Root systems

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1. (05/07/2011)

1.1. **Root systems.** Let V be a finite dimensional $\mathbb{R}$-vector space. A *reflection* is a linear map $s_{\alpha,H}$ on V satisfying $s_{\alpha,H}(x) = x$ for all $x \in H$ and $s_{\alpha,H}(\alpha) = -\alpha$, where H is a hyperplane and $\alpha \notin H$.

Clearly $H = \ker f$ for some $0 \neq f \in V^*$ and we can choose f such that $f(\alpha) = 2$, then $s_{\alpha,H} = s_{\alpha,f}$ where $s_{\alpha,f}(v) = v - f(v)\alpha$.

Lemma 1.1. *Let R be a finite subset of V which spans V. Let $0 \neq \alpha \in V$ then there exists at most one reflection s on V such that s of V such that $s(\alpha) = -\alpha$ and $s(R) = R$.*

Proof. If there are two such s_{α,H_1} and s_{α,H_2} then consider $t = s_{\alpha,H_1}s_{\alpha,H_2}$ is of finite order and is identity on α and on $H_1 \cap H_2$ so by determinant being one, $t = 1$. □

Definition 1.2. A *root system* R is a finite subset of V such that

(1) $0 \notin R$, R spans V,
(2) $\alpha \in R \implies \exists$ a reflection with respect to α, i.e., a reflection of the form $s_{\alpha,\alpha^\vee}$ for $\alpha^\vee \in V^*$ (such that $\langle \alpha^\vee, \alpha\rangle := \alpha^\vee(\alpha) = 2$), such $s_{\alpha,\alpha^\vee}(R) = R$.
(3) $\alpha^\vee(\beta) \in \mathbb{Z}$ for all $\alpha, \beta \in R$.

Note that above lemma guarantees uniqueness of $s_{\alpha,\alpha^\vee}$.

Elements of R are called *roots*, the dimension of V is called the *rank of* R. Let $A(R) = \{T \in \mathrm{GL}(V) : T(R) = R\}$ and $W(R) = \langle s_{\alpha,\alpha^\vee} : \alpha \in R\rangle \subseteq A(R)$.

Examples 1.3. 1. Let $V = \mathbb{R}$ and let $0 \neq \alpha \in \mathbb{R}$, then $\alpha^\vee$ is determined by $\langle \alpha, \alpha^\vee\rangle = 2$. If $\alpha \in R$ then $-\alpha \in R$ and $\{\alpha, -\alpha\}$ forms a root system in V. It is called a root system of type A_1.

2. Another root system on $V = \mathbb{R}$ is $\{\pm\alpha, \pm 2\alpha\}$. It is called BC_1.

We now assume an inner product defined on V, standard inner product, because it would then be easy to describe the hyperplane orthogonal to each vector.

3. For $V = \mathbb{R}^2$ with the standard inner product we get several root systems. One of them, called A_2, is described as $\{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ where $\alpha^\vee(\beta) = -1$. Other root systems on $\mathbb{R}^2$ are $A_1 \times A_1, B_2, G_2$ and BC_2.

Let us now fix a root system $R \subset V$.

Lemma 1.4. If $(\cdot|\cdot)$ is a symmetric, bilinear, non-degenerate $W(R)$-invariant form then $(\alpha|\alpha) \neq 0$ for every $\alpha \in R$ and $\psi^{-1}(\alpha^\vee) = \frac{2\alpha}{(\alpha|\alpha)}$.

The non-degeneracy of $(\cdot|\cdot)$ gives an isomorphism $V \xrightarrow{\psi} V^*$ which is used in the statement above.

Proof. Use $s_{\alpha,\alpha^\vee} \in W(R)$ and so $(s_{\alpha,\alpha^\vee}(\beta)|s_{\alpha,\alpha^\vee}(\alpha)) = (\beta|\alpha)$. □

Proposition 1.5. Such a $W(R)$-invariant form exists.

Proof. Averaging trick! □

Exercise 1.6. Show that if R is a root system then so is $R^\vee = \{\alpha^\vee : \alpha \in R\} \subset V^*$.

Exercise 1.7. The form $(x|y) = \sum_{\alpha\in R}\langle \alpha^\vee, x\rangle\langle \alpha^\vee, y\rangle$ is an $A(R)$-invariant form on V.

Definition 1.8. A direct sum of root systems $R_i \subset V_i, 1 \leq i \leq n$, is a subset $R := \coprod R_i \in \oplus V_i$. It will be written as $R = \oplus R_i$.

