Midterm Exam #1

(1) Suppose G is a simple group, i.e., has no normal subgroups other than the identity and itself. Let S be a G-set on which the group action is non-trivial, in other words, there exists a pair $(\sigma, a) \in G \times S$ such that $\sigma \cdot a \neq a$.

(a) Prove that given any $g \in G$, $g \neq 1$, there exists $s \in S$ such that $g \cdot s \neq s$.

(b) If G is not simple, then (a) fails. Construct a counter-example.

(2) A subgroup H of a group G is said to be a *characteristic subgroup* if for all automorphisms φ of G, we have $\varphi(H) = H$. Let G be a finite group and P be a p-Sylow subgroup of G for some prime p. Show that:

(a) P is a characteristic subgroup of $N(P)$.

(b) $N(N(P)) = N(P)$ (normalizers are taken in G).

(3) Consider the following real valued functions on $\mathbb{R}$:

$$\alpha(x) = x^3, \quad \beta(x) = -x, \quad \gamma(x) = x + 1$$

(observe they all define bijections $\mathbb{R} \to \mathbb{R}$). Consider the free group F on three generators a, b, c and let N denote the normal subgroup of F generated by $\{bb,\ baba^{-1},\ bcbc\}$. Let $G = F/N$. Prove that there exists an action of G on the set $S = \mathbb{R}$ such that $\overline{a}, \overline{b}, \overline{c}$ (the images of a, b, c under the natural projection map $F \to F/N$) act as follows:

$$\overline{a} \cdot x = \alpha(x)$$
$$\overline{b} \cdot x = \beta(x)$$
$$\overline{c} \cdot x = \gamma(x)$$

for all $x \in \mathbb{R}$.

(4) Let a finite group G act on a finite set S. Let A_g denote the number of fixed points in S of the element $g \in G$:

$$A_g = |\{s \in S : g \cdot s = s\}|$$

Prove that $|G|$ divides $\sum_{g \in G} (A_g)^d$ for each $d \geq 1$.

Hint: Apply Burnside's lemma to a suitable action of G on some set.