THE WEYL GROUP

JULY 8, 2011

Let W be the Weyl group of a root system R. Fix a decomposition $R = R^+ \cup R^-$ and let $B = \{\alpha_i : i = 1 \cdots l\}$ be the corresponding basis of R. Let $S := \{s_i : i = 1 \cdots l\}$ be the simple reflections in W, where we denote $s_i := s_{\alpha_i}$; recall that S generates W. Define the length of an element of W by

$$l(w) := \min\{k \geq 0 : w = s_{i_1} s_{i_2} \cdots s_{i_k} \text{ for some } 1 \leq i_j \leq l\}$$

In other words, this is the smallest number k such that w can be written as a product of k simple reflections.

(1) Prove the following simple properties of length:
  (a) $l(w^{-1}) = l(w)$ for all $w \in W$.
  (b) $l(w_1 w_2) \leq l(w_1) + l(w_2)$ for all $w_1, w_2 \in W$.
  (c) $l(w_1 w_2) \geq |l(w_1) - l(w_2)|$.

(2) Prove that there is well defined sign homomorphism $\epsilon : W \to \{\pm 1\}$ such that $\epsilon(s_i) = -1$ for all i.

(3) Prove that $l(ws_i) = l(w) \pm 1$ for all $w \in W$, $s_i \in S$.

(4) **Theorem:** If $w \in W$ and $s_i \in S$, then (a) $l(ws_i) = l(w) - 1 \iff w\alpha_i \in R^-$ and (b) $l(ws_i) = l(w) + 1 \iff w\alpha_i \in R^+$.

Prove this theorem using the following steps:
  (a) Assume $w\alpha_i \in R^-$. Write $w = s_{i_k} s_{i_{k-1}} \cdots s_{i_1}$ where $k = l(w)$. Define the *right subwords*, $w_0 = 1$, $w_1 := s_{i_1}$, $w_2 := s_{i_2} s_{i_1}$, $\cdots$, $w_k := s_{i_k} s_{i_{k-1}} \cdots s_{i_1} = w$. Now $w_0 \alpha_i \in R^+$ while $w_k \alpha_i \in R^-$. There is a smallest j such that $w_j \alpha_i \in R^+$ but $w_{j+1}\alpha_i \in R^-$. Prove now that $w_j \alpha_i$ must be a simple root (which one ?).
  (b) If $w\beta = \gamma$ for $w \in W$, $\beta, \gamma \in R$, prove that $s_\gamma = w s_\beta w^{-1}$.
  (c) Use this to obtain an expression for ws_i as a product of $k-1$ simple reflections.
  (d) Finally show that all other assertions of the theorem can be deduced from what has been proved above (by replacing w with ws_i).

(5) Recall that the inversion set $I(w) = \{\alpha \in R^+ : w\alpha \in R^-\}$. Show that if $l(ws_i) = l(w) + 1$, then $I(ws_i) = \{\alpha_i\} \cup s_i(I(w))$. Hence show (by induction) that $l(w) = |I(w)|$ for all $w \in W$.

(6) (a) Show that if C is a chamber, then so is $-C := \{-x : x \in C\}$.
  (b) By the simple transitivity of the W-action on the set of chambers, there is a unique $w_0 \in W$ such that $w_0(C) = -C$. Prove that $w_0^2 = 1$.
  (c) Prove that w_0 is the unique longest element of the Weyl group W (length being measured wrt the simple reflections obtained from the basis corresponding to C).
  (d) Prove that $l(w_0 \sigma) = l(w_0) - l(\sigma)$ for all $\sigma \in W$.
  (e) For the root system A_{n-1} constructed in lecture, recall $W \cong S_n$. Find w_0, and compute its length.

---

for more problems, see Bourbaki's *Lie Groups and Lie algebras, Chapters 4-6*.