An example of such direct sum is $A_1 \times A_1$.

Exercise 1.9. If $R = R_1 \oplus R_2$ then $W(R) = W(R_1) \times W(R_2)$.

Definition 1.10. A root system R is called *irreducible* if it is not a direct sum $R_1 \oplus R_2$ for root systems R_1, R_2.

Proposition 1.11. A root system R is irreducible if and only if the action of $W(R)$ on V is irreducible, i.e., V is an irreducible representation of $W(R)$.

Proof. $\Longrightarrow$ If V is not an irrep of $W(R)$, then $V = V_1 \oplus V_2$ for nontrivial $W(R)$-invariant subspaces V_1, V_2. We define $R_i = R \cap V_i$ for $i = 1, 2$. One observes that if $\alpha \in R, v \in V_i \implies s_\alpha(v) \in V_i$ hence R_1, R_2 are root systems and finally $R = R_1 \oplus R_2$. □

2. (06/07/2011)

2.1. **An aside on non-degenerate forms.** Let V be a finite dimensional real vector space and let $(\cdot|\cdot)$ be a symmetric non-degenerate bilinear form on V. Having such a form is the same as having an isomorphism $\psi = \psi_{(\cdot|\cdot)} : V \to V^*, v \mapsto (v|\cdot)$, i.e., $\psi(v)(v') = (v|v')$. Having $(\cdot|\cdot)$ non-degenerate means that ψ is injective and since $\dim V < \infty$ this is enough.

There exists such a form on V^*: for $f, g \in V^*$ we define $(f|g)_{V^*} := (\psi^{-1}(f)|\psi^{-1}(g))_V = \langle f, \psi^{-1}(g)\rangle$.

2.2. **Configurations of pairs of roots.** Let (V, R) be a root system and let $(\cdot|\cdot)$ be a $W(R)$-invariant form on V giving $\psi : V \to V^*$. Observe that $\psi^{-1}(\alpha^\vee) = \frac{2\alpha}{(\alpha|\alpha)}$.

For roots $\alpha, \beta \in R$ we define the *Cartan integer* $n(\alpha, \beta)$ to be the integer $\langle \alpha, \beta^\vee\rangle = \frac{2(\alpha|\beta)}{(\beta|\beta)}$.

Lemma 2.1. 1. $n(\alpha, \alpha) = 2$.

2. $s_\beta(\alpha) = \alpha - n(\alpha, \beta)\beta$.

3. (a) If $(\alpha|\beta) \neq 0$ then $\frac{n(\alpha,\beta)}{n(\beta,\alpha)} = \frac{(\alpha,\alpha)}{(\beta,\beta)}$.

(b) $n(\alpha, \beta)n(\beta, \alpha) = \frac{4(\alpha|\beta)^2}{\|\alpha\|^2\|\beta\|^2} = 4\cos^2\theta$.

Then the possibilities for a pair α, β, with $\|\beta\| \geq \|\alpha\|$, are as follows:

$n(\alpha,\beta)$	$n(\beta,\alpha)$		θ	
0	0	$*$	$\frac{\pi}{2}$	$A_1 \times A_1$
1	1	$\|\beta\| = \|\alpha\|$	$\frac{\pi}{3}$	A_2
-1	-1	$\|\beta\| = \|\alpha\|$	$\frac{2\pi}{3}$	A_2
1	2	$\|\beta\| = \sqrt{2}\|\alpha\|$	$\frac{\pi}{4}$	B_2
-1	-2	$\|\beta\| = \sqrt{2}\|\alpha\|$	$\frac{3\pi}{4}$	B_2
1	3	$\|\beta\| = \sqrt{3}\|\alpha\|$	$\frac{\pi}{6}$	G_2
-1	-3	$\|\beta\| = \sqrt{2}\|\alpha\|$	$\frac{5\pi}{6}$	G_2
2	2	$\|\beta\| = \|\alpha\|$	0	$\alpha = \beta$
-2	-2	$\|\beta\| = \|\alpha\|$	π	$\alpha = -\beta$
1	4	$\|\beta\| = 2\|\alpha\|$	0	$\beta = 2\alpha$
-1	-4	$\|\beta\| = 2\|\alpha\|$	π	$\beta = -2\alpha$

We call a root system *reduced* if $\alpha \in R$ then $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$.

Corollary 2.2. Let $\alpha, \beta \in R$. If $n(\alpha, \beta) \gtrless 0$ then $\alpha \mp \beta \in R$.

