parabola.