Assignment 11

Unless otherwise specified, V denotes a finite-dimensional vector space over the field F, with char $F \neq 2$.

(1) Let A denote an F-algebra. We say A is $\mathbb{Z}_+$-graded (or simply graded) if it can be written as a direct sum of subspaces

$$A = \bigoplus_{n=0}^{\infty} A_n$$

such that $A_m A_n \subset A_{n+m}$ for all $n, m \geq 0$, i.e., $a \in A_m, b \in A_n$ implies $ab \in A_{m+n}$. An element $a \in A$ is said to be *homogeneous* if $a \in A_n$ for some n. A subspace $W \subset A$ is said to be *graded* if $W = \oplus_{n \geq 0} (W \cap A_n)$, i.e., given $w = \sum_n w_n$ with $w_n \in A_n$ (finitely nonzero), we have $w_n \in W$ for all n. Now suppose A is graded and S is a subset of A each of whose elements is homogeneous, then prove that the two-sided ideal I generated by S is graded (*hint:* consider $\oplus_{n \geq 0} (I \cap A_n)$).

(2) Let A be a graded F-algebra as above and I a graded two-sided ideal. Denote $I \cap A_n$ by I_n. (a) Show that $\overline{A} = A/I$ becomes a graded F-algebra as follows: let $\pi : A \to A/I$ be the quotient map and $\overline{A}_n = \pi(A_n)$. Then $\overline{A} = \oplus_n \overline{A}_n$. (b) Now $\pi(A_n) = \frac{A_n + I}{I} \cong \frac{A_n}{A_n \cap I} = \frac{A_n}{I_n}$ as subspaces. Show that the direct sum $\oplus_n A_n / I_n$ can be given the structure of an F-algebra isomorphic to A/I.

(3) Let TV be the tensor algebra of V and let I denote the 2-sided ideal of TV generated by the subset $\{v \otimes w - w \otimes v : v, w \in V\}$. Show that $(T^0 V \oplus T^1 V) \cap I = (0)$ and I is graded. Hence the symmetric algebra $\operatorname{Sym} V = TV/I$ is graded. Prove that:
  - (a) $T^0 V \oplus T^1 V$ generate TV, and their images under the projection $TV \twoheadrightarrow \operatorname{Sym} V$ generate $\operatorname{Sym} V$.
  - (b) $\operatorname{Sym} V$ is commutative.
  - (c) Consider the maps: $V \hookrightarrow TV \twoheadrightarrow \operatorname{Sym} V$. Show that their composition is injective, and one thereby obtains a map $V \overset{i}{\hookrightarrow} \operatorname{Sym} V$.
  - (d) Show that $\operatorname{Sym} V$ has the following universal property. Given a *commutative* F-algebra A together with a F-linear map $f : V \to A$, there exists a unique F-algebra homomorphism $\tilde{f} : \operatorname{Sym} V \to A$ such that $\tilde{f} \circ i = f$.

(4) Retain the notation of the preceding problem.
  - (a) For $n \geq 2$, let I'_n denote the F-span of elements of the form

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$$

    where $v_i \in V$ and σ is a permutation of $1, 2, \cdots, n$. Show that $I' = \bigoplus_{n \geq 2} I'_n$ is a two-sided ideal of TV.
  - (b) $I' = I$, and hence $I_n := I \cap T^n V = I'_n$.
  - (c) Let $\operatorname{Sym}^n V = T^n V / I_n$. We have the maps $V \times V \times \cdots \times V$ (n copies) $\longrightarrow V^{\otimes n} \twoheadrightarrow \operatorname{Sym}^n V$; let $j : V^{\times n} \to \operatorname{Sym}^n V$ denote their composition. Prove the following universal property: given a multilinear map $f : V^{\times n} \to W$ to a F-vector space W such that f is symmetric (i.e.,

$f(v_1, v_2, \cdots, v_n) = f(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(n)})$ for all permutations σ), there exists a unique F-linear map $\tilde{f} : \mathrm{Sym}^n V \to W$ such that $\tilde{f} \circ j = f$.

(d) Given $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in T^n V$, we denote its image under the projection $T^n V \twoheadrightarrow \mathrm{Sym}^n V$ by $v_1 v_2 \cdots v_n$. If $\{e_1, e_2, \cdots, e_d\}$ is a basis of V, prove that the following elements form a basis of $\mathrm{Sym}^n V$:

$$e_{i_1} e_{i_2} \cdots e_{i_n},\ 1 \le i_1 \le i_2 \le \cdots \le i_n \le d$$

Hint: For linear independence, define a symmetric multilinear map $V^{\times n} \to F$ which is nonzero on the tuple $(e_{i_1}, e_{i_2}, \cdots, e_{i_n})$ and all its permutations, and zero on all other $(e_{j_1}, e_{j_2}, \cdots, e_{j_n})$.