2.3. **Chambers, basis and $W(R)$.** Let (V, R) be a reduced root system. Let $\mathcal{H}$ be the set of all hyperplanes $\ker(\alpha^\vee)$ for $\alpha \in R$.

Lemma 2.3. $W(R)$ acts on $\mathcal{H}$.

Proof. $w(H_\alpha) = H_{w(\alpha)}$ as $H_\alpha = \ker(\alpha^\vee) = \{x \in V : (x|\alpha) = 0\}$. □

Now we define the *regular* part of the space V by $V^{\text{reg}} := V - \cup_{\mathcal{H}} H$. The vectors that lie in V^{reg} are called *regular vectors* and the connected components of V^{reg} are called *chambers*. Note that the action of $W(R)$ on V preserves V^{reg} and takes a chamber to another chamber.

If $x \in V^{\text{reg}}$ then $(x|\alpha) \neq 0$ for any $\alpha \in R$. This allows us to divide the roots in two sets, called *positive* and *negative* roots, as follows:

$$R^{\pm}(x) = \{\alpha \in R : (x|\alpha) \gtrless 0\}.$$

Observe that $R = R^+ \coprod R^-$, $R^- = -R^+$ and $R^+ \cap R^- = \emptyset$.

Lemma 2.4. For $x, y \in V^{\text{reg}}$, $R^+(x) = R^+(y) \iff x, y$ lie in the same chamber.

If C is a chamber in V then $R^{\pm}(C) := R^{\pm}(x)$ for some $x \in C$. Above lemma says that $R^{\pm}(C)$ is well-defined. We further abbreviate this notation by letting $R^{\pm} := R^{\pm}(C)$.

Now, a root $\beta \in R^+$ is called *decomposable* if $\beta = \beta_1 + \beta_2$ for some $\beta_1, \beta_2 \in R^+$, and a root is called *indecomposable* if it is not decomposable.

We define $B(C)$ to be the set of indecomposable roots in $R^+ = R^+(C)$.

Proposition 2.5. 1. $B(C)$ is a basis for V.
2. $R^+(C) \subseteq \mathbb{Z}_{\geq 0}(B(C))$ and $R^-(C) \subseteq \mathbb{Z}_{\leq 0}(B(C)$.

Proof. We need the notion of *height* for this proof. Fix an element $x_0 \in C$. Then $\alpha \in R^+ \iff (x_0|\alpha) > 0$, so we define $ht(\alpha) := (x_0|\alpha)$. Observe that ht is linear. We use this real number to compare different roots.

By noting that ht attains a minimum over R^+ it follows that $R^+ \subseteq \mathbb{Z}_{\geq 0}(B(C))$.

Further, the Cartan integer $n(\alpha, \beta)$ is always negative for $\alpha, \beta \in B(C)$! Then the linear independence of $B(C)$ follows and so $B(C)$ must be a basis of V. ☐

Lemma 2.6. If V is an inner product space and $S \subseteq V$ such that S lies in some half-space and $(\alpha|\beta) \leq 0$ for all $\alpha, \beta \in S$ then S is linearly independent.

This proof is left as an exception.

3. (07/07/2011)

3.1. **Root system of type A_n.** We now do an explicit example of a general root system.

Consider $\mathbb{R}^n$ with the standard inner product $(\cdot|\cdot)$ and let us fix a basis $\epsilon_i, 1 \leq i \leq n$, of $\mathbb{R}^n$. We define $V = \{\sum c_i\epsilon_i : \sum c_i = 0\} = (1, 1, \ldots, 1)^{\perp}$ and $R := \{\epsilon_i - \epsilon_j : i \neq j\}$. The pair (V, R) is a root system. Observe that $|R| = 2\binom{n}{2}$.

Let $\alpha = \epsilon_i - \epsilon_j$. We observe that the reflection in $\mathbb{R}^n$ wrt the vector α switches ϵ_i and ϵ_j and fixes all other basis elements. So it induces a linear map, denoted by $s_{\epsilon_i - \epsilon_j}$, on V. Further $s_{\epsilon_i - \epsilon_j}(\epsilon_k - \epsilon_l) = \epsilon_{\sigma(k)} - \epsilon_{\sigma(l)}$ where σ is the transposition (i, j).

Exercise 3.1. Verify that the Cartan integers, $n(\alpha, \beta)$, are indeed integers.

We remark here that this root system corresponds to the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ and is called the root system of type A_{n-1}.

