Assignment 3

1. Let $\mathfrak{h}$ be an abelian Lie algebra and V be a diagonalizable representation of $\mathfrak{h}$, i.e, $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ where $V_\lambda := \{v \in V : Hv = \lambda(H)v \text{ for all } H \in \mathfrak{h}\}$. If U is a $\mathfrak{h}$-invariant subspace of V, prove that U is *graded*, i.e.,
$$U = \oplus_{\lambda \in \mathfrak{h}^*} (U \cap V_\lambda).$$

2. Prove that the Lie algebra $\mathfrak{sl}_n\mathbb{C}$ is simple. *Hint:* Use the root space decomposition, and the above exercise.

3. Let (ρ, V) be a finite dimensional representation of $\mathfrak{sl}_2$, with weight space decomposition $V = \oplus_{j \in \mathbb{Z}} V_j$. Define the following operator on V:
$$w := (\exp \rho(F))\,(\exp -\rho(E))\,(\exp \rho(F))$$
Prove that w is an invertible operator on V which maps V_j to V_{-j} for all $j \in \mathbb{Z}$.

4. Let $\mathfrak{g} := \mathfrak{sl}(V)$ for some finite dimensional vector space V. Let P be an invertible operator on V. Show that the map $X \mapsto PXP^{-1}$ is a Lie algebra automorphism of $\mathfrak{g}$.

5. Consider the Lie algebra $\mathfrak{g} := \mathfrak{sl}_2$. Let $H' := \begin{bmatrix} 3 & 1 \\ 0 & -3 \end{bmatrix}$. Show that H' is a semisimple element of $\mathfrak{g}$. Further, find elements E' and F' such that $[H', E'] = 2E', [H', F'] = 2F', [E', F'] = H'$. More generally, given an arbitrary diagonalizable matrix H' in $\mathfrak{sl}_2$, how would you find the E' and F' ?