Assignment 7

Unless specified otherwise, R will denote an arbitrary (not necessarily commutative) ring.

(1) Let M be a left R-module. Define $\operatorname{ann} M = \{a \in R : ax = 0 \text{ for all } x \in M\}$. Prove that $\operatorname{ann} M$ is a two-sided ideal of R. Further, M can be made into a left module over the ring $R/\operatorname{ann} M$ via:

$$(a + \operatorname{ann} M)\, x = ax \text{ for } a \in R.$$

(2) Show that a cyclic R-module is isomorphic to R/I for some left ideal I of R.

(3) If R is a commutative ring and M is a cyclic R-module, prove that $\operatorname{ann} M = \operatorname{ann} x$ for any generator $x \in M$. Is this true if R is not commutative ?

(4) Let M be a left R-module and I a left ideal of R. Let $N = \{x \in M : ax = 0 \text{ for all } a \in I\}$. Prove that N is a $\mathbb{Z}$-submodule of M, which is isomorphic as $\mathbb{Z}$-modules to $\operatorname{Hom}_R(R/I, M)$.

(5) Let $V = F^n$ be a finite dimensional vector space over a field F and let $T : V \to V$ be the operator

$$T((a_1, a_2, \cdots, a_n)) = (a_n, a_1, \cdots, a_{n-1})$$

for all $(a_1, a_2, \cdots, a_n) \in F^n$. Viewing V as an $F[x]$-module, determine $\operatorname{ann} V$ (an ideal of $F[x]$).

(6) Recall that if M is an R-S bimodule and N is an R-T bimodule, then $\operatorname{Hom}_R(M, N)$ has the structure of an S-T bimodule. Now let X denote an R-S bimodule. We know R^n is an R-R bimodule. Prove that $\operatorname{Hom}_R(R^n, X)$ and X^n are isomorphic as R-S bimodules (where the module structure on X^n is componentwise).

(7) A nonzero R-module M is said to be *simple* if the only submodules of M are 0 and M. Show that M is simple iff it is cyclic with every nonzero element as generator.

(8) (*Schur's lemma*) If M, N are simple R-modules, prove that every $\varphi \in \operatorname{Hom}_R(M, N)$ is either zero or an isomorphism. In particular, $\operatorname{End}_R(M)$ is a division ring, i.e., every nonzero element is a unit.

(9) Prove that $\operatorname{Hom}_R(M_1 \oplus M_2, N) \cong \operatorname{Hom}_R(M_1, N) \oplus \operatorname{Hom}_R(M_2, N)$ (as $\mathbb{Z}$-modules).

(10) Prove that the $\mathbb{Z}$-module $\mathbb{Z}/(p^e)$ for a prime p and $e \geq 1$ cannot be written as the (internal) direct sum of two nonzero submodules.

(11) Let R be a commutative ring and $A, B \in M_n(R)$. Prove that

$$\det(AB) = \det A \det B$$

(*see page 456 of Michael Artin's* Algebra *for one possible approach - the principle of permanence of identities*).