ASSIGNMENT 7

(1) Recall that the *free Lie algebra* generated by a vector space V is a Lie algebra $F(V)$ together with a linear map $i : V \to F(V)$ which has the following universal property: given a Lie algebra L and a linear map $f : V \to L$, there is a unique *Lie algebra homomorphism* $\tilde{f} : F(V) \to L$ such that $\tilde{f}i = f$.

Prove that the following gives a construction of $F(V)$. Take the tensor algebra TV and let $F(V)$ be the Lie subalgebra of TV generated by V (note that V sits inside TV as the first tensor component). Further prove that the universal enveloping algebra of $F(V)$ is TV, by showing the appropriate universal property.

(2) Let M be an abelian group and let $\mathfrak{g}$ be an M-graded Lie algebra, i.e., $\mathfrak{g}$ admits a decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in M} \mathfrak{g}_\alpha$$

satisfying $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in M$. Prove that this induces an M-grading on $U\mathfrak{g}$ as well (where for an associative algebra A, an M-grading would be a decomposition $A = \oplus_{\alpha \in M} A_\alpha$ satisfying $A_\alpha A_\beta \subset A_{\alpha+\beta}$ for all $\alpha, \beta \in M$).

(3) (a) A Lie algebra homomorphism $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ induces an algebra homomorphism $\tilde{f} : U\mathfrak{g}_1 \to U\mathfrak{g}_2$.
(b) A Lie algebra automorphism of $\mathfrak{g}$ induces an algebra automorphism of $U\mathfrak{g}$.
(c) A Lie algebra antihomomorphism $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ (i.e, $f([x, y] = [f(y), f(x)]$ for all $x, y \in \mathfrak{g}_1$) induces an algebra antihomomorphism $\tilde{f} : U\mathfrak{g}_1 \to U\mathfrak{g}_2$ (i.e., $\tilde{f}(uv) = \tilde{f}(v)\tilde{f}(u)$ for all $u, v \in U\mathfrak{g}_1$).

(4) Let $\mathfrak{g}$ be a Lie algebra. Prove that $U\mathfrak{g}$ has no zero divisors.

(5) Let (V, Δ) be a root system and let S be a set of simple roots (for some choice of chamber). A *diagram automorphism* of Δ is an element $\sigma \in \mathrm{GL}(V)$ such that $\sigma(S) = S$ and $n_{\alpha,\beta} = n_{\sigma\alpha,\sigma\beta}$ for all $\alpha, \beta \in S$ (here the $n_{\alpha,\beta}$ denotes the Cartan integers). Use Serre's theorem to show that if $\mathfrak{g}$ is a finite dimensional simple Lie algebra, and σ is a diagram automorphism of its root system, then σ induces a Lie algebra automorphism of $\mathfrak{g}$. Further, for each Dynkin diagram (types A-G), find its group of diagram automorphisms.