Assignment 2

(1) Consider the alternating group A_n, consisting of even permutations of $1, 2, \cdots, n$. Prove that A_n is isomorphic to the group F/N where F is the free group on $n-2$ generators $x_1, x_2, \cdots, x_{n-2}$ and N is the normal subgroup of F generated by the elements: $x_1^3,\ \ x_i^2\ (2 \le i \le n-2),\ \ (x_i\, x_{i+1})^3\ (1 \le i \le n-3)$ and $(x_i\, x_j)^2\ (1 \le i, j \le n-2$ with $i > j+1)$.

(2) Let G be a group acting on a set S. We say the action is *faithful* if the defining homomorphism $G \to \operatorname{Sym} S$ is injective. In other words, $gs = s$ for all s implies $g = 1$. Let H be a subgroup of G and consider the G-action on the left coset space $S = G/H$. Prove that this is faithful iff H does not contain any normal subgroup of G other than (1).

(3) Let G act on S, T. Let $\mathcal{F}(S, T)$ denote the set of all functions from S to T. Prove that the following defines a G-action on $\mathcal{F}(S, T)$:

$$(g \bullet f)(s) := g \cdot (\, f(g^{-1} \cdot s\,))$$

for $g \in G, f \in \mathcal{F}(S, T)$ and $s \in S$.

(4) Let A, B be groups. Suppose we are given an action of A on B, i.e., a homomorphism $\varphi : A \to \operatorname{Sym} B$. We say A acts on B by automorphisms if $\varphi(a)$ is a group automorphism of B (rather than just a set bijection) for each $a \in A$; in other words the image of φ is a subgroup of $\operatorname{Aut} B$. Given such an action, we can define a new group C as follows: $C = B \times A$ with multiplication

$$(b_1, a_1)(b_2, a_2) = (b_1\, \varphi(a_1)(b_2),\ a_1 a_2)$$

Prove that: (i) C is a group. (ii) C has a subgroup B' isomorphic to B such that the quotient group C/B' is isomorphic to A. We call C the semidirect product of A and B via φ.

(5) Let $G = S_3$ the symmetric group on 3 elements. It acts on itself by conjugation and there is thus an induced action on its power set. Let $\mathcal{P}_k$ denote the set of k-element subsets of G. Determine the orbits for the G-action on $\mathcal{P}_k$ for $k = 1, 2, \cdots, 6$. Do the same for the left translation action of G on itself.

(6) Repeat the above problem for $G = S_4$ and $k = 2, 3$.

(7) An action of G on S is said to be *transitive* if there is only one G-orbit, i.e., for each pair $s, s' \in S$, there is a $g \in G$ such that $s' = gs$. Suppose S is a transitive G-set and $U \subset S$,

prove that the subsets $gU := \{gu : u \in U\}$ evenly cover S, i.e., each s in S belongs to the same number of sets gU.

(8) Let G be a finite group acting on itself (and thereby on its power set) by conjugation. Let $U \subset G$ such that $|U|$ is relatively prime to $|G|$. Prove or disprove: the stabilizer of U is trivial.[1]

[1]For more problems, see Michael Artin's *Algebra*, Chapter 6.