Assignment 4

Let V be a finite dimensional vector space over a field F and let $\mathrm{BL}(V)$ denote the space of bilinear forms on V.

(1) Let β be a nondegenerate bilinear form on V. Prove that:
   (a) Given a bilinear form B on V, there exists a unique linear operator $T : V \to V$ such that:
   $$B(x, y) = \beta(Tx, y) \text{ for all } x, y \in V$$
   (b) The association $B \mapsto T$ defines an isomorphism of vector spaces $\mathrm{BL}(V) \cong \mathrm{End}_F(V)$.
   (c) B is nondegenerate iff T is invertible.
   (d) There exists an invertible operator T on V such that $\beta(y, x) = \beta(Tx, y)$.
   (e) Given a basis $\mathcal{X}$ of V, what relation holds between the matrices of B, β and T relative to $\mathcal{X}$?

(2) Let V, W be finite dimensional vector spaces over F. For a bilinear map $B : V \times W \to F$, define its left and right radicals by: $\mathrm{Lrad}(B) = \{v \in V : B(v, x) = 0 \text{ for all } x \in W\}$ and $\mathrm{Rrad}(B) = \{w \in W : B(x, w) = 0 \text{ for all } x \in V\}$. Let $V' = V/\mathrm{Lrad}(B)$ and $W' = W/\mathrm{Rrad}(B)$. Prove that:
   (a) B induces a bilinear map $B' : V' \times W' \to F$ such that the left and right radicals of B' are both zero.
   (b)
   $$\dim V - \dim \mathrm{Lrad}(B) = \dim W - \dim \mathrm{Rrad}(B)$$
   (c) If B is a bilinear form on V, then the left and right radicals of B have the same dimension.

(3) Suppose B_1, B_2 are bilinear forms on V. Prove that there exists a linear operator $T : V \to V$ such that $B_2(x, y) = B_1(Tx, y)$ for all $x, y \in V$ iff $\mathrm{Rrad}(B_1) \subseteq \mathrm{Rrad}(B_2)$.

(4) Let $\mathrm{char}\, F \neq 2$. Let B be a bilinear form on V. Prove that there are unique bilinear forms B_1, B_2 with B_1 symmetric and B_2 alternating such that $B(x, y) = B_1(x, y) + B_2(x, y)$ for all $x, y \in V$.

(5) If B is a bilinear form on V, prove that there exist bases $\{e_i\}$ and $\{f_j\}$ of V such that the matrix M defined by $M_{ij} = B(e_i, f_j)$ is a diagonal matrix, with diagonal entries 0 or 1.

(6) Assume $F = \mathbb{R}$, so V can be thought of as some $\mathbb{R}^d$. Give V the Euclidean topology of $\mathbb{R}^d$. Let B be a bilinear form on V such that for all $x, y \in V$, we have $B(x, y) = 0$ iff $B(y, x) = 0$.

(a) Let $P = \{y \in V : B(y, y) \neq 0\}$. Show that P is an open subset of V.

(b) Given $x \in V,\ y \in P$, define

$$x' = x - \frac{B(x, y)}{B(y, y)}\, y$$

Show that $B(x', y) = 0$. Hence conclude that $B(x, y) = B(y, x)$.

(c) For $x \in V$, define $f_x \in V^*$ by $f_x(y) = B(x, y) - B(y, x)$. Show that $\ker f_x$ is either V or has empty interior.

(d) Prove that $P \subset \ker f_x$, and hence one of the following must hold: (i) $\ker f_x = V$ or (ii) $P = \emptyset$.

(e) Show that B is either symmetric or alternating.