Midterm Exam #2

(1) Let V be a finite dimensional vector space, of dimension $2d$, over a field F. Let B be a nondegenerate symmetric bilinear form on V. Let U be an isotropic subspace of V, i.e., $B(u, u') = 0$ for all $u, u' \in U$. If $\dim U = d$, prove that there exists a basis of V relative to which the matrix of the form B becomes:

$$\begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}$$

(the $2d \times 2d$ matrix with 1's on the antidiagonal and zeros elsewhere).

Hint: Use the isomorphism $V \to V^*$ induced by B to show $V/U \cong U^*$. Now consider a basis of U and its dual basis in U^* (ignore this hint if you have an alternate approach).

(2) Consider the real symmetric matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Find an invertible, *upper triangular* real matrix P such that $P^T A P$ is a diagonal matrix with *integer entries*. Justify your steps.

(3) Recall that an integral domain R is a Euclidean domain if there exists a function $\delta : R \to \mathbb{Z}_{\geq 0}$ such that given any pair $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ such that $a = bq + r$ with $\delta(r) < \delta(b)$. Let $\mathbb{Z}[i]$ denote the set of complex numbers of the form $x + iy$ with $x, y \in \mathbb{Z}$, where $i = \sqrt{-1}$. Prove that $\mathbb{Z}[i]$ is a Euclidean domain by constructing a suitable function δ.

(4) Let R, S be integral domains and $\varphi : R \to S$ be a surjective ring homomorphism. If R is a UFD, show that it is not necessary that S be a UFD.

Hint: You can use the fact from the assignment that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.