THE WEYL GROUP

Let W be the Weyl group of a root system Δ. Fix a decomposition $\Delta = \Delta^+ \cup \Delta^-$ and let $B = \{\alpha_i : i = 1 \cdots l\}$ be the corresponding basis of Δ. Let $S := \{r_i : i = 1 \cdots l\}$ be the simple reflections in W, where we denote $r_i := r_{\alpha_i}$; recall that S generates W. Define the length of an element of W by

$$l(w) := \min\{k \geq 0 : w = r_{i_1} r_{i_2} \cdots r_{i_k} \text{ for some } 1 \leq i_j \leq l\}$$

In other words, this is the smallest number k such that w can be written as a product of k simple reflections.

(1) Prove the following simple properties of length:
  (a) $l(w^{-1}) = l(w)$ for all $w \in W$.
  (b) $l(w_1 w_2) \leq l(w_1) l(w_2)$ for all $w_1, w_2 \in W$.
  (c) $l(w_1 w_2) \geq |l(w_1) - l(w_2)|$.

(2) Prove that there is a well-defined sign homomorphism $\epsilon : W \to \{\pm 1\}$ such that $\epsilon(r_i) = -1 \, \forall i$.

(3) Prove that $l(wr_i) = l(w) \pm 1$ for all $w \in W$, $r_i \in S$.

(4) **Theorem:** If $w \in W$ and $r_i \in S$, then (a) $l(wr_i) = l(w) - 1 \iff w\alpha_i \in \Delta^-$ and (b) $l(wr_i) = l(w) + 1 \iff w\alpha_i \in \Delta^+$.

Prove this theorem using the following steps:
  (a) Assume $w\alpha_i \in \Delta^-$. Write $w = r_{i_k} r_{i_{k-1}} \cdots r_{i_1}$ where $k = l(w)$. Define the *right subwords*, $w_0 = 1$, $w_1 := r_{i_1}$, $w_2 := r_{i_2} r_{i_1}$, $\cdots$, $w_k := r_{i_k} r_{i_{k-1}} \cdots r_{i_1} = w$. Now $w_0 \alpha_i \in \Delta^+$ while $w_k \alpha_i \in \Delta^-$. There is a smallest j such that $w_j \alpha_i \in \Delta^+$ but $w_{j+1} \alpha_i \in \Delta^-$. Prove now that $w_j \alpha_i$ must be a simple root (which one ?).
  (b) If $w\beta = \gamma$ for $w \in W$, $\beta, \gamma \in \Delta$, prove that $r_\gamma = w r_\beta w^{-1}$.
  (c) Use this to obtain an expression for wr_i as a product of $k - 1$ simple reflections.
  (d) Finally show that all other assertions of the theorem can be deduced from what has been proved above (by replacing w with wr_i).

(5) The *inversion set* $I(w)$ of $w \in W$ is defined to be:

$$I(w) := \{\alpha \in \Delta^+ : w\alpha \in \Delta^-\}.$$

Show that if $l(wr_i) = l(w) + 1$, then $I(wr_i) = \{\alpha_i\} \cup r_i(I(w))$. Hence show (by induction) that $l(w) = |I(w)|$ for all $w \in W$.

(6) (a) Show that if C is a chamber, then so is $-C := \{-x : x \in C\}$.
  (b) By the simple transitivity of the W-action on the set of chambers, there is a unique $w_0 \in W$ such that $w_0(C) = -C$. Prove that $w_0^2 = 1$.
  (c) Prove that w_0 is the unique longest element of the Weyl group W (length being measured wrt the simple reflections obtained from the basis corresponding to C).
  (d) Prove that $l(w_0 \sigma) = l(w_0) - l(\sigma)$ for all $\sigma \in W$.
  (e) For the root system A_{n-1} constructed in lecture, recall $W \cong S_n$. Find w_0, and compute its length.