# Assignment 8

Unless specified otherwise, $R$ will denote a PID.

(1) Let $a_i \in R$ for $1 \leq i \leq n$. Let $d$ be a gcd of $a_1, a_2, \cdots, a_n$.

(a) Prove that there exists a matrix $Q \in GL_n(R)$ such that

$$[a_1\ a_2\ \cdots\ a_n]\, Q = [d\ \ 0\ \ \cdots\ 0]$$

(b) If $d$ is a unit, prove that there exists $A \in GL_n(R)$ whose first row is $[a_1\ a_2\ \cdots\ a_n]$

(2) Suppose $A$ is a $\mathbb{Z}$-matrix of size $4 \times 4$ with $\det A = 360$. Write down the possibilities for the normal form of $A$. Using this or otherwise, show that the gcd of the entries of $A$ is 1, i.e.,

$$\gcd(a_{ij} : 1 \leq i, j \leq 4) = 1$$

(3) Let $A, B \in M_n(R)$. If the normal forms of $A, B$ are $C, D$ respectively, is it true that the normal form of $AB$ is $CD$ ?

(4) Let $R = \mathbb{Z}$ and suppose

$$M = \frac{\mathbb{Z}}{24\mathbb{Z}} \oplus \frac{\mathbb{Z}}{20\mathbb{Z}} \oplus \frac{\mathbb{Z}}{150\mathbb{Z}} \oplus \frac{\mathbb{Z}}{28\mathbb{Z}}$$

Find the invariant factors and elementary divisors of $M$.

(5) Let $R = \mathbb{Q}[x]$ and suppose $M = M_1 \oplus M_2 \oplus M_3$ where $M_i$ are cyclic $R$-modules with $\operatorname{ann} M_i = (g_i)$ where $g_1(x) = (x-1)^3$, $g_2(x) = (x-1)(x^2+2)^2$ and $g_3(x) = (x^4 - 4)$.

(a) Find the invariant factors and elementary divisors of $M$.

(b) Redo the problem assuming $R = \mathbb{R}[x]$ and $R = \mathbb{C}[x]$ instead.

(6) Determine the number of non-isomorphic abelian groups of order: (i) 360, (ii) $p^n$ where $p$ is prime and $n \geq 1$, (iii) $p^n q^m$ where $p, q$ are primes and $n, m \geq 1$.

(7) Let $M$ be a finitely generated torsion module over $R$. Then:

(a) $M$ is simple (*see Problem 7 of Assignment 7*) iff it is isomorphic to $R/(p)$ where $p$ is a prime in $R$.

(b) $M$ is *indecomposable*, i.e., cannot be written as a direct sum of two proper submodules, iff $M$ is isomorphic to $R/(p^e)$ for some prime $p \in R$ and some $e \geq 1$.

(8) For a finitely generated $R$-module $M$, we define $\operatorname{rank} M$ to be the rank of the free $R$-module $M/M_{\text{tor}}$, i.e., $M/M_{\text{tor}} \cong R^{\operatorname{rank} M}$. Suppose $M$ is isomorphic to $R^n/K$ for some submodule $K$ of $R^n$; recall that $K$ is free, of rank $k$ say. Prove that $\operatorname{rank} M = n - k$.

(9) Suppose $M$ is a finitely generated $R$-module and $N \subset M$ is a submodule. Prove that $N$ and $M/N$ are finitely generated. Using the earlier problem, show that

$$\operatorname{rank} M = \operatorname{rank} N + \operatorname{rank} M/N$$