Assignment 3

(1) Write down the possible cycle types of elements of S_5. Determine the sizes of the conjugacy classes of S_5 and use this to prove that the only normal subgroups of S_5 are S_5, A_5 and (1).

(2) Let a finite group G act on a finite set S with $|S| > 1$. Suppose each $g \in G$ has at least one fixed point in S, i.e., there exists $s \in S$ such that $gs = s$. Show that G cannot act transitively on S (i.e., there must be two or more orbits).

(3) Let H be a proper subgroup of a finite group G. Prove (using the previous problem or otherwise) that

$$\bigcup_{g \in G} gHg^{-1} \neq G$$

(4) Suppose G acts on S. Consider the group of G-automorphisms of S:

$$A = \mathrm{Aut}_G(S) = \{\varphi \in \mathrm{Sym}\, S : \varphi(g \cdot s) = g \cdot \varphi(s) \text{ for all } (g, s) \in G \times S\}$$

Show that S admits an action of the direct product $G \times A$ of these groups via:

$$(g, \varphi) \cdot s = g \cdot \varphi(s) \text{ for all } g \in G, \varphi \in A, s \in S$$

Work this out explicitly for $S = G$ with the left translation action. What does A become in this case ?

(5) Let K, H be subgroups of G, with $H \subset N_G(K)$. Suppose G acts on a set S. This defines actions of K and H on S by restriction. Let $T \subset S$ be the subset of K-fixed points, i.e.,

$$T = \{s \in S : \sigma \cdot s = s \text{ for all } \sigma \in K\}$$

Show that T is H-stable.

(6) Consider necklaces with 5 beads. Each bead can be one of three possible colours R, B, G. How many distinct necklaces can be formed ?

(7) Let $(K, +)$ be an *abelian group* and let H be a group that acts on K by (group) automorphisms. Let $\varepsilon : H \times H \to K$ be a map. Let G denote the set $K \times H$ and define the following multiplication operation on G:

$$(k_1, h_1) \bullet (k_2, h_2) = (k_1 + h_1 \cdot k_2 + \varepsilon(h_1, h_2),\ h_1 h_2)$$

where $h_1 \cdot k_2$ denotes the action of h_1 on k_2. Find the conditions on ε under which this operation makes G into a group. Such an ε is called a 2-cocyle, and we let G_ε denote the group G with this multiplication operation.

(8) Retain the notation of the previous problem. Fix a map $\alpha : H \to K$ and define $\delta : H \times H \to K$ as follows:

$$\delta(h_1, h_2) = h_1 \cdot \alpha(h_2) - \alpha(h_1 h_2) + \alpha(h_1)$$

Prove that δ is a 2-cocycle, and that $G_\delta \cong K \rtimes H$. More generally, if ε is a 2-cocycle, then so is $\varepsilon + \delta$ (the pointwise sum of the maps ε and δ). Prove further that $G_{\varepsilon+\delta} \cong G_\varepsilon$.

(9) Consider the free group F on three generators a, b, c and let N denote the normal subgroup of F generated by $a\,b\,a^{-1}\,b^{-1}$ and $a^2\,b\,c^{-1}$. Let $G = F/N$. Prove that there exists an action of G on the set $S = \mathbb{R}^2$ such that $\overline{a}, \overline{b}, \overline{c}$ (the images of a, b, c under the natural projection map $F \to F/N$) act as follows:

$$\begin{aligned}
\overline{a} \cdot (x, y) &= (x + 1, y) \\
\overline{b} \cdot (x, y) &= (x, y + 1) \\
\overline{c} \cdot (x, y) &= (x + 2, y + 1)
\end{aligned}$$

for all $(x, y) \in \mathbb{R}^2$.