Assignment 5

By a *skew-symmetric* matrix, we will mean a matrix M satisfying $M_{ij} = -M_{ji}$ for all i, j and $M_{ii} = 0$ for all i (this is the correct definition in characteristic 2 as well).

(1) Let M be a skew-symmetric matrix over a field F. Show that $\det M$ is a square in F.

(2) If M has integer entries, then $\det M$ is the square of an integer.

(3) Let M be a symmetric complex matrix such that $\det M \neq 0$. Prove that $M = P^T P$ for some invertible matrix P.

(4) Let V be the vector space of 2×2 real matrices. Define forms β_1, β_2 on V by:

$$\beta_1(A, B) = \text{trace}(AB) \text{ and } \beta_2(A, B) = \text{trace}(AB^T)$$

(a) Show that both forms are symmetric. Determine their signatures.
(b) For each of these forms, find a basis of V with respect to which its matrix becomes diagonal (with $0, \pm 1$ entries).

(5) Can you repeat the above problem for $n \times n$ matrices for any $n > 2$?

(6) Let B be a symmetric bilinear form of signature (p, n, z) on a real vector space V. Let U be a subspace such that $B|_U$ is nondegenerate. Let (p', n', z') denote the signature of $B|_U$. Prove that:

$$p' \leq p, \; n' \leq n \text{ and } z' \leq z$$

Does this remain true if $B|_U$ is degenerate ?

(7) Let B be a sesquilinear form on a complex vector space V. Let M_1, M_2 denote the matrices of B with respect to two bases $\mathcal{X}_1, \mathcal{X}_2$ of V. Prove that $M_1 = P M_2 P^*$ for some invertible complex matrix P, where P^* is the conjugate transpose of P.

(8) A Hermitian matrix M is a complex matrix satisfying $M^* = M$. A $d \times d$ Hermitian matrix is *positive definite* if $x^* M x > 0$ for all nonzero column vectors $x \in \mathbb{C}^d$. Prove that every positive definite M is of the form PP^* for some invertible complex matrix P.

(9) Let B be an alternating or symmetric bilinear form on V. A subspace U of V is said to be *isotropic* for B if $B|_U$ is identically zero, i.e., $B(u, u') = 0$ for all $u, u' \in U$.
(a) If U is an isotropic subspace of V of maximal dimension, prove that U contains $\text{rad}\, B$.
(b) If B is nondegenerate (on V) and U is an isotropic subspace, prove that $\dim U \leq \dim V/2$. More generally, show that $\dim U \leq (\dim V + \dim(\text{rad}\, B))/2$.

(10) If B is a nondegenerate alternating form on V, determine

$$\max\{\dim U : U \text{ is an isotropic subspace of } V\}$$

(11) Let B be a nondegenerate symmetric form on a real vector space V of dimension d. Suppose

$$\max\{\dim U : U \text{ is an isotropic subspace of } V\} = m$$

Determine the possible signatures of B in terms of d, m.