Homework 8. March 21, 2017.

1. Compute the Chern number of the universal line bundle L on $\mathbb{CP}^1$. The Chern number is the integer $\int_{\mathbb{CP}^1} c_1(L)$.
   SUGGESTED APPROACH:

   - In our case the line bundle L is itself a complex manifold and the projection $\pi : L \to \mathbb{P}^1$ is holomorphic. Such a complex line bundle is called a *holomorphic line bundle*. A section $\sigma : \mathbb{P}^1 \to L$ is holomorphic, if it is a holomorphic map of complex manifolds.
   - The Chern number of the trivial line bundle is zero. Since $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ for line bundles L_1, L_2, we can say that $c_1(L) = -c_1(L^*)$. To solve our problem, we will find the Chern number of $c_1(L^*)$ by looking at the number of zeros of a holomorphic section of L^*.
   - A section $\sigma : \mathbb{P}^1 \to L^*$ is a linear functional on each fiber L_x ($x \in \mathbb{P}^1$) of L. A section $\sigma : \mathbb{P}^1 \to L^*$ is holomorphic, if for every holomorphic section $f : \mathbb{P}^1 \to L$, the pairing $\sigma(f) : \mathbb{P}^1 \to \mathbb{C}$ is holomorphic. To solve the problem, produce a holomorphic section of L^*. Hint: Use co-ordinate functions on $\mathbb{C}^2$, and lift them to L via the blow-down map $L \to \mathbb{C}^2$.

2. Show that the Chern number of $T\mathbb{P}^1 \to \mathbb{P}^1$ is 2 using one of the two suggested approaches, or some other method you like.

   (a) Find a vector field s on $\mathbb{P}^1$ which vanishes at two points, the zeros are transverse intersections of s with the zero section of $T\mathbb{P}^1$, and the index of each zero point is 1.

   (b) Produce a meromorphic 1-form on $\mathbb{P}^1$, and use $c_1(T^*\mathbb{P}^1) = -c_1(T\mathbb{P}^1)$. An additional result is required, which I state here.
   Suppose $\sigma : \mathbb{P}^1 \to L$ is a meromorphic section of a holomorphic line bundle L, then

   $$c_1(L) = \text{number of zeros of } \sigma - \text{number of poles of } \sigma$$

   counted with multiplicity.

3. Consider the map

   $$u : \mathbb{P}^1 \to \mathbb{P}^n, \quad [x : y] \mapsto [x : y : 0 : \cdots : 0].$$

   Show that $c_1(u^*T\mathbb{P}^n) = n + 1$.
   HINT: $u^*T\mathbb{P}^n$ decomposes into n holomorphic line bundles, one of which is image$(du) \simeq T\mathbb{P}^1$. Use the result of Problem 2 for this line bundle. For the others, produce holomorphic sections, with one zero.