Homework 6. March 2, 2017.

1. Suppose M is a complex manifold. In class we showed if $\omega \in \Omega^2(M, \mathbb{C})$ is a Kähler form, then,

   - ω is in $\Omega^{1,1}(M)$,
   - and in a local chart $(U, (z_1, \dots, z_n))$ if we write $\omega = i \sum_{j,k} h_{jk}(z)$, the matrix $(h_{jk}(p))$ is Hermitian at all points $p \in U$.

   Use ω-tameness of the complex structure to show that the matrix is positive definite.

2. In this problem we will finish the proof of the fact that

$$\Phi : \mathbb{C}^n \to \mathfrak{u}(n)^\vee, \quad z \mapsto \frac{i}{2} z z^*$$

   is a moment map for the standard action of $U(n)$ on $\mathbb{C}^n$. Here $\mathfrak{u}(n)$ is identified to its dual via the Ad-invariant inner product on $\mathfrak{u}(n)$: $(A, B) = \operatorname{tr}(A^* B)$. In class we showed that when the action is restricted to the subgroup T of diagonal matrices in $U(n)$, Φ projects to a moment map for the action. Use $U(n)$-equivariance to finish the proof. You will need the fact that $\operatorname{Ad}_{U(n)} \mathfrak{t} = \mathfrak{u}(n)$.

3. Suppose (M, ω) is an exact symplectic manifold, i.e. $\omega = d\lambda$ is exact. Suppose a Lie group G acts on M and λ is invariant under the G-action. Show that the action is Hamiltonian with moment map given by $\Phi_\xi(p) = \lambda(\xi_M)$, where $p \in M$ and $\xi \in \mathfrak{g}$.

4. Problems 6,7 in Homework 12 of Cannas da Silva on Fubini-Study form.