Assignment 9

Unless otherwise specified, V will denote a finite dimensional vector space over a field F, T a linear operator on V and we consider V as a $F[x]$-module by letting x act on V by T.

(1) Prove that the minimal polynomial of T equals its characteristic polynomial iff V is a cyclic $F[x]$-module.

(2) Let $g(x) \in F[x]$ be a monic polynomial of degree ≥ 1 and let $A(g)$ denote its companion matrix. Using the above problem or otherwise, show that the minimal polynomial and characteristic polynomial of $A(g)$ are both equal to $g(x)$.

(3) Suppose $\varphi(x) \in F[x]$ is an irreducible polynomial such that $\varphi(T)$ is not invertible on V. Show that $\varphi(x)$ divides the minimal polynomial of T. *Hint: For* $v \in \ker \varphi(T)$, *find* $\operatorname{ann} v$.

(4) Recall that a Jordan block $J(\lambda, m)$ is a square matrix of size m of the form: $\begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & 1 & \ddots & & \\ & & \ddots & \lambda & \\ & & & 1 & \lambda \end{pmatrix}$ where $\lambda \in F$.
If the matrix of T wrt some basis of V is $J(\lambda, m)$, show that V is isomorphic to $F[x]/((x-\lambda)^m)$ as $F[x]$-modules.

(5) If the matrix of T wrt some basis of V is in Jordan form, i.e., a block diagonal matrix with blocks $J(\lambda_i, m_i)$ for $1 \leq i \leq r$, where $\lambda_i \in F$ and $m_i \geq 1$, show that the elementary divisors of the $F[x]$-module V are:

$$(x-\lambda_1)^{m_1}, (x-\lambda_2)^{m_2}, \cdots, (x-\lambda_r)^{m_r}$$

Show that the converse is also true.

(6) Let F be algebraically closed. Show that the only irreducibles in $F[x]$ are linear polynomials. Use this to prove that there exists a basis of V relative to which the matrix of T is in Jordan form.

(7) Let $A \in M_n(F)$. Let $g(x)$ denote the monic gcd of the $(n-1) \times (n-1)$ minors of $xI - A$. Prove that $p_A(x) = g(x)\, m_A(x)$ where p_A, m_A denote respectively the characteristic and minimal polynomials of A.

(8) Prove that two matrices $A, B \in M_n(F)$ are similar iff there exist $P, Q \in GL_n(F[x])$ such that:

$$xI - A = P\,(xI - B)\,Q$$

(9) Let L be a subfield of F. Let $A \in M_n(L) \subset M_n(F)$. Prove that the invariant factors of A over L coincide with its invariant factors over F. Does this hold for elementary divisors ?

(10) Let L be a subfield of F. Let $A, B \in M_n(L)$. We may also think of them as elements of $M_n(F)$. Prove that A and B are similar in $M_n(F)$ iff they are similar in $M_n(L)$.

(11) Let $A \in M_n(F)$. Prove that A is similar to A^T.