Assignment 6

(1) Let R_1, R_2 be integral domains with fields of fractions F_1, F_2 respectively. Suppose $\phi : R_1 \to R_2$ is a ring homomorphism, does it induce a field homomorphism $\widetilde{\phi} : F_1 \to F_2$ (which makes the obvious diagram commute) ?

(2) Let F be a field and $f(x) \in F[x]$. Show that the quotient ring $R := F[x]/(f(x))$ is a field iff $f(x)$ is irreducible.

(3) Let F be a field which is a subring of a commutative ring R, and let $u \in R$. Prove that the subring of R generated by $F \cup \{u\}$ is isomorphic to $F[x]/(f(x))$ for some polynomial $f(x) \in F[x]$.

(4) Let R, S be integral domains and $\phi : R \to S$ be a surjective ring homomorphism. If R is a PID, is it true that S is a PID ?

(5) An integral domain R is a Euclidean domain if there exists a function $\delta : R \to \mathbb{Z}_{\geq 0}$ such that given any pair $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ such that $a = bq + r$ with $\delta(r) < \delta(b)$. Prove that a Euclidean domain is a PID. Show that $\mathbb{Z}$ and $F[x]$ are Euclidean domains (F a field).

(6) Let $\mathbb{Z}[\sqrt{2}]$ denote the set of real numbers of the form $a + b\sqrt{2}$ with $a, b \in \mathbb{Z}$. Show that this is a Euclidean domain, with respect to $\delta(a + b\sqrt{2}) = |a^2 - 2b^2|$.

(7) Let $\mathbb{Z}[\sqrt{-5}]$ denote the set of complex numbers of the form $a + b\sqrt{-5}$ with $a, b \in \mathbb{Z}$. Show that this is not a UFD (*see Jacobson: section 2.14*). However, show that the divisor chain condition holds in this ring.

(8) Give an example of an integral domain R and an element $a \in R$ such that a is irreducible but not prime.

(9) Let $f(x)$ be a monic polynomial with integer coefficients. Prove that any rational root of $f(x)$ must in fact be an integer.

(10) Prove Eisenstein's criterion: if $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$ and p is a prime such that (i) $p | a_i$ for $0 \leq i < n$ (ii) $p \nmid a_n$ (iii) $p^2 \nmid a_0$, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

(11) Let R be an integral domain. Prove that $R[x]$ is a UFD iff R is a field.