ROOT SYSTEMS

Let Δ denote a root system in the Euclidean space $(V, (\cdot,\cdot))$, with Weyl group W.

(1) Let α, β be non-proportional roots in Δ. Prove that r_α and r_β commute $\iff$ $n_{\alpha,\beta} = n_{\beta,\alpha} = 0$. More generally, determine the order of the element $r_\alpha r_\beta \in W$ as a function of $n_{\alpha,\beta}\, n_{\beta,\alpha}$.

(2) If Δ is an irreducible root system, then V is an irreducible representation of the group W. As a corollary, show that given any root α, the W-orbit of α spans V. *Hint:* Show that if $U \subset V$ is a W-invariant subspace, then each root is either in U or in $U^\perp$.

(3) (a) If $\alpha, \beta \in \Delta$ are such that $n_{\alpha,\beta} = n_{\beta,\alpha} = -1$, then $\exists\, w \in W$ such that $\beta = w\alpha$.
(b) If Δ is an irreducible root system, and α, β are roots of the same length, then $\exists\, w \in W$ such that $\beta = w\alpha$ (*Hint:* use (a) and the preceding problem).

(4) If Δ is an irreducible root system and $\alpha, \beta \in \Delta$, show that $\frac{(\beta,\beta)}{(\alpha,\alpha)}$ can only take one of the values $1, 2, 1/2, 3, 1/3$. Further, at most two root lengths occur in Δ.

(5) Let Δ be a root system with simple roots $\{\alpha_i : i = 1 \cdots l\}$. If $\alpha = \sum_i c_i \alpha_i$ is a root, prove that $c_i(\alpha_i, \alpha_i)/(\alpha, \alpha) \in \mathbb{Z}$ for all i.

(6) Let Δ be an irreducible root system, C a chamber of Δ, Δ^+ the corresponding set of positive roots, and B the set of simple roots. Let

$$\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

Prove that:
(a) $r_\alpha(\rho) = \rho - \alpha$ for all $\alpha \in B$.
(b) $\frac{2(\rho,\alpha)}{(\alpha,\alpha)} = 1$ for all $\alpha \in B$.
(c) $\rho \in C$.

(7) Prove that if there is an inclusion of Dynkin diagrams $X \hookrightarrow Y$, then the root system of X can be obtained from the root system $(V(Y), \Delta(Y))$ of Y as follows: take $V(X)$ to be the subspace of $V(Y)$ spanned by the simple roots corresponding to the vertices of X, and $\Delta(X) := V(X) \cap \Delta(Y)$ (i.e., show that $\Delta(X)$ is indeed a root system, with Dynkin diagram X).

(8) **The Exceptional Root systems**: In each part below, do the following:
- Find the elements of Δ explicitly, and check that Δ is a root system.
- Choose a chamber C appropriately, and find the corresponding Δ^+ and $B(C)$.
- Check that the configuration of roots in $B(C)$ gives the required Dynkin diagram.

Consider the vector space $V_n := \mathbb{R}^n$ with the standard inner product $(\cdot,\cdot)$, and standard orthonormal basis $\{\epsilon_i : i = 1 \cdots n\}$. Let $x_n := \sum_{i=1}^n \epsilon_i \in V_n$. Define the lattice $L_n := \oplus_{i=1}^n \mathbb{Z}\epsilon_i$. Finally, let $\tilde{V}_n := x_{n+1}^{\perp} \subset V_{n+1}$.

(a) E_8: Let L be the following lattice in V_8:

$$L := \{\sum_{i=1}^{8} a_i\epsilon_i : a_i \in \frac{1}{2}\mathbb{Z},\ \ a_i \equiv a_j \pmod{\mathbb{Z}},\ \ \sum_{i=1}^{8} a_i \in 2\mathbb{Z}\}$$

Then $\Delta(E_8) := \{\alpha \in L : (\alpha,\alpha) = 2\}$.

(b) E_6 and E_7: Obtain these using the diagram inclusions $E_6 \hookrightarrow E_7 \hookrightarrow E_8$.

(c) F_4: Let $L := L_4 \oplus \mathbb{Z}(\frac{1}{2}x_4) \subset V_4$. Then $\Delta(F_4) := \{\alpha \in L : (\alpha,\alpha) = 1 \text{ or } 2\}$.

(d) G_2: $\Delta(G_2) := \{\alpha \in L_3 \cap \tilde{V}_2 : (\alpha,\alpha) = 2 \text{ or } 6\} \subset \tilde{V}_2$.

(9) A *root subsystem* of a root system (V,Δ) is a subset $\Delta' \subset \Delta$ such that Δ' is invariant under the reflections r_α for $\alpha \in \Delta'$. In this case, Δ' becomes a root system in $V' := \operatorname{span}\Delta'$.

As observed earlier, if there is a diagram inclusion $X \hookrightarrow Y$, then $\Delta(X)$ occurs as a root subsystem of $\Delta(Y)$. For each X and Y below, show that $\Delta(X)$ occurs as a root subsystem of $\Delta(Y)$, though there is no diagram inclusion.

(a) $X = D_l$, $Y = B_l$ $(l \geq 4)$.
(b) $X = D_8$, $Y = E_8$.
(c) $X = B_4$, $Y = F_4$.
(d) $X = A_2$, $Y = G_2$.

In fact, this implies that there is an inclusion of the corresponding Lie algebras. For example D_l occurring inside B_l corresponds to the obvious inclusion $\mathfrak{so}_{2l} \hookrightarrow \mathfrak{so}_{2l+1}$.