ROOT SYSTEMS - CLASSIFICATION

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Recall that the possible Dynkin diagrams that can arise from an irreducible, reduced root system are $A_l\,(l \geq 1)$, $B_l\,(l \geq 2)$, $C_l\,(l \geq 2)$, $D_l\,(l \geq 4)$, E_6, E_7, E_8, F_4, G_2. In the exercises below, we will construct a root system correponding to each diagram from this list.

In each problem below, do the following:

(1) Find the elements of R explicitly, and check that R is a root system.
(2) Choose a chamber C appropriately, and find the corresponding $R^{\pm}$ and B.
(3) Check that the configuration of roots in B gives the required Dynkin diagram in each case.
(4) Compute the action of the simple reflections, and determine the Weyl group, for the diagrams $A - D$.
(5) Find the highest root (recall from first problem sheet on root systems), and compute the Coxeter number h.
(6) Check in each case that $|R| = lh$.

1. General principles

Prove the following:

(1) Let $V, (\cdot \mid \cdot)$ be a finite dimensional real inner product space, and $L \subset V$ be a lattice (i.e a discrete subgroup of V). Let $N \subset \mathbb{R}_{>0}$ be a finite set. Define $R := \{\alpha \in L : (\alpha \mid \alpha) \in N\}$. If R spans V and $\frac{2(\alpha|\beta)}{(\beta|\beta)} \in \mathbb{Z}$ for all $\alpha, \beta \in R$, then R is a root system (in other words, the second root system axiom comes for free).

(2) If there is an inclusion of Dynkin diagrams $X \hookrightarrow Y$, then the root system of X can be obtained from the root system $(V(Y), R(Y))$ of Y as follows: take $V(X)$ to be the subspace of $V(Y)$ spanned by the simple roots corresponding to the vertices of X, and $R(X) := V(X) \cap R(Y)$.

2. Classical Types A-D

Consider the vector space $V_n := \mathbb{R}^n$ with the standard inner product $(\cdot \mid \cdot)$, and standard orthonormal basis $\{\epsilon_i : i = 1 \cdots n\}$. Let $x_n := \sum_{i=1}^n \epsilon_i \in V_n$. Define the lattice $L_n := \oplus_{i=1}^n \mathbb{Z}\epsilon_i$. Finally, let $\tilde{V}_n := x_{n+1}^{\perp} \subset V_{n+1}$.

A_n: $R(A_n) := \{\alpha \in L_{n+1} \cap \tilde{V}_n : (\alpha \mid \alpha) = 2\}$ (as a subset of $\tilde{V}_n$).

B_n: $R(B_n) := \{\alpha \in L_n : (\alpha \mid \alpha) = 1 \text{ or } 2\}$.

C_n: $R(C_n) := \{\frac{2\alpha}{(\alpha|\alpha)} : \alpha \in R(B_n)\}$.

D_n: $R(D_n) := \{\alpha \in L_n : (\alpha \mid \alpha) = 2\}$.

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for more problems, see Bourbaki's *Lie Groups and Lie algebras, Chapters 4-6.*

3. Exceptional types E, F, G

E_8: Let L be the following lattice in V_8:

$$L := \{\sum_{i=1}^{8} a_i\epsilon_i : a_i \in \frac{1}{2}\mathbb{Z},\ \ a_i \equiv a_j \pmod{\mathbb{Z}},\ \ \sum_{i=1}^{8} a_i \in 2\mathbb{Z}\}$$

Then $R(E_8) := \{\alpha \in L : (\alpha \mid \alpha) = 2\}$.

E_6 and E_7: Obtain these using the diagram inclusions $E_6 \hookrightarrow E_7 \hookrightarrow E_8$.

F_4: Let $L := L_4 \oplus \mathbb{Z}(\frac{1}{2}x_4) \subset V_4$. Then $R(F_4) := \{\alpha \in L : (\alpha \mid \alpha) = 1 \text{ or } 2\}$.

G_2: $R(G_2) := \{\alpha \in L_3 \cap E_2 : (\alpha \mid \alpha) = 2 \text{ or } 6\}$.

4. More problems

(1) **Definition:** A *root subsystem* of (V, R) is a subset $R' \subset R$ such that R' is invariant under the reflections s_α for $\alpha \in R'$. In this case, R' becomes a root system in $V' := \operatorname{span} R'$.

Observe if there is a diagram inclusion $X \hookrightarrow Y$, then $R(X)$ occurs as a root subsystem of $R(Y)$. For each X and Y below, show that $R(X)$ occurs as a root subsystem of $R(Y)$, though there is no diagram inclusion.

(a) $X = D_l$, $Y = B_l$.
(b) $X = D_8$, $Y = E_8$.
(c) $X = B_4$, $Y = F_4$.
(d) $X = A_2$, $Y = G_2$.

Eventually, we will see that this implies there is an inclusion of the corresponding Lie algebras. For example D_l corresponds to the Lie algebra $\mathfrak{so}_{2l}$ while B_l corresponds to $\mathfrak{so}_{2l+1}$; the inclusion is more transparent here in this case.

(2) Recall the definition of the *Weyl vector* $\rho := \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ from the first problem sheet. Compute ρ explicitly in the *simply laced* cases A, D, E using our construction of these root systems. For these cases, prove :

$$(\rho \mid \rho) = \frac{(|R| + l)|R|}{12l}$$

where R is the root system of rank l.