For a suitable chamber C we get $R = R^+ \coprod R^-$ where $R^{\pm} = \{\epsilon_i - \epsilon_j : i \lessgtr j\}$.

Exercise 3.2. Find $x \in V^{\text{reg}}$ such that $R^{\pm} = R^{\pm}(x)$.

The basis in this R^+ is $B = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq n - 1\}$ and the Weyl group $W(R)$ is the symmetric group S_n.

3.2. **Action of $W(R)$ on roots.** Let R be a root system of rank l and let $B = \{\alpha_1, \ldots, \alpha_l\}$ be its simple roots wrt a chamber C.

Theorem 3.3. $W = W(R) = \langle s_{\alpha_i} : 1 \leq i \leq l\rangle \subset \mathrm{GL}(V)$.

We need two lemmas for proving this theorem, the first of which is clear and the second one is proved by height reduction. The theorem follows easily from these lemmas.

Lemma 3.4. For all $1 \leq i \leq l$, $s_{\alpha_i}(R^+ - \{\alpha_i\}) = (R^+ - \{\alpha_i\})$.

Lemma 3.5. $\beta \in R^+$ then there is some $w \in \langle s_{\alpha_i} : 1 \leq i \leq n \rangle$ such that $w\beta \in B$.

Corollary 3.6. $\cup W\alpha_i = R$.

The point here is that to describe a root system R completely, it is enough to know the set of simple roots, B, and the Cartan integers $n(\alpha_i, \alpha_j)$ for $\alpha_i, \alpha_j \in B$.

3.3. **Action of $W(R)$ on chambers.** Let $\mathcal{C}$ denote the set of chambers in V.

Theorem 3.7. $W(R)$ acts simply transitively on $\mathcal{C}$.

Let us denote s_{α_i} by s_i now where α_i are simple roots.

For an element $w \in W$ we define the *length* of w, $l(w)$, as the minimal length of an expression for w in terms of s_i. For example, $l(s_i) = 1$.

Lemma 3.8. $l(w)$ is the cardinality of the *inversion* set of w, $I(w) := \{\alpha \in R^+ : w\alpha \in R^-\}$.

The proof is left as an exercise.

Proof of the theorem. Let C, C' be chambers and B be the system of simple roots wrt C. Then $C = \{v \in V : (v|\alpha_i) > 0$ for all $\alpha_i \in B\}$.

Let us fix $x_0 \in C$. Now we choose $v \in C'$. If $(v|\alpha_i) > 0$ for every simple root α_i then $v \in C$. Otherwise $(v|\alpha_i) < 0$ for some simple α_i and then $ht(v) := (v|x_0) < ht(s_{\alpha_i}(v))$. So if we take the element attaining the maximum height in the W-orbit of v, we get an element of C.

That the action of W is simple follows by the above lemma. □

3.4. If (V, R) is a root system and B is a system of simple roots, we form a matrix, called *Cartan matrix*, consisting of the entries $n(\alpha_i, \alpha_j)$ where α_i, α_j are simple roots. This $l \times l$ matrix, where l is the rank of R, is a positive definite matrix that encodes information about the form $(\cdot|\cdot)$ on V. It looks like as follows:

$$\begin{pmatrix} 2 & & & \\ & 2 & \leq 0 & \\ & \leq 0 & \ddots & \\ & & & 2 \end{pmatrix}$$

4. (08/07/2011)

Today we will work towards classification of root systems.

4.1. Gram matrix. Let (V, R) be a reduced root system of rank l and let $(\cdot|\cdot)$ be a positive definite $A(R)$-invariant form on V. Recall $R = R^+ \coprod R^-$ wrt some chamber C and one has $B = B(C) \subset R^+$.

Lemma 4.1. If R is an irreducible root system then the set of simple roots, B, is not a disjoint union of two non-empty subsets B_1 and B_2 such that $(\alpha|\beta) = 0$ for any $\alpha \in B_1, \beta \in B_2$.

Proof is left as an exercise.

Let $B = \{\alpha_1, \ldots, \alpha_l\}$. If $e_i = \frac{\alpha_i}{\|\alpha_i\|}$ then $(e_i|e_j) = \cos\theta_{i,j}$ where $\theta_{i,j}$ is the angle between α_i and α_j. Observe that $\cos\theta_{i,j} = -\cos\frac{\pi}{m_{i,j}}$ where $m_{i,j} \in \{2, 3, 4, 6\}$. Then the *Gram* matrix $G = [(e_i|e_j)] = [-\cos(\pi/m_{i,j})]$ is a positive definite, symmetric matrix with $m_{i,i} = 1$ and for $i \neq j$, $m_{i,j} \in \{2, 3, 4, 6\}$.

