Assignment 4

1. Let V be an irreducible representation (not necessarily finite dimensional) of $\mathfrak{sl}_3$. If V contains a nonzero vector v such that $E_{13}v = E_{21}v = E_{32}v = 0$, then prove that V must be the one dimensional trivial representation.

2. Let $\mathfrak{g} := \mathfrak{sl}_n$ for some $n \geq 2$. Let $\mathfrak{n}^- :=$ span $\{E_{ji} : i < j\}$ denote the set of strictly lower triangular matrices in $\mathfrak{g}$. Let

$$M := \{[X_1, [X_2, [\cdots, [X_k, E_{1n}]]] : X_i \in \mathfrak{n}^-, \, i = 1 \cdots k, \, k \geq 1\}.$$

Prove that span $M = \mathfrak{g}$. *Hint: Show that E_{1n} is the highest weight vector for the adjoint representation.*

3. Let V be a finite dimensional representation of $\mathfrak{sl}_3$. Show that V contains a nonzero weight vector v (i.e an eigenvector for $\mathfrak{h}$) such that $E_{13}v = E_{23}v = E_{21}v = 0$.

4. (a) Let $\mathfrak{g} := \mathfrak{sl}_n$ for some $n \geq 2$ and let $\mathfrak{h}$ be the set of diagonal matrices (Cartan subalgebra). Find all elements $H \in \mathfrak{h}$ for which $C(H) = \mathfrak{h}$, where $C(H) := \{X \in \mathfrak{g} : [X, H] = 0\}$ is the centralizer of H in $\mathfrak{g}$.

   (b) More generally, let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, and let Δ denote the set of roots of $\mathfrak{g}$. For $H \in \mathfrak{h}$, prove that

$$C(H) = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Delta \\ \alpha(H)=0}} \mathfrak{g}_\alpha$$

   Hint: First use problem 1 of assignment 3.

5. Prove *Schur's lemma* in our context: Let V, W be irreducible finite dimensional representations of a Lie algebra $\mathfrak{g}$ (over $\mathbb{C}$).

   (a) Any $\mathfrak{g}$-linear map $T : V \to W$ must be zero or an isomorphism.

   (b) Any $\mathfrak{g}$-linear map $T : V \to V$ must be of the form cI for some $c \in \mathbb{C}$.

   (c) The space $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ of $\mathfrak{g}$-linear maps from V to W is at most one dimensional.

6. Let V be a finite dimensional vector space and $f : V \times V \to \mathbb{C}$ be a nondegenerate (not necessarily symmetric) bilinear form. Recall from lecture that the corresponding linear map $T_f : V \to V^*$ defined by $v \mapsto f(v, \cdot)$ is an isomorphism. Now, suppose V is a representation of a Lie algebra $\mathfrak{g}$. We say that f is $\mathfrak{g}$-invariant if it satisfies

$$f(Xv, w) + f(v, Xw) = 0 \text{ for all } v, w \in V, X \in \mathfrak{g}.$$

   (a) f is $\mathfrak{g}$-invariant $\iff$ T_f is $\mathfrak{g}$-linear.

   (b) V admits a nondegenerate $\mathfrak{g}$-invariant bilinear form $\iff$ V is isomorphic to V^* as $\mathfrak{g}$-representations.

   (c) If V is irreducible, so is V^*.

   (d) If V is irreducible, and $V \cong V^*$, then there is a unique $\mathfrak{g}$-invariant nondegenerate bilinear form on V, upto scaling.

   (e) Finally, if $\mathfrak{g}$ is a finite dimensional simple Lie algebra, prove that the Killing form on $\mathfrak{g}$ is the unique (upto scaling) associative form on $\mathfrak{g}$.