# Assignment 10

(1) Let $R^{\mathrm{op}}$ denote the opposite ring of $R$. Let $M$ be a right $R$-module (hence a left $R^{\mathrm{op}}$-module) and $N$ a left $R$-module (hence a right $R^{\mathrm{op}}$-module). Prove that the map $m \otimes n \mapsto n \otimes m$ defines a $\mathbb{Z}$-linear isomorphism of $M \otimes_R N$ with $N \otimes_{R^{\mathrm{op}}} M$.

(2) Let $R$ be commutative and $M, N$ be $R$-modules. If $E \subset M$ generates $M$ and $F \subset N$ generates $N$ as $R$-modules, show that $\{e \otimes f : e \in E, f \in F\}$ generates $M \otimes_R N$ as an $R$-module.

(3) Let $m, n \geq 1$ and consider the $\mathbb{Z}$-modules:

$$M = \mathbb{Z}/m\mathbb{Z} \text{ and } N = \mathbb{Z}/n\mathbb{Z}$$

(a) If $m$ and $n$ are relatively prime, prove that $M \otimes_{\mathbb{Z}} N = 0$.
(b) More generally, prove that $M \otimes_{\mathbb{Z}} N \cong \mathbb{Z}/d\mathbb{Z}$ where $d$ is the gcd of $m, n$.

(4) Let $R$ be a PID and let $p, q$ denote prime elements in $R$. Let $a, b \geq 1$. Prove that $R/p^aR \otimes R/q^bR$ is isomorphic to: (i) 0 if $p \neq q$, (ii) $R/p^cR$ if $p = q$, where $c = \min(a, b)$. Now suppose $M, N$ are finitely generated torsion modules over $R$, determine the elementary divisors of $M \otimes_R N$ in terms of those of $M, N$. Do the same for invariant factors. What if $M, N$ are not necessarily torsion modules (i.e., they may have a free part) ?

(5) Given collections $\{M_i : i \in I\}$ of right $R$-modules and $\{N_j : j \in J\}$ of left $R$-modules, prove that there is a $\mathbb{Z}$-linear isomorphism:

$$\left(\bigoplus_{i\in I} M_i\right) \otimes \left(\bigoplus_{j\in J} N_j\right) \xrightarrow{\sim} \bigoplus_{\substack{i\in I \\ j\in J}} (M_i \otimes N_j)$$

(6) Fix a commutative ring $R$. Define the following notions for $R$-algebras: *subalgebra, ideal, homomorphism, quotient algebra* and prove the analogue of the first isomorphism theorem in this setting.

(7) Let $F$ be a field and $V$ a finite dimensional vector space over $F$. Show that $\mathrm{End}_F V$ is an $F$-algebra. Prove that:

$$\mathrm{End}_F V \otimes_F \mathrm{End}_F W \xrightarrow{\sim} \mathrm{End}_F(V \otimes_F W)$$

as $F$-algebras, where $W$ is a finite dimensional vector space over $F$ (*hint:* use the universal property of the tensor product of algebras).

(8) Let $F$ be a field. Show that $F[x]$ is an $F$-algebra. Prove that:

$$F[x] \otimes_F F[y] \xrightarrow{\sim} F[x, y]$$

as $F$-algebras.

(9) If $R$ is a commutative ring and $A, B, C$ are $R$-algebras, prove that there exist isomorphisms of $R$-algebras:

(a) $A \otimes_R B \cong B \otimes_R A$ via the map $a \otimes b \mapsto b \otimes a$

(b) $(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C)$ via the map:

$$(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$$

(10) Let $R$ be a commutative ring and $S$ be a $R$-algebra. Construct a homomorphism of $R$-algebras:

$$S \otimes_R S^{\mathrm{op}} \to \mathrm{End}_R(S)$$

where $\mathrm{End}_R(S)$ denotes the space of $R$-linear endomorphisms of $S$ (*hint:* bimodules).