Exercise 4.2. Show that if B' is another basis of V then the Gram matrix G' is the same as G upto reordering of indices.

The classification problem can now be rephrased as the following problem.

Problem 4.3. Let $l \geq 1$. Let $m_{i,j} \in \mathbb{N}$ such that $m_{i,i} = 1$, $m_{i,j} \geq 2$ for $i \neq 2$ and $m_{i,j} = m_{j,i}$. For what choices of $m_{i,j}$ is $G = [-\cos(\pi/m_{i,j})]$ positive definite?

As an example, if $l = 2$ we need to choose only $m_{1,2}$. Let $m_{1,2} = p \geq 2$. Then G_p is always positive definite and we get the (general) root system corresponding to dihedral group.

4.2. Coxeter graph. This is a graph labelled by the simple roots α_i and we put an edge between α_i and α_j if (and only if) $m_{i,j} \geq 3$. Further, if $m_{i,j} \geq 4$ then we also write $m_{i,j}$ on the edge.

Fact 4.4. R irreducible if and only if the corresponding Coxeter graph is connected.

Now we have reduced the classification problem to the following problem:

Problem 4.5. Classify all connected, positive definite Coxeter graphs.

We now state three main steps of this classification programme. Let X be a Coxeter graph, G the corresponding matrix $[g_{i,j}]$.

Lemma 4.6. If X is positive definite then there are no cycles in X.

Lemma 4.7. If X is positive definite and i a vertex in X let $N(i) = \{j : j \text{ is connected to } i\}$ then $\sum g_{i,j}^2 < 1$.

Lemma 4.8. a) If X is positive definite then any subgraph Y of X is again positive definite.
b) (Shrinking lemma) If X is positive definite and if there are two subgraphs Y, Z of X such that Y and Z are connected by a simply laced chain, the union of Y, Z and the simply laced chain being the graph X, then the graph $\tilde{X}$ obtained by shrinking the chain to a vertex is also positive definite.

Now we list a number of corollaries.

Corollary 4.9. Let X be a positive definite Coxeter graph.
1. For any vertex i, $\deg(i) \leq 3$.
2. If $\deg(i) = 3$ then $m_{i,j} = 3$ all for j connected to i.
3. If there exist i, j for which $m_{i,j} \geq 6$, then $X = \{i, j\}$ with an edge between i and j labeled by $m_{i,j}$.
4. If there is a vertex i with $\deg(i)$ then all the subgraphs on the three sides of i are simply laced chains.
5. If all the vertices have degree 2 then $m_{i,j} \geq 4$ for at most one $m_{i,j}$.

4.3. The main theorem. Now we completely solve the classification problem.

Theorem 4.10. The connected positive definite graphs of rank l are of the following types:
1. A_l: a simply laced chain, no labels, $m_{i,i+1} = 3$, $l \geq 1$
2. B_l: simply laced chain with last label 4, $l \geq 2$
3. D_l: a forked chain, no labels, $l \geq 4$
4. F_4: $m_{1,2} = 3, m_{2,3} = 4, m_{3,4} = 3$.
5. G_2: $m_{1,2} = 6$.
6. E_6:
7. E_7:
8. E_8:
9. H_3: $m_{1,2} = 3, m_{2,3} = 5$
10. H_4: $m_{1,2} = m_{2,3} = 3, m_{3,4} = 5$.
11. $I_2(p)$: $m1, 2 = p$.

4.4. Dynkin diagram of R. This is obtained from the Coxeter graph by indicating the longer root among α_i, α_j whenever $m_{i,j} = 4$ or 6.

Theorem 4.11. The Dynkin diagram of an irreducible root systems is of the following type.

1. A_l:
2. B_l: last root smallest
3. C_l: last root longest
4. D_l:
5. G_2, F_4, E_6, E_7, E_8

Theorem 4.12. Each of the Dynkin diagrams listed above is the Dynkind diagram of a reduced irreducible root system.

One has the following correspondence for the first four Dynkin diagrams:

$$A_l \longleftrightarrow \mathfrak{sl}_{l+1}, B_l \longleftrightarrow \mathfrak{so}_{2l+1}, C_l \longleftrightarrow \mathfrak{sp}_l, D_l \longleftrightarrow \mathfrak{so}_{2l}.$$