THE TENSOR PRODUCT

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(1) The tensor product $V \otimes W$ of two vector spaces V and W is a vector space equipped with bilinear map $\alpha : V \times W \to V \otimes W$ satisfying the following *universal property*: given any pair (U, f) with U a vector space, and f a bilinear map $f : V \times W \to U$, there exists a unique linear map $\phi : V \otimes W \to U$ such that $f = \phi \circ \alpha$. We usually write $v \otimes w$ for the vector $\alpha(v, w)$ in $V \otimes W$.
  (a) Prove existence and uniqueness (up to unique isomorphism) of tensor product.
  (b) If V and W are finite dimensional, prove that $\dim(V \otimes W) = \dim V \, \dim W$.
  (c) If $\{v_i : i \in I\}$ and $\{w_j : j \in J\}$ are bases of V and W respectively, prove that $\{v_i \otimes w_j : i \in I, j \in J\}$ is a basis of $V \otimes W$.

(2) (a) Formulate and prove an analogous result for the n-fold tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_n$, using multilinear maps instead of bilinear ones.
  (b) Prove that the n-fold tensor product is canonically isomorphic to iterated two fold tensor products. This reduces to showing that $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$.

(3) (a) Prove that there is a linear map $\phi : V^* \otimes W \to \mathrm{Hom}(V, W)$ satisfying $\phi(f \otimes w)(v) := f(v)\, w$.
  (b) If V, W are finite dimensional, prove that ϕ is an isomorphism.

(4) (a) Let A be a linear operator on V and B be a linear operator on W. Prove that there exists a linear operator C on $V \otimes W$ satisfying $C(v \otimes w) = Av \otimes Bw$ for all $v \in V, w \in W$. We usually denote C by $A \otimes B$.
  (b) Find the matrix representation of $A \otimes B$ in terms of matrix representations of A and B (choosing appropriate bases).
  (c) Prove that $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\, \mathrm{tr}(B)$.
  (d) If A and B are diagonalizable operators, then so is $A \otimes B$.
  (e) If A or B is nilpotent, then so is $A \otimes B$.

(5) Let k be the base field. Let $V^{\otimes n}$ denote $V \otimes V \otimes \cdots \otimes V$ (n times), where for $n = 0$, this is defined to be k. Define the *tensor algebra* $T(V) := \oplus_{n=0}^{\infty} V^{\otimes n}$. Prove that the tensor product defines an associative multiplication on $T(V)$. Further, prove that the tensor algebra has the following universal property: Given a k-algebra R and a vector space map $f : V \to R$, there is a unique *algebra homomorphism* $\tilde{f} : T(V) \to R$ such that $f = \tilde{f} \circ i$ where i is the natural inclusion of V in $T(V)$.

(6) Let $S(V)$ be the quotient of $T(V)$ by the ideal I generated by $\{v \otimes w - w \otimes v : v, w \in V\}$. Prove that $S(V)$ is a commutative, associative k-algebra, isomorphic to the ring of polynomials in n variables, where $n = \dim V$. Prove that $I = \oplus_{n=0}^{\infty} I \cap V^{\otimes n}$ and hence $S(V) = \oplus_{n=0}^{\infty} S^n(V)$ where $S^n(V) = V^{\otimes n}/(I \cap V^{\otimes n})$. Thus, $S(V)$ is a graded algebra.

for more exercises, see `http://math.berkeley.edu/~serganov/math252`.