(e) Find a closed formula for $\dim(\mathrm{Sym}^n V)$.

(5) Let J denote the 2-sided ideal of TV generated by the subset $\{v \otimes v : v \in V\}$. Show that $(T^0 V \oplus T^1 V) \cap J = (0)$ and J is graded. Hence the exterior algebra $\bigwedge(V) = TV/J$ is graded. We denote the product in $\bigwedge(V)$ by $\wedge$.

(a) Consider the maps: $V \hookrightarrow TV \twoheadrightarrow \bigwedge(V)$. Show that their composition is injective, and one thereby obtains a map $V \overset{i}{\hookrightarrow} \bigwedge(V)$. We identify V with its image in $\bigwedge(V)$.

(b) Show that $\bigwedge(V)$ has the following universal property. Given a F-algebra A together with a F-linear map $f : V \to A$ such that $f(v)^2 = 0$ for all $v \in V$, there exists a unique F-algebra homomorphism $\tilde{f} : \bigwedge(V) \to A$ such that $\tilde{f} \circ i = f$.

(c) Show that the image of $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ under the projection $TV \twoheadrightarrow \bigwedge(V)$ is $v_1 \wedge v_2 \wedge \cdots \wedge v_n$.

(d) Given $\xi = v_1 \wedge v_2 \wedge \cdots \wedge v_n$ and $\eta = w_1 \wedge w_2 \wedge \cdots \wedge w_m$, show that $\xi \wedge \eta = \pm(\eta \wedge \xi)$. Determine the sign in terms of m, n.

(e) For $n \ge 2$, let J'_n denote the F-span of elements of the form

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n$$

where $v_i \in V$ and $v_i = v_j$ for some pair $i \ne j$. Show that $J' = \bigoplus_{n \ge 2} J'_n$ is a two-sided ideal of TV.

(f) $J' = J$, and hence $J_n := J \cap T^n V = J'_n$.

(g) Let $\bigwedge^n(V) = T^n V / J_n$. Let $j : V^{\times n} \to V^{\otimes n} \twoheadrightarrow \bigwedge^n(V)$ denote the natural map. Prove the following universal property: given a multilinear map $f : V^{\times n} \to W$ to a F-vector space W such that f is alternating (i.e., $f(v_1, v_2, \cdots, v_n) = 0$ for all tuples in which $v_i = v_j$ for some $i \ne j$), there exists a unique F-linear map $\tilde{f} : \bigwedge(V) \to W$ such that $\tilde{f} \circ j = f$.

(h) If $\{e_1, e_2, \cdots, e_d\}$ is a basis of V, prove that the following elements form a basis of $\bigwedge^n(V)$:

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n},\ 1 \le i_1 < i_2 < \cdots < i_n \le d$$

Hint: For linear independence, define an alternating multilinear map $V^{\times n} \to F$ which is nonzero on the tuple $(e_{i_1}, e_{i_2}, \cdots, e_{i_n})$ and all its permutations, and zero on all other $(e_{j_1}, e_{j_2}, \cdots, e_{j_n})$.

(i) Show that $\dim \bigwedge^n(V) = \binom{d}{n}$.

(6) Prove that $V \mapsto TV$, $V \mapsto \operatorname{Sym} V$, $V \mapsto \bigwedge(V)$ are all functors from the category of F-vector spaces to the category of F-algebras. Similarly, $V \mapsto T^n V$, $V \mapsto \operatorname{Sym}^n V$, $V \mapsto \bigwedge^n(V)$ are functors from the category of F-vector spaces to itself.

(7) Given a linear operator $P : V \to V$, consider the induced map $\bigwedge^d(V) \xrightarrow{\varphi_P} \bigwedge^d(V)$ where $d = \dim V$. Since $\bigwedge^d(V)$ is one-dimensional, the map φ_P is multiplication by a scalar. Prove that this scalar is $\det P$.

(8) Let $\{e_i\}_{i=1}^d$ denote a basis of V and given $v \in V$, we write $v = \sum_i x_i(v) e_i$. A function $p : V \to F$ is said to be a *polynomial function* if $p(v)$ is a polynomial in the coordinates $x_1(v), x_2(v), \cdots, x_d(v)$.
  (a) Show that this definition is independent of the chosen basis.
  (b) Let $\mathcal{P}(V)$ denote the set of polynomial functions on V. Show that this is an F-algebra under pointwise operations.
  (c) Show that $V \mapsto \mathcal{P}(V)$ is a contravariant functor from F-vector spaces to F-algebras.
  (d) Construct a natural isomorphism: $\operatorname{Sym} V^* \to \mathcal{P}(V)$ (i.e., such that it is a natural transformation of the two functors).

(9) Prove that the projection $TV \twoheadrightarrow \operatorname{Sym} V$ is an isomorphism iff $\dim V = 1$.