Assignment 1

(1) Let $\phi : M \to M'$ be a surjective homomorphism of monoids.

(a) Prove that there is a 1-1 correspondence between congruences on M' and congruences on M containing $\equiv_\phi$ (the kernel of ϕ).

(b) Let $\equiv$ and $\equiv'$ be congruences on M, M' (respectively) that are paired under this correspondence. Prove that $M/\equiv$ is isomorphic to $M'/\equiv'$.

(2) Let X, Y be sets and let (M, i) and (N, j) be the free monoids on X, Y respectively.

(a) Given a map of sets $f : X \to Y$, prove that there exists a unique monoid homomorphism $\widetilde{f} : M \to N$ such that $\widetilde{f} \circ i = j \circ f$.

(b) f is a bijection of sets iff $\widetilde{f}$ is a monoid isomorphism.

(3) Redo the above problem with *monoids* replaced by *groups* everywhere.

(4) Let X be a set and let (M, i) and (G, j) denote the free monoid and free group (respectively) on X. Prove that there is a monoid homomorphism $\phi : M \to G$ such that $j = \phi \circ i$. Prove that ϕ is injective (assume X finite if you like).

(5) Let $X = \{p, q\}$ and let (G, i) denote the free group on X. We write $a = i(p)$ and $b = i(q)$. Prove that the following elements of G are not equal to the identity element of G:

(a) a^n for all $n \geq 1$.

(b) $(ab)^n$ for all $n \geq 1$.

(c) $aba^{-1}b^{-1}$

(6) Consider the symmetric group S_n. Using the following transpositions as generators

$$(12),\ (23),\ \cdots, (n-1\, n)$$

prove that S_n is isomorphic to the group F/N where F is the free group on $n-1$ generators $x_1, x_2, \cdots, x_{n-1}$ (say) and N is the normal subgroup of F generated by the elements x_i^2 $(1 \leq i \leq n-1)$, $(x_i\, x_{i+1})^3$ $(1 \leq i \leq n-2)$ and $(x_i\, x_j)^2$ $(1 \leq i, j \leq n-1$ with $i > j+1)$.